$$\varphi([X\otimes f, Y\otimes g]) = \varphi([X,Y]\otimes fg) = \left([X,Y]\otimes \tau_1(fg), \dots, [X,Y]\otimes \tau_n(fg)\right)$$ 11

$$\overset{?}{\parallel}$$

$$[\varphi(X\otimes f), \varphi(Y\otimes g)] = [(X\otimes \tau_1 f, \dots, X\otimes \tau_n f), (Y\otimes \tau_1 g, \dots, Y\otimes \tau_n g)] =$$

$$= \left([X,Y]\otimes \tau_1 f\,\tau_1 g, \dots, [X,Y]\otimes \tau_n f\,\tau_n g\right) - \sum_{i=1}^{n} K_0(X,Y)\,\mathrm{Res}_{t_i=0}\left(\tau_i f\,(\tau_i g)'\right)$$

BUT $\sum_{i=1}^{n} \mathrm{Res}_{t_i=0}\left(\tau_i f\cdot(\tau_i g)'\right) = 0$ on $\mathbb{C}P^1$!
by the Residue thm.

$k\in\mathbb{N}$, $\lambda_1,\dots,\lambda_n\in P_k$ $\Rightarrow$ H_{k,λ_i} $\hat{\mathfrak{g}}$-mod

$\mathfrak{g}(x_1,\dots,x_n)$ acts on $\bigotimes_{i=1}^{n} H_{k,\lambda_i}$ by the embedding $\mathfrak{g}(\lambda_1\dots\lambda_n)\hookrightarrow \bigoplus_{i=1}^{n}\mathfrak{g}((t_i)) \oplus \mathbb{C}$

[def] The space of conformal blocks $\mathcal{H}(x_1,\dots,x_n,\lambda_1,\dots,\lambda_n)$ is the space of linear forms

$$\varphi: \bigotimes_{i=1}^{n} H_{k,\lambda_i} \longrightarrow \mathbb{C}$$

invariant w. respect to $\mathfrak{g}(x_1,\dots,x_n)$: $\varphi(\gamma\cdot v)=0$ $\forall \gamma\in\mathfrak{g}(x_1,\dots x_n)$
$\forall v\in\bigotimes_{i=1}^{n} H_{k,\lambda_i}$

next time: • $\mathcal{H}(\dots)$ is finite dimensional

• Verlinde formula $\Rightarrow$ dim of $\mathcal{H}(\dots)$

Segal – Sugawara construction

$\{J^a\}_{a=1}^{d}$, $d=\dim\mathfrak{g}$ a basis of $\mathfrak{g}$

$\{J_a\}$ dual basis w.r.t. K_0

$C=\frac{1}{2}\sum_{a=1}^{d} J^a J_a\in U(\mathfrak{g})$ the Casimir

notation: $X\in\mathfrak{g}$ $\quad X_n := X\otimes t^n$, $n\in\mathbb{Z}$

or any smooth $\hat{\mathfrak{g}}$-module

$$X(z) = \sum_{n\in\mathbb{Z}} X_n z^{-n-1} \in \mathrm{End}(H_{\ell,\lambda})[[z,z^{-1}]]$$

called field

(!) $\forall v \in H_{\ell,\lambda} \quad \exists n_0 \in \mathbb{Z} \quad \forall n \geq n_0 \quad X_n v = 0$

(because $H_{\ell,\lambda} \cong \mathcal{U}(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \underset{\hat{\mathfrak{g}}}{\otimes} V_{\ell,\lambda}$)

(*) $\boxed{A_n v = 0 \ \forall n \ll 0}$

normal ordering:

$A(z), B(z)$ ... fields

$$:A(z)B(z): \; = A(z)_+ B(z) + B(z) A(z)_-$$

$$A(z)_+ = \sum_{n\geq 0} A_n z^n$$

$$A(z)_- = \sum_{n<0} A_n z^n$$

The condition (!) guarantees that the products of power series are well defined:

$$:A(z)B(z): \; = \sum_{n\geq 0}\sum_m A_n B_m z^{m+n} + \sum_m \sum_{n<0} B_m A_n z^{m+n}$$

$$= \sum_k \Big(\underbrace{\sum_{n\geq 0} A_n B_{k-n} + \sum_{n<0} B_n A_{k-n}}_{C_k} \Big) z^k$$

evaluate on v: $\quad C_k v = \sum_{n\geq 0} A_n B_{k-n} v + \sum_{n<0} B_n A_{k-n} v$

finite sums due to (!) or (*)

Sugawara operator

$$S(z) = \frac{1}{2(\ell+h^\vee)} \sum_{a=1}^{d} :J^a(z) J_a(z): \; = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad L_n \in \mathrm{End}(H_{\ell,\lambda})$$

the dual Coxeter

$$S(z) = \frac{1}{2(\ell+h^\vee)} \sum_{n\in\mathbb{Z}} \underbrace{\sum_{a=1}^{d} \Big(\sum_{k<0} J^a_k J_{a,n-k} + \sum_{k\geq 0} \underbrace{J_{a,n-k} J^a_k}_{\text{commute!}} \Big)}_{L_n} z^{-n-2}$$

$$n\neq 0 \quad [J_{a,n-k}, J^a_k] = [J_a, J^a]_n + (n-k) K_0 (J_a\, J^a)\, \underbrace{\delta_{n-k,-k}}_{0 \Leftarrow n\neq 0} = 0$$

$$\Rightarrow L_n = \frac{1}{2(\ell+h^\vee)} \sum_{a=1}^{d} \Big(\sum_{k\in\mathbb{Z}} J^a_k J_{a,n-k} \Big)$$

$$n=0 \quad L_0 = \frac{1}{2(\ell+h^\vee)} \sum_{a=1}^{d} \Big(\sum_{k<0} J^a_k J_{a,-k} + \sum_{k\geq 0} J_{a,-k} J^a_{-k} \Big)$$

compare to: $\displaystyle \frac{1}{2(l+h^\vee)}\sum_{a=1}^{d}\Big(\underbrace{\sum_{k\in\mathbb{Z}} J^a_k J_{a,-k}}_{\text{naive definition}} + \sum_{k\geq 0} kc\Big)$ (13)

$\boxed{\text{Thm}}$ $[L_n, X_m] = -m X_{n+m}$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c_g(l)}{12}(n^3-n)\delta_{n,-m}$$

where $c_g(l) = \dfrac{l \dim g}{2(l+h)}$ ... the <u>central charge</u>.

$\boxed{\text{Lemma}}$ $\displaystyle \sum_{a=1}^{d}\left([X,J^a]_m J_{a,n} + J_{a,m}[X,J_a]_n\right) = 0$, $X\in g$

Pf: $\displaystyle \sum_a [X,J^a]_m J_{a,m} = \sum_a\sum_{b=1}^{d} K_0([X,J^a],J^b) J_{b,m} J_{a,n}$

$\displaystyle [X,J^a] = \sum_b K_0([X,J^a],J^b) J_b$

$\displaystyle = -\sum_a\sum_b K_0(J^a,[X,J^b]) J_{b,m} J_{a,n} = -\sum_b J_{b,m}[X,J^b]_n$ $\boxed{\text{Ch}}$

Pf of the Thm.:

$$[X_m, 2(l+h^\vee)L_n] = \sum_a\Big(\underbrace{\Big(\sum_{k<0}[X_m,J^a_k]J_{a,n-k} + \sum_{k<0}J^a_k[X_m,J_{a,n-k}]\Big)}_{1} + \underbrace{\sum_{k\geq 0}(\ldots)}_{2}\Big)$$

$$1 = \ldots = \sum_a\sum_{-m\leq k<0} J^a_{k+m}[X,J_a]_{n-k} + \sum_{k<0}(mX_{n-k}\delta_{m,-k} + mX_k\delta_{m,-n+k})c$$

$$2 = \ldots = \sum_a\sum_{m>k\geq 0}[X,J_a]_{n+m-k}J^a_k + \sum_{k\geq 0}(mX_k\delta_{m,-n+k} + mX_{n-k}\delta_{m,k})c$$

$1 + 2$ by the lemma

$$= 2mlX_{n+m} + m\sum_{a=1}^{d}[J^a,[J_a,X]]_{n+m} = \overset{2mlX_{n+m}+}{\operatorname{ad}(C)}X_{n+m} = 2mh^\vee X_{n+m} + 2mlX_{n+m} = 2m(l+h^\vee)X_{n+m}$$

the <u>Casimir</u> C acts on irred. g mult. by a number

$(\lambda,\lambda)+2(\lambda,\rho)$

$\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ $\boxed{\text{Ch}}$