4 Quantum Coordinate Algebras

In this lecture we will introduce the Hopf algebra $\mathbf{C}_q[SL_2]$, which is a direct generalisation of the classical coordinate algebra of the Lie group SL_2. We construct $\mathbf{C}_q[SL_2]$ as a distinguished subalgebra of the vector space dual of $U_q(\mathfrak{sl}_2)$, and in so doing give a natural presentation of the well-known pairing between the two algebras. This important pairing generalises the classical pairing between the universal enveloping algebra of $\mathfrak{sl}_2$ and the coordinate algebra of SL_2.

4.1 Coordinate Algebras as Hopf Algebras

As we saw in the first lecture, for any Lie algebra $\mathfrak{g}$, its enveloping algebra $U(\mathfrak{g})$ has a natural Hopf algebra structure. As we will now see, the same is also true for the coordinate algebra of any algebraic group.

Let G be a variety in $\mathbf{C}^n$, with a given algebraic group structure. We denote by $\mathbf{C}[G]$ the subalgebra of the complex-valued function algebra of G generated by the canonical coordinate functions of $\mathbf{C}^n$ restricted to G. We will refer to $C[G]$ as the *coordinate algebra* of G. Now for $f \in C[G]$, consider the map

$$\Delta(f) : G \otimes G \to G, \qquad\qquad g \otimes h \mapsto f(gh).$$

As a little thought will confirm, the fact that G is an algebraic group implies that $\Delta(f)$ can be considered as an element of $C[G] \otimes C[G]$. Thus, the multiplication of G induces a well-defined map

$$\Delta : C[G] \to C[G] \otimes C[G].$$

The identity of G by $\mathbf{1}$ also induces a canonical map, which is given by

$$\varepsilon : C[G] \to \mathbf{C}, \qquad\qquad f \mapsto f(\mathbf{1}).$$

Finally, the existence of inverses in G, gives us a third canonical map $S : C[G] \to C[G]$ defined by

$$S(f)(g) = f(g^{-1}).$$

Again, S being well-defined is guaranteed by the fact that G is an algebraic group.

A natural question to ask is if there exist some type quantum deformations of these Hopf algebra structures for the compact semi-simple Lie groups. More specifically, one might ask if $U_q(\mathfrak{sl}_2)$ can be used to find a deformation of $\mathbf{C}_q[SL_2]$. To answer this question we will need to introduce some new general structures.

4.2 The Hopf Dual of a Hopf Algebra

In this section we present some basic facts about dual spaces and coalgebras. For sake of clarity, let us first recall some standard notation: For V, W two vector spaces, we write their vector space duals by V^* and W^* respectively. Moreover, for $L : V \to W$, a linear map, we denote by $L^* : W^* \to V^*$ the linear map determined by

$$L^*(f) = f \circ L, \qquad (f \in V^*).$$

Now if C is a coalgebra, then C^* has the structure of an algebra with multiplication given by the restriction of $\Delta^* : (C \otimes C)^* \to C^*$ to a map from $C^* \otimes C^*$ to C^*, and unit given by ε^*. The multiplication in C^* is called the *convolution* of C^*, and one usually denotes $f * f' := \Delta^*(f \otimes f')$, for $f, f' \in C^*$. Explicitly, $f * f'$ acts according to

$$(f * f')(c) = f(c_{(1)})f'(c_{(2)}), \qquad (c \in C).$$

This offers some motivation for the term *co*algebra.

The obvious dual version of this construction fails, which is to say, the dual of an algebra does not have an automatic coalgebra structure. To see why this is so, first note that if m is the multiplication of A, then the dual of m has domain and codomain as

$$m^* : A^* \to (A \otimes A)^*.$$

Now when A is infinite dimensional, $A^* \otimes A^*$ is a proper subset of $(A \otimes A)^*$, and we have no guarantee that the image of m^* will lie in $A^* \otimes A^*$. We remedy this situation by defining the *finite dual* of an algebra A to be

$$A^o := \{f \in A^* \mid f(I) = 0, \text{ for some ideal } I \text{ of } A \text{ with } \dim_{\mathbf{C}}(A/I) < \infty\}.$$

This definition is motivated by the following result:

Lemma 4.1 *If A is an algebra with multiplication denoted by m and unit 1, then A^o is a coalgebra with $\Delta = m^*$, and $\varepsilon = 1^*$. If A is commutative, then A^o is cocommutative.*

4.3 Quantum Coordinate Functions

We are now ready to take our first look at the quantum analogue of coordinate functions:

Definition 4.2. Let M be a left-module over a Hopf algebra H. For any $f \in M^*$, define the *coordinate function* $c^M_{f,v}$ by the rule

$$c^M_{f,v}(h) = f(h.v), \qquad\qquad (h \in H).$$

The *coordinate space* of M is the subspace of M^* given by

$$C(M) := \{c^M_{f,v} \mid f \in M^*, v \in M\}.$$

We see that if M is finite dimensional with a basis $\{e_i\}_{i=1}^n$, then a basis of $C(M)$ is given by

$$\{c^M_{\widehat{e}_i,e_j} \mid i,j = 1, \cdots, n\},$$

where $\{\widehat{e}_i\}_{i=1}^n$ is the dual basis of M^*.
We leave the proof of the following important lemma as an instructive exercise.

Lemma 4.3 *Let M and N be finite dimensional modules over a Hopf algebra H. Let $f \in M^*, g \in N^*$, and $w \in N$. Then*

$$c^M_{f,v} + c^N_{g,w} = c^{M\oplus N}_{(f,g),(v,w)}, \qquad c^M_{f,v}c^N_{g,w} = c^{M\otimes N}_{f\otimes g,v\otimes w},$$
$$\varepsilon(c^M_{f,v}) = f(v), \qquad S(c^M_{f,v}) = c^{M^*}_{v,f}.$$

If $\{e_i\}_i$ and $\{\widehat{e}_i\}_i$ are dual basis for M and M^*, then

$$\Delta(c^M_{f,v}) = \sum_i c^M_{f,v_i} \otimes c^M_{f_i,v}.$$

This result tells us that for any Hopf algebra H, the subset of H^o containing the coordinate functions of the finite dimensional representations of H is a Hopf subalgebra of H^o. In fact as the following result tells us, the two algebras are equal:

Proposition 4.4 *For any Hopf algebra H, its Hopf dual is equal to its Hopf algebra of coordinate functions.*

Our motivation for considering coordinate functions is given by the following classical result (which we have expressed in a form suited to our needs):

Theorem 4.5 (Peter–Weyl) *For a simply-connected semi-simple complex algebraic group G, an isomorphism of Hopf algebras is given by*

$$\mathbf{C}[G] \simeq \bigoplus_{\mathcal{C}} \mathcal{C}(V),$$

where, for $U(\mathfrak{g})$ the universal enveloping algebra of the Lie algebra of G, we have used $\mathcal{C}$ to denote the family of finite-dimensional irreducible representations of $U(\mathfrak{g})$.

4.4 The Hopf Algebra $\mathbf{C}_q[SL(2)]$

We will now return to the specific case of $U_q(\mathfrak{sl}_2)$, and use the general theory outlined above to find a Hopf algebra deformation of the coordinate algebra of SL_2. Motivated by the Peter–Weyl theorem we make the following definition.

Definition 4.6. The Hopf algebra $\mathbf{C}_q[SL_2]$ is defined by

$$\mathbf{C}_q[SL_2] := \bigoplus_{l \in \frac{1}{2}\mathbf{N}_0} \mathcal{C}(T_{1,l}).$$

The following proposition gives us some basic facts about $\mathbf{C}_q[SL_2]$, as well as a workable presentation of the algebra. We will not prove the result here, postponing this to the general Drinfeld–Jimbo case discussed in the next lecture.

Proposition 4.7 *The Hopf algebra $\mathbf{C}_q[SL_2]$ has the following properties:*

1. *It is generated as an algebra by the coordinate functions of the representation* $T_{1,\frac{1}{2}}$.

2. *Denoting by* u^i_j, *for* $i,j = 1,2$ *the choice of basis of* $C(T_{1,\frac{1}{2}})$ *implied by the choice basis of* $T_{1,\frac{1}{2}}$ *given in Definition 3.1, we have that*

$$\begin{aligned} u^1_1u^1_2 &= qu^1_2u^1_1, & u^1_1u^2_1 &= qu^2_1u^1_1, & u^1_1u^2_2 &= u^2_2u^1_1 + (q-q^{-1})u^1_2u^2_1, \\ u^1_2u^2_2 &= qu^2_2u^1_2, & u^2_1u^2_2 &= qu^2_2u^2_1, & u^1_2u^2_1 &= u^2_1u^1_2. \end{aligned}$$

Moreover, these relations completely determine $\mathbf{C}_q[SL_2]$.

3. *The Hopf algebra structure of* $\mathbf{C}_q[SL_2]$ *is determined by:*

$$\Delta(u^i_j) = u^i_1 \otimes u^1_j + u^i_2 \otimes u^2_j, \qquad S(u^i_j) = -q^{i-j}u^i_h, \qquad \varepsilon(u^i_j) = \delta_{ij}.$$

4.5 Hopf Algebra Dual Pairings

We finish the lecture by introducing a concept that is central to the theory of Hopf algebras . Moreover, we will see that it generalises a very familiar concept from classical Lie theory.

Definition 4.8. A *dual pairing* of two Hopf algebras s G and H is a bilinear map $(\cdot,\cdot) : G \times H \to \mathbf{C}$ such that for all $g_1, g_2 \in G$, and $h_1, h_2 \in H$, it holds that

$$\begin{aligned} (h, gg') &= (h_{(1)}, g)(h_{(2)}, g'), \\ (hh', g) &= (h, g_{(1)})(h', g_{(2)}), \\ (h, 1_G) &= \varepsilon_H(h), \quad (1_H, g) = \varepsilon_G(g), \\ (h, S_G g) &= (S_H h, g) \end{aligned}$$

In fact, it can be shown that the fourth condition follows from the other three

A very important, and useful, fact about dually paired Hopf algebras is given in the following result, whose proof is left as an exercise.

A *(right) comodule* of H is a pair (V, Δ_R), where V is a vector space, and

$$\Delta_R : V \to V \otimes H,$$

is a linear map for which it holds that

$$(\mathrm{id} \otimes \Delta) \circ \Delta_R = (\Delta_R \otimes \mathrm{id}) \circ \Delta_R, \qquad (\mathrm{id} \otimes \varepsilon) \circ \Delta_R = \mathrm{id}.$$

(A left comodule is defined analogously.) In an extension of Sweedler notation, we will usually denote

$$\Delta_R(v) = \sum_i v_i \otimes h_i =: v_{(0)} \otimes v_{(1)}.$$

Lemma 4.9 *Let H and G be a dually pair of Hopf algebras with dual pairing $\langle \cdot, \cdot \rangle$. It holds that every right (left) comodule of H induces a right (left) module of G. Explicitly, if V is a right H-comodule with coaction Δ_R, then the corresponding action of G on V is given by*

$$V \times G \to V, \qquad (v, g) \mapsto v_{(0)} \left\langle S(g), v_{(1)} \right\rangle.$$

As one might have guessed, the two Hopf algebras $U_q(\mathfrak{sl}_2)$ and $\mathbf{C}_q[SL_2]$ are dually paired. This generalises the classical pairing between the Lie algebra $\mathfrak{sl}_2$ and the coordinated algebra of SL_2

Proposition 4.10 *A dual pairing for the Hopf algebras $U_q(\mathfrak{sl}_2)$ and $\mathbf{C}_q[SL_2]$ is given by*

$$\langle \cdot, \cdot \rangle : U_q(\mathfrak{sl}_2) \times \mathbf{C}_q[SL_2] \to \mathbf{C}, \qquad (X, f) \to f(X).$$

Moreover, this pairing is non-degenerate in the standard sense.

References

[1] A. Klimyk, K. Schmüdgen, *Quantum Groups and their Representations*, Springer Verlag, Heidelberg–New York, 1997