Charles University Prague Faculty of Mathematics and Physics

# Properties of invariant differential operators 

PhD. thesis

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## CHAPTER 1

## Introduction

The topic of the presented dissertation belongs to global analysis on manifolds, the attention is concentrated to a study of properties of differential operators acting between various vector bundles over a given manifold. The class of differential operators considered here is, however, very much restricted by the requirement that the operators are invariant with respect to a big group of symmetries. This is possible because we consider operators acting on manifolds, which are homogeneous spaces of the form $G / H$, where $G$ is a semisimple Lie group and $H$ is its subgroup. Operators are acting on homogeneous bundles, which are given by a choice of an $H$-module $\mathbb{V}$ and are defined as associated bundles $V=G \times_{H} \mathbb{V}$. Sections of these bundles are $G$-modules (sometimes called the induced representations) and a differential operator $D$ acting between two such homogeneous bundles is called invariant (homogeneous), if it intertwines the action of the group $G$.

There are two different types of symmetries considered. The first type is formed by the so called parabolic geometries. Homogeneous models for parabolic geometries are given by homogeneous spaces $M=$ $G / P$, where $G$ is a real semisimple Lie group and $P$ is its parabolic subgroup. Usual notation for such homogeneous space is the Satake diagram for $G$ together with some crosses on white nodes of the diagram, specifying the parabolic subgroup $P$. The key role in parabolic geometries is played by the Cartan connection. On the homogeneous model $M=G / P$, the Cartan connection is just the Maurer-Cartan form $\omega \in \Omega(G, \mathfrak{g})$. Following ideas of Élie Cartan going back to the beginning of the last century, a parabolic geometry of type $(G, P)$ is a couple $(\mathcal{G}, \omega)$, where $\mathcal{G}$ is a $P$-principal bundle over $M$, dimension of $M$ is equal to dimension of $G / P$ and $\omega \in \Omega(\mathcal{G}, \mathfrak{g})$ is the so called Cartan connection (its properties imitate properties of the Maurer-Cartan form). Invariant differential operators on homogeneous models of parabolic geometries were extensively studied in past fifty years and a huge amount of knowledge were accumulated during this period (for references, see for example [CD01, RCD03, CSS01, Die99, GJMS92, HS01, Rei01, BasE89, CSS97a, CSS97b, CSS00c, SloPG]). Curved version of parabolic geometries describes quite a few basic geometric structure, including projective, conformal, quaternionic, CR, projective contact, quaternionic contact and path geometries. This field is at present a very active field of research with a broad international cooperation.

Invariant differential operators on homogeneous models of parabolic geometries can be equivalently reformulated as homomorphisms of the so called generalized Verma modules. Their study were iniciated in 70's by J. Bernstein, I. M. Gelfand and S. I. Gelfand (for more detailed information, see e.g. [Die99]). This belongs to a broad field of classical (infinite dimensional) representation theory which was developed after the second world war (mainly under the influence of I. M. Gelfand and Harish-Chandra) into a beautiful and extended field with many remarkable results. A similar formulation for invariant differential operators on curved manifolds with a given parabolic structure was developed by M. Eastwood and J. Slovák (see [EaSl97]). Invariant differential operators on manifolds with a given parabolic structure are rare beings, it is difficult to construct them, they are typically coming in discrete series (the structure of this series is described by the so called Hasse diagrams).

The second type of invariant differential operators considered in the dissertation consists of invariant differential operators on symmetric spaces. Symmetric spaces themselves are homogeneous spaces of the form $M=G / H$, where $H$ is the set of fixed points of a suitable Cartan involution. Invariant differential operators on symmetric spaces acting on functions (densities) were intensively studied in representation theory and in harmonic analysis (for citations, see for example [KR98, KR00, Oer00, Ped99, GoodW98]). There is also a growing amount of information on the vector bundles cases.

There are two main new results in the dissertation. The first one is the computation of the complete form of the Hasse diagram and the BGG diagram for the parabolic geometry for the orthogonal group in even dimension, the parabolic subgroup is given by the second crossed node in the Satake diagram notation. This parabolic geometry has not yet been studied in many details. Its importance is related to interesting open problems in Clifford analysis. The Clifford analysis itself is a study of solutions of the Dirac equation on Euclidean space and it is a nice generalization of complex analysis to higher dimensions. An analogue of CR equation for several complex variables consists of a set of several Dirac equations defined on a Cartesian product of several copies of the Euclidean space. Holomorphic functions of several complex variables are elements of the kernel of the $\partial$ operator, which is overdetermined. The operator $\partial$ is hence the first operator in an exact sequence, which is called the Dolbeault complex. Important properties of holomorphic functions of several complex variables are consequences of this fact (e.g.,the Hartogs phenomenon). Similarly, the set of several Dirac operators is an overdetermined system of equation and an analogue of the Dolbeault sequence in this case is already described (in particular, this is true in dimension four, see [Bas92]). In dimensions bigger than four, the appropriate symmetry of the set of several Dirac equations is
exactly corresponding to the parabolic geometry mentioned at the beginning of this paragraph. This is why the analogue of the Dolbeault sequence should be related to the BGG sequences for this parabolic geometries. The corresponding sequence has a singular infinitesimal character, hence not too much is known for this case. The study of the full BGG diagram for a general regular case makes it possible to deduce the form of it in various singular characters. This application of the result of the dissertation is discussed in a paper by L. Krump [Kru04]. In the complex case, the form of the BGG diagram is independent of a choice of $G$-module, which specifies the corresponding BGG sequence of invariant operators and it is described by the corresponding Hasse diagram. Situation is more complicated in the real case. Here, depending on the chosen $G$-module, the form of the BGG diagram may change, some vertices of the graph can be glued together. The form of the BGG diagram in the studied parabolic geometries is described in all allowed real forms.

Further new results of the dissertation are contained in the last chapter on Poisson transform. Its role is to connect invariant differential operators in both types of homogeneous spaces described above. In general, homogeneous models for parabolic geometries are contained in the boundary (sometimes even equal to the boundary) of an appropriate symmetric space. Typical examples studied here in detail is the case of the hyperbolic space (realized as a ball in $\mathbb{R}^{n}$ ) and the conformal sphere, which is its boundary. In work of Koranyi, Reimann and Ørsted (see [KR98, KR00, Oer00]), the relation between invariant differential operators on the (part of) boundary and inside the symmetric space is studied and described for first order operators. The problem solved in the disseration is to desribe similar relations for higher order operators.

To construct invariant operators on parabolic geometries is not an easy task. The classification and construction of first order operators on homogeneous models for parabolic geometries is well understood (see [Oer00, SS99]) and was used in the description of relations to invariant operators inside the symmetric space. For higher order operators, we are studying this relation for a broad class of invariant differential operators on homogeneous models for parabolic geometries, constructed in [RCD03]. First order invariant operators have a simple form, but higher order ones are very complicated with a lot of lower order correction terms. Fortunately, the formulae for correction terms are uniform, independent of the parabolic geometry or the $G$-module inducing the BGG sequence. They are given by suitable projections of expressions below (hence the form depends only on the order)

$$
\begin{aligned}
& \mathcal{D}_{1} s=\nabla s \\
& \mathcal{D}_{2} s=\nabla^{2} s+r^{\nabla} s
\end{aligned}
$$

$$
\begin{aligned}
& \mathcal{D}_{3} s=\nabla^{3} s+2 \nabla\left(r^{\nabla} s\right)+2 r^{\nabla} \nabla^{2} s \\
& \mathcal{D}_{4} s=\nabla^{4} s+3 \nabla^{2}\left(r^{\nabla} s\right)+4 \nabla\left(r^{\nabla} \nabla s\right)+r^{\nabla} \nabla^{2} s+9\left(r^{\nabla}\right)^{2} s
\end{aligned}
$$

Here, $\nabla$ denotes a suitable connection (a Levi-Civita connection for conformal geometry, for example).

It seems hence to be a difficult task to show a connection to such operators to invariant operators inside the symmetric spaces for higher order operators. The corresponding result is proved in the last chapter of the dissertation. Both types of operators are related using the Poisson transform.

Invariant operators on parabolic geometries are coming in whole sequences (the so called BGG sequences). A natural question is whether a sequence of invariant operators inside the symmetric space can be found in such a way that the Poisson transform intertwines both of them. It is shown in the last chapter that for the case of real hyperbolic geometry (with the conformal sphere as a boundary), it is true 'locally'. It means that we can do that after dividing the BGG sequence on the boundary into several pieces.

The structure of the thesis is the following. In the first chapter we define the directed graph of a Weyl group and the Hasse diagram of a pair ( $\mathfrak{g}, \mathfrak{p}$ ), where $\mathfrak{g}$ is semisimple Lie group and $\mathfrak{p}$ its parabolic subgroup. We prove some properties of these diagrams, above all the Cap criterion that allows to construct the Hasse diagram inductively. Then we define the BGG diagram as the Hasse diagram with labels over vertices.

The second chapter deals with invariant differential operators in parabolic geometries and symmetric spaces. We define the invariant derivative and following [CSS01, RCD03, SS99] we define the strongly invariant operators in parabolic geometries. We recall a theorem from [RCD03] that leads to an explicit construction of many of these operators in terms of a Weyl connection and the corresponding Ricci curvature.

Real forms of simple Lie algebras and their real representations are discussed in the third part. The last two chapters of the dissertation contain then new results, described above.

## CHAPTER 2

## Hasse diagrams

The Bruhat ordering of the Weyl group $W_{\mathfrak{g}}$ of a semisimple Lie algebra $\mathfrak{g}$ allows one to represent $W_{\mathfrak{g}}$ in the form of a directed graph. A subgraph of $W_{\mathfrak{g}}$ of elements whose affine action turns a $\mathfrak{g}$-dominant weight into a $\mathfrak{p}$-dominant one is called the Hasse diagram. In this section we shall say more about how to define and construct Hasse diagrams. The references for this chapter are [KS03, SloPG, GoodW98, BasE89].

## 1. Preliminaries

Definition 1. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ its Cartan subalgebra, $\Delta$ its root system, $\Delta^{ \pm}$the subset of positive and negative roots, respectively, $\Pi$ the set of simple roots.

For $\lambda \in \mathfrak{h}^{*}, \alpha \in \Delta$, the reflection of $\lambda$ with respect to $\alpha$ is given by

$$
\begin{equation*}
\sigma_{\alpha} \lambda:=\lambda-\left(\lambda, \alpha^{\vee}\right) \alpha \tag{1}
\end{equation*}
$$

where $\alpha^{\vee}:=2 \alpha /(\alpha, \alpha)$ is the co-root of $\alpha$ and $(\cdot, \cdot)$ is a positive definite bilinear form on $\mathfrak{h}_{\mathbb{R}}$, i.e. a multiple of the Killing form. The Weyl group $W_{\mathfrak{g}}$ of a semisimple Lie algebra is a group of isometries of $\mathfrak{h}^{*}$ generated by reflections with respect to simple roots. The length $|w|$ of $w \in W_{\mathfrak{g}}$ is the minimal $k$ such that $w=\sigma_{1} \ldots \sigma_{k}$, where $\sigma_{i}$ are simple reflections. Such an expression is called the reduced expression. We shall define a structure of a directed graph on $W_{\mathfrak{g}}$ as follows: there is an arrow $w \rightarrow w^{\prime}$ between $w, w^{\prime} \in W_{\mathfrak{g}}$ if and only if $\left|w^{\prime}\right|=|w|+1$ and there is an $\alpha \in \Delta^{+}$such that $w^{\prime}=\sigma_{\alpha} w$. The Bruhat ordering is then given by the requirement that $w_{1} \preceq w_{2}$ if and only if there is a directed path from $w_{1}$ to $w_{2}$.

The structure of the directed graph can be calculated directly but a more comfortable way is to calculate it as an orbit of any regular weight (i.e. not fixed by any $w \in W_{\mathfrak{g}}$ or equivalently not lying on a wall of a Weyl chamber). The elements of the orbit are in one-toone correspondence with elements of $W_{\mathfrak{g}}$. A usual choice is the weight $\rho=\frac{1}{2} \sum_{\alpha \in \Delta^{+}} \alpha$. We shall write it in the basis of fundamental weights $\Pi^{*}:=\left\{\lambda_{i}\right\}$ defined as the dual basis to the basis of simple roots, i.e. $\left(\lambda_{i}, \alpha_{j}^{\vee}\right)=\delta_{i j}$. Regular weights are precisely these, which have all coefficients in their expression in fundamental weights non-zero.

Lemma 1. $\forall \alpha_{i} \in \Pi,\left(\rho, \alpha_{i}^{\vee}\right)=1$

Proof. The reflection $\sigma_{\alpha_{i}}$ is an isometry of $\Delta$. If $\beta \in \Delta^{+}$then $\beta$ can be expressed as $\sum_{\Pi} c_{k} \alpha_{k}$ for $c_{k} \in \mathbb{Z}_{0}^{+}$. Then $\beta^{\prime}:=\sigma_{\alpha_{i}} \beta=$ $\sum_{\Pi} b_{k}\left(\alpha_{k}-\left(\alpha_{k}, \alpha_{i}^{\vee}\right) \alpha_{i}\right)=\sum_{\Pi} b_{k}^{\prime} \alpha_{k}$. The Cartan integers $c_{k j}:=\left(\alpha_{k}, \alpha_{i}^{\vee}\right)$ are non-positive for $k \neq i$, hence $b_{k}^{\prime} \geq b_{k} \geq 0$ for $k \neq i$. Unless $\beta=\alpha_{i}$ this means that $\beta^{\prime}$ is a positive root, and $\sigma_{\alpha_{i}} \alpha_{i}=-\alpha_{i}$. Thus $\sigma_{\alpha_{i}}$ is a permutation of $\Delta^{+} \backslash\left\{\alpha_{i}\right\}$. This means that $\sigma_{\alpha_{i}} \rho=$ $\sigma_{\alpha_{i}}\left(\frac{1}{2} \sum_{\Delta^{+} \backslash\left\{\alpha_{i}\right\}} \alpha\right)+\sigma_{\alpha_{i}}\left(\frac{1}{2} \alpha_{i}\right)=\frac{1}{2} \sum_{\Delta_{+} \backslash\left\{\alpha_{i}\right\}} \alpha-\frac{1}{2} \alpha_{i}=\rho-\alpha_{i}$. From the definition of a reflection (1) it follows that $\left(\rho, \alpha_{i}^{\vee}\right)=1$.

In other words $\rho=\sum_{\Pi^{*}} \lambda_{i}$ and so it is a regular weight. It is usual to express a weight $\lambda=\sum_{\Pi^{*}} l_{k} \lambda_{k}$ by means of a labelled Dynkin diagram like ${ }_{l_{1}}^{l_{0}} l_{0}^{l_{2}} l_{3} \quad l_{3} \quad l_{4} \quad l_{5} \quad l_{6}$ In this notation a reflection with respect to a simple root changes only labels over the corresponding node and the nodes that are linked with it. It happens in the following way:


These simple rules provide a way of calculating the orbit of $\rho$ :

and from this we may extract the directed graph of $W$ :


The two additional arrows correspond to the reflection $(3) \equiv(121) \equiv$ (212) with respect to the third, non-simple root $\alpha_{1}+\alpha_{2}$. We see that the construction of the directed graph is two-step, first constructing the nodes and some arrows and second filling in the rest of the arrows. In the sequel we vill state some equivalent reformulations of the directed graph that provide us with better methods to attain these goals and translate into the level of Hasse diagrams.

## 2. Properties of directed graphs

Each $w \in W$ defines a set $\Phi_{w} \equiv \Delta^{+} \cap w\left(\Delta^{-}\right)$. For $w=\sigma_{\alpha}$ we denote for convenience $\Phi_{\alpha}=\Phi_{w}$.

Lemma 2. $\forall w \in W$ it holds
(1) If $\alpha_{1}, \alpha_{2} \in \Phi_{w}$ and $\alpha_{1}+\alpha_{2} \in \Delta^{+}$, then $\alpha_{1}+\alpha_{2} \in \Phi_{w}$.
(2) If $\alpha \in \Phi_{w}$ and $\alpha=\alpha_{1}+\alpha_{2}$ for $\alpha_{1}, \alpha_{2} \in \Delta^{+}$then $\alpha_{1} \in \Phi_{w}$ or $\alpha_{2} \in \Phi_{w}$.

Proof. For $\alpha_{1}, \alpha_{2}$ satisfying the premise of the first claim, $\alpha_{i}=$ $w \alpha_{i}^{\prime}, i=1,2$ for some $\alpha_{i}^{\prime} \in \Delta^{-}$. We see that $\alpha_{1}+\alpha_{2}=w\left(\alpha_{1}^{\prime}+\alpha_{2}^{\prime}\right) \in$ $w\left(\Delta^{-}\right)$which verifies condition (1). If we have $\alpha, \alpha_{i}$ satisfying the premise of condition (2), then $\alpha^{\prime}:=w^{-1}(\alpha) \in \Delta^{-}$and $\alpha^{\prime}=w^{-1} \alpha_{1}+$ $w^{-1} \alpha_{2}$, hence at least one of the summands must be in $\Delta^{-}$.

Sets satisfying the first condition are called saturated, sets satisfying both conditions $(*)$-saturated, i.e. they are saturated and so is their complement in $\Delta^{+}$.

Lemma 3. For any (*)-saturated set $\Phi$ there is a $w \in W_{\mathfrak{g}}$ such that $\Phi=\Phi_{w}$. Moreover $\left|\Phi_{w}\right|=|w|$.

Proof. First we prove the claim on existence of $w$ by induction on $|\Phi|$. If $\Phi=\emptyset$ then $w=$ Id satisfies $\Phi_{w}=\Phi$. Suppose that for $|\Phi| \leq k$ there is a $w \in W$ such that $\Phi=\Phi_{w}$. Let $\Phi$ be $(*)$-saturated and $|\Phi|=k+1$. If we apply condition (2) of lemma 2 to $\Phi$, we can find an $\alpha \in \Phi \cap \Pi$, so $\Phi=\left\{\alpha_{1}, \ldots, \alpha_{k}, \alpha\right\}$. We shall verify that $\tilde{\Phi}:=\sigma_{\alpha}(\Phi) \backslash\{-\alpha\}=\left\{\sigma_{\alpha} \alpha_{1}, \ldots, \sigma_{\alpha} \alpha_{k}\right\}$ satisfies conditions (1) and (2) of lemma 2.
(1): Let $\tilde{\alpha}_{1}, \tilde{\alpha}_{2} \in \tilde{\Phi}$ such that $\tilde{\alpha}_{1}+\tilde{\alpha}_{2} \in \Delta^{+}$. There exist $\alpha_{i_{1}}, \alpha_{i_{2}} \in$ $\Phi \backslash\{\alpha\}$ such that $\sigma_{\alpha} \alpha_{i_{1}}=\tilde{\alpha}_{1}, \sigma_{\alpha} \alpha_{i_{1}}=\tilde{\alpha}_{1}$. Condition (1) for $\Phi$ implies that $\alpha_{i_{3}}:=\alpha_{i_{1}}+\alpha_{i_{2}} \in \Phi$ and since $\alpha \in \Pi$, even $\alpha_{i_{3}} \in \Phi \backslash\{\alpha\}$. Therefore $\tilde{\alpha}_{1}+\tilde{\alpha}_{2}=\sigma_{\alpha} \alpha_{i_{3}} \in \tilde{\Phi}$.
(2): Let $\tilde{\beta} \in \tilde{\Phi}$ such that $\tilde{\beta}=\tilde{\beta}_{1}+\tilde{\beta}_{2}$ for $\tilde{\beta}_{1}, \tilde{\beta}_{2} \in \Delta^{+}$. There is an $\alpha_{i} \in \Phi \backslash\{\alpha\}$ such that $\sigma_{\alpha} \alpha_{i}=\tilde{\beta}$, hence $\alpha_{i}=\sigma_{\alpha} \tilde{\beta}_{1}+\sigma_{\alpha} \tilde{\beta}_{2}$. Since $\alpha \in \Delta^{+}$, at least one of $\sigma_{\alpha} \tilde{\beta}_{1}, \sigma_{\alpha} \tilde{\beta}_{2}$ is in $\Delta^{+}$. If both are then condition (2) for $\Phi$ implies that one of them, say $\sigma_{\alpha} \tilde{\beta}_{1}$, is in $\Phi$, or, since $\tilde{\beta}_{1} \in \Delta^{+} \backslash\{\alpha\}$, even in $\Phi \backslash\{\alpha\}$. Thus $\tilde{\beta}_{1}=\sigma_{\alpha}\left(\sigma_{\alpha} \tilde{\beta}_{1}\right) \in \tilde{\Phi}$. On the other hand, if say $\sigma_{\alpha} \tilde{\beta}_{1} \in \Delta^{-}$, then $\sigma_{\alpha} \tilde{\beta}_{1}=-\alpha$ and $\alpha_{i}+\alpha=\sigma_{\alpha} \tilde{\beta}_{2} \in \Phi \backslash\{\alpha\}$, hence as before $\tilde{\beta}_{2}=\sigma_{\alpha}\left(\sigma_{\alpha} \tilde{\beta}_{2}\right) \in \tilde{\Phi}$.

Induction hypothesis now says that there exists a $\tilde{w} \in W_{\mathfrak{g}}$ such that $\tilde{\Phi}=\Phi_{\tilde{w}}$. We shall show that for $w=\sigma_{\alpha} \tilde{w}$ it is $\Phi=\Phi_{w}$. If $\beta=\alpha_{i} \in \Phi \backslash\{\alpha\}$ then $\sigma_{\alpha} \alpha_{i} \in \Phi_{\tilde{w}}$ and $w^{-1} \alpha_{i}=\tilde{w}^{-1}\left(\sigma_{\alpha} \alpha_{i}\right) \in \Delta^{-}$and so $\beta \in \Phi_{w}$. If $\beta=\alpha \in \Phi$ then $w^{-1} \alpha=\tilde{w}^{-1} \sigma_{\alpha} \alpha=-\tilde{w}^{-1} \alpha \in \Delta^{-}$and so $\alpha \in \Phi_{w}$. Conversely if $\beta \in \Phi_{w}, \beta \neq \alpha$, then $\tilde{w}^{-1} \sigma_{\alpha} \beta=w^{-1} \beta \in \Delta^{-}$. Since $\beta \neq \alpha$, we know that $\sigma_{\alpha} \beta \in \Delta^{+}$, thus $\sigma_{\alpha} \beta \in \Phi_{\tilde{w}}$ and $\sigma_{\alpha}\left(\sigma_{\alpha} \beta\right) \in$ $\Phi \backslash\{\alpha\}$. Clearly $\alpha \in \Phi$ and so we have shown also that $\Phi_{w} \subset \Phi$.

Now we shall prove that $|w|=\left|\Phi_{w}\right|$, again by induction. For $w=\mathrm{Id}$ the statement is trivial. Suppose that $w$ has a reduced expression $\sigma_{\alpha_{1}} \ldots \sigma_{\alpha_{k}}$ where $\alpha_{i} \in \Pi$, then $\tilde{w}=\sigma_{\alpha_{1}} w=\sigma_{\alpha_{2}} \ldots \sigma_{\alpha_{k}}$ is of length $k-1$ and by induction hypothesis $\left|\Phi_{\tilde{w}}\right|=k-1$. Since $w^{-1} \alpha=-\tilde{w}^{-1} \alpha$, either $\alpha \in \Phi_{w}$ and $\alpha \notin \Phi_{\tilde{w}}$ or $\alpha \notin \Phi_{w}$ and $\alpha \in \Phi_{\tilde{w}}$. In the former case we
have seen in the first part of the proof that $\left|\Phi_{w}\right|=\left|\Phi_{\tilde{w}}\right|+1$ and in the latter similarly $\left|\Phi_{\tilde{w}}\right|=\left|\Phi_{w}\right|+1$. The first case gives us $\left|\Phi_{w}\right|=k$, the second $\left|\Phi_{w}\right|=k-2$. But if the second were possible, we could choose $\beta_{1} \in \Pi \cap \Phi_{w}$ so that $\left|\Phi_{\sigma_{\beta_{1}} w}\right|=k-3$ and inductively $\sigma_{\beta_{2}}, \ldots, \sigma_{\beta_{k-2}}$ such that $\Phi_{\sigma_{\beta_{k-2}} \ldots \sigma_{\beta_{1}} w}=\emptyset$. Yhis would mean that $w=\sigma_{\beta_{1}} \ldots \sigma_{\beta_{k-2}}$ which contradicts $|w|=k$.

We shall now state and prove several properties of saturated sets. We introduce some notation that will make these propositions better readable. First for a $(*)$-saturated set $\Phi$ the sum if its elements will be denoted in a standard way by $\langle\Phi\rangle$. Moreover we shall define $\operatorname{sgn}(\alpha)=$ $\pm 1$ according to whether $\alpha \in \Delta^{ \pm}$.

Lemma 4. Let $w \in W_{\mathfrak{g}} \alpha \in \Delta^{+}$. Then
(1) $w^{\prime}=\sigma_{\alpha} w \Leftrightarrow w^{\prime} \rho-w \rho=k \alpha$ where $k=\left(\alpha^{\vee}, w \rho\right)$.
(2) $w(\rho)=\rho-\left\langle\Phi_{w}\right\rangle$.
(3) $\forall w, w^{\prime} \in W_{\mathfrak{g}}, w \neq w^{\prime} \Rightarrow \Phi_{w} \neq \Phi_{w^{\prime}}$
(4) There is an arrow $\alpha$ in the directed graph going from $w$ to $w^{\prime}$ iff $\left\langle\Phi_{w^{\prime}}\right\rangle-\left\langle\Phi_{w}\right\rangle$ is a multiple of $\alpha$ and $\left|\Phi_{w^{\prime}}\right|=\left|\Phi_{w}\right|+1$.
(5) $\alpha \in \Phi_{w^{\prime}} \Rightarrow \sigma_{\alpha}\left(\Phi_{w} \backslash \Phi_{w^{\prime}}\right) \subset\left(\Phi_{w^{\prime}} \backslash\{\alpha\}\right) \backslash \Phi_{w}$
(6) There is an arrow $\alpha$ in the directed graph going from $w$ to $w^{\prime}$ iff $\alpha \in \Phi_{w^{\prime}}$ and $\left(\Phi_{w^{\prime}} \backslash\{\alpha\}\right) \backslash \Phi_{w}=\sigma_{\alpha}\left(\Phi_{w} \backslash \Phi_{w^{\prime}}\right)$

Proof.
(1) The orbit of $\rho$ under $W_{\mathfrak{g}}$ is in a bijection with $W_{\mathfrak{g}}$ itself. The left hand side of the claim is then equivalent to $w^{\prime} \rho=w \rho-$ $\left(w \rho, \alpha^{\vee}\right) \alpha$ which is exactly the right hand side.
(2) Since any $w \in W_{\mathfrak{g}}$ is a permutation of $\Delta$, it holds

$$
\rho=\frac{1}{4} \sum_{\Delta} \operatorname{sgn}(\alpha) \alpha=\frac{1}{4} \sum_{\Delta} \operatorname{sgn}(w \alpha) w \alpha
$$

Thus

$$
\rho-w \rho=\frac{1}{4} \sum_{\Delta}(\operatorname{sgn}(w \alpha)-\operatorname{sgn}(\alpha)) w \alpha
$$

The summands are nonzero only for $w \alpha \in \Phi_{w}$ or $w \alpha \in-\Phi_{w}$, the sum is then $\frac{1}{4}\left[2\left\langle\Phi_{w}\right\rangle+(-2)\left(-\left\langle\Phi_{w}\right\rangle\right)\right]=\left\langle\Phi_{w}\right\rangle$.
(3) If $w \neq w^{\prime}$, then $0 \neq w \rho-w^{\prime} \rho=\left\langle\Phi_{w^{\prime}}\right\rangle-\left\langle\Phi_{w}\right\rangle$, so $\Phi_{w}$ cannot be equal $\Phi_{w^{\prime}}$.
(4) If there is an arrow from $w$ to $w^{\prime}$ labelled by $\alpha$ then $\left|w^{\prime}\right|=$ $|w|+1$ and from the second statement of lemma 3 we see that $\left|\Phi_{w^{\prime}}\right|=\left|\Phi_{w}\right|+1$. Moreover, according to 1 and $2, w^{\prime} \rho-w \rho=$ $\left\langle\Phi_{w}\right\rangle-\left\langle\Phi_{w^{\prime}}\right\rangle$ is a multiple of $\alpha$. On the other hand, if $w^{\prime} \rho-w \rho$ is a multiple of $\alpha$ then $w^{\prime} \rho$ must be a reflection of $w \rho$ since they are on the same Weyl orbit and thus have the same Euclidean norm. From $\left|\Phi_{w^{\prime}}\right|=\left|\Phi_{w}\right|+1$ we see that $\left|w^{\prime}\right|=|w|+1$.
(5) We can write $w^{-1} \sigma_{\alpha} \beta=w^{\prime-1} \beta$ and conversely $w^{\prime-1} \sigma_{\alpha} \beta=$ $w^{-1} \beta$. This and $\alpha \in \Phi_{w^{\prime}}$ means that $\alpha \notin \Phi_{w} \backslash \Phi_{w^{\prime}}$. Therefore for $\beta \in \Phi_{w} \backslash \Phi_{w^{\prime}}$ we know that $\beta \neq \alpha$ and thus $\sigma_{\alpha} \beta \in \Delta^{+} \backslash\{\alpha\}$. $w^{-1} \sigma_{\alpha} \beta=w^{\prime-1} \beta$ for $\beta, \sigma_{\alpha} \beta \in \Delta^{+}$gives that $\sigma_{\alpha}\left(\Phi_{w} \backslash \Phi_{w^{\prime}}\right)$ is a subset of $\Phi_{w^{\prime}} \backslash \Phi_{w}$ and since $\sigma_{\alpha} \beta \neq \alpha$, even of $\left(\Phi_{w^{\prime}} \backslash\{\alpha\}\right) \backslash \Phi_{w}$
(6) From statement 3 we know that if $\alpha \in \Phi_{w^{\prime}}$ then $\left|\Phi_{w} \backslash \Phi_{w^{\prime}}\right| \leq$ $\left|\Phi_{w^{\prime}} \backslash\{\alpha\} \backslash \Phi_{w}\right|$, hence $\left|\Phi_{w}\right|<\left|\Phi_{w^{\prime}}\right|$. Exchanging $w$ and $w^{\prime}$, we see that $\alpha \in \Phi_{w} \Rightarrow\left|\Phi_{w}\right|>\left|\Phi_{w^{\prime}}\right|$. We know from $w^{-1} \alpha=-w^{\prime-1} \alpha$ that either $\alpha \in \Phi_{w} \backslash \Phi_{w^{\prime}}$ or $\alpha \in \Phi_{w^{\prime}} \backslash \Phi_{w}$. If $\alpha$ labels the arrow from $w$ to $w^{\prime}$, then $\left|\Phi_{w^{\prime}}\right|=\left|\Phi_{w}\right|+1$ by lemma 3 and so $\alpha \in \Phi_{w^{\prime}}$. Hence $\left|\Phi_{w} \backslash \Phi_{w^{\prime}}\right|=\left|\Phi_{w^{\prime}} \backslash\{\alpha\} \backslash \Phi_{w}\right|$ and from 3 we see that $\sigma_{\alpha}\left(\Phi_{w} \backslash \Phi_{w^{\prime}}\right)=\Phi_{w^{\prime}} \backslash\{\alpha\} \backslash \Phi_{w}$. Conversely, $\sigma_{\alpha}\left(\Phi_{w} \backslash \Phi_{w^{\prime}}\right)=\Phi_{w^{\prime}} \backslash\{\alpha\} \backslash \Phi_{w}$ and $\alpha \in \Phi_{w^{\prime}}$ give that $\left|\Phi_{w^{\prime}}\right|=\left|\Phi_{w}\right|+1$ and $\Phi_{w^{\prime}}=\left\{\alpha_{1}, \ldots, \alpha_{j}, \beta_{1}, \ldots, \beta_{k}, \alpha\right\}$, $\Phi_{w}=\left\{\alpha_{1}, \ldots, \alpha_{j}, \sigma_{\alpha}, \ldots, \sigma_{\alpha}\right\}$. Since $\sigma_{\alpha} \beta-\beta$ is a multiple of $\alpha$, we see that so is $\left\langle\Phi_{w^{\prime}}\right\rangle-\left\langle\Phi_{w}\right\rangle$ and by 4 there is an arrow from $w$ to $w^{\prime}$ labelled by $\alpha$.

Statement 3 is the last bit that we need to make sure that the set of $(*)$-saturated sets is in bijection with $W_{\mathfrak{g}}$, hence a node of the directed graph can be regarded either as a Weyl element $w$ or as the corresponding set $\Phi_{w}$ as one wishes. Statements 4 and 6 allow us to determine all arrows in the directed graph solely in terms of $(*)$-saturated sets. However, both aren't very convenient. In the computations we shall use the following result called Čap criterion that appears in the monograph [SloPG]:

Lemma 5. Let $\alpha \in \Delta^{+}$and $\Phi_{w}$ be a (*)-saturated set. There is an arrow labeled by $\alpha$ from $\Phi_{w}$ iff there is a $k \in \mathbb{Z}_{0}^{+},\left|\Phi_{\alpha}\right|=2 k+$ $1,\left|\Phi_{w} \cap \Phi_{\alpha}\right|=k$. The endpoint of this arrow is $\Phi_{w^{\prime}}$ the set $\Phi_{w^{\prime}}=$ $\{\alpha\} \cup\left(\Phi_{w} \cap \Phi_{\alpha}\right) \cup \sigma_{\alpha}\left(\Phi_{w} \backslash \Phi_{\alpha}\right)$, where $w^{\prime}=\sigma_{\alpha} w$.

Proof. Since $\sigma_{\alpha}$ is a reflection, its expression in terms of simple reflections must contain an odd number of terms. Thus a set $\Phi_{\alpha}^{\prime}:=\Phi_{\alpha} \backslash$ $\{\alpha\}$ has $2 k$ elements for some $k$. If $w^{\prime}=\sigma_{\alpha} w$, then $w^{\prime-1}=w^{-1} \sigma_{\alpha}$. This means that $w^{\prime-1} \alpha=-w^{-1} \alpha$, i.e. $\alpha \in \Phi_{w^{\prime}} \Leftrightarrow \alpha \notin \Phi_{w}$. Thus we can assume $\alpha \notin \Phi_{w} . \Delta^{+}$splits into five disjoint subsets: $\Phi^{++}:=\Phi_{w} \cap \Phi_{\alpha}^{\prime}$, $\Phi^{+-}:=\Phi_{w} \cap \Phi_{\alpha}^{c}, \Phi^{-+}:=\Phi_{w}^{c} \cap \Phi_{\alpha}^{\prime}, \Phi^{--}:=\Phi_{w}^{c} \cap \Phi_{\alpha}^{c}$ and $\{\alpha\}$, where the superscript $c$ means the complement.

First we observe that $-\sigma_{\alpha}\left(\Phi_{\alpha}^{\prime}\right) \subset \Phi_{\alpha}^{\prime}$ and $\sigma_{\alpha}\left(\Phi_{\alpha}^{c}\right) \subset \Phi_{\alpha}^{c}$. Since $\sigma_{\alpha}$ is injective, it is an automorphism of $\Phi_{\alpha}^{c}$. Let $\beta \in \Phi_{\alpha}^{c}, \beta^{\prime}:=\sigma_{\alpha} \beta$, then $w^{-1} \beta=w^{-1} \sigma_{\alpha} \beta^{\prime}=w^{\prime-1} \beta^{\prime}$ and hence $\beta \in \Phi_{w} \Leftrightarrow \beta^{\prime} \in \Phi_{w^{\prime}}$ or otherwise stated $\Phi_{w^{\prime}} \cap \Phi_{\alpha}^{c}=\sigma_{\alpha}\left(\Phi^{+-}\right)$.

On the other hand, if $\beta \in \Phi_{\alpha}^{\prime}$, first assume that $\beta \in \Phi_{w}$ and consider $w^{\prime-1} \beta=w^{-1} \sigma_{\alpha} \beta=w^{-1} \beta-\langle\beta, \alpha\rangle w^{-1} \alpha$. Since $\alpha, \beta \in \Delta^{+}$and $\sigma_{\alpha} \beta=$ $\beta-\langle\beta, \alpha\rangle \alpha \in \Delta^{-}$, the number $\langle\beta, \alpha\rangle$ must be positive. Because $\alpha \notin \Phi_{w}$
by assumption, we see that $w^{\prime-1} \beta \in \Delta^{-}$, i.e. $\Phi^{++} \subset \Phi_{w^{\prime}}$. Moreover it follows that for any $\beta \in \Phi^{++} w^{-1}\left(-\sigma_{\alpha}(\beta)\right)=-w^{\prime-1} \beta \in \Delta^{+}$and thus $-\sigma_{\alpha}\left(\Phi^{++}\right) \subset \Phi^{-+}$. This means that $\left|\Phi^{++}\right| \leq\left|\Phi^{-+}\right|$. But the union of these two sets is $\Phi_{\alpha}^{\prime}$, thus the former must have $k-e$ elements and the latter $k+e$ elements for some non-negative integer $e$.

We see that $\Phi^{-+}$can be written as a union of two sets: $-\sigma_{\alpha}\left(\Phi^{++}\right)$ and its complement which is invariant with respect to $-\sigma_{\alpha}$. For $\beta \in$ $-\sigma\left(\Phi^{++}\right)$thus $-w^{\prime-1} \beta=w^{-1}\left(-\sigma_{\alpha} \beta\right) \in \Delta^{-}$and thus $\beta \in \Phi_{w^{\prime}}^{c}$. On the other hand $\beta \notin-\sigma_{\alpha}\left(\Phi^{++}\right)$implies $\beta \in \Phi_{w^{\prime}}$. Hence $\Phi_{w^{\prime}} \cap \Phi_{\alpha}^{\prime}=$ $\left(\Phi^{-+} \backslash-\sigma_{\alpha}\left(\Phi^{++}\right)\right) \cup \Phi^{++}$.

Finally, $\alpha \in \Phi_{w^{\prime}}$ by assumption. Bringing all this together we get

$$
\Phi_{w^{\prime}}=\sigma_{\alpha}\left(\Phi^{+-}\right) \cup\left(\Phi^{-+} \backslash-\sigma_{\alpha}\left(\Phi^{++}\right)\right) \cup \Phi^{++} \cup\{\alpha\} .
$$

which is a union of disjoint sets. Clearly $\left|\Phi_{w}\right|=\left|\Phi^{+-} \cup \Phi^{++}\right|=$ $\left|\sigma_{\alpha}\left(\Phi^{+-}\right) \cup \Phi^{++}\right|$.

Since $\left|-\sigma_{\alpha}\left(\Phi^{++}\right)\right|=k-e,\left|\Phi^{-+} \backslash-\sigma_{\alpha}\left(\Phi^{++}\right)\right|$must be $2 e$, thus $\left|\Phi_{w^{\prime}}\right|=\left|\Phi_{w}\right|+2 e+1$. Since $e$ is non-negative, we see that the length $l(w):=\left|\Phi_{w}\right|$ must increase going from $w$ to $w^{\prime}$ and it will increase only by one iff $e=0$. But this means that $\left|\Phi^{++}\right|=k$ as stated in the lemma and the expression for $\Phi_{w^{\prime}}$ is given by omiting the second term from the expression upstairs.

The advantage of Čap criterion is that knowing all saturated sets $\Phi_{\alpha}$ for $\alpha \in \Delta^{+}$we can construct all arrows from any given node, calculate their endpoints and inductively reveal the whole directed graph.

## 3. Hasse and BGG diagrams

Let us consider now a standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with the Levi decomposition $\mathfrak{g}_{0} \oplus \mathfrak{p}_{+}$, where the Cartan subalgebra of the reductive part $\mathfrak{g}_{0}$ is arranged to be $\mathfrak{h}=\mathfrak{h}_{s} \oplus \mathfrak{z}$, where $\mathfrak{h}_{s}$ is a Cartan subalgebra of the semisimple part $\mathfrak{g}_{0 s}$ of $\mathfrak{g}_{0}$ and $\mathfrak{z}$ is the center of $\mathfrak{g}_{0}$. It is usual to denote weights by means of a labelled Dynkin diagram of $\mathfrak{g}$ where $\operatorname{dim} \mathfrak{z}$ nodes are crossed and the rest of the nodes form the Dynkin diagram of $\mathfrak{g}_{0 s}$. We shall denote by $\Sigma \subset \Pi$ the subset of simple roots that are crossed. Finite-dimensional irreducible representations of $\mathfrak{p}$ are classified by $\mathfrak{p}$-dominant weights, i.e. weights that are nonnegative on $\mathfrak{h}_{s}$.

There is an element $E \in \mathfrak{z}$ called the grading element, whose adjoint action on $\mathfrak{g}$ is diagonal and integer-valued. Its action splits the set of positive roots $\Delta^{+}$into a disjoint union $\Delta^{+}\left(\mathfrak{g}_{0}\right) \cup \Delta\left(\mathfrak{p}_{+}\right)$, where $\Delta(\tilde{\mathfrak{g}})$ denotes the set of roots whose root spaces are contained in $\tilde{\mathfrak{g}}$.

Definition 2 (Hasse diagram). Hasse diagram of a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is a subgraph of the directed graph of $W_{\mathfrak{g}}$. Its set of vertices is the set of all $w$ such that $\Phi_{w} \subset \Delta\left(\mathfrak{p}_{+}\right)$. There is an arrow
between two vertices of the Hasse graph iff there is an arrow between the corresponding vertices in the directed graph.

We clearly see from Čap criterion that for any $\alpha \in \Delta\left(\mathfrak{p}_{+}\right)$and $\Phi_{w} \subset \Delta\left(\mathfrak{p}_{+}\right), \Phi_{w^{\prime}}$ where $w^{\prime}=\sigma_{\alpha} w$ is a subset of $\Delta\left(\mathfrak{p}_{+}\right)$too. On the other hand, an arrow between any two saturated subsets of $\Delta\left(\mathfrak{p}_{+}\right)$must be labelled by a root $\alpha \in \Delta\left(\mathfrak{p}_{+}\right)$, since the target saturated set contains $\alpha$.

In the following we show two alternative characterisations of the Hasse graph. The first one, in terms of acceptable subset of a weight graph comes from [KS03] and gives a quick and systematic way of finding the shape of the Hasse diagram. The second is the standard one explained in the classical book [BasE89] and provides us with a recipe analogous to the one with which the vertices of the directed graph are constructed as the orbit of $\rho$.

Definition 3 (Weight graph). The weight graph of $\mathfrak{p}$ is the set of vertices $\Delta\left(\mathfrak{p}_{+}\right)$and an arrow goes from $\beta$ to $\beta^{\prime}$ iff $\beta^{\prime}=\beta-\alpha$ for some simple root $\alpha \in \Delta\left(\mathfrak{g}_{0}\right)$. The arrow is labelled by $\alpha$.

Definition 4 (Acceptable subset). A subset $\mathcal{V} \subset \Delta\left(\mathfrak{p}_{+}\right)$is called acceptable iff the following three conditions are satisfied:
(1) $\forall \gamma \in \mathcal{V}, \beta \in \Delta\left(\mathfrak{p}_{+}\right), \gamma-\beta \in \Delta^{+} \Rightarrow \beta \in \mathcal{V}$
(2) $\forall \beta, \beta^{\prime} \in \mathcal{V}, \beta+\beta^{\prime} \in \Delta\left(\mathfrak{p}_{+}\right) \Rightarrow \beta+\beta^{\prime} \in \mathcal{V}$
(3) $\forall \beta, \beta^{\prime} \notin \mathcal{V}, \beta+\beta^{\prime} \in \Delta\left(\mathfrak{p}_{+}\right) \Rightarrow \beta+\beta^{\prime} \notin \mathcal{V}$

Theorem 1. A subset $\mathcal{V} \subset \Delta\left(\mathfrak{p}_{+}\right)$is acceptable iff there exists $w \in \Phi_{w}$ such that $\mathcal{V}=\Phi_{w}$.

Proof. We must prove that $\mathcal{V}$ is acceptable iff it is $(*)$-saturated. For the proof and some examples illustrating this attempt see [KS03]

Corollary 1. Suppose that the algebra $\mathfrak{g}$ is $|1|$-graded. Then $\mathcal{V} \subset$ $\Delta\left(\mathfrak{p}_{+}\right)$is $(*)$-saturated iff the condition (1) of the definition of acceptability holds and there is an arrow $w \xrightarrow{\alpha} w^{\prime}, \alpha \in \Delta^{+}$iff $\Phi_{w^{\prime}}=\Phi_{w} \cup\{\alpha\}$.

Proof. If $\mathfrak{p}_{+}=\mathfrak{g}_{1}$, then for $\beta, \beta^{\prime} \in \Delta\left(\mathfrak{p}_{+}\right), \beta+\beta^{\prime}$ is not a root, thus the second and third condition of acceptability are vacuous. Also from lemma 5 we see that a part of the target set of an arrow $\Phi_{w} \xrightarrow{\alpha} \Phi_{w^{\prime}}$, $\alpha \in \Delta\left(\mathfrak{p}_{+}\right)$has the form $\sigma_{\alpha} \beta$ where $\beta \in \Phi_{w} \subset \Delta\left(\mathfrak{p}_{+}\right)$which is also a sum of two roots in $\mathfrak{g}_{1}$ and thus this part must be empty.

The preceding corollary shows that the construction of Hasse diagrams is considerably easier in the $|1|$-graded case. However, our interest will be the standard contact gradation of $\mathfrak{s o}(2 n)$, which is a $|2|$-grading and thus we will rather have to rely upon the Cap criterion.

Now we shall state a lemma that will summarize the theoretical background for the most standard method of constructing (vertices
of) Hasse diagrams. Let us denote by $\rho_{\mathfrak{p}}=\sum \lambda_{\Delta+\left(\mathfrak{g}_{0}\right)} \lambda_{i}$ the sum of fundamental weights over the indices of the roots whose root vectors are in $\Delta^{+}$. Such an element is usually denoted by a Dynkin diagram with 0's over uncrossed nodes and 1's over crossed nodes, like
 the Weyl group of the reductive factor $\mathfrak{g}_{0} \subset \mathfrak{p}$, i.e. generated by simple reflections corresponding to uncrossed nodes. The subset $W^{\mathfrak{p}} \subset W_{\mathfrak{g}}$ is then the set of vertices of the Hasse graph, i.e. elements $w$ such that $\Phi_{w} \subset \Delta\left(\mathfrak{p}_{+}\right)$.

## Lemma 6.

(1) (Kostant lemma) Any $w \in W_{\mathfrak{g}}$ admits a unique decomposition $w=w_{\mathfrak{p}} w^{\mathfrak{p}}$ with $w_{\mathfrak{p}} \in W_{\mathfrak{p}}$ and $w^{\mathfrak{p}} \in W^{\mathfrak{p}}$. Moreover $|w|=$ $\left|w_{\mathfrak{p}}\right|+\left|w^{\mathfrak{p}}\right|$.
(2) The map $w \rightarrow w^{-1} \rho_{\mathfrak{p}}$ restricts to a bijection between $W^{\mathfrak{p}}$ and the orbit of $\rho_{\mathfrak{p}}$ under $W_{\mathfrak{g}}$.
(3) Let $w \in W_{\mathfrak{p}}, \alpha \in \Pi, \alpha \notin \Phi_{w^{-1}}$ such that $\sigma_{\alpha}\left(w^{-1} \rho_{\mathfrak{p}}\right) \neq w^{-1} \rho_{\mathfrak{p}}$, then $w \sigma_{\alpha} \in W^{\mathfrak{p}}, \Phi_{w \sigma_{\alpha}}=\Phi_{w} \cup\{w(\alpha)\}$ and the arrow from $w$ to $w \sigma_{\alpha}$ has label wa.
(4) $W^{\mathfrak{p}}$ is precisely the subset of $W_{\mathfrak{g}}$ of elements that send $a \mathfrak{g}$ dominant weight into $a \mathfrak{p}$-dominant weight.

## Proof.

(1) See $[$ SloPG]
(2) For $\alpha \in \Pi \backslash \Sigma, \sigma_{\alpha}$ preserves $\rho_{\mathfrak{p}}$ because $\left(\rho_{\mathfrak{p}}, \alpha\right)=0$. These reflections generate $W_{\mathfrak{p}}$ and so $W_{\mathfrak{p}}$ fixes $\rho_{\mathfrak{p}}$ too. By (1), any $w \in W_{\mathfrak{g}}$ can be written as $w_{\mathfrak{p}} w^{\mathfrak{p}}$, where $w_{\mathfrak{p}} \in W_{\mathfrak{p}}, w^{\mathfrak{p}} \in W^{\mathfrak{p}}$. Since $w^{-1} \rho_{\mathfrak{p}}=\left(w^{\mathfrak{p}}\right)^{-1} \rho_{\mathfrak{p}}$, the map from $W^{\mathfrak{p}}$ os onto the orbit of $\rho_{\mathfrak{p}}$. On the other hand, if $w^{\mathfrak{p}} \in W^{\mathfrak{p}}, w^{\mathfrak{p}} \neq \mathrm{Id}$, then there is an $\alpha \in \Delta^{-}$such that $w^{\mathfrak{p}} \alpha \in \Phi_{w}$. This means that $\left(\rho_{\mathfrak{p}}, \alpha^{\vee}\right) \leq 0$ and $\left(\left(w^{\mathfrak{p}}\right)^{-1} \rho_{\mathfrak{p}}, \alpha^{\vee}\right)=\left(\rho_{\mathfrak{p}}, w^{\mathfrak{p}}\left(\alpha^{\vee}\right)\right)=\left(\rho_{\mathfrak{p}}, w^{\mathfrak{p}}(\alpha)^{\vee}\right)>0$, where we have used $w\left(\alpha^{\vee}\right)=w(\alpha)^{\vee}$ guaranteed by $w$ being isometry, hence $\left(w^{\mathfrak{p}}\right)^{-1} \rho_{\mathfrak{p}} \neq \rho_{\mathfrak{p}}$. Thus if $w^{-1} \rho_{\mathfrak{p}}=\rho_{\mathfrak{p}}$, then by (1) it must be $w \in W_{\mathfrak{p}}$. Hence if there are $\$ w_{1} \rho_{\mathfrak{p}}=w_{2} \rho_{\mathfrak{p}}$ for $w_{1}, w_{2} \in W^{\mathfrak{p}}$, then $w_{1}^{-1} w_{2} \in W_{\mathfrak{p}}$ implies that $w_{1}=w_{2}$ by the uniqueness of the decomposition of (1).
(3) We have seen in the final part of the proof of lemma 3 that for $\alpha \in \Pi, \alpha \notin \Phi_{w^{-1}}$ it is $\left|\sigma_{\alpha} w^{-1}\right|=\left|w^{-1}\right|+1$. This means also that $\left|w \sigma_{\alpha}\right|=|w|+1$ and we claim that $w=\sigma_{w \alpha} w \sigma_{\alpha}$ which means that there is an arrow labelled $w \alpha \in \Delta^{+}$from $w$ to $w \sigma_{\alpha}$. To verify the claim it is enough to check that for any weight $\lambda$ it is $\sigma_{w \alpha} \lambda=w\left(\sigma_{\alpha}\left(w^{-1} \lambda\right)\right)$. This is because

$$
\begin{aligned}
& w\left(\sigma_{\alpha}\left(w^{-1} \lambda\right)\right)=w\left[w^{-1} \lambda-\left(\alpha^{\vee}, w^{-1} \lambda\right) \alpha\right] \\
& \quad=\lambda-\left(w\left(\alpha^{\vee}\right), \lambda\right) \alpha=\lambda-\left(w(\alpha)^{\vee}, \lambda\right)=\sigma_{w \alpha} \lambda
\end{aligned}
$$

with the help of $w\left(\alpha^{\vee}\right)=w(\alpha)^{\vee}$ as in (2). If $\beta \in \Delta^{+}$, then $\left.\left(w \sigma_{\alpha}\right)^{-1}\right) \beta=\sigma_{\alpha} w^{-1} \beta \in \Delta^{-}$happens iff $w^{-1} \beta=\alpha$ or $w^{-1} \beta \in$ $\Delta^{-} \backslash\{-\alpha\}$, i.e. $\Phi_{w \sigma_{\alpha}}=\Phi_{w} \cup\{\alpha\}$. It remains to be proved that $w \alpha \in \Delta\left(\mathfrak{p}_{+}\right)$. Since $\sigma_{\alpha}$ does not fix $w^{-1} \rho_{\mathfrak{p}}$ it must be $\left(w^{-1} \rho_{\mathfrak{p}}, \alpha^{\vee}\right) \equiv\left(\rho_{\mathfrak{p}},(w \alpha)^{\vee}\right) \neq 0$, moreover we have assumed that $w \alpha \in \Delta^{+}$, hence $w \alpha \in \Delta\left(\mathfrak{p}_{+}\right)$.
(4) If $\lambda$ is $\mathfrak{g}$-dominant, then $\forall \alpha \in \Delta^{+}$it is $(\lambda, \alpha) \geq 0$ and hence if $\alpha \in \Phi_{w}$ then $\left(\lambda, w^{-1} \alpha\right) \leq 0$. Thus $(w \lambda, \alpha) \leq 0$. If $w \lambda$ is $\mathfrak{p}$-dominant then $\alpha$ cannot be in $\Delta^{+}\left(\mathfrak{g}_{0}\right)$, therefore $\alpha \in \Delta\left(\mathfrak{p}^{+}\right)$, hence $\Phi_{w} \subset \Delta\left(\mathfrak{p}_{+}\right)$. Conversely, for $w \lambda$ not $\mathfrak{p}$-dominant, we can find an $\alpha \in \Delta^{+}\left(\mathfrak{g}_{0}\right)$ such that $(w \lambda, \alpha)<0$, so $w^{-1} \alpha \in \Delta^{-}$ and hence $\Phi_{w} \not \subset \Delta\left(\mathfrak{p}_{+}\right)$

The recipe for constructing the Hasse diagram of a given standard parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is then as follows:
(1) Calculate the orbit of $\rho_{\mathfrak{p}}$ under $W_{\mathfrak{g}}$ : Start from $\rho_{\mathfrak{p}}$ and on any $w^{-1} \rho_{\mathfrak{p}}$ make all non-trivial reflections that do not lead backwards. According to (2) of lemma 6 , this way one gets all vertices of the Hasse diagram. Denote any arrow with the (temporary) label of the corresponding reflection $\sigma_{\alpha}$.
(2) Construct the sets $\Phi_{w}$ of each $w \in W^{\mathfrak{p}}$ : According to (3) of lemma 6, the bijection

gives a "bijection of labels" $\sigma_{\alpha} \leftrightarrow w \alpha$. We can thus calculate the (final) label $w \alpha$ by applying $w=\sigma_{1} \ldots \sigma_{k}$, i.e. the series of reflections on any path from the starting point of the arrow backwards to the identity. The statement (3) tells us also that $\Phi_{w \sigma_{\alpha}}=\Phi_{w} \cup\{w \alpha\}$. This way we can inductively construct all sets $\Phi_{w}$.
(3) Fill in the remaining arrows: If $\Phi_{w^{\prime}}$ and $\Phi w$ differ only by one root, we can write there the arrow labelled by this root. There may however remain some arrows that cannot be constructed this way. We have to apply the Čap criterion, i.e. check if there is some $\alpha$ which is neither contained in nor emanating from the set $\Phi_{w}$ such that $2\left|\Phi_{\alpha} \cap \Phi_{w}\right|=\left|\Phi_{\alpha} \backslash\{\alpha\}\right|$.
Definition 5 (BGG diagram). A BGG diagram is the Hasse diagram whose vertices are labeled with integral dominant weights of $\mathfrak{p}$. It is defined by the choice of the label $\lambda$ of the first vertex and the affine Weyl action

$$
w \cdot \lambda=w(\lambda+\rho)-\rho
$$

which allows us to compute the label $w \cdot \lambda$ over the vertex $w$.

## CHAPTER 3

## Invariant operators in parabolic geometries and symmetric spaces

We shall consider a symmetric space where $G$ is a semisimple Lie group and $K$ its maximal compact subgroup. Then Cartan involution $\theta$ gives the Cartan decomposition $\mathfrak{g}=\mathfrak{k} \oplus \mathfrak{q}$ onto $\pm 1$ eigenspaces. The tangent bundle can be written as $T(G / K)=G \times_{K} \mathfrak{q}$.
$G$ has the Iwasawa decomposition $K A N$ which translates into the Lie algebra level as $\mathfrak{g}=\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{q}$. If we denote the centralizer of $A$ in $K$ by $M$ then $P:=M A N$ is a minimal parabolic subgroup of $G$, i.e. $G / P$ is the maximal boundary of the symmetric space $G / K$.

The Lie algebra $\mathfrak{g}$ then splits as $\overline{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ where $\overline{\mathfrak{n}}=\theta \mathfrak{n}$. The tangent bundle of the boundary is then $T(G / P)=G \times_{P} \overline{\mathfrak{n}}$. Sometimes $\mathfrak{m} \oplus \mathfrak{a}$ is denoted also by $\mathfrak{g}_{0}, \mathfrak{n}$ by $\mathfrak{g}_{+}$and $\overline{\mathfrak{n}}$ by $\mathfrak{g}_{-}$. In fact there is an element $E \in \mathfrak{a}$ that has integral eigenvalues acting on $\mathfrak{g}$ such that $[E, X]=j X, j$ is positive for $X \in \mathfrak{g}_{+}$and negative for $X \in \mathfrak{g}_{-}$. This element induces for certain $k \in \mathbb{N}$ a $|k|$-grading of $\mathfrak{g}$, i.e. a splitting

$$
\mathfrak{g}=\mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{1} \oplus \mathfrak{g}_{0} \oplus \mathfrak{g}_{1} \oplus \ldots \oplus \mathfrak{g}_{k}
$$

that satisfies $\left[\mathfrak{g}_{i}, \mathfrak{g}_{j}\right] \subset \mathfrak{g}_{i+j}$, hence $E$ is called the grading element.

## 1. Invariant operators in parabolic geometries

Definition 6 (Cartan connection). A Cartan connection of type $(\mathfrak{g}, P)$ on a manifold $N$ is a 1-form $\eta: T \mathcal{G} \rightarrow \mathfrak{g}$ defined on a principal $P$-bundle $\pi: \mathcal{G} \rightarrow N$ such that
(1) $(\forall y \in \mathcal{G}) \quad \eta_{y}: T_{y} \mathcal{G} \rightarrow \mathfrak{g}$ is an isomorphism
(2) $(\forall y \in \mathcal{G}, X \in \mathfrak{p}) \quad \eta_{y}^{-1}(X)=\xi_{X, y}$ where $\xi_{X}$ is the left invariant vector field generated by $X$
(3) $(\forall p \in P) \quad A d(p) \cdot r_{p}^{*} \eta=\eta$ where $r_{p}$ stands for the right $P$ action on $\mathcal{G}$

Let $\mathbb{V}$ be a finite dimensional $P$-module carrying a representation $\lambda$ and $V:=\mathcal{G} \times{ }_{P} \mathbb{V}$ the corresponding associated vector bundle. As usual, we will identify the space of sections $C^{\infty}(M, V)$ with the space of $P$-equivariant maps $C^{\infty}(\mathcal{G}, \mathbb{V})^{P}$ where the action is $(p \cdot f)(y):=$ $\lambda(p)\left(f \circ r_{p}\right)(y)$.

Definition 7 (Invariant derivative). Let $f \in C^{\infty}(\mathcal{G}, \mathbb{V})^{P}, X \in \overline{\mathfrak{n}}$. Then

$$
\begin{gathered}
\nabla^{\eta}: C^{\infty}(\mathcal{G}, \mathbb{V})^{P} \rightarrow C^{\infty}\left(\mathcal{G}, \overline{\mathfrak{n}}^{*} \otimes \mathbb{V}\right) \\
\nabla_{X}^{\eta} f:=d f\left(\eta^{-1}(X)\right)
\end{gathered}
$$

is called the invariant derivative on $V$.
Our interest will be focused on the flat (homogeneous) case where $\eta$ is the Maurer-Cartan form. Then the invariant derivative is given by the expression

$$
\nabla_{X}^{\eta} f=\left.\frac{d}{d t}\right|_{0}\left(f \circ r_{\exp t X}\right),
$$

where $r_{\exp t X}$ is the group multiplication from right on $G$.
The problem with the invariant derivative is that it does not map to $C^{\infty}\left(\mathcal{G}, \overline{\mathfrak{n}}^{*} \otimes \mathbb{V}\right)^{P}$. To remedy this, we start to work with semiholonomic jet modules. First we define a $P$-module structure on $\bar{J}^{1} \mathbb{V}=\mathbb{V} \oplus(\overline{\mathfrak{n}} \otimes \mathbb{V})$ such that for $\left(\phi_{0}, \phi_{1}\right) \in \bar{J}^{1} \mathbb{V}$

$$
\left[Z \cdot\left(\phi_{0}, \phi_{1}\right)\right](X)=\left(Z \cdot \phi_{0}, Z \cdot \phi_{1}(X)-\phi_{1}\left([Z, X]_{\overline{\mathfrak{n}}}\right)-\lambda\left([Z, X]_{\mathfrak{p}}\right) \phi_{0}\right)
$$

where $Z \in \mathfrak{p}$. It is easy to show that the map $j_{\eta}^{1}: f \rightarrow\left(f, \nabla^{\eta} f\right)$ then maps $C^{\infty}(\mathcal{G}, \mathbb{V})^{P}$ to $C^{\infty}\left(\mathcal{G}, \mathbb{V} \oplus\left(\overline{\mathfrak{n}}^{*} \otimes \mathbb{V}\right)\right)^{P}$. This can be a beginning of an inductive definition

Definition 8 (Semiholonomic jet modules). The first order semiholonomic jet module $\bar{J}^{1} \mathbb{V}$ is a set of pairs $\left(\phi_{0}, \phi_{1}\right) \in \mathbb{V} \oplus(\overline{\mathfrak{n}} \otimes \mathbb{V})$ with the action of $\mathfrak{p}$ given by the formula above. Let us suppose that $\bar{J}^{k-1}$ is well defined and consider the action of $\mathfrak{p}$ given by the first order action on

$$
\left(\left(\phi_{0}, \phi_{1}, \ldots, \phi_{k-1}\right),\left(\phi_{1}^{\prime}, \phi_{2}^{\prime}, \ldots, \phi_{k-1}^{\prime}, \phi_{k}\right)\right) \in \bar{J}^{1}\left(\bar{J}^{k-1} \mathbb{V}\right)
$$

The $k$-th order semiholonomic jet module $\bar{J}^{k} \mathbb{V}:=\oplus_{0}^{k}\left(\overline{\mathfrak{n}}^{*}\right)^{i} \otimes \mathbb{V}$ is a set of vectors ( $\phi_{0}, \phi_{1}, \ldots, \phi_{k}$ ) identified with a $P$-submodule of $\bar{J}^{1} \bar{J}^{k-1} \mathbb{V}$ of elements satisfying $\phi_{i}=\phi_{i}^{\prime}$ for $1 \leq i \leq k-1$.

The inclusion $j_{\eta}^{k} f$ of $J^{k} V$ into $\mathcal{G} \times{ }_{P} \bar{J}^{k} \mathbb{V}$ maps $j^{k} f$ to $\left(f, \nabla^{\eta} f, \ldots\right.$, $\left.\left(\nabla^{\eta}\right)^{k} f\right)$ and the image of $f \in C^{\infty}(\mathcal{G}, \mathbb{V})^{P}$ is in $C^{\infty}\left(\mathcal{G}, \bar{J}^{1} \mathbb{V}\right)^{P}$, thus it is a well defined map of sections of bundles.

Definition 9 (Strongly invariant operators). Let $\mathbb{V}$ and $\mathbb{W}$ be $P$ modules and $\Phi: \bar{J}^{k} \mathbb{V} \rightarrow \mathbb{W}$ be a $P$-homomorphism. Then $\Phi$ induces a bundle map $\Phi: \mathcal{G} \times{ }_{P} \bar{J}^{k} \mathbb{V} \rightarrow \mathcal{G} \times{ }_{P} \mathbb{W}$ and thus an invariant differential operator $\Phi \circ j_{\eta}^{k}$ from $V$ to $W$.

The usual strategy for constructing strongly invariant operators ([CSS01],[RCD03]) is to consider $\Phi$ as a $G_{0}$-morphism first and then to find algebraic conditions for it being also a $P$-morphism. $\mathbb{W}$ is an irreducible $P$-module, so the action of the nilpotent part of $P$ is trivial. On the other hand $\bar{J}^{k} \mathbb{V}$ is $G_{0}$-reducible and the $P$-action maps between its $G_{0}$-components. The image of the $P$-action is a $G_{0}$-submodule of
$\bar{J}^{k} \mathbb{V}$ that must be anihilated by $\Phi$ to obtain $P$-equivariancy. This leads in [SS99] to the following lemma for first order operators:

Lemma 7. Let $\mathbb{V}$, $\mathbb{W}$ be irreducible $P$-modules. Then a $G_{0}$-module homomorphism $\Phi: \bar{J}^{1} \mathbb{V} \rightarrow \mathbb{W}$ is a $P$-module homomorphism iff $\Phi$ factors through $\mathbb{V} \oplus\left(\mathfrak{g}_{1}^{*} \otimes \mathbb{V}\right)$ and for all $Z \in \mathfrak{g}_{1}, v \in \mathbb{V}$

$$
\Phi\left(\sum_{\alpha} Y^{\alpha} \otimes\left[Z, X_{\alpha}\right] \cdot v\right)=0
$$

where $Y^{\alpha}$ is a basis of $\mathfrak{g}_{1}$ and $X_{\alpha}$ is the dual basis of $\mathfrak{g}_{-1}$.
Further considerations in [SS99] consisting mainly in computing Casimir operator eigenvalues on irreducible $G_{0}$-components of the so called restricted jet module $\mathbb{V} \oplus\left(\mathfrak{g}_{1}^{*} \otimes \mathbb{V}\right)$ give us the characterization of invariant first order operators:

Theorem 2. Let $\mathfrak{g}$ be a graded Lie algebra and $\mathfrak{g}^{\mathbb{C}}$ its graded complexification. Then $\mathfrak{g}_{j}=\mathfrak{g} \cap \mathfrak{g}_{j}^{\mathbb{C}}$. Let $\mathbb{V}_{\lambda}$ be a complex irreducible representation of $\mathfrak{g}_{0}$ with highest weight $\lambda$ and let $\mathfrak{g}_{1}^{\mathbb{C}}=\sum_{j} \mathfrak{g}_{1}^{j}$ be a decomposition of $\mathfrak{g}_{1}^{\mathbb{C}}$ into irreducible $\mathfrak{g}_{0}$-submodules and let $\alpha_{j}$ be highest weights of $\mathfrak{g}_{1}^{j}$. Suppose that

$$
\mathfrak{g}_{1} \otimes_{\mathbb{R}} \mathbb{V}_{\lambda}=\mathfrak{g}_{1}^{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{V}_{\lambda}=\sum_{j} \sum_{\mu_{j}} \mathbb{V}_{\mu_{j}}^{j}
$$

be a decomposition of the product into irreducible $\mathfrak{g}_{0}$-modules and let $\pi_{\lambda, \mu_{j}}$ be the corresponding projections. Let us denote by $\rho_{0}$ the half-sum of positive roots for $\mathfrak{g}_{0}$ and let us define constants $c_{\lambda, \mu_{j}}$ by

$$
c_{\lambda, \mu_{j}}=\frac{1}{2}\left[\left(\mu_{j}, \mu_{j}+2 \rho_{0}\right)-\left(\lambda, \lambda+2 \rho_{0}\right)-\left(\alpha_{j}, \alpha_{j}+2 \rho_{0}\right)\right]
$$

Then the operator $D_{j, \mu_{j}}: \pi_{\lambda, \mu_{j}} \circ \nabla^{\eta}$ is an invariant first order differential operator iff $c_{\lambda, \mu_{j}}=0$. Moreover, all first order invariant differential operators acting on sections of $V_{\lambda}$ are obtained (modulo a scalar multiple and curvature terms) in such a way.

This result was generalized in [RCD03] for a certain class of operators of higher order. Before we state it, we shall define $\mathbb{V}_{b}$ for a $k$-tuple $b=\left(b_{1}, \ldots, b_{k}\right)$. It is simply the $b_{k}$-th irreducible component of $\otimes^{k} \mathfrak{g}_{1}^{*} \otimes \mathbb{V}_{b_{k-1}}$ where $\mathbb{V}_{b_{k-1}}$ is a $b_{k-1}$-th component of $\otimes^{k-1} \mathfrak{g}_{1}^{*} \otimes \mathbb{V}_{b_{k-2}}$ and so on, $\mathbb{V}_{1}$ is a $b_{1}$-th component of $\mathfrak{g}_{1}^{*} \otimes \mathbb{V}$ for an arbitrarily chosen numbering of components.

Theorem 3. Let $\alpha$ be a positive root in $\mathfrak{g}_{1}^{*}$. In the case that $\mathfrak{g}$ has roots of different lengths, we shall suppose that $\alpha$ is a long root. Let $\lambda, \mu$ be two dominant integral weights of $\mathfrak{g}_{0}$ satisfying

$$
\mu+\rho_{0}=\sigma_{\alpha}\left(\lambda+\rho_{0}\right)=\lambda+\rho_{0}-\left\langle\lambda+\rho_{0}, \alpha\right\rangle \alpha .
$$

Interchanging $\lambda$ and $\mu$ if necessary we can suppose that the integer $k:=-\left\langle\lambda+\rho_{0}, \alpha\right\rangle$ is positive. Then:
(1) There is a unique irreducible component $\mathbb{V}_{\mu}$ with highest weight $\mu$ in $\left(\otimes^{k} \overline{\mathfrak{n}}^{*}\right) \otimes \mathbb{V}_{\lambda}$. Furthemore, this component belongs to $S^{k} \mathfrak{\mathfrak { n }}^{*} \otimes \mathbb{V}_{\lambda}$ and is of the form $\mathbb{V}_{b}$ where $\mathbb{V}_{b_{j}}=\mathbb{V}_{\lambda+j \alpha}$ for $b=\left(b_{1}, \ldots, b_{k}\right)$.
(2) If $\pi: \bar{J}^{k} \mathbb{V}_{\lambda} \rightarrow \mathbb{V}_{\mu}$ is the corresponding $G_{0}$-invariant projection, then $\pi$ is in fact a P-homomorphism and the operator $\pi \circ\left(\nabla^{\eta}\right)^{k}$ is an invariant differential operator of order $k$ from sections of $V_{\lambda}$ to sections of $V_{\mu}$.

Remark 1. This theorem is in fact in [RCD03] formulated in terms of the differential operators $D^{(j)}$ called there Ricci corrected derivatives. This involves defining an isomorphism of vector bundles mapping $\left(f, \nabla^{\eta} f, \ldots,\left(\nabla^{\eta}\right)^{k} f\right) \in \mathcal{G} \times_{P} \bar{J}^{k} \mathbb{V}$ to $\left(f, D^{(1)} f, \ldots, D^{(k)} f\right) \in$ $\oplus_{0}^{k}\left(\otimes^{j} T^{*} M\right) \otimes V$. This mapping depends on the Weyl structure that we choose but the operator $\pi \circ D^{(k)}$ that we obtain from this is an invariant operator independent of the Weyl structure.

## 2. Invariant operators in the symmetric space

Next we turn to the invariant differential operator on the interior symmetric space $G / K$. For our purposes it would be convenient to consider a certain simple subclass of them, which nevertheless contains all first order invariant differential operators between any two vector bundles $G \times_{K} E_{1}$ and $G \times_{K} E_{2}$, for $\mathbb{E}_{1}, \mathbb{E}_{2}$ irreducible $K$-modules, as shown in [Rei01].

Definition 10 (Generalized Stein-Weiss gradient). Let $\mathbb{E}$ be a $K$ module, $f \in C^{\infty}(G, \mathbb{E})^{K}$, $K$ be acting on $\mathfrak{q}$ by Ad. Then there is a first order differential operator

$$
\nabla f(g)(X):=X f(g)=\left.\frac{d}{d t}\right|_{0} f(g \exp (t X))
$$

where $X \in \mathfrak{q}$. Moreover $\nabla f \in C^{\infty}\left(G, \mathfrak{q}^{*} \otimes \mathbb{E}\right)^{K}$.
We can simply check that $\nabla f$ is really $K$-equivariant. If we denote by $\rho$ the representation of $K$ on $\mathbb{E}$, then for $g \in G, k \in K, f \in$ $C^{\infty}(G, \mathbb{E})^{K}$ we have

$$
\begin{aligned}
\nabla_{X} f(g k) & =\left.\frac{d}{d t}\right|_{0} f\left(g k \mathrm{e}^{t X}\right) \\
& =\left.\frac{d}{d t}\right|_{0} f\left(g \exp ^{\operatorname{Ad}(k) X} k\right) \\
& =\left.\rho\left(k^{-1}\right) \frac{d}{d t}\right|_{0} f\left(g \mathrm{e}^{\operatorname{Ad}(k) X}\right) \\
& =\rho\left(k^{-1}\right) \nabla_{\operatorname{Ad}(k) X} f(g)
\end{aligned}
$$

The following proposition from [Oer00] says that $\nabla$ is in fact the Levi-Civita connection:

Proposition 1. The Stein-Weiss gradient defines a $G$-equivariant covariant derivative with zero torsion. In particular the canonical metric on $G / K$ is parallel, so that on tensors or spinors $\nabla$ is the canonical Levi-Civita connection.

All first order differential operators between vector bundles associated to irreducible $K$-modules are then obtained by projection onto irreducible components of $\mathfrak{q}^{*} \otimes \mathbb{E}$. Moreover, the operator $\operatorname{proj}_{j} \circ(\nabla)^{k}$, where $\operatorname{proj}_{j}$ is a projection onto the $j$-th component of $\left(\otimes^{k} \mathfrak{q}^{*}\right) \otimes \mathbb{E}$ is an invariant differential operator.

## CHAPTER 4

## Real forms

Real simple Lie algebras can be classified in an analogous way as complex simple Lie algebras. In fact, to each complex simple Lie algebra corresponds a finite set of real forms. In the beginning of this chapter we introduce a common way of denoting the real forms in terms of Satake diagrams. Next we turn our attention to real irreducible representations of real simple Lie algebras. The references for this chapter are $[$ OniV88], [Oni03], [Sil01], [Sil03], [Wis01].

## 1. Real and complex Lie algebras

Definition 11. A complexification $V_{0}(\mathbb{C})$ of a real vector space $V_{0}$ is a complex vector space $V_{0} \oplus i V_{0}$. A real form of a complex vector space $V$ is a real vector space $V_{0}$ such that $V_{0}(\mathbb{C})=V$. A realification $V_{\mathbb{R}}$ of a complex vector space $V$ is the same vector space regarded as a real vector space. A real structure in a complex vector space is an antiinvolution, i.e. a complex conjugation with respect to any real form. A complex structure in a real vector space is an automorphism $J$ satisfying $J^{2}=-\mathrm{Id}$. A quaternionic structure in a complex vector space is an antiautomorphism $J$ satisfying, $J^{2}=-\mathrm{Id}$, i.e. a second complex structure anticommuting with the default one. A complex vector space $V$ can be regarded as a pair $\left(V_{\mathbb{R}}, J\right)$, where $J: v \rightarrow i v$. A complex conjugate vector space $\bar{V}$ is then $\left(V_{\mathbb{R}},-J\right)$.

A complexification of a Lie algebra is a complexification of the underlying vector space compatible with the Lie bracket. Real forms of a complex Lie algebra $\mathfrak{g}$ are in 1-1 correspondence with real structures on it, where to a real structure $\sigma$ corresponds its fixed-point space $\mathfrak{g}^{\sigma}$. Two real forms $\mathfrak{g}^{\sigma_{0}}, \mathfrak{g}^{\sigma_{1}}$ of $\mathfrak{g}$ are isomorphic iff there is an $\alpha \in$ Aut $\mathfrak{g}$ such that $\sigma_{0}=\alpha \sigma_{1} \alpha^{-1}$.

The "rough" classification of real simple Lie algebras is given by the following theorem from [Oni03]:

Theorem 4. Let $\mathfrak{g}$ be a complex simple Lie algebra. Then any real form of $\mathfrak{g}$ and, whenever $\mathfrak{g}$ is non-commutative, the realification $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$, are real simple Lie algebras. Conversely any real simple Lie algebra is either a real form of a complex simple Lie algebra or a realification of a non-commutative complex simple Lie algebra.

## 2. Types of real forms

There are always at least two real forms of any complex semisimple Lie algebra, the split form and the compact form. Their corresponding real structures are given by their action on the cannonical generators. The set of canonical generators of $\mathfrak{g}$ with respect to a set of simple roots $\Pi=\left\{\alpha_{i}\right\}_{1}^{l}$ is given as $\mathbb{U}_{\Pi}\left\{h_{i}, e_{i}, f_{i}\right\}$, where $h_{i}$ is the co-root of $\alpha_{i}$ and $e_{i}, f_{i}$ are the root vectors of $\alpha_{i},-\alpha_{i}$ respectively. Each of the triples $\left\{h_{i}, e_{i}, f_{i}\right\}$ satisfies the commutation relations of the standard generators of $\mathfrak{s l}(2, \mathbb{C})$.

The real span of $\mathbb{U}_{\Pi}\left\{h_{i}, e_{i}, f_{i}\right\}$ is called the split (or normal) real form of $\mathfrak{g}$. The corresponding real structure $\sigma$ is the identity on generators. For example for $X \in \mathfrak{s l}(n, \mathbb{C})$ it is the complex conjugation $\sigma(X)=\bar{X}$.

If we choose $\Pi^{\prime}=-\Pi$ then we get a new system of canonical generators $\mathbb{U}_{\Pi}\left\{-h_{i},-f_{i},-e_{i}\right\}$. An $\omega \in$ Aut $\mathfrak{g}$ satisfying

$$
\begin{aligned}
\omega\left(h_{i}\right) & =-h_{i} \\
\omega\left(e_{i}\right) & =-f_{i} \\
\omega\left(f_{i}\right) & =-e_{i}
\end{aligned}
$$

is called the Weyl involution.
The compact real structure is given as $\tau:=\omega \sigma$. It is characterised by the property that the Hermitian form $B_{\sigma}(x, y):=-B(x, \sigma y)$, where $x, y \in \mathfrak{g}, B$ is the Killing form and $\sigma$ is a real form, is positive definite iff $\sigma$ is a compact real structure with respect to some system of simple roots $\Pi$. For any two compact real structures $\tau_{0}, \tau_{1}$ of $\mathfrak{g}$ there is an $\alpha \in \operatorname{Int} \mathfrak{g}$ such that $\tau_{0}=\alpha \tau_{1} \alpha^{-1}$, in other words, the compact form is essentially unique.

The following theorem from [Oni03] shows that classification of conjugacy classes of real structures can be reduced to classification of involutions:

THEOREM 5. Let $\sigma$ be a real structure on a complex semisimple Lie algebra $\mathfrak{g}, \tau$ be a fixed compact real structure. Then there is $\alpha \in \operatorname{Int} \mathfrak{g}$ such that for $\sigma^{\prime}=\alpha \sigma \alpha^{-1}$ the real structures $\sigma^{\prime}$ and $\tau$ are compatible, i.e. $\sigma^{\prime} \tau=\tau \sigma^{\prime}$. The mapping $\sigma \rightarrow \theta:=\sigma^{\prime} \tau$ determines a bijection between conjugacy classes of antiinvolutions and involutions by inner automorphisms (or by arbitrary automorphisms) of $\mathfrak{g}$. This bijection does not depend on the choice of $\tau$.

The involution $\theta$ is called the Cartan involution of a real form. For a compact real form $\theta=\mathrm{Id}$ and for the split form $\theta=\omega$. Let us as an example state a list of the real forms of $\mathfrak{s l}(n, \mathbb{C})$ together with their
real structures and Cartan involutions:

$$
\begin{array}{lll}
\mathfrak{s l}(n, \mathbb{R}) & \sigma(X)=\bar{X} & \theta(X)=-X^{T} \\
\mathfrak{s u}(p, q) & \sigma(X)=-I_{p, q} \bar{X}^{T} I_{p, q} & \theta(X)=\operatorname{Ad}\left(I_{p, q}\right)(X) \\
\mathfrak{s l}(m, \mathbb{H}) & \sigma(X)=S_{m} \bar{X} S_{m}^{-1} & \theta(X)=\operatorname{Ad}\left(S_{m}\right)\left(-X^{T}\right)
\end{array}
$$

where the last real form exists only for $n=2 m$ even and

$$
I_{p, q}=\left(\begin{array}{cc}
-I_{p} & 0 \\
0 & I_{q}
\end{array}\right) \quad S_{m}=\left(\begin{array}{cc}
0 & -I_{m} \\
I_{m} & 0
\end{array}\right)
$$

where $I_{m}$ is the identity matrix of rank $m$.

## 3. Involutions

Definition 12. Let Aut $\Pi$ be the set of bijections of $\Pi$ preserving the Cartan matrix. Then there is a map $\Psi:$ Aut $\Pi \rightarrow$ Aut $\mathfrak{g}$ such that $\Psi(s)=\hat{s}$ iff

$$
\begin{aligned}
& \hat{s}\left(h_{i}\right)=h_{s(i)} \\
& \hat{s}\left(e_{i}\right)=e_{s(i)} \\
& \hat{s}\left(f_{i}\right)=f_{s(i)}
\end{aligned}
$$

on canonical generators. Such $\hat{s}$ is called the diagram automorphism of $\mathfrak{g}$ corresponding to $s$.

Theorem 6. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then

$$
\text { Aut } \mathfrak{g}=\operatorname{Int} \mathfrak{g} \rtimes \Psi(\operatorname{Aut} \Pi)
$$

This theorem gives an explanation, why the list of real forms of $\mathfrak{s l}(n, \mathbb{C})$ is complete. For $n \geq 3$ we know that $\operatorname{Aut} \Pi=\mathbb{Z}_{2}=\{\operatorname{Id}, s\}$ and, since $X$ and $-X^{T}$ generally have different eigenvalues, the Weyl involution $\omega(X)=-X^{T}$ is an outer automorphism which must be conjugated to $\Psi(s)$.

Any outer involution $\theta$ of $\mathfrak{g}$ can thus be written as $\theta=\phi \omega$, where $\phi=\operatorname{Ad} C, C \in S L(n, \mathbb{C})$. It is easy to show that due to involutivity of $\theta$ the matrix $C$ can be either symmetric or antisymmetric. The former case leads to $\theta$ defining $\mathfrak{s l}(n, \mathbb{R})$ and the latter works only for $n=2 m$ and leads to $\mathfrak{s l}(m, \mathbb{H})$. On the other hand, one can show that conjugacy classes of inner involutions are classified by $\operatorname{Ad} I_{p, q}, p \leq q$. For $n=2$ there is no outer isomorphism, so either $\theta=\operatorname{Id}$ or $\theta=\operatorname{Ad} I_{1,1}$.

There is a unique element $w_{0} \in W_{\mathfrak{g}}$ of the Weyl group such that $w_{0}(D)=-D$, where $D$ is the dominant Weyl chamber with respect to $\Pi$. The element $-w_{0}$ is an isometry of $D$ and so it induces an involutive automorphism $\nu \in$ Aut $\Pi$. Since the Weyl involution $\left.\omega\right|_{\Pi}=-\mathrm{Id}$, the decomposition according to theorem 6 which reads $\omega=\phi \mu$, where $\phi \in \operatorname{Int} \mathfrak{g}$ and $\mu \in \Psi(\operatorname{Aut} \Pi)$, can be written as

$$
\omega=\phi \hat{\nu}
$$

since $\phi=w_{0}$, when $w_{0}$ is regarded as an inner automorphism of the whole $\mathfrak{g}$, and $\left.\hat{\nu}\right|_{\Pi}=-w_{0}$, this time $-w_{0}$ regarded as a permutation on the set of simple roots.

By case by case analysis we can prove the following theorem on $\nu$ from [Oni03]:

## Theorem 7.

(1) If $\mathfrak{g}$ is a non-commutative complex simple Lie algebra, then $\nu$ is non-trivial precisely in the cases $\mathfrak{g}=A_{l}, l \geq 2 ; D_{2 m+1}$, $m \geq 1 ; E_{6}$. In these cases, $\nu$ is the only non-trivial symmetry of the Dynkin diagram.
(2) If $\mathfrak{g}$ is semisimple, then $\nu$ induces the automorphism of the previous assertion on each of the simple components of $\mathfrak{g}$.
(3) $\nu$ lies in the center of Aut $\Pi$.

The Cartan involution $\theta=\sigma \tau$, where $\sigma$ is the real structure defining a real form $\mathfrak{g}_{0}:=\mathfrak{g}^{\sigma}$ defines the Cartan decomposition $\mathfrak{g}=\mathfrak{g}_{+} \oplus \mathfrak{g}_{-}$, where $\mathfrak{g}_{ \pm}$are $\pm 1$ eigenspaces of $\theta$. Since $\theta$ commutes with $\sigma$ and $\tau$, the real form $\mathfrak{g}_{0}$ and the compact real form $\mathfrak{u}:=\mathfrak{g}^{\tau}$ decompose as

$$
\mathfrak{g}_{0}=\mathfrak{k} \oplus \mathfrak{p} \quad \mathfrak{u}=\mathfrak{k} \oplus i \mathfrak{p}
$$

where $\mathfrak{k}=\left(\mathfrak{g}_{0}\right)_{+}$and $\mathfrak{p}=\left(\mathfrak{g}_{0}\right)_{-}$. Let us choose a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ and find an abelian subalgebra $\mathfrak{t}_{0}^{+} \subset \mathfrak{k}$ such that $\mathfrak{t}_{0}:=\mathfrak{t}_{0}^{+} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}_{0}$.

The roots from the subset

$$
\Delta_{c}:=\left\{\alpha \in \Delta \mid \theta^{T}(\alpha)=\alpha\right\}=\left\{\alpha|\alpha|_{\mathfrak{a}}=0\right\}
$$

are called compact roots and these from $\Delta_{n c}:=\Delta \backslash \Delta_{c}$ are non-compact roots. It is possible to choose a system of positive roots $\Delta^{+}$in such a way that the following lemma holds:

Lemma 8. There exists an involution $\kappa: \Pi_{n c} \rightarrow \Pi_{n c}$ such that for any $\alpha \in \Pi_{n c}$ we have

$$
\theta^{T}(\alpha)=-\kappa(\alpha)-\sum_{\alpha_{i} \in \Pi_{c}} c_{i} \alpha_{i}
$$

where $c_{i}$ are non-negative integers.

## 4. Satake diagrams

The Satake diagram of $\mathfrak{g}_{0}$ is defined as the Dynkin diagram corresponding to $\Pi$ where the compact roots are denoted by black circles, non-compact roots by white circles and whenever $\alpha \neq \kappa(\alpha)$, the corresponding two white circles are joined by an arrow.

Let us denote by $\nu$ the symmetry of the Dynkin diagram induced by $-w_{0}$ as above, by $\nu_{c}$ the similar symmetry of the sub-diagram of black vertices and by $s$ the symmetry defined by the decomposition
$\theta=\phi \hat{s}$ according to theorem 6. Then there is the following relationship between the various involutions of Aut $\Pi$ :

Theorem 8. The involution $s_{0}:=s \nu$ of the Satake diagram leaves invariant both the subset of white vertices and the subset of black ones. On the white vertices, it induces the involution $\kappa$ depicted by the arrows, while on the black ones it coincides with $\nu_{c}$. If $\mathfrak{g}_{0}$ is simple, then $\nu_{c}=\mathrm{Id}$ except for the following cases: $\mathfrak{g}_{0}=\mathfrak{s u}(n), \mathfrak{s u}(k, n-k), \mathfrak{s o}(k, 2 n-k)$ ( $n-k$ odd), and the compact form of $E_{6}$.

The complete list of real forms and Satake diagrams for $D_{n}$ is given below:

and the compact form $\mathfrak{s o}(2 n)$ which does not have a Satake diagram or could be written with all nodes black.

## 5. Real representations

Now let us turn our attention to representations.
Definition 13. Let $\mathfrak{g}_{0}$ be a real form of $\mathfrak{g}$. A real representation of $\mathfrak{g}_{0}$ is a homomorphism $\rho: \mathfrak{g}_{0} \rightarrow \mathfrak{g l}\left(V_{0}\right)$, where $V_{0}$ is a real vector space. A complex representation of $\mathfrak{g}_{0}$ is a homomorphism $\rho: \mathfrak{g}_{0} \rightarrow \mathfrak{g l}(V)$ for $V$ a complex vector space. A homomorphism $\rho: \mathfrak{g} \rightarrow \mathfrak{g l}(V)$ will be also called a complex representation. For a real representation of $\mathfrak{g}_{0}$ on $V_{0}$ we have a two-step complexification process: $\rho^{\mathbb{C}}$ is an extension of $\rho$ to a complex linear operator on $V_{0}(\mathbb{C})$ and by $\rho(\mathbb{C})$ we shall denote the homomorphism of complex Lie algebras $\rho(\mathbb{C}): \mathfrak{g} \rightarrow \mathfrak{g l}(V)$. A realification $\rho_{\mathbb{R}}$ of a complex representation $\rho$ of $\mathfrak{g}_{0}$ on $V$ is the same representation considered as a representation on $V_{\mathbb{R}}$. We can realify a representation of $\mathfrak{g}$ on $V$, too, this will be the representation of $\mathfrak{g}_{\mathbb{R}}$ on $V_{\mathbb{R}}$.

A complex structure $J$ in a real vector space $V_{0}$ is said to be invariant under a real representation $\rho$ iff $\forall x \in \mathfrak{g}_{0} \rho(x) J=J \rho(x)$. Then $\rho$ can be regarded as a complex representation on $\left(V_{0}, J\right)$ and the original $\rho$ is its realification.

A real structure $S$ in a complex vector space $V$ is invariant under a complex representation $\rho$ iff $\forall x \in \mathfrak{g}_{0} \rho(x) S=S \rho(x)$. Then $V_{0}=V^{S}$ is an invariant subspace that carries a real subrepresentation $\rho_{0}$ such that $\rho_{0}^{\mathbb{C}}=\rho$.

Similarly an invariant quaternionic structure on a complex vector space $V$ makes a complex representation $\rho$ a homomorphism $\mathfrak{g}_{0} \rightarrow$ $\mathfrak{g l}_{m}(\mathbb{H})$ where $m$ is a half of the complex dimension of $V$.

Let $\rho$ be a complex representation in a vector space $V$, whose action in a chosen basis of $V$ is given by a matrix multiplication by a matrix function $x \rightarrow C_{i j}(x)$. Then the same representation in a vector space $\bar{V}$ will be given as multiplication by $x \rightarrow C_{i j} \bar{j}(x)$ and denoted by $\bar{\rho}$. These two representations are not necesarilly equivalent. It holds for any complex representation $\rho: \mathfrak{g}_{0} \rightarrow \mathfrak{g l}(V)$ that $\left(\rho_{\mathbb{R}}\right)^{\mathbb{C}}$ is equivalent to $\rho+\bar{\rho}$.

Theorem 9. There are two disjoint classes of irreducible real representations.
(1) The class I consists of all irreducible real representations that admit no invariant complex structure. In this case, $\rho^{\mathbb{C}}$ is an irreducible complex representation admitting an invariant real structure.
(2) The class II consists of all irreducible real representations that admit an invariant complex structure. Then they have the form $\rho=\rho_{\mathbb{R}}^{\prime}$ where $\rho^{\prime}$ is an irreducible complex structure admitting no invariant real structure.

## 6. Types of real representations

An invariant real structure under a complex representation $\rho: \mathfrak{g}_{0} \rightarrow$ $\mathfrak{g l}(V)$ can be regarded as an isomorphism $S: V \rightarrow \bar{V}$ satisfying $S \rho(x)=$ $\bar{\rho}(x)$ for $x \in \mathfrak{g}_{0}$, i.e. as an isomorphism between $\rho$ and $\bar{\rho}$. We shall thus call a representation $\rho$ satisfying $\rho \sim \bar{\rho}$ a self-conjugate representation. A complex irreducible representation that is not self-conjugate must be a component of a complexification of a real irreducible representation of class II, according to the two previous theorems. A self-conjugate representation either admits an invariant real structure and then it is a complexification of a real irreducible representation of class I, or it does not and then one can show that it admits a quaternionic structure. The indicator of whether a self-conjugate representation admits either a real or a quaternionic structure is called the Cartan index:

Definition 14. Let $\rho: \mathfrak{g}_{0} \rightarrow \mathfrak{g l}(V)$ be a self-conjugate irreducible complex representation and $S$ be an antiautomorphism of $V$ commuting with $\rho$. The Cartan index $\varepsilon(\rho)=\operatorname{sgn} c$ where $S^{2}=c \mathrm{Id}$.

We have seen that for $n$ even, there are two kinds of outer Cartan involutions of $\mathfrak{s l}(n, \mathbb{C})$, namely $\theta_{1}(X):=\omega(X)=-X^{T}$ and $\theta_{2}(X):=$
$\left[\left(\operatorname{Ad} S_{m}\right) \omega\right](X)=S_{m} X^{T} S_{m}$. Let us write them in a unified way $\theta_{i}=\left(\operatorname{Ad} B_{i}\right) \omega$, where $B_{0}=\operatorname{Id}$ and $B_{1}=S_{m}$, and denote by $\theta_{i}^{\prime}=\alpha \theta_{i} \alpha^{-1}$ different representatives of these two classes of Cartan involutions, where $\alpha:=\operatorname{Ad} U, U \in S L(n, \mathbb{C})$ is an inner automorphism of $\mathfrak{s l}(n, \mathbb{C})$. Then

$$
\begin{aligned}
& \theta_{i}^{\prime}(X)=(\operatorname{Ad} U)\left(\operatorname{Ad} B_{i}\right) \omega(\operatorname{Ad} U)^{-1}(X) \\
&=-U B_{i} U^{T} X^{T} U^{T-1} B_{i}^{-1} U^{-1}=\operatorname{Ad}\left(U B_{i} U^{T}\right) \omega(X)
\end{aligned}
$$

The matrice $B_{i}$ is symmetric for $i=0$ and antisymmetric for $i=1$ and so is $U B_{i} U^{T}$. Thus any outer automorphism of $\mathfrak{s l}(n, \mathbb{C})$ looks like $(\operatorname{Ad} B) \omega$ where $B$ is either symmetric or antisymmetric, the latter case ruled out for $n$ odd.

Definition 15. Let $\rho: \mathfrak{g} \rightarrow \mathfrak{s l}(n, \mathbb{C})$ be a complex irreducible representation of a complex semisimple Lie algebra $\mathfrak{g}$ and $\theta$ be a Cartan involution of $\mathfrak{g}$. Then according to theorem 6.2 of [Oni03] there can be only one outer involution $\theta^{\prime}$ such that $\rho \theta=\theta^{\prime} \rho$. We denote by $j(\mathfrak{g}, \theta, \rho)$ the Karpelevich index, which has the value +1 if $\theta^{\prime}=(\operatorname{Ad} B) \omega$ for $B$ symmetric, and -1 if $B$ is antisymmetric.

Theorem 10. Let $\rho_{0}: \mathfrak{g}_{0} \rightarrow \mathfrak{s l}(V)$ be an irreducible complex representation of a real semisimple Lie algebra $\mathfrak{g}_{0}$ and let $\Lambda, \bar{\Lambda}$ be highest weights of the representations $\rho_{0}, \bar{\rho}_{0}$, respectively. Then $\bar{\Lambda}=s_{0}(\Lambda)$, where $s_{0}$ is the involution of the Satake diagram of $\mathfrak{g}_{0}$. In particular, $\rho_{0}$ is self-conjugate if and only if $s_{0}(\Lambda)=\Lambda$. In this case the Cartan index of $\rho_{0}$ is expressed by $\varepsilon\left(\mathfrak{g}_{0}, \rho_{0}\right)=j(\mathfrak{g}, \theta, \rho(\mathbb{C}))$.

Provided we have a way how to calculate $j(\mathfrak{g}, \theta, \rho(\mathbb{C}))$ we obtain the following classification of real irreducible representations.

Definition 16. Let $\mathfrak{g}_{0}$ be a real form of a semisimple Lie algebra $\mathfrak{g}$ and $\rho_{0}: \mathfrak{g}_{0} \rightarrow \mathfrak{g l}\left(V_{0}\right)$ be its real irreducible representation, $\rho_{0}^{\mathbb{C}}: \mathfrak{g}_{0} \rightarrow$ $\mathfrak{g l}\left(V_{0}(\mathbb{C})\right)$ its complexification.
(I) We say that $\rho_{0}$ is of the real type iff it is of class I, i.e. $\rho_{0}^{\mathbb{C}}$ is irreducible. This happens precisely when the highest weight $\Lambda$ of $\rho_{0}^{\mathbb{C}}$ satisfies $s_{0}(\Lambda)=\Lambda$ and the Cartan index $\varepsilon\left(\mathfrak{g}_{0}, \rho_{0}^{\mathbb{C}}\right)$ is equal to 1 .
(II) We say that $\rho_{0}$ is of the quaternionic type iff it is of class II and $\rho_{0}^{\mathbb{C}}$ is self-conjugate. Then $\rho_{0}^{\mathbb{C}}=\rho+\rho$. This happens when the highest weight $\Lambda$ of $\rho$ satisfies $s_{0}(\Lambda)=\Lambda$ and $\varepsilon\left(\mathfrak{g}_{0}, \rho_{0}^{\mathbb{C}}\right)=-1$.
(III) Finally $\rho_{0}$ is of the complex type iff it is of class II and $\rho_{0}^{\mathbb{C}}$ is not self-conjugate. Then $\rho_{0}^{\mathbb{C}}=\rho+\bar{\rho}$. This happens when the highest weight $\Lambda$ of $\rho$ satisfies $s_{0}(\Lambda) \neq \Lambda$.

Real irreducible representations are thus classified by highest weights of irreducible components of their complexifications. Each self-conjugate dominant weight labels a unique real irreducible representation either
of the real or quaternionic type, representations of the complex type have two labels, $\Lambda$ and $s_{0}(\Lambda)$.

The following theorem from [ZhiDa82] allows us to calculate the Karpelevich index.

THEOREM 11. Let $\rho$ be a self-conjugate complex irreducible representation of a real semisimple Lie algebra corresponding to the Cartan involution $\theta$ of $\mathfrak{g}$. Then the Karpelevich index is given as

$$
j(\mathfrak{g}, \theta, \rho(\mathbb{C}))=(-1)^{T(\rho)}
$$

where $T(\rho)$ is called the Maltsev height

$$
T(\rho)=\sum_{\Delta_{c}} r_{i} \Lambda_{i}
$$

where the sum is taken over black vertices of the Satake diagram, $\Lambda_{i}$ are koefficients of the weight $\Lambda$ of $\rho$ expressed in the basis of fundamental weights, i.e. numbers over the black vertices, and

$$
r_{i}=2 \sum_{j}\left(A^{-1}\right)_{i j}
$$

where $A$ is the Cartan matrix of the subdiagram of black vertices.
Let us calculate the indices of representations of the real forms of $\mathfrak{s o}(2 n, \mathbb{C})$. For $\mathfrak{s o}(l, 2 n-l), 0 \leq l \leq n-2$ the index of the representation

is given by $(-1)^{\sum_{i=l+1}^{n} r_{i} \Lambda_{i}}$, where $r_{i}$ is twice the sum of the inverse of

$$
A=\left(\begin{array}{ccccccc}
2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & & 0 & 0 & 0 \\
\vdots & & & \ddots & & & \vdots \\
0 & 0 & 0 & & 2 & 0 & -1 \\
0 & 0 & 0 & & -1 & 2 & 0 \\
0 & 0 & 0 & \cdots & -1 & 0 & 2
\end{array}\right)
$$

which is

$$
A^{-1}=\left(\begin{array}{ccccccc}
1 & 1 & 1 & \cdots & 1 & \frac{1}{2} & \frac{1}{2} \\
1 & 2 & 2 & & 2 & \frac{1}{3} & \frac{1}{2} \\
1 & 2 & 3 & & 3 & \frac{3}{2} & \vdots \\
\vdots & & & \ddots & & & \vdots \\
1 & 2 & 3 & & n-l-2 & \frac{n-l-2}{2} & \frac{n-l-2}{2} \\
\frac{1}{2} & 1 & \frac{3}{2} & & \frac{n-l-2}{2} & \frac{n-l}{2} & \frac{n-l-2}{2} \\
\frac{1}{2} & 1 & \frac{3}{2} & \cdots & \frac{n-l-2}{2} & \frac{n-l-2}{2} & \frac{n-l}{2}
\end{array}\right)
$$

the sums over the all but two last rows are integral, thus the corresponding $r_{i}$ is even and $r_{i} \Lambda_{i}$ does not contribute to the index. On the other hand $r_{n}=r_{n-1}=1+2+3+\ldots+(n-l-2)+(n-l-1)=$
$(n-l-1)(n-l) / 2$, so the index depends on the parity of $\Lambda_{n}+\Lambda_{n-1}$ and $n-l \bmod 4$.

For the real forms $\mathfrak{u}^{*}(n, \mathbb{H})$ the Cartan matrix is $\operatorname{diag}(2, \ldots, 2)$ and so for all $i$ corresponding to a black vertex $r_{i}=1$. Hence the index is $(-1)^{\Sigma \cdot \Lambda_{i}}$, where the sum is over the black vertices.

We shall resume this chapter with the issue of real parabolic subalgebras. According to [Yam93], standard parabolic subalgebras of a real simple Lie algebra are classified by Satake diagrams with crossed vertices. There is a restriction that we can put crosses only instead of a group of white vertices which is stable under the symmetry $s_{0}$ of the Satake diagram.

## CHAPTER 5

## Hasse and BGG diagrams for $\mathfrak{s o}(2 n)$

This chapter contains computations that apply the theory of Hasse diagrams and real representations of real semisimple Lie algebras to the case of the standard contact gradation of $D_{n}, n \geq 4$, which is given by crossing out the second vertex. First we draw the Hasse diagram and prove its generality and completeness for all $n$. Then we determine the labels over the vertices turning the Hasse diagram into the BGG diagram for the complex standard contact parabolic geometry. Finally we determine the BGG diagram for the real standard contact parabolic geometries using the fact that complexification of Lie algebra cohomology corresponding to a real representation $\mathbb{V}$ is the Lie algebra cohomology of the complexification of $\mathbb{V}$.

## 1. Preliminaries

In the complex case we can choose any regular matrix as the matrix of the defining quadratic form of $\mathfrak{s o}(2 n) \equiv D_{n}$. According to [Yam93],

$$
Q=\left(\begin{array}{cc}
0 & K  \tag{2}\\
K & 0
\end{array}\right),
$$

is a suitable choice, where $K$ is the matrix of rank $n$ with 1 's on the antidiagonal and zeros elsewhere. The Lie algebra $\mathfrak{s o}(2 n)$ then consists of matrices $X$ satisfying $X^{T} Q+Q X=0$. Such a matrix can be written in a block diagonal form

$$
X=\left(\begin{array}{rr}
A & B  \tag{3}\\
C & -A^{\prime}
\end{array}\right) \quad \text { where } B=-B^{\prime}, C=-C^{\prime}
$$

where the apostrophe means transposition with respect to the antidiagonal.

As a Cartan subalgebra $\mathfrak{h}$, the subalgebra of diagonal matrices can be chosen. Let $\lambda_{i} \in \mathfrak{h}^{*}$ take a value $a_{i}$ on $\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n},-a_{n}, \ldots\right.$, $\left.-a_{2},-a_{1}\right) \in \mathfrak{h}$. The positive roots of $\mathfrak{s o}(2 n)$ are $\lambda_{i}-\lambda_{j}$ and $\lambda_{i}+\lambda_{j}$, $1 \leq i<j \leq n$. The corresponding root vectors are antidiagonal skew matrices with only one nonzero entry above the antidiagonal. The
position of the nonzero entry is as follows:

|  | $1-2$ | $\mathbf{1 - 3}$ | $\mathbf{1 - 4}$ | $\ldots$ | $\mathbf{1 - n}$ | $\mathbf{1 + n}$ | $\ldots$ | $\mathbf{1 + 4}$ | $\mathbf{1 + 3}$ | $1+2$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |

where $2+5$ is a shorthand for $\lambda_{2}+\lambda_{5}$ etc. This is the upper part of the matrix $X \in \mathfrak{s o}(2 n)$, the white entries on the left belong to the Cartan subalgebra, the ones on the right are zeros because $X$ is skew with respect to the antidiagonal. Thus the denoted entries contain complete information about elements of the maximal nilpotent subalgebra $\mathfrak{n}$ that is a complement of the standard Borel subalgebra. The boldfaced root vectors are elements of the first graded part $\mathfrak{g}_{1}$ of the chosen parabolic subalgebra $\mathfrak{p}=\mathfrak{g}_{1} \oplus \mathfrak{g}_{2}$ and $1+2$ written in italic is the only root in the second graded part $\mathfrak{g}_{2}$. Let us denote by $\Delta^{+}$the set of positive roots of $\mathfrak{g} \equiv \mathfrak{s o}(2 n)$ and by $W$ the Weyl group of $\mathfrak{g}$.

There are two kinds of roots in $\mathfrak{s o}(2 n)$, the "plus" and "minus" ones. A root $\beta=\lambda_{k} \pm \lambda_{l}$ is in $\Phi_{\alpha} \equiv \Phi(\alpha)=\Phi\left(\lambda_{i} \pm \lambda_{j}\right)$ iff

$$
\begin{equation*}
\sigma_{\alpha} \beta=\beta-2 \frac{(\beta, \alpha)}{(\alpha, \alpha)} \alpha=\lambda_{k} \pm \lambda_{l}-\left(\lambda_{k} \pm \lambda_{l}, \lambda_{i} \pm \lambda_{j}\right)\left(\lambda_{i} \pm \lambda_{j}\right) \tag{4}
\end{equation*}
$$

is a negative root.
Lemma 9. The sets $\Phi_{\alpha}, \alpha \in \Delta^{+}$of the standard contact gradation of $\mathfrak{s o}(2 n)$ are

$$
\begin{array}{rrl}
\Phi\left(\lambda_{i}-\lambda_{j}\right)=\left\{\lambda_{i}-\lambda_{j}\right\} & \Phi\left(\lambda_{i}+\lambda_{j}\right) & =\left\{\lambda_{i}+\lambda_{j}\right\} \\
\cup\left\{\lambda_{i}-\lambda_{m} \mid i<m<j\right\} & & \cup\left\{\lambda_{i}-\lambda_{m} \mid i<m<j\right\} \\
\cup\left\{\lambda_{m}-\lambda_{j} \mid i<m<j\right\} & & \cup\left\{\lambda_{m}+\lambda_{j} \mid i<m<j\right\} \\
& \cup\left\{\lambda_{j} \pm \lambda_{m} \mid j<m\right\} \\
& & \cup\left\{\lambda_{i} \pm \lambda_{m} \mid j<m\right\}
\end{array}
$$

and thus $\forall \alpha \in \mathfrak{g}_{m}\left|\Phi_{\alpha} \cap \Delta_{\mathfrak{p}}^{+} \backslash\{\alpha\}\right|=m . q$, where $\left|\Phi_{\alpha}\right|=2 q+1, m=1,2$.
Proof. One has to check all the possible orderings of the values of $k, l, i, j$. If $\{k, l\} \cap\{i, j\}=\emptyset$ then the reflection is trivial. The remaining orderings fall into one of the following cases

$$
(k, l)=\quad \begin{array}{llll} 
& (x, i)_{x<i} & (x, j)_{x<i} & (x, j)_{i<x<j} \\
(i, j) & (i, x)_{j<x} & (j, x)_{j<x} & (i, x)_{i<x<j}
\end{array}
$$

and one has to check them all for the four different combinations of signs in $\alpha=\lambda_{i} \pm \lambda_{j}, \beta=\lambda_{k} \pm \lambda_{l}$ using the orthogonality of $\lambda_{i}$ 's.

The second part follows from considering the three cases： $\mathfrak{g}_{2} \equiv$ $\left\{\lambda_{1}+\lambda_{2}\right\}, \mathfrak{g}_{1}^{-} \equiv\left\{\lambda_{i}-\lambda_{j} \mid j \geq 3\right\}$ and $\mathfrak{g}_{1}^{+} \equiv\left\{\lambda_{i}+\lambda_{j} \mid j \geq 3\right\}$ ．Denote $\Phi_{\mathfrak{p}}(\alpha):=\Phi_{\alpha} \cap \Delta_{\mathfrak{p}}^{+} \backslash\{\alpha\}$ ，then we can visualise the result by

$$
\begin{aligned}
& \Phi_{\mathfrak{p}}\left(\lambda_{1}-\lambda_{j}\right)=\square \boldsymbol{\square + \pi \square} \\
& \Phi_{\mathfrak{p}}\left(\lambda_{2}-\lambda_{j}\right)=\text { ローローロ } \square
\end{aligned}
$$

where the rectangle denotes the positions in a general matrix from $\mathfrak{s o}(2 n)$ that were previously boldfaced，the white square is at the po－ sition of the defining root $\alpha$ and the black squares denote the other elements of $\Phi_{\alpha}$ ．

## 2．Hasse diagram

Now we have enough information to write down the Hasse diagram of $D_{4} \equiv \mathfrak{s o}(8)$ ：


The reader can readily check that in an arbitrary vertex the out－ going arrows correspond precisely to the inscribed roots．We have written the $\Phi_{\mathfrak{p}}(\alpha)$＇s on the right side for convenience．The rectangles
representing saturated sets should be understood in an obvious way black rectangles represent elements of the corresponding saturated set.

Theorem 12. Hasse diagram for the standard contact gradation of $D_{n}$ has the shape

where the number of vertices in the middle two rows is $n \equiv p+2$. Two arrows in the same rectangle which are parallel correspond to the same root, thus all the arrows fall into $4(n-2)+1$ disjoint families similar to the one denoted by double arrows and labeled the same as its unique member from the "outer belt".

Remark 2. The vertices are denoted according to the type of their respective saturated set:

where $0 \leq i \leq j \leq n$. Black squares are roots that are in every set of the corresponding type, white places are ones that are in none of them. Pictures for the type $A_{j}^{i}$ contain $i$ framed squares in the first row and $j$ in the second, for $\tilde{A}_{i}^{j} j$ dashed squares in the first and $i$ in the second, for $B_{j}^{i} i$ framed in the first and $j$ dashed in the second and for $\tilde{B}_{i}^{j} j$ framed in the first and $i$ dashed in the second. This is because the roots in the right half of the table are numbered in the opposite direction than in the left and also for the sake of symmetry: $A_{j}^{i}$ and $\tilde{A}_{i}^{j}$ are related by operation of complement and reflection with respect to the center of the table and similarly for $B$ 's. The framed squares are roots that are in the respective set for $i, j$ and dashed are ones that aren't.

We do not know yet that these sets are saturated but still we can apply the Čap criterion and decide which outgoing arrows would they allow, if they were saturated. For the root $1+2$ we see from lemma 9 that there must be exactly $2 p$ roots in the saturated set and this is satisfied by $B_{0}^{0}, B_{1}^{1}, \ldots B_{p}^{p}, A_{p}^{p}$. Similarly for every root $\alpha$ from the first graded part we see that there must be all but one element of $\Phi_{\mathfrak{p}}(\alpha)$ in the saturated set where the corresponding arrow starts. Going carefully through the list of the roots and through our four types of sets, we arrive at the following classification.

Lemma 10. If the sets $A_{j}^{i}, B_{j}^{i}, \tilde{A}_{i}^{j}, \tilde{B}_{i}^{j}$ are saturated then they allow arrows of the following type, ordered according to their labels. ${ }^{1}$

The "generic" roots ( $1 \leq k \leq p-1$ ):

$$
\begin{aligned}
\lambda_{1}-\lambda_{k+2}: & \left.A_{l}^{k-1} \rightarrow A_{l}^{k}\right|_{l \geq k} & \lambda_{2}+\lambda_{k+2}: & \left.\tilde{A}_{k}^{l} \rightarrow \tilde{A}_{k-1}^{l}\right|_{l \geq k} \\
& \tilde{B}_{k}^{k-1} \rightarrow B_{k}^{k} \tilde{B}_{l \geq k}^{l} & & B_{k}^{k} \rightarrow \tilde{B}_{k-1}^{k-1} \\
& B_{k-1}^{k-1} \rightarrow \tilde{B}_{k}^{k} & & \left.B_{k}^{k} \rightarrow B_{k-1}^{l}\right|_{l<k} ^{l} \\
& \left.\tilde{B}_{l}^{k-1} \rightarrow \tilde{B}_{l}^{k}\right|_{l<k} & & B_{k}^{l} \\
\lambda_{2}-\lambda_{k+2}: & \left.A_{k-1}^{l} \rightarrow A_{k}^{l}\right|_{l<k} & \lambda_{1}+\lambda_{k+2}: & \left.\tilde{A}_{l}^{k} \rightarrow \tilde{A}_{l}^{k-1}\right|_{l<k}
\end{aligned}
$$

[^0]The "middle" roots:

$$
\begin{aligned}
& \lambda_{1}-\lambda_{p+2}: A_{p}^{p-1} \rightarrow A_{p}^{p} \quad \lambda_{2}+\lambda_{p+2}: \quad \tilde{A}_{p}^{p} \rightarrow \tilde{A}_{p-1}^{p} \\
& \left.\tilde{B}_{l}^{p-1} \rightarrow \widetilde{A}_{l}^{p}\right|_{l<p} \\
& \begin{array}{l}
B_{p p-1}^{p-1} \rightarrow \tilde{A}_{p}^{p} \\
\left.\tilde{B}_{l}^{p} \rightarrow \tilde{A}_{l}^{p-1}\right|_{l<p}
\end{array} \\
& \begin{array}{lll}
\lambda_{2}-\lambda_{p+2}: & \left.A_{p-1}^{l} \rightarrow A_{p}^{l}\right|_{l<p} ^{l} & \lambda_{1}+\lambda_{p+2}: \\
\left.\left.A_{p-1}^{l} \rightarrow \tilde{A}_{l}^{p} \rightarrow \tilde{A}_{l}^{p-1}\right|_{l<p} ^{l}\right|_{l<p} \\
\tilde{B}_{l}^{p}
\end{array} \\
& \left.B_{p}^{l} \rightarrow B_{p-1}^{l}\right|_{l<p} \\
& \left.\tilde{B}_{l}^{p-1} \rightarrow \tilde{B}_{l}^{p}\right|_{l<p} \\
& \tilde{\sim}_{p}^{p} \rightarrow \tilde{B}_{p-1}^{p-1} \\
& B_{p-1}^{p-1} \rightarrow \tilde{B}_{p}^{p} \\
& \tilde{B}_{p}^{p} \rightarrow \tilde{B}_{p-1}^{p} \\
& B_{p}^{p-1} \rightarrow B_{p}^{p}
\end{aligned}
$$

and the root $\lambda_{1}+\lambda_{2}$ labels arrows $B_{k}^{k} \rightarrow \tilde{B}_{k}^{k}, 0 \leq k \leq p$ and $A_{p}^{p} \rightarrow \tilde{A}_{p}^{p}$.
Proof. We have together nine types of roots and four types of sets $A_{j}^{i}, B_{j}^{i}, \tilde{B}_{i}^{j}, \tilde{A}_{i}^{j}$ and we must go through all these 36 cases. Take for example the root $\alpha=\lambda_{1}-\lambda_{k+2}$ and a set of the type $A_{j}^{i}$. The
 and $\Phi_{\mathfrak{p}}(\alpha)=$ "世"ロ
Čap criterion and the last part of lemma 9 imply that the set $A_{j}^{i}$ must contain all but one element of $\Phi_{\mathfrak{p}}(\alpha)$. Since there is no set $A$ that can contain the white square of $\Phi_{\mathfrak{p}}(\alpha)$ and not contain the black one under it, we see that a candidate for the starting point of the seeked arrow must contain all the black squares of $\Phi_{\mathfrak{p}}(\alpha)$. The only such sets are $A_{l}^{k-1}$ where $l>k-1$. Čap criterion gives us the shape of the endpoint - it is $A_{l}^{k}$. The proof for other roots from $\mathfrak{g}_{1}$ and other sets goes the same way and we will omit it.

For $\lambda_{1}+\lambda_{2} \in \mathfrak{g}_{2}$ we see from lemma 9 that the starting set must have $2 p$ roots from $\mathfrak{g}_{1}$ and not contain $\lambda_{1}+\lambda_{2}$. This is satisfied by the sets $A_{p}^{p}$ and $B_{k}^{k}, 0 \leq k \leq p$. The endpoints are result of Čap criterion again.

Proof of Theorem. We have already calculated the Hasse diagram for $D_{4}, H\left(D_{4}\right)$ for briefness. We can easily check that the sets and arrows are precisely the ones given in the statement. Assume that we know all the specified data for $D_{n}, n \leq 4$. We will construct $H\left(D_{n+1}\right)$ as the union of $H\left(D_{n}\right)$ and the "outer belt" and fill in the missing arrows.

First, let $\Phi$ be a saturated set of $D_{n}$. We define an inclusion $i$ : $\Delta_{\mathfrak{p}}^{+}\left(D_{l}\right) \rightarrow \Delta_{\mathfrak{p}}^{+}\left(D_{l+1}\right)$ by $i\left(\lambda_{k} \pm \lambda_{j}\right)=\lambda_{k} \pm \lambda_{j+1}, k=1,2,3 \leq j \leq p$ and $i\left(\lambda_{1}+\lambda_{2}\right)=\lambda_{1}+\lambda_{2}$. We claim that $\Phi^{\prime} \equiv i(\Phi) \cup\left\{\lambda_{1}-\lambda_{3}, \lambda_{2}-\lambda_{3}\right\}$ is a saturated set for $D_{l+1}$ of the same type.

If $\Phi=\emptyset$, then $\Phi^{\prime}=\left\{\lambda_{1}-\lambda_{3}, \lambda_{2}-\lambda_{3}\right\}$, which is clearly saturated, because it is connected with the origin of $H\left(D_{n+1}\right)$ (the empty set) by consecutive arrows $\lambda_{2}-\lambda_{3}$ and $\lambda_{1}-\lambda_{3}$, as can be simply checked by Čap criterion. Assume that for every $\Phi \in H\left(D_{n}\right),|\Phi|=q$, the set $\Phi^{\prime}$ is saturated. Take an arbitrary such $\Phi$ and arbitrary root $\alpha \in \mathfrak{g}_{1}$. We see from lemma 9 and Čap criterion that $\alpha$ is a label of an outgoing
arrow from $\Phi$ iff $\left|\Phi_{\mathfrak{p}}(\alpha)\right|-\left|\Phi_{\mathfrak{p}}(\alpha) \cap \Phi\right|=1$. If $\alpha$ is a root of the first row, then $\Phi_{\mathfrak{p}}(i(\alpha))=i\left(\Phi_{\mathfrak{p}}(\alpha)\right) \cup\left\{\lambda_{1}-\lambda_{3}\right\}$ and if it is of the second row, $\Phi_{\mathfrak{p}}(i(\alpha))=i\left(\Phi_{\mathfrak{p}}(\alpha)\right) \cup\left\{\lambda_{2}-\lambda_{3}\right\}$ Hence $\left|\Phi_{\mathfrak{p}}(i(\alpha))\right|-\left|\Phi_{\mathfrak{p}}(i(\alpha)) \cap \Phi^{\prime}\right|=$ $\left|\Phi_{\mathfrak{p}}(\alpha)\right|+1-\left(\left|\Phi_{\mathfrak{p}}(\alpha) \cap \Phi\right|+1\right)$ and thus $\alpha$ is an arrow from $\Phi$ iff $i(\alpha)$ is an arrow from $\Phi^{\prime}$. If $\alpha=\lambda_{1}+\lambda_{2}$ then it is an arrow iff $|\Phi|=2 p$ but this happens if and only if $\left|\Phi^{\prime}\right|=2(p+1)$. Traversing between $\Phi$ to $\Phi^{\prime}$ does not change the type of the set $(A, B, \tilde{A}, \tilde{B})$, only adds one to both the indices, therefore the arrow $\Phi, \alpha$ is of the same type from the list of lemma 10 as the arrow $\Phi^{\prime}, i(\alpha)$ and the endpoint of the latter is $\Psi^{\prime} \equiv i(\Psi) \cup\left\{\lambda_{1}-\lambda_{3}, \lambda_{2}-\lambda_{3}\right\}$, where $\Psi$ is the endpoint of the former. Hence we get an inclusion of $H\left(D_{n}\right)$ into $H\left(D_{n+1}\right)$.

Now we claim and prove the shape of the "outer belt" of $H\left(D_{n+1}\right)$. It consists of four series of sets $\left(\emptyset,\left\{\lambda_{2}-\lambda_{3}\right\},\left\{\lambda_{2}-\lambda_{3}, \lambda_{2}-\lambda_{4}\right\}, \ldots, \Delta_{\mathfrak{p}}^{+}\right)$ $=\left(A_{0}^{0}, A_{1}^{0}, \ldots, A_{p+1}^{0}, B_{p+1}^{0}, B_{p}^{0}, \ldots B_{0}^{0}, \tilde{B}_{0}^{0}, \tilde{B}_{0}^{1} \ldots \tilde{B}_{0}^{p+1}, \tilde{A}_{0}^{p+1}, \tilde{A}_{0}^{p}, \ldots \tilde{A}_{0}^{0}\right)$. $A_{0}^{0}$ is clearly saturated and allows only the arrow $\lambda_{2}-\lambda_{3}$ of $\Delta_{p+1}^{+}$. We see from lemma 10 that if $A_{i-1}^{0}, 0<i-1<p$ is saturated, it allows an arrow $\lambda_{2}-\lambda_{i+2}$ to $A_{i}^{0}$, thus all the members of the series are saturated sets. Moreover, for $0<i \leq p+1$ there is an arrow $\lambda_{1}-\lambda_{3}$ from $A_{i}^{0}$ to $A_{i}^{1}=\{1-3,2-3\} \cup i\left(A_{i-1}^{0}\right)$ and also there are two arrows $\lambda_{2}+\lambda_{p+3}$ from $A_{p}^{0}$ to $B_{p+1}^{0}$ and from $A_{p+1}^{0}$ to $B_{p}^{0}$. No other arrows exist from the $A_{i}^{0}$-series of sets. We see that $B_{p+1}^{0}$ is also a saturated set of $H\left(D_{n+1}\right)$ and similarly we can show that the whole series $B_{p+1}^{0}, \ldots, B_{0}^{0}$ is in $H\left(D_{n+1}\right)$ and that only the connecting arrows $\lambda_{2}+\lambda_{i+2}$, the mutually parallel arrows $\lambda_{1}-\lambda_{3}$ and the arrow $\lambda_{1}+\lambda_{2}$ from $B_{0}^{0}$ to $\tilde{B}_{0}^{0}$ exist. Sets in the series $\tilde{B}_{0}^{0}, \ldots, \tilde{B}_{0}^{p+1}$ and $\tilde{A}_{0}^{p+1}, \ldots, \tilde{A}_{0}^{0}$ allow no other but the connecting arrows.

We have now a full control over arrows inside $i\left(H\left(D_{n}\right)\right)$ labeled by $i(\alpha), \alpha \in \Delta_{\mathfrak{p}}^{+}$, all arrows inside the "outer belt" and all arrows going from the "outer belt" to $i\left(H\left(D_{n}\right)\right)$. The only arrows that we could have left unnoticed are ones labeled by $\left\{\lambda_{1}-\lambda_{3}, \lambda_{2}-\lambda_{3}, \lambda_{1}+\lambda_{3}, \lambda_{2}+\lambda_{3}\right\}$ that are leaving $i\left(H\left(D_{n}\right)\right)$. These roots are "generic with $k=1$ " in the language of lemma 10 . We see that all the initial sets in the left column of first table of the lemma and the last two in the right column have at least one index zero, so they are not in $i\left(H\left(D_{n}\right)\right)$. What leaves are arrows $\lambda_{2}+\lambda_{3}: B_{1}^{1} \rightarrow \tilde{B}_{0}^{0}, \tilde{B}_{1}^{1} \rightarrow \tilde{B}_{0}^{1}, \ldots, \tilde{B}_{1}^{p+1} \rightarrow \tilde{B}_{0}^{p+1}, \tilde{A}_{1}^{p+1} \rightarrow$ $\tilde{A}_{0}^{p+1}, \ldots, \tilde{A}_{1}^{1} \rightarrow \tilde{A}_{0}^{1}$ that all lead into sets of the "outer belt".

We have shown that no arrows leave the union of $i\left(H\left(D_{n}\right)\right)$ and the "outer belt", can there be ones that enter it from a yet unknown saturated set outside the union? According to the remark after definition 2 any set of $H\left(D_{n+1}\right)$ must be connected with $\emptyset$ by a series of arrows labeled by elements of $p$. Since we know all arrows leaving $\emptyset$ this series would have to leave the union somewhere which we have proved is not possible.

## 3. BGG diagram for the complex case

The Hasse diagram gives us first part of the information hidden in the BGG diagram that encodes the structure of invariant differential operators in the appropriate parabolic geometry. Every point of the Hasse diagram corresponds to an irreducible representation and what remains to be determined are the highest weights of these representations. They can be calculated from the highest weight of the representation corresponding to $\emptyset$ by means of this well known recipe:

Recipe 1. Let $p=n-2$ and

be the weight of a representation represented by the node $\Phi \in H\left(D_{n}\right)$ expressed by its coordinates in the basis of fundamental weights increased by 1. Let $\Phi \xrightarrow{\alpha} \Phi^{\prime}$ be an outgoing arrow in the direction of $\alpha \in \Delta_{\mathfrak{p}}^{+}$. Then the representation represented by the node $\Phi^{\prime}$ is given by the affine Weyl action, i.e. its coordinates increased by 1 are given by reflection of $\widehat{\lambda}$ with respect to $\alpha$.

We introduce a notation that allows to write the weights occuring in the BGG diagram in a compact way $(i \leq j)$ :

$$
\begin{aligned}
& \left|{ }_{i}^{j}\right|=\sum_{k=i}^{j} a_{k} \\
& \left|\left.\right|_{i} ^{j}\right|_{+}=\left|{ }_{i}^{j}\right|_{+}+a_{+} \quad\left|{ }_{i}^{j}\right|_{-}=\left|{ }_{i}^{j}\right|+a_{-} \quad\left|{ }_{i}^{j}\right|_{ \pm}=\left|{ }_{i}^{j}\right|+a_{+}+a_{-} \\
& +=a_{+} \quad-=a_{-} \quad \pm=a_{+}+a_{-}
\end{aligned}
$$

and $\left|{ }_{i, j}^{k, l}\right|_{\bullet}=\left|{ }_{i}^{k}\right|+\left|{ }_{j}^{l}\right|_{\bullet}$, where • stands for,+- or $\pm$. When $i=j$, we will write also simply $i, i_{+}, i_{-}, i_{ \pm}$. We will also not write full Dynkin diagrams but only the numbers over the nodes, i.e. $123 \ldots p-1 p_{+}^{-}$is a shorthand for the labeled diagram in the recipe.

Theorem 13. If $123 \ldots p-1 p_{+}^{-}$is over the node $A_{0}^{0}$ in $H\left(D_{n}\right)$, then

$$
\begin{aligned}
& A_{j}^{0}=\left|{ }_{1}^{j+1}\right|-\left|\begin{array}{c}
j+1 \\
2
\end{array}\right| 2 \ldots j\left|\begin{array}{c}
j+2 \\
j+1
\end{array}\right| j+3 \ldots p-1 p_{+}^{-}
\end{aligned}
$$

$$
\begin{aligned}
& A_{p}^{0}=\left|\left.\right|_{1} ^{p}\right|_{-}-\left|\left.\right|_{2} ^{p}\right|_{-} \quad 2 \ldots p-2 p_{p-1}^{p} p_{ \pm} \\
& \left.A_{j}^{i}=\left|\begin{array}{|c}
j+1 \\
i+1
\end{array}\right|-\left.\right|_{1} ^{j+1}|1 \ldots i-1| \begin{array}{c}
i+1 \\
i
\end{array}|i+2 \ldots j| \begin{array}{c}
j+2 \\
j+1
\end{array} \right\rvert\, j+3 \ldots p-1 p_{+}^{-} \\
& A_{j}^{j}={ }_{j+1}-\left|\begin{array}{c}
p \\
1
\end{array}\right| 1 \ldots j-\left.1\right|_{j} ^{j+2}\left|{ }_{j}\right| \begin{array}{l}
j+\ldots p-1
\end{array} p_{+}^{-} \\
& A_{p-1}^{i}=\left|\begin{array}{|c}
p+1 \\
i+1
\end{array}\right|-\left|\begin{array}{l}
p \\
1
\end{array}\right| \ldots i-1\left|\begin{array}{c}
i+1 \\
i
\end{array}\right| i+2 \ldots p-2 p_{p-1}^{p} p_{-} \\
& A_{p-1}^{p-1}=p-\left|\left.\right|_{1} ^{p}\right| 1 \ldots p-3 p-2 \left\lvert\, \begin{array}{c}
\left.\left.\right|_{p-1} ^{p}\right|_{-} \\
\left.\left.\right|_{p-1} ^{p-1}\right|_{+}
\end{array}\right.
\end{aligned}
$$

$$
\begin{aligned}
& A_{p}^{i}=\left.\left.\right|_{i+1} ^{p}\right|_{-}-\left|{ }_{1}^{p}\right|_{-} \quad 1 \ldots i-1\left|{ }_{i}^{i+1}\right| i+2 \ldots p-2 p-1 \underset{p}{p} \\
& A_{p}^{p-1}=p_{-}--\left|\left.\right|_{1} ^{p}\right|_{-} 1 \ldots p-3 p-2 \left\lvert\, \begin{array}{l}
\left.\left.\right|_{p} ^{p}{ }_{p}\right|^{p} \mid \\
\left|p_{p-1}^{p}\right|_{ \pm}
\end{array}\right. \\
& A_{p}^{p}=--\left|\left.\right|_{1} ^{p}\right|_{-} 1 \ldots p-3 p-\left.\left.2{ }_{p}^{(p-1)_{ \pm}}\right|_{p, p-1}\right|^{(p,} \\
& B_{0}^{0}=\left|\begin{array}{|c|}
p, p \\
1,2 \\
\pm
\end{array}-_{2,3}^{p, p}\right|_{ \pm} 34 \ldots p-1 p_{-}^{+} \\
& B_{j}^{0}=\left|\left.\right|_{1, j+2} ^{p, p}\right|_{ \pm}-\left.\left.\right|_{2, j+2} ^{p, p}\right|_{ \pm} 2 \ldots j| |_{j+1}^{j+2} \mid j+3 \ldots p-1 p_{-}^{+} \\
& B_{p-1}^{0}=\left|\left.\right|_{1} ^{p}\right|_{ \pm}-\left.\left.\right|_{2} ^{p}\right|_{ \pm} \quad 2 \ldots p-2 p-1{ }_{p}^{p_{+}}{ }_{p_{-}} \\
& B_{p}^{0}=\left|\left.\right|_{1} ^{p}\right|_{+}-\left|\left.\right|_{2} ^{p}\right|_{+} 2 \ldots p-2 p-1 \underset{p}{p_{ \pm}} \\
& \left.B_{j}^{i}=\left|\begin{array}{|c|}
p, p+1, j+2
\end{array}\right|_{ \pm}-\left.\left.\right|_{1, j+2} ^{p, p}\right|_{ \pm} 1 \ldots i-\left.\left.1\right|_{i} ^{i+1}|i+2 \ldots j|\right|_{j+1} ^{j+2} \right\rvert\, j+3 \ldots p-1 p{ }_{-}^{+} \\
& B_{j}^{j}=\left|\begin{array}{|c|c|}
p, p+1, j+2
\end{array}\right|_{ \pm}-\left.\left.\right|_{1, j+2} ^{p, p}\right|_{ \pm} 1 \ldots j-1\left|{ }_{j}^{j+2}\right| j+3 \ldots p-1 p+ \\
& B_{p-1}^{i}=\left.\left.\right|_{i+1} ^{p}\right|_{ \pm}-\left|\begin{array}{l}
p \\
1
\end{array}\right|_{ \pm} 1 \ldots i-1\left|\begin{array}{c}
i+1 \\
i
\end{array}\right| i+2 \ldots p-2 p-1 p_{p}^{p_{+}} \\
& B_{p-1}^{p-1}=p_{ \pm}-\left|{ }_{1}^{p}\right|_{ \pm} 1 \ldots p-3 p-2 \left\lvert\, \begin{array}{c}
\left.\left.\right|_{p-1} ^{p}\right|_{+} \\
\left.\left.\right|_{p-1} ^{p}\right|_{-}
\end{array}\right.
\end{aligned}
$$

$$
\begin{aligned}
& B_{p}^{p-1}=p_{+}-\left|\begin{array}{c}
p \\
1
\end{array}\right|_{+}{ }^{1 \ldots p-3} \begin{array}{l}
p-2
\end{array} \begin{array}{c}
\left.\left.\right|_{p-1} ^{p}\right|_{ \pm} \\
\left.\left.\right|_{p-1} ^{p}\right|^{p}
\end{array}
\end{aligned}
$$

where $0<i<j<p-1$. The representation over $\tilde{A}_{i}^{j}$ differs from the one over $A_{j}^{i}$ only in the number over the second node and similarly for $\tilde{B}_{i}^{j}$ and $B_{j}^{i}$. This number equals

$$
\begin{aligned}
& -\sum_{1}^{p} a_{k}-\sum_{3}^{p} a_{k}+a_{+}+a_{-} \quad \text { for } \quad \tilde{B}_{0}^{0}, \tilde{A}_{0}^{0} \\
& -\sum_{1}^{p} a_{k}-\sum_{j+1}^{p} a_{k}+a_{+}+a_{-} \quad \text { for } \quad \tilde{A}_{0}^{j}, \tilde{B}_{0}^{j} \quad 0<j \leq p \\
& -\sum_{1}^{p} a_{k}-\sum_{i+1}^{p} a_{k}+a_{+}+a_{-} \quad \text { for } \quad \tilde{A}_{i}^{j}, \tilde{B}_{i}^{j} \quad 0<i \leq j<p-1 \\
& -\sum_{1}^{p} a_{k}+a_{+}+a_{-} \quad \text { for } \quad \tilde{A}_{p}^{p}, \tilde{B}_{p}^{p}
\end{aligned}
$$

Proof. To determine the reflection $\sigma_{\alpha} \widehat{\lambda}=\widehat{\lambda}-\langle\widehat{\lambda}, \alpha\rangle \alpha$ we must first know the scalar products $\langle\widehat{\lambda}, \alpha\rangle=(\widehat{\lambda}, \alpha)$ and express $\alpha$ in the basis of fundamental weights. It is a straightforward calculation to show how the roots in $\Delta_{\mathfrak{p}}^{+}$look in bases of simple roots and of fundamental
weights:

$$
\begin{array}{|l|l|l|l|}
\hline 110 . .0_{0}^{0} 1110 . .0_{0}^{0} \ldots 11 . .1_{0}^{0} 11 . .1_{0}^{1} & 11 . .1_{1}^{0} 11 . .1_{1}^{1} 11 . .2_{1}^{1} \ldots 112 . .2_{1}^{1}
\end{array} 12 . .2_{1}^{1}
$$

$$
\begin{array}{|cccc|ccccc|}
\hline 11-10 . .0_{0}^{0} & 101-10 . .0_{0}^{0} & \ldots & 10 . .01_{-1}^{-1} & 10 . .0_{-1}^{1} & 10 . .0_{1}^{-1} & 10 . .0-11_{1}^{1} & 10 . .0-11_{0}^{0} & \ldots \\
\hline-12-10 . .0_{0}^{0}-111-10 . .0_{0}^{0} & \ldots & -110 . .01_{-1}^{-1} & -110 . .0_{-1}^{1} & -110 . .0_{1}^{-1} & -110 . .0-11_{1}^{1} & -110 . .0-11_{0}^{0} & \ldots & -1010 . .0_{0}^{0} \\
\hline
\end{array}
$$

Since the expression of $\lambda_{1}+\lambda_{2}$ in fundamental weights has the only nonzero number over the crossed root, the weights over $B_{0}^{0}$ and $\tilde{B}_{0}^{0}$ can differ only in the number over the crossed node. Since $\left(\lambda_{1}-\lambda_{k}\right)+\left(\lambda_{2}+\right.$ $\left.\lambda_{k}\right)=\left(\lambda_{1}+\lambda_{k}\right)+\left(\lambda_{2}-\lambda_{k}\right)=\lambda_{1}+\lambda_{2}$ we see that for every arrow in the left part of $H\left(D_{n}\right)$ its counterpart in the right part differs only over the crossed root. We conclude that for every $B_{j}^{i}$ and $\tilde{B}_{i}^{j}$, the weights differ only over the crossed node and similarly for $A$ 's. Hence it suffices to determine the weights in the left part and the coeffitients over crossed nodes in the right part.

What follows is a mere calculation. We can join every saturated set of the left part with the empty set with a path that contains first $x$ arrows of the outer belt $2-3,2-4, \ldots, \begin{gathered}2-n \\ 2+n \\ 2+n \\ 2+n\end{gathered}, 2+(n-1), \ldots, 2+3$ and then it turns up and goes along the arrows labeled $1-3,1-4, \ldots, 1-n$. What bothers us is the variety of cases that must be verified, namely ( $n=p+2$ )

$$
\begin{array}{rl}
A_{0}^{0} \xrightarrow{2-3} A_{1}^{0} & \\
A_{j}^{0} \xrightarrow{2-(j+3)} A_{j+1}^{0} & 0<j<p-2 \\
A_{p-2}^{0} \xrightarrow{2-(p+1)} A_{p-1}^{0} & \\
A_{p-1}^{0} \xrightarrow{2-(p+2)} A_{p}^{0} & 0<j<p-1 \\
A_{j}^{0} \xrightarrow{1-3} A_{j}^{1} & 0<i<j+1<p \\
A_{j}^{i} \xrightarrow{1-(i+3)} A_{j}^{i+1} & 0<i<p-1 \\
A_{i}^{i-1} \xrightarrow{1-(i+2)} A_{i}^{i} & \\
A_{p-1}^{0} \xrightarrow{1-3} A_{p-1}^{1} & 0<i<p-2 \\
A_{p-1}^{i} \xrightarrow{1-(i+3)} A_{p-1}^{i+1} & \\
A_{p-1}^{p-2} \xrightarrow{1-(p+1)} A_{p-1}^{p-1} & 0<i<p-1 \\
A_{p}^{0} \xrightarrow{1-3} A_{p}^{1} & \\
A_{p}^{i} \xrightarrow{1-(i+3)} A_{p}^{i+1} & \\
A_{p}^{p-1} \xrightarrow{1-(p+2)} A_{p}^{p} &
\end{array}
$$

The number of cases reflects the number of different "kinds" of simple roots, i.e. the most left one and the two most right with only one neighbor, the series of ones with two neighbors and the ramifying one.

Assume that the weight over $A_{j}^{0}$ is equal to

$$
\widehat{\lambda}=\left|{ }_{1}^{j+1}\right|-\left|\begin{array}{c}
j+1 \\
2
\end{array}\right| 2 \ldots j\left|\begin{array}{c}
j+1 \\
j+1
\end{array}\right| j+3 \ldots p-1 p_{+}^{-}
$$

The scalar product of this weight with the root $\alpha=2-(j+3)=$ $011 . .10 . .0_{0}^{0}$ (there is $j+1$ nodes with number 1 ) is given as the ordinary scalar product

$$
(\widehat{\lambda}, \alpha)=\sum_{i} \widehat{\lambda}_{i} \alpha_{i}=-\sum_{k=2}^{j+1} a_{k}+a_{2}+\ldots+a_{j}+\sum_{k=j+1}^{j+2} a_{k}=a_{j+2}
$$

because the basis of fundamental weights (in which $\hat{\lambda}$ is expressed) is dual to the basis of simple roots (in which $\alpha$ is written). Then the weight over $A_{j+1}^{0}$ equals

$$
\begin{aligned}
& \left(\left.\left|\begin{array}{c}
j+1 \\
1
\end{array}\right|-\left.\left|{ }_{2}^{j+1}\right| 2 \ldots\right|_{j+1} ^{j+2} \right\rvert\, j+3 \ldots p--\right) \\
& -(-(j+2)(j+2) 0 \ldots 0(j+2)-(j+2) 0 \ldots 000) \\
& =\left|{ }_{1}^{j+2}\right|-\left|{ }_{2}^{j+2}\right| 2 \ldots j+1\left|\begin{array}{l}
j+3 \\
j+2
\end{array}\right| j+4 \ldots p_{+}^{-}
\end{aligned}
$$

where the zeros in the second term mean of course actual zeros, not (a nonexisting) $a_{0}$ and there is $j-1$ of them between the two $j+2$ 's. This is precisely what we supposed. We omit the other cases.

From these arrows we can establish a path leading to every node of $H\left(D_{n}\right)$ of the $A$ kind. In the same way we can calculate what the $2+n$ does to the weight over $A_{p-1}^{0}$ and get access to the nodes of the $B$ kind. Since this proceeds mechanically we omit it as well.

Finally we address the question of numbers over crossed nodes in the right part of $H\left(D_{n}\right)$. Since the roots $1-4,1-5, \ldots 1+5,1+4$ and $2+3$ have zero at the second position if expressed in fundamental weights and therefore do not change the number over the crossed node, we see that excepting the miscellaneous cases $\tilde{A}_{p}^{p}, \tilde{B}_{p}^{p}, \tilde{A}_{0}^{0}, \tilde{B}_{p}^{p}$ the number over $\tilde{A}_{i}^{j}, \tilde{B}_{i}^{j}$ does not depend upon $i$ nor the kind $\tilde{A}, \tilde{B}$. Thus we can calculate all but $\tilde{A}_{0}^{0}$ just by reflecting weights over $B_{j}^{j}, 0 \leq j \leq p$ and $A_{p}^{p}$ with respect to $1+2$. Finally we determine $\tilde{A}_{0}^{0}$ by reflecting $\tilde{A}_{0}^{1}$.

To become more comfortable with what happens we shall draw an example of a BGG diagram for $D_{6}$. Although we use the most compact notation, still some labels had to be written aside from the diagram.


## 4. BGG diagram for the real case

Real (standard) contact gradations are given by crossing an $s_{0^{-}}$ invariant subset of white roots [Yam93]. This excludes from our considerations the real forms $\mathfrak{s o}(2 n)$ and $\mathfrak{s o}(1,2 n-1)$ and the semisimple part of the parabolic subalgebra in other cases is a sum of a real form of $\mathfrak{s l}(2, \mathbb{C})$ and of $\mathfrak{s o}(2 n-2, \mathbb{C})$. The condition $\Lambda=s_{0} \Lambda$ is nontrivial only for the second and fifth Satake diagram from our list and possibly for the first, since the arrow does not encode information about the action of $s_{0}$ on the black nodes. According to [GoGro78] (Theorem (8.6.6) and Proposition (8.7.2)) and [OniV88], $s$ is nontrivial only for $\mathfrak{s o}(2 i+1,2 n-(2 i+1)), i<n$ and $\nu$ is nontrivial only for $n=2 j+1$. Thus $s_{0}=s \nu$ is nontrivial for $\mathfrak{s o}(i, 2 n-i)$ iff the number of black nodes is odd.

To compute the indices we can refer again to [OniV88], [GoodW98] (sect 5.1.7, 5.1.8) and [ZhiDa82]. For $\mathfrak{u}^{*}(n, \mathbb{H})$ the index of a representation $\rho=\left(\lambda_{1}, \ldots, \lambda_{n}\right)$ expressed in the basis of fundamental weights is $(-1)^{\Sigma \boldsymbol{\bullet} \lambda_{i}}$ (sum over the set of black nodes). For $\mathfrak{s o}(i, 2 n-i)$ it is $(-1)^{r\left(\lambda_{n-1}+\lambda_{n}\right)}$, where $r=1$ if $n-i \bmod 4=2$ or 3 and $r=0$ otherwise.

Definition 17. Let $\mathfrak{g}$ be real simple Lie algebra with a given parabolic subalgebra $\mathfrak{p}$. Then the BGG diagram for an irreducible representation $V$ of $\mathfrak{g}$ is defined as follows. Vertices of order $k$ are irreducible pieces of the cohomology group $H^{k}\left(\mathfrak{g}_{-}, V\right)$. There is an arrow from an irreducible piece $V \subset H^{k}\left(\mathfrak{g}_{-}, V\right)$ to and irreducible piece $W \subset H^{k+1}\left(\mathfrak{g}_{-}, V\right)$ iff there is an arrow in the corresponding complex BGG sequence from an irreducible component of $V_{\mathbb{C}}$ to an irreducible component of $W_{\mathbb{C}}$.

Theorem 14. Let $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right)$ be a highest weight of an irreducible representation of $\mathfrak{s o}(2 n, \mathbb{C})$ expressed in the basis of fundamental weights. Let $\mathfrak{g}$ be a real form of $\mathfrak{s o}(2 n, \mathbb{C})$. Suppose $\lambda_{n-1}=\lambda_{n}$ and one of the following cases occurs:
(i) $\mathfrak{g}=\mathfrak{s o}(i, 2 n-i)$ for $n-i$ odd
(ii) $\mathfrak{g}=\mathfrak{s o}(n-1, n+1)$
(iii) $\mathfrak{g}=\mathfrak{u}^{*}(2 k+1, \mathbb{H})$ with $\sum_{i=0}^{k-1} \lambda_{2 i+1}$ even

Then the real $B G G$ sequence has the form

that was constructed from the complex case by merging the nodes $A_{p}^{i}$ and $B_{p}^{i}$ for $i=0 \ldots p$. It looks the same as in the complex case otherwise.

Proof. Complex BGG sequences are determined by the Dynkin diagram and the highest weight $\lambda$ of the representation $V$. Given the weight $\lambda$ several cases may occur with different Satake diagrams.

Let us take the real forms with $s_{0}=\mathrm{Id}$. Then all representations in the BGG are either of the real or quaternionic type. Moreover, the index of a representation is preserved by any arrow. This is because the endpoint $\widehat{\lambda}^{\prime}$ of an arrow $\alpha$ differs from the starting point $\widehat{\lambda}$ by an integral multiple of $\alpha$ and all $\alpha$ 's expressed in the basis of fundamental weights have $\alpha_{n}+\alpha_{n-1}$ and $\sum_{i=0}^{k-1} \alpha_{2 i+1}$ even, hence do not change the index for $\mathfrak{s o}(i, 2 n-i), 2 \leq i \leq n-2$ and $\mathfrak{u}^{*}(2 k, \mathbb{H})$ respectively. Thus for both cases the real BGG sequence has the same shape as its complexification, and it is isomorphic to the complexification or to a direct sum of two copies of the complexification in the real and quaternionic cases, respectively.

If $s_{0} \neq \mathrm{Id}$, then $V$ can be of the complex, quaternionic or real type. If $V$ is of the complex type, then it is a component of a complexification of a real representation, and the conjugate representation is $s_{0}(\lambda)$. We clearly see from the structure of the complex BGG that for any $w_{\tilde{\sim}} \in$ $W_{\mathfrak{p}},\left(w \cdot s_{0} \lambda\right)_{n}=w \cdot \lambda_{n-1}$ everywhere except over $A_{p}^{i}, B_{p}^{i}, \tilde{A}_{i}^{p}, \tilde{B}_{i}^{p}$, $0 \leq i \leq p$, where labels of $\left(A_{p}^{i}, B_{p}^{i}\right)$ and $\left(\tilde{A}_{i}^{p}, \tilde{B}_{p}^{i}\right)$ are pairs of correlative representations. The direct sum of the BGGs for $\lambda$ and for $s_{0}(\lambda)$ is thus a complexification of a real BGG diagram. If $V$ is of a real or quaternionic type, then $\widehat{\lambda}_{n-1}=\widehat{\lambda}_{n}$. The index is preserved for the same reason as above for $\mathfrak{s o}(i, 2 n-i), 2 \leq i \leq n-1$, but for $\mathfrak{u}^{*}(n, \mathbb{H})$, $n=2 k+1$ the arrows $1+n, 1-n, 2+n, 2-n$ multiply the index by $(-1)^{a_{+}}=(-1)^{a_{-}}$, as we see from the BGG diagram. Thus if $V$ is of a real or quaternionic type, all representations corresponding to the nodes $A_{j}^{i}, j<p$ are of the same type, labels over $A_{p}^{i}$ and $B_{p}^{i}$, $0 \leq i \leq p$ are two correlative representations of the complex type and the representations corresponding to $B_{j}^{i}$ and $\tilde{B}_{i}^{j}, j<p$ are of the same type as $V$ since the index of the label of $B_{p-1}^{i}$ is $(-1)^{a_{+}+a_{-}}$times the index of $A_{p-1}^{i}$. For the same reason the representations corresponding to $\tilde{A}_{i}^{p}$ and $\tilde{B}_{i}^{p}$ are correlative and are of the same type as $V$ over $\tilde{A}_{i}^{j}$, $j<p$. This gives the shape of the BGG stated in the theorem for $V$ of the real type and the same shape as in the complex case for $V$ of the quaternionic type. The index calculated for $\mathfrak{s o}(i, 2 n-i), 2 \leq i \leq n-1$ (the first and the second of the Satake diagrams) is 1 regardless of $\lambda$ and for $\mathfrak{u}^{*}(2 k+1, \mathbb{H})$ it is $1 \mathrm{iff} \sum_{\boldsymbol{\bullet}} \lambda_{i}$ is even, hence $V$ is real precisely in these cases.

Finally we have to verify that for a representation of a quaternionic or complex type we cannot pair it with a different representation than the one we have found. But this is clear since a BGG diagram is an
orbit of the Weyl group $W_{\mathfrak{p}}$ and thus each weight $\lambda$ sits in a unique place and in a unique BGG.

## CHAPTER 6

## Poisson transform

The aim of this chapter is to establish a correspondence between invariant differential operators on vector bundles over symmetric spaces and parabolic geometries. First we generalize a theorem from [Oer00] that an integral operator between sections of vector bundles over $G / P$ and $G / K$ exists that intertwines first order differential operators. Then we discuss the possibility that invariant differential operators on $G / K$ form sequences that are intertwined with the BGG sequences on $G / P$.

## 1. Poisson transform and first order operators

Let us recall first the notation. We have a semisimple Lie group $G$ with a maximal compact subgroup $K$ and the corresponding Cartan decomposition $\mathfrak{g}=\mathfrak{k} \oplus \mathfrak{q}$. The minimal parabolic subgroup $P$ of $G$ has the Langlands decomposition $M A N$ and on the Lie algebra level $\mathfrak{g}=\overline{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. If we choose a positive root system in $\mathfrak{a}$ then $\mathfrak{n}$ decomposes into a sum $\sum \mathfrak{n}_{\alpha}$ over the restricted root spaces with respect to $\mathfrak{a}$.

Let $\mathbb{V}_{\lambda, \nu}$ be a $P$-module that restricts to an $M$-representation with the highest weight $\lambda$ and $a \in A$ acts on $\mathbb{V}_{\lambda, \nu}$ by multiplication by $a^{\nu}$, where $\nu \in \mathfrak{a}_{\mathbb{C}}^{*}$. The same result as 2 is in [Oer00] formulated in this form:

Theorem 15. Let $\mathbb{V}=\mathbb{V}_{\lambda}$ be an irreducible representation of $M$ and $\mathbb{W}=\mathbb{V}_{\lambda^{\prime}}$ be an irreducible $M$-module occuring in $\mathfrak{n}_{\alpha} \otimes \mathbb{V}$ such that the value of the Casimir operator $C(\mathbb{W})$ differs from the values on other constituents of $\mathfrak{n}_{\alpha} \otimes \mathbb{V}$ and $\alpha$ is a simple root. If the weight $\nu$ satisfies

$$
2\langle\alpha, \nu\rangle=C(\mathbb{W})-C\left(\mathfrak{n}_{\alpha}\right)-C(\mathbb{V})
$$

Then $D=\operatorname{proj}_{\mathbb{W}} \circ \nabla: C^{\infty}\left(G \times_{P} \mathbb{V}\right) \rightarrow C^{\infty}\left(G \times_{P} \mathbb{W}\right)$ is a non-trivial first-order equivariant differential operator, where the weight of $\mathfrak{a}$ on $\mathbb{F}$ is $\nu^{\prime}=\nu+\alpha$.

Here

$$
\nabla: C^{\infty}\left(G \times_{P} \mathbb{V}\right) \rightarrow C^{\infty}\left(G \times_{P}(\mathfrak{n} \otimes \mathbb{V})\right)
$$

is the invariant derivative

$$
\left(\nabla_{X} f\right)(g)=\left.\frac{d}{d t}\right|_{0} f(g \exp t X)
$$

where $X \in \overline{\mathfrak{n}}, g \in G$ and $f$ is a $P$-equivariant function from $G$ to $\mathbb{V}$.

Analogously if $\mathbb{E}$ is a $K$-module and $\tilde{f}$ a $K$-equivariant $\mathbb{E}$ valued function on $G$, then the Stein-Weiss gradient

$$
\tilde{\nabla}: C^{\infty}\left(G \times_{K} \mathbb{E}\right) \rightarrow C^{\infty}\left(G \times_{K}\left(\mathfrak{q}^{*} \otimes \mathbb{E}\right)\right)
$$

is defined by

$$
\left(\tilde{\nabla}_{X} f\right)(g)=\left.\frac{d}{d t}\right|_{0} f(g \exp t X)
$$

where $X \in \mathfrak{q}$.
The Poisson transform is defined as follows:
Definition 18 (Poisson transform). Let $\mathbb{V}_{\lambda, \nu}$ be a $P$-module, $\mathbb{E}_{\sigma}$ a $K$-module and $I_{\lambda}: \mathbb{V}_{\lambda, \nu} \rightarrow \mathbb{E}_{\sigma}$ be an $M$-invariant map. Then there is a map $\mathcal{P}: C^{\infty}\left(G, \mathbb{V}_{\lambda, \nu}\right)^{P} \rightarrow C^{\infty}\left(G, \mathbb{E}_{\sigma}\right)^{K}$ given by

$$
(\mathcal{P} f)(g)=\int_{K} \sigma(k) I_{\lambda}(f(g k)) d k
$$

Taking into account invariance of the Haar measure, we can readily check that the Poisson transform maps into $K$-equivariant maps:

$$
\left((\mathcal{P} f) \circ r_{k^{\prime}}\right)(g)=\int_{K} \sigma\left(k^{\prime-1}\right) \sigma\left(k^{\prime} k\right) I_{\lambda}\left(f\left(g k^{\prime} k\right)\right) d k=\sigma\left(k^{\prime-1}\right)(\mathcal{P} f)(g)
$$

moreover regardless of the $P$-equivariance of $f$.
Let $\mathbb{V}_{\lambda, \nu}$ be a $P$-module and $\mathbb{E}_{\sigma}$ a $K$-module such that there is an $M$-invariant inclusion $I_{\lambda}: \mathbb{V}_{\lambda, \nu} \rightarrow \mathbb{E}_{\sigma}$ needed for the Poisson transform. If we define $I_{\theta}(X):=X-\theta(X)$ for $X \in \overline{\mathfrak{n}}$ then we have a projection of $\overline{\mathfrak{n}}$ into $\mathfrak{q}$ since $\theta I_{\theta}(X)=-I_{\theta}(X)$. By duality there is also $I_{\theta}: \mathfrak{n} \rightarrow \mathfrak{q}^{*}$. Suppose now that $\mathbb{V}_{\lambda^{\prime}}$ is an irreducible constituent of $\mathfrak{n}_{\alpha} \otimes \mathbb{V}_{\lambda}$ for $\alpha$ a simple root, $\mathbb{E}_{\sigma^{\prime}}$ an irreducible constituent of $\mathfrak{q}^{*} \otimes \mathbb{E}_{\sigma}$ and there is a commutative diagram

where $I_{\lambda^{\prime}}$ is an $M$-invariant map and the horizontal maps are inclusions of irreducible constituents as $M$-modules or $K$-modules, respectively. Under such conditions it is proved in [Oer00] that if $\nu$ and $\nu^{\prime}$ are such that there is a $P$-invariant operator between the $\mathbb{V}_{\lambda, \nu}$-valued and $\mathbb{V}_{\lambda^{\prime}, \nu^{\prime}}$ valued sections, then this operator is intertwined with the Stein-Weiss gradient by the Poisson transform:

Theorem 16. Let $G$ be a semisimple Lie group, $K$ its maximal compact subgroup and $P$ a parabolic subgroup of $G$. Let $\mathbb{V}_{\lambda, \nu}$ be a $P$-module, $\mathbb{E}_{\sigma}$ a $K$-module, such that there exists a Poisson transform $\mathcal{P}: C^{\infty}\left(G \times_{P} \mathbb{V}_{\lambda, \nu}\right) \rightarrow C^{\infty}\left(G \times_{K} E_{\sigma}\right)$. Then we can define the gradients

$$
\begin{aligned}
& D=\operatorname{proj}_{\lambda^{\prime}} \circ \nabla: C^{\infty}\left(G \times_{P} \mathbb{V}_{\lambda, \nu}\right) \rightarrow C^{\infty}\left(G \times_{P} \mathbb{V}_{\lambda^{\prime}, \nu^{\prime}}\right) \\
& \tilde{D}=\operatorname{proj}_{\sigma^{\prime}} \circ \tilde{\nabla}: C^{\infty}\left(G \times_{K} \mathbb{E}_{\sigma}\right) \rightarrow C^{\infty}\left(G \times_{K} \mathbb{E}_{\sigma^{\prime}}\right)
\end{aligned}
$$

where $\nu^{\prime}=\nu+\alpha$, $\alpha$ simple with the root space $\mathfrak{n}_{\alpha}$, and

$$
2\langle\alpha, \nu\rangle=C\left(\mathbb{V}_{\lambda^{\prime}}\right)-C\left(\mathfrak{n}_{\alpha}\right)-C\left(\mathbb{V}_{\lambda}\right)
$$

Then (up to a normalizing constant) the maps $\mathcal{P}, D, \tilde{D}$ form a commutative diagram of $G$-invariant maps, i.e.

$$
\mathcal{P} \circ D=\tilde{D} \circ \mathcal{P}
$$

Øersted in [Oer00] gives an example of an application of theorem 16. Let $G=S O_{0}(n+1,1), K=S O(n+1), P=C O(n) \ltimes\left(\mathbb{R}^{*}\right)^{n}$. Then $G / K=H^{n+1}$, the hyperbolic ball, and $G / P=S^{n}$. If we consider the defining representations $\mathbb{R}^{n}$ of $M=S O(n)$ and $\mathbb{R}^{n+1}$ of $K$ with the embedding $I: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n+1}$ which also defines the embedding $\mathbb{R}^{n} \otimes \mathbb{R}^{n} \rightarrow \mathbb{R}^{n+1} \otimes \mathbb{R}^{n+1}$. This embedding clearly respects the decomposition of the tensor products into antisymmetric tensors, symmetric tracefree tensors and pure traces. The projections on symmetric tracefree parts are called Ahlfors operators $S, \tilde{S}$ on $S^{n}$ and $H^{n+1}$ respectively. According to theorem 16 we have $\mathcal{P} S X=\tilde{S} \mathcal{P} X$ for $X$ a vector field and $\mathcal{P}$ a Poisson transform corresponding to the embedding of the vector representations on the right side and of the symmetric tracefree tensors on the left side. If $X$ is conformal, then $S X=0$ and the extension $Y=\mathcal{P} X$ from the boundary $G / P$ to the interior $G / K$ satisfies $\tilde{S} X=0$. If $X$ is quasi-conformal, i.e. there is a certain estimate on $S X$, then $Y$ is also quasi-conformal. Further references can be found in [KR00].

## 2. Generalization to higher order operators

In this section we shall generalize theorem 16 to higher order operators. Let us now consider a representation $\left(\lambda^{\prime}, \nu^{\prime}\right)$ such that there exists an invariant differential operator of order $j$ between $C^{\infty}\left(G, \mathbb{V}_{\lambda, \nu}\right)^{P}$ and $C^{\infty}\left(G, \mathbb{V}_{\lambda^{\prime}, \nu^{\prime}}\right)^{P}$, and that there exists a $M$-invariant $I_{\lambda^{\prime}}: \mathbb{V}_{\lambda^{\prime}, \nu^{\prime}} \rightarrow \mathbb{E}_{\sigma^{\prime}}$ such that the diagram

where the horizontal maps are injections of irreducible constituents, is commutative.

Theorem 17. The diagram

$$
\begin{array}{cccc}
C^{\infty}\left(G, \mathbb{V}_{\lambda, \nu}\right)^{P} \xrightarrow{j_{\eta}^{i}} C^{\infty}\left(G, \bar{J}^{i} \mathbb{V}_{\lambda, \nu}\right)^{P} & \longrightarrow C^{\infty}\left(G, \mathbb{V}_{\lambda^{\prime}, \nu^{\prime}}\right)^{P} \\
\mathcal{P} \downarrow & \mathcal{P} \downarrow & \downarrow \mathcal{P} \\
C^{\infty}\left(G, \mathbb{E}_{\sigma}\right)^{K} & \longrightarrow & & C^{\infty}\left(G, \mathbb{E}_{\sigma^{\prime}}\right)^{K}
\end{array}
$$

is commutative. In particular its outer square expresses the fact that Poisson transform intertwines the two invariant operators $\operatorname{proj}_{\lambda^{\prime}, \nu^{\prime}}\left(\nabla^{\eta}\right)^{i}$ and $\operatorname{proj}_{\sigma}(\nabla)^{i}$.

Proof. Let us denote by $\sigma_{(\mathfrak{n})^{i} \mathbb{E}}$ the tensor product representation $\otimes^{i} \mathrm{Ad}^{*} \otimes \sigma$ on $\otimes^{i} \mathfrak{q}^{*} \otimes \mathbb{E}_{\sigma}$ and suppose $g \in G, X_{1}, \ldots X_{i} \in \overline{\mathfrak{n}}$. Then

$$
\begin{aligned}
& \mathcal{P}\left(\left(\nabla^{\eta}\right)^{i} f\right)\left(g, X_{1}, \ldots, X_{i}\right) \\
& =\int_{K} \sigma_{(\mathfrak{n})^{i} \mathbb{E}}(k)\left(I_{\theta}\right)^{i} \otimes I_{\lambda}\left(\left(\nabla^{\eta}\right)^{i} f\right)(g k) d k\left(X_{1}, \ldots, X_{i}\right) \\
& =\left.\int_{K} \sigma_{(\mathfrak{n})^{i} \mathbb{E}}(k)\left[\left(I_{\theta}\right)^{i} \otimes I_{\lambda}\right] \frac{d}{d t}\right|_{0} f\left(g k \mathrm{e}^{t_{1} X_{1}} k^{-1} k \ldots k^{-1} k \mathrm{e}^{t_{i} X_{i}} k^{-1} k\right) d k \\
& =\left.\int_{K} \sigma(k) I_{\lambda} \frac{d}{d t}\right|_{0} f\left(g \mathrm{e}^{t_{1} \operatorname{Ad}\left(k^{-1}\right) I_{\theta} \operatorname{Ad}(k) X_{1}} \ldots \mathrm{e}^{t_{i} \operatorname{Ad}\left(k^{-1}\right) I_{\theta} \operatorname{Ad}(k) X_{i}} k\right) d k \\
& =\left.\int_{K} \sigma(k) I_{\lambda} \frac{d}{d t}\right|_{0} f\left(g \mathrm{e}^{t_{1} I_{\theta} X_{1}} \ldots \mathrm{e}^{t_{i} I_{\theta} X_{i}} k\right) d k \\
& =\left((\nabla)^{i}(\mathcal{P} f)\right)\left(g, I_{\theta} X_{1}, \ldots, I_{\theta} X_{i}\right)
\end{aligned}
$$

Here we write shortly $\left.\frac{d}{d t}\right|_{0}$ for $\left.\left.\frac{d}{d t_{1}}\right|_{0} \ldots \frac{d}{d t_{1}}\right|_{0}$ and use $\left(\operatorname{Ad}^{*}(k) \omega\right)(X)=$ $\omega\left(\operatorname{Ad}\left(k^{-1}\right) X\right)$ for a 1-form $\omega$ and commutativity of $\theta$ and $\operatorname{Ad}(k)$. This gives the commutativity of the left square and the right one is commutative already on the algebraic level.

## 3. An example - complex hyperbolic space

An example of such higher order operators intertwined by the Poisson transform is given in [Rei01]. There $G=S U(n+1,1), K=$ $S(U(n+1) \times U(1))$, so $G / K$ is the complex hyperbolic space. On the Lie algebra level $\mathfrak{g}=\mathfrak{k} \oplus \mathfrak{q}$ where

$$
\begin{aligned}
& \mathfrak{k}=\left\{\left.\left(\begin{array}{cc}
A & 0 \\
0 & i \alpha
\end{array}\right) \right\rvert\, \bar{A}^{T}=-A, \alpha \in \mathbb{R}\right\} \\
& \mathfrak{p}=\left\{\left.\left(\begin{array}{cc}
0 & x \\
\bar{x}^{T} & 0
\end{array}\right) \right\rvert\, x \in \mathbb{C}^{n+1}\right\}
\end{aligned}
$$

Then there is the root space decomposition

$$
\begin{aligned}
\mathfrak{g} & =\mathfrak{g}_{-2 \alpha}+\mathfrak{g}_{-\alpha}+\mathfrak{a}+\mathfrak{m}+\mathfrak{g}_{\alpha}+\mathfrak{g}_{2 \alpha} \\
& =\overline{\mathfrak{n}}+\mathfrak{a}+\mathfrak{m}+\mathfrak{n}
\end{aligned}
$$

with

$$
\begin{aligned}
\mathfrak{a} & =\left\{\left.\left(\begin{array}{ccc}
0 & 0 & t \\
0 & 0 & 0 \\
t & 0 & 0
\end{array}\right) \right\rvert\, t \in \mathbb{R}\right\} \\
\mathfrak{m} & =\left\{\left.\left(\begin{array}{ccc}
i \gamma & 0 & 0 \\
0 & A & 0 \\
0 & 0 & i \delta
\end{array}\right) \right\rvert\, \bar{A}^{T}=-A, \gamma, \delta \in \mathbb{R}\right\} \\
\mathfrak{g}_{\alpha} & =\left\{\left.\left(\begin{array}{ccc}
0 & -\bar{x}^{T} & 0 \\
x & 0 & y \\
0 & \bar{y}^{T} & 0
\end{array}\right) \right\rvert\, x, y \in \mathbb{C}^{n}\right\} \\
\mathfrak{g}_{2 \alpha} & =\left\{\left.\left(\begin{array}{ccc}
i s & 0 & i s \\
0 & 0 & 0 \\
-i s & 0 & i s
\end{array}\right) \right\rvert\, s \in \mathbb{R}\right\}
\end{aligned}
$$

i.e. a contact parabolic geometry on the boundary.

It is possible to define invariant differential operators

$$
\begin{aligned}
& D: C^{\infty}\left(G \times_{K} \mathbb{R}\right) \rightarrow C^{\infty}\left(G \otimes_{K} \mathfrak{q}\right) \\
& S: C^{\infty}\left(G \times_{K} \mathfrak{q}\right) \rightarrow C^{\infty}\left(G \otimes_{K} \mathfrak{q}^{[2,0]}\right)
\end{aligned}
$$

in the interior and

$$
\begin{aligned}
& D^{S}: C^{\infty}\left(G \times_{P} \mathfrak{g}_{-2 \alpha}\right) \rightarrow C^{\infty}\left(G \otimes_{P} \overline{\mathfrak{n}}\right) \\
& T^{S}: C^{\infty}\left(G \times_{P} \mathfrak{g}_{-2 \alpha}\right) \rightarrow C^{\infty}\left(G \otimes_{P} \mathfrak{g}_{-2 \alpha}^{[2,0]}\right)
\end{aligned}
$$

Here the superscript $[2,0]$ means projection onto the symmetric traceless part of the tensor product. If we define $T=S \circ D$ and calculate with the explicit expressions of the operators (see [KR98]), we get the following formulas

$$
D \mathcal{P}=\frac{2(n+2)}{2 n+1} \mathcal{P} D^{S}
$$

$$
T \mathcal{P}=\frac{(n+2)(n+3)}{n(n+1)} \mathcal{P} T^{S}
$$

for suitable Poisson transforms. In the latter formula the Poisson transform intertwines two operators of second order (constants occur due to a different normalisation in the definition of the differential operators in [KR98]). However it seems that there is no invariant "Ahlfors operator" $S^{S}$ such that $T^{S}=S^{S} \circ D^{S}$ and the intertwining of $T$ and $T^{S}$ thus cannot be reduced to succesive application of theorem 16 for first order operators.

## 4. Sequences of differential operators in the conformal case

Differential operators between homogeneous bundles on $G / P$ come in complexes, so it is natural to ask whether there are some sequences also on $G / K$ that are intertwined by the Poisson transform as in theorem 17 , that under suitable conditions become complexes as well. We shall show some examples of such sequences for the special case of hyperbolic space whose boundary is a conformal sphere. In this case there is an advantage that the hyperbolic space can be taken both as a symmetric space and as a parabolic geometry. We may thus construct a BGG sequence and view it as a sequence of differential operators on the symmetric space.

We shall restrict now to the case of a hyperbolic ball bounded by a conformal sphere. The basic setting is a pseudo-euclidean space $\mathbb{R}^{n, 1}$ with a scalar product $(\cdot, \cdot)$ of signature $(n, 1)$. The projectivisation $P \mathcal{L}$ of the lightcone $\mathcal{L}=\left\{v \in \mathbb{R}^{n, 1} \mid(v, v)=0\right\}$ is the conformal sphere $S^{n}$ and can be regarded as a parabolic geometry $G / P$, where $G=$ $S O(n, 1)$ and $P=C O(n-1) \ltimes\left(\mathbb{R}^{*}\right)^{n-1}$. The projectivisation of the interior of the lightcone is $S O(n, 1) / S O(n)$, where $K:=S O(n)$ is the maximal compact subgroup of $G$. Operators on $G / P$ and on $G / K$ are intertwined by the Poisson transform according to theorem 17.

The hyperbolic space $H^{n}:=G / K$ carries a conformal structure, too. It is conformally equivalent to the upper half of the conformal sphere $S^{n}$, which can be written as $G^{\prime} / P^{\prime}$ where $G^{\prime}=S O(n+1,1)$ and $P^{\prime}=C O(n) \ltimes\left(\mathbb{R}^{*}\right)^{n}$, or in the standard notation

for $n$ even or odd respectively.
Let us suppose first that $n=2 k+1$ is odd, the even case will be dealt analogously. Then the boundary $G / P$ is a parabolic geometry, which for a representation

$$
\lambda=\begin{array}{ccccc}
b & d_{1} & d_{2} \\
\times \longrightarrow-\cdots \cdots & d_{k-2}
\end{array}
$$

has the BGG diagram

where we write weights shifted as usual, each weight $\mu$ is represented in the diagram by labels of $\mu+\rho$, whose we write tilded as opposed to labels of $\mu$. Moreover, $\tilde{d}=\sum_{1}^{k-2} \tilde{d}_{i}$ and the numbers by the arrows denote orders of the operators between the corresponding bundles.

We want to form commutative squares of representations like 5 such that the upper row leads to an operator from the above-mentioned BGG sequence and the lower row to a projection of a power of a

Stein-Weiss gradient. The vertical embeddings $I_{\lambda}$ and $I_{\lambda^{\prime}}$ map an $M$-representation $M$-equivariantly into a $K$-representation and their possibility and number depends upon the decomposition of the $K$ representation into $M$-representations, i.e. the branching rules for $K=S O(n)$ and $M=S O(n-1)$. They are most easily writen in terms of the highest weight expressed in the standard basis $\left\{e_{1}, e_{2}, \ldots, e_{k}\right\}$, where $k$ is the rank. We shall write coordinates with respect to the standard basis in square brackets to distinguish them from coordinates with respect to the basis of fundamental weights. Since fundamental weights have coordinates $[1,0 \ldots, 0],[1,1,0, \ldots, 0], \ldots,[1, \ldots, 1,0,0],\left[\frac{1}{2}, \ldots, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \ldots,-\frac{1}{2}\right]$ for $\mathfrak{s o}(2 k)$ and $[1,0 \ldots, 0],[1,1,0, \ldots, 0], \ldots,[1, \ldots, 1,0]$, $\left[\frac{1}{2}, \ldots, \frac{1}{2}\right]$ for $\mathfrak{s o}(2 k+1)$, it is clear that a dominant weight $\left[x_{1}, \ldots, x_{k}\right]$ is characterized precisely by the condition $\forall i x_{i} \in \mathbb{Z} \vee \forall i x_{i} \in \mathbb{Z}+\frac{1}{2}$ $x_{1} \geq x_{2} \geq \ldots x_{k} \geq 0$ for $\mathfrak{s o}(2 k+1)$ and $x_{1} \geq x_{2} \geq \ldots\left|x_{k}\right|$ for $\mathfrak{s o}(2 k)$ and it is integral iff $\forall i x_{i} \in \mathbb{Z} \vee \forall i x_{i} \in \mathbb{Z}+\frac{1}{2}$.

Theorem 18. Let $\mathfrak{g}=\mathfrak{s o}(n)$ and $\mathfrak{h}=\mathfrak{s o}(n-1)$. Then $a \mathfrak{h}$-module $\mathbb{V}$ is either present in a decomposition of $a \mathfrak{g}$-module $\mathbb{E}$ with a multiplicity one or is not present there. The former is true iff

- $n$ is odd, $n=2 k+1, \mathbb{E}$ and $\mathbb{V}$ have a highest weight $\left[x_{1}, \ldots, x_{k}\right]$ and $\left[y_{1}, \ldots, y_{k}\right]$, respectively, and

$$
x_{1} \geq y_{1} \geq x_{2} \geq \ldots \geq y_{k-1} \geq x_{k} \geq\left|y_{k}\right| \text { and } x_{i}-y_{i} \in \mathbb{Z}
$$

- $n$ is even, $n=2 k, \mathbb{E}$ and $\mathbb{V}$ again have a highest weight $\left[x_{1}, \ldots, x_{k}\right]$ and $\left[y_{1}, \ldots, y_{k-1}\right]$, respectively, and

$$
x_{1} \geq y_{1} \geq \ldots \geq x_{k-1} \geq y_{k-1} \geq\left|x_{k}\right| \text { and } x_{i}-y_{i} \in \mathbb{Z}
$$

The relationship between coordinates in the standard basis and fundamental basis is as follows:

- $\mathfrak{s o}(2 k)$ :
- $\mathfrak{s o}(2 k+1)$

We shall now write the BGG diagram above in standard coordinates, or more precisely just the $\mathfrak{s o}(2 k)$-representations arising when we delete
the crossed node:


We keep to use the coordinates $b, d_{i}, a, c$ (of $\lambda$ ) with respect to the basis of fundamental weights wherever it makes notation simpler. It can be clearly seen that if

$$
\left[x_{1}, x_{2}, \ldots, x_{k}\right] \xrightarrow{h}\left[x_{1}^{\prime}, x_{2}^{\prime}, \ldots, x_{k}^{\prime}\right]
$$

is an arrow in the BGG diagram, then the difference $\left[x_{1}^{\prime}-x_{1}, x_{2}^{\prime}-\right.$ $\left.x_{2}, \ldots, x_{k}^{\prime}-x_{k}\right]$ is a $h$-th multiple of weights $[1,0, \ldots, 0],[0,1,0, \ldots, 0]$, $\ldots,[0, \ldots, 1],[1,0, \ldots, 0],[0,1,0, \ldots, 0], \ldots,[0, \ldots, 1]$, i.e. weights of the vector representation $\overline{\mathfrak{n}}$. In the first part of the BGG diagram only weights with +1 occur, in the final part only weights with -1 do and in the middle diamond there is a mix of them.

Lemma 11. Let $\mathbb{V}_{\lambda}$ be a representation of $M=S O(2 k)$ with standard coordinates $\left[x_{1}, x_{2}, \ldots, x_{k}\right]$ and $\mathbb{V}_{\lambda^{\prime}}$ with coordinates $\left[x_{1}^{\prime}, x_{2}^{\prime}, \ldots, x_{k}^{\prime}\right]$ be a weight such that there is an arrow $\mathbb{V}_{\mu} \xrightarrow{h} \mathbb{V}_{\nu}$ in some $B G G$ diagram and $\left[x_{1}^{\prime}-x_{1}, x_{2}^{\prime}-x_{2}, \ldots, x_{k}^{\prime}-x_{k}\right]$ contains no negative number. Suppose that $\mathbb{E}_{\sigma}$ is a representation of $K=S O(2 k+1)$ with standard coordinates $\left[x_{1}, x_{2}, \ldots,\left|x_{k}\right|\right]$ and let $I_{\lambda}$ be the embedding $V_{\lambda} \rightarrow \mathbb{E}_{\sigma}$ (uniquely defined since there is a unique $M$-component $\left[x_{1}, x_{2}, \ldots,\left|x_{k}\right|\right]$ in the $K$-representation $\left[x_{1}, x_{2}, \ldots, x_{k}\right]$. Then there is a unique component $\mathbb{E}_{\sigma^{\prime}}=\left[x_{1}^{\prime}, x_{2}^{\prime}, \ldots, x_{k}^{\prime}\right]$ in $\otimes^{h} \mathfrak{q}^{*} \otimes \mathbb{E}_{\sigma}$.

Proof. The representation $\mathfrak{q}^{*}$ has weights $[1,0, \ldots, 0],[0,1,0, \ldots, 0]$, $\ldots[0, \ldots, 0,1],[-1,0, \ldots, 0],[0,-1,0, \ldots, 0], \ldots[0, \ldots, 0,-1],[0, \ldots, 0]$. Let us denote by $\mathbb{E}_{\sigma^{\prime}}=\left[y_{1}, \ldots, y_{k}\right]$ a general component of $\otimes^{h} \mathfrak{q}^{*}$. Branching rules imply that for $I_{\lambda^{\prime}}$ to exist there must be $y_{1} \geq x_{1}^{\prime} \geq y_{2} \geq$ $\ldots x_{k-1} \geq y_{k} \geq\left|x_{k}\right|$. If $x_{i}^{\prime}-x_{i}$ are all non-negative, then in fact one of them must be $h$ and others zero, suppose therefore that $x_{j}^{\prime}=x_{j}+h$. This means that $y_{j} \geq x_{j}+h$ and since the weight $\left[0, \ldots, h^{\prime}, \ldots, 0\right]$ where $h^{\prime} \geq h$ is at the $j$-th place, occurs with multiplicity 1 in $\otimes^{h} \mathfrak{q}^{*}$ for $h^{\prime}=h$ and with multiplicity 0 for $h^{\prime}>h$ then there is a unique component $\mathbb{E}_{\sigma^{\prime}}$ in $\otimes^{h} \mathfrak{q}^{*} \otimes \mathbb{E}_{\sigma}$ such that $\mathbb{V}_{\lambda^{\prime}}$ can be embedded into it.

This means in particular that for the part of the BGG sequence on the boundary that consists of arrows which do not lower any of the standard coordinates, i.e. for

$$
\begin{aligned}
& {\left[x_{1}, \ldots, x_{k}\right] \xrightarrow{b+1}\left[x_{1}+b+1, x_{2}, \ldots, x_{k}\right] \xrightarrow{d_{1}+1}\left[x_{1}+b+1, x_{1}+1, x_{3}, \ldots, x_{k}\right] \xrightarrow{d_{2}+1} \ldots} \\
& \ldots \xrightarrow{d_{k-2}+1}\left[x_{1}+b+1, \ldots, x_{k-2}+1, x_{k}\right] \xrightarrow{a+1}\left[x_{1}+b+1, \ldots, x_{k-2}+1, x_{k-1}+1\right]
\end{aligned}
$$

there is a sequence of invariant differential operators on $G / K$ which are intertwined with the operators on the boundary by the Poisson transform. The lemma ensures that the Poisson transforms are well defined. The representations of the sequence have the same standard coordinates:

$$
\begin{aligned}
& {\left[x_{1}, \ldots, x_{k}\right] \xrightarrow{b+1}\left[x_{1}+b+1, x_{2}, \ldots, x_{k}\right] \xrightarrow{d_{1}+1}\left[x_{1}+b+1, x_{1}+1, x_{3}, \ldots, x_{k}\right] \xrightarrow{d_{2}+1} \ldots } \\
& \ldots \xrightarrow{d_{k-2}+1}\left[x_{1}+b+1, \ldots, x_{k-2}+1, x_{k}\right] \xrightarrow{a+1}\left[x_{1}+b+1, \ldots, x_{k-2}+1, x_{k-1}+1\right]
\end{aligned}
$$

and of course also the orders of the operators are the same.
We can interpret the operators in the sequence in the interior in an interesting way. As we noted before, the hyperbolic space $G / K$ is not only a symmetric space but also a conformal parabolic geometry $G^{\prime} / P^{\prime}$, with $K$ the semisimple part of $P^{\prime}$. There is a BGG sequence on $G^{\prime} / P^{\prime}$ of conformally invariant operators between sections of vector bundles associated to representations of $P^{\prime}$, but these can be also interpreted as operators invariant with respect to the Riemannian structure between bundles associated to representations of $K$. The representations in the sequence that we constructed on $G / K$ are precisely these that occur in the BGG resolution of


Moreover, it is the maximal connected part of the BGG sequence such that it contains only operators corresponding to long roots. According to [RCD03], the operators in the BGG sequence that correspond to a long root can be expressed by an inductive prescription in terms of a chosen Weyl connection $\nabla$, the corresponding normalized Ricci curvature $r^{\nabla}$ and a suitable projection on an irreducible component. If we choose $\nabla$ to be the Levi-Civita connection, then the projections of the symbol parts $\nabla^{k}$ of these operators give powers of the generalized Stein-Weiss gradient by theorem 1 . To sum it up, the sequence on $G / K$ can be interpreted as the beginning of a BGG sequence on $G^{\prime} / P^{\prime}$ with the operators projected to their symbol parts with respect to the Levi-Civita connection.

For $n=2 k$ even, the BGG sequence on the boundary looks in standard coordinates like

$$
\begin{aligned}
& {\left[x_{1}, \ldots, x_{k-1}\right] } \\
& \xrightarrow{b+1}\left[x_{1}+b+1, x_{2}, \ldots, x_{k-1}\right] \xrightarrow{d_{1}+1} \ldots \\
& {\left[x_{1}+b+1, x_{1}+1, \ldots, x_{k-3}+1, x_{k-1}\right] \xrightarrow{d_{k-2}+1}\left[x_{1}+b+1, \ldots, x_{k-2}+1\right] \xrightarrow{a+1}\left[x_{1}+b+1, \ldots, x_{k-2}+1\right] } \\
& \xrightarrow{d_{k-2}+1}\left[x_{1}+b+1, x_{1}+1, x_{2}+1, \ldots, x_{k-3}+1, x_{k-2}\right] \xrightarrow{d_{k-3}+1} \ldots \xrightarrow{b+1}\left[x_{1}, x_{2}, \ldots, x_{k-1}\right]
\end{aligned}
$$

where we write again only weights of $M$-representations and the defining representation of $\mathfrak{s o}(2 k+1)$ has highest weight

We can state an analogue of lemma 11:
Lemma 12. Let $\mathbb{V}_{\lambda}$ be a representation of $M=S O(2 k-1)$ with standard coordinates $\left[x_{1}, x_{2}, \ldots, x_{k-1}\right]$ and $\mathbb{V}_{\lambda^{\prime}}$ with standard coordinates $\left[x_{1}^{\prime}, x_{2}^{\prime}, \ldots, x_{k-1}^{\prime}\right]$ be a weight such that there is an arrow $\mathbb{V}_{\mu} \xrightarrow{h} \mathbb{V}_{\nu}$ in some $B G G$ diagram.
(1) Let $\left[x_{1}^{\prime}-x_{1}, \ldots, x_{k-1}^{\prime}-x_{k-1}\right]=[0, \ldots, 0, h, 0, \ldots, 0]$ and $\mathbb{E}_{\sigma}$ be a representation of $K=S O(2 k)$ with standard coordinates $\left[x_{1}, \ldots, x_{k-1}, x\right]$ where $|x| \leq x_{k-1}$ and let $I_{\lambda}$ be the (unique) embedding $\mathbb{V}_{\lambda} \rightarrow \mathbb{E}_{\sigma}$. Then there is a unique component $\mathbb{E}_{\sigma^{\prime}}=$ $\left[x_{1}^{\prime}, x_{2}^{\prime}, \ldots, x_{k-1}^{\prime}, x\right]$ in $\otimes^{h} \mathfrak{q}^{*} \otimes \mathbb{E}_{\sigma}$.
(2) Let $\left[x_{1}^{\prime}-x_{1}, \ldots, x_{k-1}^{\prime}-x_{k-1}\right]=[0, \ldots, 0,-h, 0, \ldots, 0]$ and $\mathbb{E}_{\sigma}$ be a representation of $K=S O(2 k)$ with standard coordinates $\left[x, x_{1}, \ldots, x_{k-1}\right]$ where $x \geq x_{1}$ and let $I_{\lambda}$ be the (unique) embedding $\mathbb{V}_{\lambda} \rightarrow \mathbb{E}_{\sigma}$. Then there is a unique component $\mathbb{E}_{\sigma^{\prime}}=$ $\left[x, x_{1}^{\prime}, x_{2}^{\prime}, \ldots, x_{k-1}^{\prime}\right]$ in $\otimes^{h} \mathfrak{q}^{*} \otimes \mathbb{E}_{\sigma}$.

## Proof.

(1) As in lemma 11 to embed $\left[x_{1}^{\prime}, \ldots, x_{k-1}^{\prime}\right]$ into $\left[y_{1}, \ldots, y_{k}\right]$ there must be $y_{1} \geq x_{1}^{\prime} \geq y_{2} \geq x_{2}^{\prime} \geq \ldots \geq y_{k-1} \geq x_{k-1}^{\prime} \geq\left|y_{k}\right|$. Since for precisely one $j, 1 \leq j \leq k-1$ it is $x_{j}^{\prime}=x_{j}+h$, so $y_{j} \geq x_{j}+h$. There is only one such component in $\otimes^{h} \mathfrak{q}^{*} \otimes \mathbb{E}_{\sigma}$, with $y_{j}=x_{j}+h$ and other $y_{i}$ equal to $x_{i}$.
(2) Here we again want to embed $\left[x_{1}^{\prime}, \ldots, x_{k-1}^{\prime}\right]$ into $\left[y_{1}, \ldots, y_{k}\right]$ with the same inequalities satisfied. We know that for precisely one $j, 1 \leq j \leq k-1$ it is $x_{j}^{\prime}=x_{j}-h$. In particular, the branching rules imply that $y_{j+1} \leq x_{j}-h$. In $\otimes^{h} \mathfrak{q}^{*} \otimes \mathbb{E}_{\sigma}$ there is only one such component, namely $\left[x, x_{1}^{\prime}, \ldots, x_{k}^{\prime}\right]$.

As before, these embeddings give rise to Poisson transforms that intertwine differential operators and sequences of them. In the first case we get intertwining of a beginning of a BGG sequence and a sequence on $G / K$ that comes from a BGG sequence on $G^{\prime} / P^{\prime}$ by the procedure of taking symbol parts of operators. In the second case however the
operators on $G / K$ do not fit in a BGG sequence. We summarize the results into a theorem:

THEOREM 19. Let $\mathbb{W}$ be a representation of $G=S O(n, 1)$. Then there is a representation $\mathbb{F}$ of $G^{\prime}=S O(n+1,1)$ such that the $B G G$ sequence on $G / P$, where $P=C O(n-1) \ltimes\left(\mathbb{R}^{*}\right)^{n-1}$, and the sequence on $G / K$ obtained from the $B G G$ sequence on $G^{\prime} / P^{\prime}$, where $P=C O(n) \ltimes$ $\left(\mathbb{R}^{*}\right)^{n}$ by taking symbol parts of the operators are intertwined by the Poisson transform.
(1) For $n=2 k+1$ and the highest weight of $\mathbb{W}$ being

the operators give rise to a sequence of commutative squares

where the upper sequence is the $B G G$ sequence on $G / P$ and the lower one is the sequence on $G / K$.
(2) For $n=2 k$ and the highest weight of $\mathbb{W}$ being

we get a sequence

and a sequence


The lower row of the first sequence of commutative squares comes from the beginning of a $B G G$ sequence on $G^{\prime} / P^{\prime}$ but the lower row of the other sequence does not.

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[^0]:    ${ }^{1}$ Note that you can get the content of the second column of the following tables out of the first by inverting the root with respect to the center point of the table, transposing and (un)tilding both the corresponding sets and reversing the arrow. This symmetry is partly a result of a symmetry of the root system $\left(\lambda_{1}+\lambda_{j}\right)+\left(\lambda_{2}-\right.$ $\left.\lambda_{j}\right)=\lambda_{1}+\lambda_{2}$ and partly of the chosen notation.

