Dynamic equations on time scales and generalized ordinary differential equations*

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Abstract

The aim of this paper is to show that dynamic equations on time scales can be treated in the framework of generalized ordinary differential equations as introduced by J. Kurzweil. We also use some known results for generalized ordinary differential equations to obtain new theorems related to stability and continuous dependence on parameters for dynamic equations on time scales.

Keywords: Variational stability, stability with respect to perturbations, continuous dependence on parameters, linear equations, integration on time scales, Kurzweil-Stieltjes integral

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1 Introduction

The name of Jaroslav Kurzweil is connected especially with the nonabsolutely convergent Henstock-Kurzweil integral, which generalizes the integrals of Riemann, Lebesgue, and Newton. The roots of this integral lie in the theory of generalized ordinary differential equations, which can be traced back to 1957 (see the paper [8]). Consider an interval $I \subset \mathbb{R}$, a set $B \subset \mathbb{R}^n$, and a function $F : B \times I \to \mathbb{R}^n$. A function $x : I \to B$ is called a solution of the generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t),$$

whenever

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau), t)$$

for each pair of values $s_1, s_2 \in I$, where the integral on the right-hand side is the generalized Kurzweil integral (see the next section).

Although generalized ordinary differential equations are not as widely known as the Kurzweil integral, they have turned out to be quite a powerful concept,

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which includes not only ordinary differential equations, but also differential equations with impulses, measure differential equations and other concepts. For example, the recent papers [3] and [4] demonstrate the possibility of converting retarded functional differential equations to generalized ordinary differential equations for functions with values in a Banach space.

Štefan Schwabik has suggested a different research direction in his work [10], which shows that discrete systems of the form

$$x_{k+1} = f(x_k), \quad k \in \mathbb{N},$$

might be also rewritten as generalized differential equations. The present paper can be considered as a continuation of his work; we concentrate on dynamic equations on time scales of the form

$$x^{\Delta}(t) = f(x(t), t), \quad t \in \mathbb{T},$$

and show a procedure which allows us to convert this dynamic equation into a generalized differential equation. Finally, to illustrate the usefulness of this procedure, we obtain some new results concerning stability and continuous dependence on parameters for dynamic equations on time scales as corollaries of known results for generalized differential equations.

Although we do not presuppose a knowledge of generalized differential equations, some familiarity with the subject might be helpful; the monograph [9] is recommended as a good starting point. On the other hand, we assume a basic knowledge of dynamic equations on time scales as presented in [1] and [2].

2 Integrals and their properties

We start with a short summary of the generalized Kurzweil integral, also known as the generalized Perron integral.

Consider a function $\delta : [a, b] \to \mathbb{R}^+$. A partition D of interval $[a, b]$ with division points $a = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_k = b$ and tags $\tau_i \in [\alpha_{i-1}, \alpha_i]$, $i = 1, \ldots, k$, is called δ-fine if

$$[\alpha_{i-1}, \alpha_i] \subset [\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)], \quad i = 1, \ldots, k.$$

A function $F : [a, b] \times [a, b] \to \mathbb{R}^n$ is called Kurzweil integrable over $[a, b]$ if there exists a vector $I \in \mathbb{R}^n$ such that given an $\varepsilon > 0$, there is a function $\delta : [a, b] \to \mathbb{R}^+$ such that

$$\left\| \sum_{j=1}^{k} \big(F(\tau_j, \alpha_j) - F(\tau_j, \alpha_{j-1})\big) - I \right\| < \varepsilon$$

for every δ–fine partition D. The vector $I \in \mathbb{R}^n$ is called the generalized Kurzweil integral of F over $[a, b]$ and will be denoted by $\int_a^b DF(\tau, t)$.

An important special case is the Kurzweil-Stieltjes integral $\int_a^b f(s)\,\mathrm{d}g(s)$, which is obtained from a pair of functions $f : [a, b] \to \mathbb{R}^n$ and $g : [a, b] \to \mathbb{R}$ by setting $F(\tau, t) = f(\tau)g(t)$. Note also that the choice $g(t) = t$ leads to the Henstock-Kurzweil integral $\int_a^b f(s)\,\mathrm{d}s$ mentioned in the introduction; this justifies the name "generalized Kurzweil integral".

If $a < b$, we let $\int_b^a DF(\tau,t) = -\int_a^b DF(\tau,t)$ provided the right-hand side exists; we also set $\int_a^b DF(\tau,t) = 0$ when $a = b$.

The following results will be needed later in our development; see Chapter 1 of [9] for more information about the generalized Kurzweil integral.

Theorem 1. *If $f : [a,b] \to \mathbb{R}^n$ is a regulated function and $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, then the integral $\int_a^b f(s)\,\mathrm{d}g(s)$ exists. Moreover, when $\|f(s)\| \le C$ for every $s \in [a,b]$, then*

$$\left\| \int_a^b f(s)\,\mathrm{d}g(s) \right\| \le C(g(b) - g(a)).$$

Proof. See Corollary 1.34 in [9]; the inequality follows easily from the definition of the integral. $\square$

Theorem 2. *Let $f : [a,b] \to \mathbb{R}^n$ and $g : [a,b] \to \mathbb{R}$ be a pair of functions such that g is regulated and $\int_a^b f(s)\,\mathrm{d}g(s)$ exists. Then the function*

$$h(t) = \int_a^t f(s)\,\mathrm{d}g(s), \quad t \in [a,b],$$

is regulated and satisfies

$$\begin{aligned} h(t+) &= h(t) + f(t)\Delta^+ g(t), \quad t \in [a,b), \\ h(t-) &= h(t) - f(t)\Delta^- g(t), \quad t \in (a,b], \end{aligned}$$

where $\Delta^+ g(t) = g(t+) - g(t)$ and $\Delta^- g(t) = g(t) - g(t-)$.

Proof. The statement is a special case of Theorem 1.16 in [9]. $\square$

Theorem 3. *Let $g : [a,b] \to \mathbb{R}$ be a nondecreasing function. Consider a sequence of functions $f_k : [a,b] \to \mathbb{R}^n$, $k \in \mathbb{N}$, such that $\int_a^b f_k(t)\,\mathrm{d}g(t)$ exists for every $k \in \mathbb{N}$. Assume there exists a function $m : [a,b] \to \mathbb{R}$ such that the integral $\int_a^b m(t)\,\mathrm{d}g(t)$ exists, and such that*

$$\|f_k(t)\| \le m(t), \quad t \in [a,b], \ k \in \mathbb{N}.$$

If $\lim_{k\to\infty} f_k(t) = f(t)$ for every $t \in [a,b]$, then $\int_a^b f(t)\,\mathrm{d}g(t)$ exists and

$$\int_a^b f(t)\,\mathrm{d}g(t) = \lim_{k\to\infty} \int_a^b f_k(t)\,\mathrm{d}g(t).$$

Proof. The statement follows easily from Corollary 1.32 in [9]. $\square$

Our next goal in this section is to show that the Riemann Δ-integral, which represents a time scale version of the classical Riemann integral, is in fact a special case of the Kurzweil-Stieltjes integral. We assume that the reader is familiar with the Riemann Δ-integral as described in Chapter 5 of [2].

Given a time scale $\mathbb{T}$ and a pair of numbers $a, b \in \mathbb{T}$, the symbol $[a,b]_{\mathbb{T}}$ will be used throughout this paper to denote a compact interval in $\mathbb{T}$, i.e. $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T}; a \le t \le b\}$. The open and half-open intervals are defined in an similar way. On the other hand, $[a, b]$ will be used to denote intervals on the real line, i.e. $[a, b] = \{t \in \mathbb{R}; a \le t \le b\}$. This notational convention should help the reader to distinguish between ordinary and time scale intervals.

Given a real number $t \le \sup \mathbb{T}$, we define

$$t^* = \inf\{s \in \mathbb{T}; s \ge t\}.$$

Since $\mathbb{T}$ is a closed set, we have $t \in \mathbb{T}$. Further, let

$$\mathbb{T}^* = \begin{cases} (-\infty, \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty, \\ (-\infty, \infty) & \text{otherwise.} \end{cases}$$

Given a function $f : \mathbb{T} \to \mathbb{R}^n$, we define a function $f^* : \mathbb{T}^* \to \mathbb{R}^n$ by

$$f^*(t) = f(t^*), \quad t \in \mathbb{T}^*.$$

Similarly, given a set $B \subset \mathbb{R}^n$ and a function $f : B \times \mathbb{T} \to \mathbb{R}^n$, we define

$$f^*(x, t) = f(x, t^*), \quad x \in B, \ t \in \mathbb{T}^*.$$

Lemma 4. *If $f : \mathbb{T} \to \mathbb{R}^n$ is a regulated function, then $f^* : \mathbb{T}^* \to \mathbb{R}^n$ is also regulated. If f is left-continuous on $\mathbb{T}$, then f^* is left-continuous on $\mathbb{T}^*$. If f is right-continuous on $\mathbb{T}$, then f^* is right-continuous at right-dense points of $\mathbb{T}$.*

Proof. Let us calculate $\lim_{t \to t_0-} f^*(t)$, where $t_0 \in \mathbb{T}^*$. If $t_0 \in \mathbb{T}$ and it is left-dense, then

$$\lim_{t \to t_0-} f^*(t) = \lim_{t \to t_0-} f(t).$$

If $t_0 \in \mathbb{T}$ and it is left-scattered, then

$$\lim_{t \to t_0-} f^*(t) = f(t_0) = f^*(t_0).$$

Finally, if $t_0 \notin \mathbb{T}$, then

$$\lim_{t \to t_0-} f^*(t) = f(t_0^*) = f^*(t_0).$$

Now consider $\lim_{t \to t_0+} f^*(t)$, where $t_0 \in \mathbb{T}^*$ and $t_0 < \sup \mathbb{T}^*$. If $t_0 \in \mathbb{T}$ and it is right-dense, then

$$\lim_{t \to t_0+} f^*(t) = \lim_{t \to t_0+} f(t).$$

If $t_0 \in \mathbb{T}$ and it is right-scattered, then

$$\lim_{t \to t_0+} f^*(t) = f(\sigma(t_0)).$$

Finally, if $t_0 \notin \mathbb{T}$, then

$$\lim_{t \to t_0+} f^*(t) = f(t_0^*) = f^*(t_0). \qquad \square$$

Theorem 5. *Let $f : \mathbb{T} \to \mathbb{R}^n$ be an rd-continuous function. Choose an arbitrary $a \in \mathbb{T}$ and define*

$$\begin{aligned} F_1(t) &= \int_a^t f(s)\, \Delta s, \quad t \in \mathbb{T}, \\ F_2(t) &= \int_a^t f^*(s)\, \mathrm{d}g(s), \quad t \in \mathbb{T}^*, \end{aligned}$$

where $g(s) = s^*$ for every $s \in \mathbb{T}^*$. Then $F_2 = F_1^*$.

Proof. Note that the functions F_1 and F_2 are well-defined; indeed, the Riemann Δ-integral in the definition of F_1 exists because f is rd-continuous, and the Kurzweil-Stieltjes integral in the definition of F_2 exists because f^* is regulated (use Lemma 4 and the fact that every rd-continuous function is regulated) and g is nondecreasing. To complete the proof, it is sufficient to prove the following two statements:

(1) $F_1(t) = F_2(t)$ for every $t \in \mathbb{T}$.

(2) If $t \in \mathbb{T}$ and $s = \sup\{u \in \mathbb{T};\, u < t\}$, then F_2 is constant on $(s,t]$.

We start with the second statement, which is easy to prove: If $u, v \in (s,t]$ and $u < v$, then

$$F_2(v) - F_2(u) = \int_u^v f^*(s)\,\mathrm{d}g(s) = 0,$$

where the last equality follows from the definition of the Kurzweil-Stieltjes integral and the fact that g is constant on $[u,v]$.

To prove the first statement, we note that $F_1(a) = F_2(a) = 0$ and it is thus sufficient to show that $F_1^\Delta(t) = F_2^\Delta(t)$ for every $t \in \mathbb{T}$ (any two functions with the same Δ-derivative differ only by a constant). It follows from the properties of the Riemann Δ-integral that $F_1^\Delta(t) = f(t)$, and it remains to calculate F_2^Δ.

When t is a right-dense point, then f is continuous at t and

$$\lim_{s \to t} f^*(s) = f^*(t) = f(t)$$

(see Lemma 4). Therefore, given an arbitrary $\varepsilon > 0$, there is a $\delta > 0$ such that $\|f^*(s) - f(t)\| < \varepsilon$ whenever $|s - t| < \delta$. Now, consider a sequence of time scale points $\{t_k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} t_k = t$. We can find a $k_0 \in \mathbb{N}$ such that $|t_k - t| < \delta$ whenever $k \geq k_0$. Thus for every $k \geq k_0$ we have

$$\left\| \frac{F_2(t_k) - F_2(t)}{t_k - t} - f(t) \right\| = \left\| \frac{1}{t_k - t} \int_t^{t_k} f^*(s)\,\mathrm{d}g(s) - f(t) \right\|$$

$$= \left\| \frac{1}{t_k - t} \int_t^{t_k} (f^*(s) - f(t))\,\mathrm{d}g(s) \right\| \leq \varepsilon\, \frac{g(t_k) - g(t)}{t_k - t} = \varepsilon,$$

since $g(t_k) = t_k$ and $g(t) = t$. It follows that

$$\lim_{k\to\infty} \frac{F_2(t_k) - F_2(t)}{t_k - t} = f(t),$$

i.e. $F_2^\Delta(t) = f(t)$.

On the contrary, when t is a right-scattered point, we have

$$F_2(\sigma(t)) = F_2(t+) = F_2(t) + f(t)\Delta^+ g(t),$$

where the first equality follows from the fact that F_2 is constant on $(t, \sigma(t)]$ and the second equality is a consequence of Theorem 2. But $\Delta^+ g(t) = g(t+) - g(t) = \sigma(t) - t = \mu(t)$, and it follows that

$$F_2^\Delta(t) = \frac{F_2(\sigma(t)) - F_2(t)}{\mu(t)} = f(t). \qquad \square$$

Note that the integral $\int_a^t f^*(s)\,\mathrm{d}g(s)$ in the definition of F_2 also exists as the Lebesgue-Stieltjes integral. We have chosen the Kurzweil-Stieltjes integral simply because it seems to be more natural in the context of generalized differential equations. On the other hand, the integral need not exist as the Riemann-Stieltjes integral, because there might be points where both f^* and g are discontinuous (this is in fact a typical behavior at right-scattered points).

3 Main result

This section describes the correspondence between dynamic equations on time scales and generalized ordinary differential equations. To obtain a reasonable theory, we restrict ourselves to differential and dynamic equations whose right-hand sides are functions satisfying the conditions given below.

Assume that $G = B \times I$, where $I \subset \mathbb{R}$ is an interval and $B \subset \mathbb{R}^n$. Given a function $F : G \to \mathbb{R}^n$, we introduce the following conditions, which play an important role in the theory of generalized ordinary differential equations:

(F1) There exists a nondecreasing function $h : I \to \mathbb{R}$ such that

$$\|F(x,t_2) - F(x,t_1)\| \le |h(t_2) - h(t_1)|$$

for every $x \in B$ and $t_1, t_2 \in I$.

(F2) There exists a continuous increasing function $\omega : [0,\infty) \to \mathbb{R}$ with $\omega(0) = 0$ such that

$$\|F(x,t_2) - F(x,t_1) - F(y,t_2) + F(y,t_1)\| \le \omega(\|x-y\|)|h(t_2) - h(t_1)|$$

for every $x, y \in B$ and $t_1, t_2 \in I$.

Now, consider a set $B \subset \mathbb{R}^n$ and a function $f : B \times \mathbb{T} \to \mathbb{R}^n$. Let us introduce the following three conditions:

(C1) f is rd-continuous, i.e. the function $t \mapsto f(x(t), t)$ is rd-continuous whenever $x : \mathbb{T} \to B$ is a continuous function.

(C2) There exists a regulated function $m : \mathbb{T} \to \mathbb{R}$ such that $\|f(x,t)\| \le m(t)$ for every $x \in B$ and $t \in \mathbb{T}$.

(C3) There exists a continuous increasing function $\omega : [0,\infty) \to \mathbb{R}$ with $\omega(0) = 0$ and a regulated function $l : \mathbb{T} \to \mathbb{R}$ such that

$$\|f(x,t) - f(y,t)\| \le l(t)\omega(\|x-y\|)$$

for every $x, y \in B$ and $t \in \mathbb{T}$.

The following lemma describes the relation between the two sets of conditions.

Lemma 6. *Consider a set $B \subset \mathbb{R}^n$. Assume that $f : B \times \mathbb{T} \to \mathbb{R}^n$ satisfies conditions* (C1)–(C3). *Define $g(s) = s^*$ for every $s \in \mathbb{T}^*$. Then for arbitrary $t_0 \in \mathbb{T}$, the function*

$$F(x,t) = \int_{t_0}^t f^*(x,s)\,\mathrm{d}g(s), \quad x \in B,\ t \in \mathbb{T}^*,$$

satisfies conditions (F1)–(F2) *on the set* $G = B \times \mathbb{T}^*$ *with*

$$h(t) = \int_{t_0}^{t} (l^*(s) + m^*(s))\,\mathrm{d}g(s).$$

Proof. For a fixed $x \in B$, condition (C1) implies that the function $t \mapsto f(x,t)$ is rd-continuous on $\mathbb{T}$, and therefore $t \mapsto f^*(x,t)$ is regulated on $\mathbb{T}^*$. The function g is nondecreasing, and thus the Kurzweil-Stieltjes integral exists and F is well defined. Similarly, the functions l^* and m^* are regulated, and thus the integral in the definition of h exists. When $t_1 \leq t_2$, we have

$$\|F(x,t_2) - F(x,t_1)\| = \left\| \int_{t_1}^{t_2} f^*(x,s)\,\mathrm{d}g(s) \right\|$$

$$\leq \int_{t_1}^{t_2} \|f^*(x,s)\|\,\mathrm{d}g(s) \leq \int_{t_1}^{t_2} m^*(s)\,\mathrm{d}g(s) \leq h(t_2) - h(t_1)$$

and

$$\|F(x,t_2) - F(x,t_1) - F(y,t_2) + F(y,t_1)\|$$

$$= \left\| \int_{t_1}^{t_2} f^*(x,s)\,\mathrm{d}g(s) - \int_{t_1}^{t_2} f^*(y,s)\,\mathrm{d}g(s) \right\| = \left\| \int_{t_1}^{t_2} (f^*(x,s) - f^*(y,s))\,\mathrm{d}g(s) \right\|$$

$$\leq \int_{t_1}^{t_2} \|f^*(x,s) - f^*(y,s)\|\,\mathrm{d}g(s) \leq \omega(\|x-y\|) \int_{t_1}^{t_2} l^*(s)\,\mathrm{d}g(s)$$

$$\leq \omega(\|x-y\|)(h(t_2) - h(t_1)).$$

The case $t_1 > t_2$ is similar and is left to the reader. □

Before proceeding to the main result, we need the following auxiliary lemmas.

Lemma 7. *Let* $G = B \times [\alpha,\beta]$, *where* $B \subset \mathbb{R}^n$. *Consider a function* $F : G \to \mathbb{R}^n$ *such that* $t \mapsto F(x,t)$ *is regulated on* I *for every* $x \in B$. *If* $x : [\alpha,\beta] \to B$ *is a step function, i.e. if there exists a partition*

$$\alpha = s_0 < s_1 < \cdots < s_k = \beta$$

and vectors $c_1, \ldots, c_k \in \mathbb{R}^n$ *such that*

$$x(s) = c_i \ \text{ for every } s \in (s_{i-1}, s_i),$$

then

$$\int_{\alpha}^{\beta} DF(x(\tau),t) = \sum_{j=1}^{k} \Bigg(F(c_j, s_j-) - F(c_j, s_{j-1}+)$$

$$+F(x(s_{j-1}), s_{j-1}+) - F(x(s_{j-1}), s_{j-1}) + F(x(s_j), s_j) - F(x(s_j), s_j-) \Bigg).$$

Proof. See the proof of Corollary 3.15 in [9]. □

Lemma 8. *Consider a set $B \subset \mathbb{R}^n$ and a function $f : B \times \mathbb{T} \to \mathbb{R}^n$ such that $t \mapsto f(x,t)$ is regulated on $\mathbb{T}$ for every $x \in B$. Define $g(t) = t^*$ for every $t \in \mathbb{T}^*$, choose an arbitrary $t_0 \in \mathbb{T}$ and let*

$$F(x,t) = \int_{t_0}^{t} f^*(x,s)\,\mathrm{d}g(s), \quad x \in B,\ t \in \mathbb{T}^*.$$

If $[\alpha,\beta] \subset \mathbb{T}^*$ and $x : [\alpha,\beta] \to B$ is a step function, then

$$\int_{\alpha}^{\beta} DF(x(\tau),t) = \int_{\alpha}^{\beta} f^*(x(t),t)\,\mathrm{d}g(t).$$

Proof. By Lemma 4, the function $t \mapsto f^*(x,t)$ is regulated on $\mathbb{T}^*$ for every $x \in B$, and thus the integral in the definition of F exists. Given a step function $x : [\alpha,\beta] \to B$, there exists a partition

$$\alpha = s_0 < s_1 < \cdots < s_k = \beta$$

and vectors $c_1, \ldots, c_k \in \mathbb{R}^n$ such that

$$x(s) = c_i \ \text{ for every } s \in (s_{i-1}, s_i).$$

The function $t \mapsto F(x,t)$ is regulated by Theorem 2 and we may use Lemma 7 to obtain

$$\int_{\alpha}^{\beta} DF(x(\tau),t) = \lim_{\varepsilon \to 0+} \sum_{j=1}^{k} \big(F(c_j, s_j - \varepsilon) - F(c_j, s_{j-1} + \varepsilon)\big) \tag{1}$$

$$+ \lim_{\varepsilon \to 0+} \sum_{j=1}^{k} \big(F(x(s_{j-1}), s_{j-1} + \varepsilon) - F(x(s_{j-1}), s_{j-1})\big) \tag{2}$$

$$+ \lim_{\varepsilon \to 0+} \sum_{j=1}^{k} \big(F(x(s_j), s_j) - F(x(s_j), s_j - \varepsilon)\big). \tag{3}$$

Now, since x is a step function, it is not difficult to see that $t \mapsto f^*(x(t),t)$ is regulated, and thus the integral $\int_{\alpha}^{\beta} f^*(x(t),t)\,\mathrm{d}g(t)$ exists. In this case, we obtain

$$\int_{\alpha}^{\beta} f^*(x(t),t)\,\mathrm{d}g(t) = \sum_{j=1}^{k} \int_{s_{j-1}}^{s_j} f^*(x(s),s)\,\mathrm{d}g(s)$$

$$= \lim_{\varepsilon \to 0+} \sum_{j=1}^{k} \int_{s_{j-1}}^{s_{j-1}+\varepsilon} f^*(x(s),s)\,\mathrm{d}g(s) \tag{4}$$

$$+ \lim_{\varepsilon \to 0+} \sum_{j=1}^{k} \int_{s_{j-1}+\varepsilon}^{s_j - \varepsilon} f^*(x(s),s)\,\mathrm{d}g(s) \tag{5}$$

$$+ \lim_{\varepsilon \to 0+} \sum_{j=1}^{k} \int_{s_j - \varepsilon}^{s_j} f^*(x(s),s)\,\mathrm{d}g(s). \tag{6}$$

Obviously, for every $i \in \{1, \dots, k\}$ we have

$$F(c_j, s_j - \varepsilon) - F(c_j, s_{j-1} + \varepsilon) = \int_{s_{j-1}+\varepsilon}^{s_j-\varepsilon} f^*(x(t), t)\,\mathrm{d}g(t),$$

and thus (1) equals (5). Theorem 2 gives

$$\lim_{\varepsilon \to 0+} F(x(s_{j-1}), s_{j-1} + \varepsilon) - F(x(s_{j-1}), s_{j-1})$$

$$= \lim_{\varepsilon \to 0+} \int_{s_{j-1}}^{s_{j-1}+\varepsilon} f^*(x(s_{j-1}), s)\,\mathrm{d}g(s) = f^*(x(s_{j-1}), s_{j-1})\Delta^+ g(s_{j-1})$$

and

$$\lim_{\varepsilon \to 0+} \int_{s_{j-1}}^{s_{j-1}+\varepsilon} f^*(x(s), s)\,\mathrm{d}g(s) = f^*(x(s_{j-1}), s_{j-1})\Delta^+ g(s_{j-1})$$

and thus (2) equals (4). Finally,

$$\lim_{\varepsilon \to 0+} F(x(s_j), s_j) - F(x(s_j), s_j - \varepsilon)$$

$$= \lim_{\varepsilon \to 0+} \int_{s_j-\varepsilon}^{s_j} f^*(x(s_j), s)\,\mathrm{d}g(s) = f^*(x(s_j), s_j)\Delta^- g(s_j)$$

and

$$\lim_{\varepsilon \to 0+} \int_{s_j-\varepsilon}^{s_j} f^*(x(s), s)\,\mathrm{d}g(s) = f^*(x(s_j), s_j)\Delta^- g(s_j)$$

and thus (3) equals (6). □

Lemma 9. *Let $G = B \times [\alpha, \beta]$, where $B \subset \mathbb{R}^n$. Assume that $F : G \to \mathbb{R}^n$ satisfies conditions* (F1)–(F2) *for some h and ω. If $x : [\alpha, \beta] \to B$ is a pointwise limit of step functions $x_k : [\alpha, \beta] \to B$, then*

$$\int_\alpha^\beta DF(x(\tau), t) = \lim_{k \to \infty} \int_\alpha^\beta DF(x_k(\tau), t)$$

Proof. See Corollary 3.15 in [9]. □

It is a known fact that given a regulated function $x : [\alpha, \beta] \to \mathbb{R}^n$ and a number $\varepsilon > 0$, there is a step function $\varphi : [\alpha, \beta] \to \mathbb{R}^n$ such that $\|x(t) - \varphi(t)\| < \varepsilon$ for every $t \in [\alpha, \beta]$. In other words, every regulated function is a uniform limit of step functions. Now suppose there is a set $B \subset \mathbb{R}^n$ such that $x(t) \in B$ for every $t \in [\alpha, \beta]$; we wish to show that x can be uniformly approximated by step functions with values in B. Assume that the above mentioned step function φ is constant on intervals (s_{i-1}, s_i), where $\alpha = s_0 < s_1 < \cdots < s_k = \beta$ is a partition of $[\alpha, \beta]$. Now, choose a $t_i \in (s_{i-1}, s_i)$ for every $i \in \{1, \dots, k\}$ and construct a function $\psi : [\alpha, \beta] \to B$ as follows:

$$\psi(s) = \begin{cases} x(s_i) & \text{for } s = s_i, \\ x(t_i) & \text{for } s \in (s_{i-1}, s_i). \end{cases}$$

It is clear that ψ is a step function. Moreover, when $s \in (s_{i-1}, s_i)$, then

$$\|x(s) - \psi(s)\| \le \|x(s) - \varphi(s)\| + \|\varphi(s) - \psi(s)\| = \|x(s) - \varphi(s)\| + \|\varphi(t_i) - x(t_i)\| < 2\varepsilon.$$

It follows that $\|x(t) - \psi(t)\| < 2\varepsilon$ for every $t \in [\alpha, \beta]$. This means that x can be uniformly approximated by step functions with values in B.

Lemma 10. *Let $B \subset \mathbb{R}^n$ and assume that $f : B \times \mathbb{T} \to \mathbb{R}^n$ satisfies conditions* (C1)–(C3)*. Define $g(t) = t^*$ for every $t \in \mathbb{T}^*$, choose an arbitrary $t_0 \in \mathbb{T}$ and let*

$$F(x,t) = \int_{t_0}^{t} f^*(x,s)\,\mathrm{d}g(s), \quad x \in B,\ t \in \mathbb{T}^*.$$

If $[\alpha,\beta] \subset \mathbb{T}^*$ and $x : [\alpha,\beta] \to B$ is a regulated function, then

$$\int_{\alpha}^{\beta} DF(x(\tau),t) = \int_{\alpha}^{\beta} f^*(x(t),t)\,\mathrm{d}g(t).$$

Proof. Given a regulated function $x : [\alpha,\beta] \to B$, there is a sequence of step functions $x_k : [\alpha,\beta] \to B$ which converge uniformly to x on $[\alpha,\beta]$. Condition (C3) implies

$$\|f^*(x_k(t),t) - f^*(x(t),t)\| \le l(t^*)\omega(\|x_k(t) - x(t)\|), \quad k \in \mathbb{N},\ t \in [\alpha,\beta],$$

and thus $\lim_{k\to\infty} f^*(x_k(t),t) = f^*(x(t),t)$ for every $t \in [\alpha,\beta]$. Using first Lemma 9 (the assumptions are satisfied by Lemma 6) and then Lemma 8, we obtain

$$\int_{\alpha}^{\beta} DF(x(\tau),t) = \lim_{k\to\infty} \int_{\alpha}^{\beta} DF(x_k(\tau),t)$$

$$= \lim_{k\to\infty} \int_{\alpha}^{\beta} f^*(x_k(t),t)\,\mathrm{d}g(t) = \int_{\alpha}^{\beta} f^*(x(t),t)\,\mathrm{d}g(t),$$

where the last equality follows from Theorem 3 (note that $\|f^*(x(t),t)\| \le m^*(t)$, m^* is regulated, and thus the assumptions are satisfied). □

Lemma 11. *If $x : [\alpha,\beta] \to \mathbb{R}^n$ is a solution of the generalized ordinary differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t),$$

then

$$\lim_{u\to t}(x(u) - F(x(t),u) + F(x(t),t)) = x(t)$$

for every $t \in [\alpha,\beta]$.

Proof. See Proposition 3.6 in [9]. □

Now we have all prerequisites necessary for the proof of the main result.

Theorem 12. *Let $X \subset \mathbb{R}^n$ and assume that $f : X \times \mathbb{T} \to \mathbb{R}^n$ is such that conditions* (C1)–(C3) *are satisfied on every set $G = B \times [\alpha,\beta]_{\mathbb{T}}$, where $B \subset X$ is bounded. If $x : \mathbb{T} \to X$ is a solution of*

$$x^{\Delta}(t) = f(x(t),t), \quad t \in \mathbb{T}, \tag{7}$$

then $x^* : \mathbb{T}^* \to X$ is a solution of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t), \quad t \in \mathbb{T}^*, \tag{8}$$

where

$$F(x,t) = \int_{t_0}^{t} f^*(x,s)\,\mathrm{d}g(s), \quad x \in X,\ t \in \mathbb{T}^*,$$

$t_0 \in \mathbb{T}$, and $g(s) = s^*$ for every $s \in \mathbb{T}^*$. Moreover, every solution $y : \mathbb{T}^* \to X$ of (8) *has the form $y = x^*$, where $x : \mathbb{T} \to X$ is a solution of* (7)*.*

Proof. Choose an arbitrary $a \in \mathbb{T}$. If $x : \mathbb{T} \to \mathbb{R}^n$ is a solution of (7), then

$$x(s) = x(a) + \int_a^s f(x(t), t)\Delta t, \quad s \in \mathbb{T}.$$

It follows that

$$x(s^*) = x(a) + \int_a^{s^*} f(x(t), t)\Delta t, \quad s \in \mathbb{T}^*.$$

Using Theorem 5, we rewrite the last equation as

$$x^*(s) = x^*(a) + \int_a^s f^*(x^*(t), t)\,\mathrm{d}g(t), \quad s \in \mathbb{T}^*. \tag{9}$$

Let I be a compact interval in $\mathbb{T}$ containing both a and s^*. Since x is continuous, it is bounded on I. Therefore it is possible to find a bounded set $B \subset X$ such that $x(t) \in B$ for every $t \in I$. The function f satisfies conditions (C1)–(C3) on $B \times I$ and we may use Lemma 10 to replace the last equality by

$$x^*(s) = x^*(a) + \int_a^s DF(x^*(\tau), t), \quad s \in \mathbb{T}^*,$$

which means that x^* is a solution of the generalized equation (8).

To prove the second assertion, let $y : \mathbb{T}^* \to X$ be a solution of (8). Then

$$y(s) = y(a) + \int_a^s DF(y(\tau), t), \quad s \in \mathbb{T}^*.$$

Fix an arbitrary $s \in \mathbb{T}^*$ and let $[\alpha, \beta]_{\mathbb{T}}$ be a time scale interval such that $a, s \in [\alpha, \beta]$. For every $\tau \in [\alpha, \beta)$, Lemma 11 implies that

$$y(\tau) = \lim_{u \to \tau+} (y(u) - F(y(\tau), u) + F(y(\tau), \tau)) =$$

$$= \lim_{u \to \tau+} \left(y(u) - \int_\tau^u f^*(y(\tau), s)\,\mathrm{d}g(s) \right) = \lim_{u \to \tau+} \left(y(u) - f^*(y(\tau), \tau)\Delta^+ g(\tau) \right),$$

and therefore $\lim_{u \to \tau+} y(u)$ exists. Similarly, for every $\tau \in (\alpha, \beta]$, we have

$$y(\tau) = \lim_{u \to \tau-} (y(u) - F(y(\tau), u) + F(y(\tau), \tau)) =$$

$$= \lim_{u \to \tau-} \left(y(u) - \int_\tau^u f^*(y(\tau), s)\,\mathrm{d}g(s) \right) =$$

$$= \lim_{u \to \tau-} \left(y(u) + f^*(y(\tau), \tau)\Delta^- g(\tau) \right) = \lim_{u \to \tau-} y(u),$$

because g is a left-continuous function. Since y is regulated and therefore bounded on $[\alpha, \beta]$, it is possible to find a bounded set $B \subset X$ such that $y(t) \in B$ for every $t \in [\alpha, \beta]$. The function f satisfies conditions (C1)–(C3) on $B \times [\alpha, \beta]_{\mathbb{T}}$ and Lemma 6 guarantees that the function F satisfies conditions (F1)–(F2) on $B \times [\alpha, \beta]$. Using Lemma 10 again, we obtain

$$y(s) = y(a) + \int_a^s f^*(y(t), t)\,\mathrm{d}g(t), \quad s \in \mathbb{T}^*.$$

But the right-hand side is constant on every interval $(s, t]$, where $t \in \mathbb{T}$ and $s = \sup\{u \in \mathbb{T};\ u < t\}$ (see the argument in the proof of Theorem 5). Thus $y = x^*$, where $x : \mathbb{T} \to B$ is the restriction of y to $\mathbb{T}$. This implies (9), and consequently also (7) (note that, according to Theorem 2, x is a rd-continuous function). $\square$

From now on, the letter g will always denote the function $g(s) = s^*$.

Let us pause for a moment to discuss conditions (C1)–(C3). Condition (C1) is fairly common in the theory of dynamic equations; its purpose is to ensure that the integral equation

$$x(s) = x(a) + \int_a^s f(x(t), t)\Delta t$$

can be differentiated to obtain $x^\Delta(t) = f(x(t), t)$. In a more general setting, we could focus our interest on the integral equation only; in this case, it would be sufficient to assume that $t \mapsto f(x(t), t)$ is regulated whenever $x : \mathbb{T} \to X$ is a regulated function.

Conditions (C2)–(C3) were used to prove that the function

$$F(x, t) = \int_{t_0}^t f^*(x, s)\,\mathrm{d}g(s), \quad x \in B,\ t \in \mathbb{T}^*,$$

satisfies conditions (F1)–(F2). Condition (C3) represents a generalization of Lipschitz-continuity with respect to x, which corresponds to the special case $\omega(r) = r$ and $l(t) = L$. Again, this is a fairly standard condition. In many cases, the function f is defined on $\mathbb{R}^n \times \mathbb{T}$ and has continuous partial derivatives with respect to $x_1, \dots, x_n$. Since we require the conditions to be satisfied only for sets of the form $G = B \times [\alpha, \beta]_{\mathbb{T}}$ with $B \subset X$ bounded, it is easy to see that both (C2) and (C3) are satisfied.

Moreover, if $B = \{x \in \mathbb{R}^n;\ \|x\| \le r\}$, condition (C3) can be weakened; in this case, it is sufficient to assume that $x \mapsto f(x, t)$ is continuous for every $t \in \mathbb{T}$ (see Chapter 5 of [9], which describes the case $\mathbb{T} = \mathbb{R}$, but the same reasoning can be used for a general time scale).

4 Linear equations

To illustrate Theorem 12 on a simple example, consider the linear dynamic equation

$$x^\Delta(t) = a(t)x(t) + h(t), \quad t \in \mathbb{T}, \tag{10}$$

where $a : \mathbb{T} \to \mathbb{R}^{n\times n}$ and $h : \mathbb{T} \to \mathbb{R}^n$ are rd-continuous functions (we use the symbol $\mathbb{R}^{n\times n}$ to denote the set of all $n \times n$ matrices). It is easy to see that the function $f(x, t) = a(t)x + h(t)$ satisfies conditions (C1)–(C3) on every set $G = B \times [\alpha, \beta]_{\mathbb{T}}$, where $B \subset X$ is bounded.

To obtain the corresponding generalized differential equation, we choose an arbitrary $\tau_0 \in \mathbb{T}$ and let

$$F(x, t) = \int_{\tau_0}^t (a^*(s)x + h^*(s))\,\mathrm{d}g(s) = A(t)x + H(t),$$

where $A(t) = \int_{\tau_0}^t a^*(s)\,\mathrm{d}g(s)$ and $H(t) = \int_{\tau_0}^t h^*(s)\,\mathrm{d}g(s)$. Now, Theorem 12 says that if $x : \mathbb{T} \to X$ is a solution of (10), then the function x^* is a solution of the linear generalized differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = D\big(A(t)x + H(t)\big), \quad t \in \mathbb{T}^*. \tag{11}$$

Conversely, every solution of this generalized equation has the form x^*, where $x : \mathbb{T} \to \mathbb{R}^n$ is a solution of the dynamic equation (10).

The monograph [9] contains a fairly complete theory of linear generalized equations. For example, the equation (11) is known to have a unique solution satisfying $x(t_0) = x_0$, whenever

$$I - (A(t) - A(t-)) \text{ and } I + A(t+) - A(t) \text{ are regular for every } t. \tag{12}$$

Let us rephrase this condition in the language of equation (10); Theorem 2 gives

$$\begin{aligned} A(t+) &= A(t) + a^*(t)\Delta^+ g(t), \\ A(t-) &= A(t) - a^*(t)\Delta^- g(t). \end{aligned}$$

First, if $t \in \mathbb{T}^* \backslash \mathbb{T}$, then $\Delta^+ g(t) = \Delta^- g(t) = 0$ and (12) is satisfied. Next, assume $t \in \mathbb{T}$. Since g is a left-continuous function, we always have $\Delta^- g(t) = 0$ and therefore $I - (A(t) - A(t-)) = I$ is regular. Finally, if t is a right-dense point, then $\Delta^+ g(t) = 0$ and $I + A(t+) - A(t) = I$ is regular; if t is right-scattered, then $\Delta^+ g(t) = \mu(t)$ and $I + A(t+) - A(t)$ is regular if and only if $I + a(t)\mu(t)$ is regular. The last condition is called regressivity and is well known in the theory of linear dynamic equations.

Let us mention one more result: Under assumption (12), there exists a function $U : \mathbb{T}^* \times \mathbb{T}^* \to \mathbb{R}^{n\times n}$ such that the function $x(t) = U(t, t_0)x_0$ represents the unique solution of the homogeneous equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = D(A(t)x), \quad t \in \mathbb{T}^*, \ x(t_0) = x_0.$$

The function U has the following properties:

(i) $U(t, t) = I$ for every $t \in \mathbb{T}^*$,

(ii) $U(t, s) = U(t, r)U(r, s)$ for every $r, s, t \in \mathbb{T}^*$,

(iii) $U(t+, s) = (I + A(t+) - A(t))U(t, s)$ for every $s, t \in \mathbb{T}^*$,

(iv) $U(t, s)$ is always a regular matrix and $U(t, s)^{-1} = U(s, t)$.

We already know that for $t \in \mathbb{T}$, the third condition might be written as $U(t+, s) = (I + a(t)\mu(t))U(t, s)$. It is easy to recognize that the restriction of U to $\mathbb{T} \times \mathbb{T}$ is the matrix exponential function, which is denoted by $e_a(t, t_0)$ in the book [1].

In his paper [11], Š. Schwabik presents the following interesting construction of the function U: He defines the Perron product integral $\prod_a^b (I + \mathrm{d}A(s))$ as a matrix $P \in \mathbb{R}^{n\times n}$ such that for every $\varepsilon > 0$, there is a function $\delta : [a, b] \to \mathbb{R}^+$ which satisfies

$$\left\| \prod_{j=k}^{1} \big(I + A(\alpha_j) - A(\alpha_{j-1})\big) - P \right\| < \varepsilon$$

for every δ–fine partition with division points $a = \alpha_0 \le \alpha_1 \le \cdots \le \alpha_k = b$ and tags $\tau_i \in [\alpha_{i-1}, \alpha_i]$, $i = 1, \ldots, k$. Now, the function U is obtained by considering the product integral as a function of its upper bound, i.e.

$$U(t, t_0) = \prod_{t_0}^{t} (I + \mathrm{d}A(s)).$$

A similar result for linear systems on time scales is given in the paper [12], which shows that the matrix exponential function $e_a(t,t_0)$ corresponding to an rd-continuous function $a:\mathbb{T}\to\mathbb{R}^{n\times n}$ can be written in the form

$$e_a(t,t_0)=\prod_{t_0}^{t}(I+a(s)\Delta s),$$

where the symbol on the right-hand side stands for the product Δ-integral. Thus our considerations imply that

$$\prod_{t_0}^{t}(I+a(s)\Delta s)=\prod_{t_0}^{t}(I+\mathrm{d}A(s)),\quad t_0,t\in\mathbb{T}$$

for every rd-continuous function $a:\mathbb{T}\to\mathbb{R}^{n\times n}$ and $A(t)=\int_{\tau_0}^{t}a^*(s)\,\mathrm{d}g(s)$.

5 Continuous dependence on a parameter

In this section, we use two known results concerning continuous dependence of generalized equations on parameters to obtain new theorems about dynamic equations on time scales. The symbol B_r will be used to denote the open ball $\{x\in\mathbb{R}^n;\ \|x\|<r\}$ and $\overline{B_r}$ stands for the corresponding closed ball.

Theorem 13. *Let $c>0$, $G=B_c\times[\alpha,\beta]$, and consider a sequence of functions $F_k:G\to\mathbb{R}^n$, $k\in\mathbb{N}_0$, such that*

$$\lim_{k\to\infty}F_k(x,t)=F_0(x,t),\quad x\in B_c,\ t\in[\alpha,\beta].$$

Assume there exist functions h and ω such that F_k satisfies conditions (F1)–(F2) *for every $k\in\mathbb{N}_0$. Finally, suppose there exist a function $x:[\alpha,\beta]\to B_c$ and a sequence of functions $x_k:[\alpha,\beta]\to B_c$ such that*

$$\frac{\mathrm{d}x_k}{\mathrm{d}\tau}=DF_k(x_k,t),\quad t\in[\alpha,\beta],\ k\in\mathbb{N},$$

$$\lim_{k\to\infty}x_k(s)=x(s),\quad s\in[\alpha,\beta].$$

Then

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}=DF_0(x,t),\quad t\in[\alpha,\beta].$$

Proof. See Theorem 8.2 in [9]. □

Theorem 14. *Consider a sequence of functions $f_k:B_c\times[\alpha,\beta]_{\mathbb{T}}\to\mathbb{R}^n$, $k\in\mathbb{N}_0$. Assume there exist functions l, m and ω such that each function f_k, $k\in\mathbb{N}_0$, satisfies conditions* (C1)–(C3). *Suppose that*

$$\lim_{k\to\infty}\int_{\alpha}^{t}f_k(x,s)\Delta s=\int_{\alpha}^{t}f_0(x,s)\Delta s \tag{13}$$

for every $x\in B_c$ and $t\in[\alpha,\beta]_{\mathbb{T}}$. Finally, suppose there exist a function $x:[\alpha,\beta]_{\mathbb{T}}\to B_c$ and a sequence of functions $x_k:[\alpha,\beta]_{\mathbb{T}}\to B_c$, $k\in\mathbb{N}$, such that

$$x_k^{\Delta}(t)=f_k(x_k(t),t),\quad t\in[\alpha,\beta]_{\mathbb{T}},$$

$$\lim_{k\to\infty} x_k(s) = x(s), \quad s \in [\alpha,\beta]_{\mathbb{T}}.$$

Then

$$x^{\Delta}(t) = f_0(x(t),t), \quad t \in [\alpha,\beta]_{\mathbb{T}}.$$

Proof. Let $G = B_c \times [\alpha,\beta]$ and

$$F_k(x,t) = \int_{\alpha}^{t} f_k^*(x,s)\,\mathrm{d}g(s), \quad x \in B_c,\ t \in [\alpha,\beta],\ k \in \mathbb{N}_0.$$

Equation (13) together with Theorem 5 imply

$$\lim_{k\to\infty} F_k(x,t) = F_0(x,t), \quad x \in B_c,\ t \in [\alpha,\beta].$$

It follows from Lemma 6 that F_k satisfies conditions (F1)–(F2) for every $k \in \mathbb{N}_0$. It is clear that

$$\lim_{k\to\infty} x_k^*(s) = x^*(s), \quad s \in [\alpha,\beta].$$

Theorem 12 implies

$$\frac{\mathrm{d}x_k^*}{\mathrm{d}\tau} = DF_k(x_k^*,t), \quad t \in [\alpha,\beta],\ k \in \mathbb{N}.$$

Thus the assumptions of Theorem 13 are satisfied and

$$\frac{\mathrm{d}x^*}{\mathrm{d}\tau} = DF_0(x^*,t), \quad t \in [\alpha,\beta].$$

The function x^* is bounded and it follows from Theorem 12 that

$$x^{\Delta}(t) = f_0(x(t),t), \quad t \in [\alpha,\beta]_{\mathbb{T}}. \qquad \square$$

Given a function $F : B \times I \to \mathbb{R}^n$ and an interval $[\alpha,\beta] \subset I$, a solution $x : [\alpha,\beta] \to B$ of the generalized differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t) \tag{14}$$

is said to be unique if every other solution $y : [\alpha,\gamma] \to B$ of (14) such that $x(\alpha) = y(\alpha)$ satisfies $x(t) = y(t)$ for every $t \in [\alpha,\gamma] \cap [\alpha,\beta]$.

Theorem 15. *Let $c > 0$, $G = B_c \times [\alpha,\beta]$, and consider a sequence of functions $F_k : G \to \mathbb{R}^n$, $k \in \mathbb{N}_0$, such that*

$$\lim_{k\to\infty} F_k(x,t) = F_0(x,t), \quad x \in B_c,\ t \in [\alpha,\beta].$$

Assume there exist a left-continuous function h and a function ω such that F_k satisfies conditions (F1)–(F2) *for every $k \in \mathbb{N}_0$. Let $x : [\alpha,\beta] \to B_c$ be a unique solution of*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF_0(x,t).$$

Finally, assume there exists a $\rho > 0$ such that $\|y - x(s)\| < \rho$ implies $y \in B_c$ whenever $s \in [\alpha,\beta]$ (i.e., a ρ-neighborhood of x is contained in B_c). Then, given an arbitrary sequence of n-dimensional vectors $\{y_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} y_k =$

$x(\alpha)$, there is a $k_0 \in \mathbb{N}$ and a sequence of functions $x_k : [\alpha, \beta] \to B_c$, $k \geq k_0$, which satisfy

$$\frac{\mathrm{d}x_k}{\mathrm{d}\tau} = DF_k(x_k, t), \quad t \in [\alpha, \beta], \quad x_k(\alpha) = y_k,$$

$$\lim_{k\to\infty} x_k(s) = x(s), \quad s \in [\alpha, \beta].$$

Proof. See Theorem 8.6 in [9]. □

In analogy with the previous case, we say that a solution $x : [\alpha, \beta]_{\mathbb{T}} \to B$ of the dynamic equation $x^\Delta(t) = f(x(t), t)$ is unique if every other solution $y : [\alpha, \gamma] \to B$ such that $x(\alpha) = y(\alpha)$ satisfies $x(t) = y(t)$ for every $t \in [\alpha, \gamma] \cap [\alpha, \beta]$.

Theorem 16. *Consider a sequence of functions $f_k : B_c \times [\alpha, \beta]_{\mathbb{T}} \to \mathbb{R}^n$, $k \in \mathbb{N}_0$. Assume there exist functions l, m, ω such that each function f_k, $k \in \mathbb{N}_0$, satisfies conditions* (C1)–(C3)*. Suppose that*

$$\lim_{k\to\infty} \int_\alpha^t f_k(x, s)\Delta s = \int_\alpha^t f_0(x, s)\Delta s \tag{15}$$

for every $x \in B_c$ and $t \in [\alpha, \beta]_{\mathbb{T}}$. Let $x : [\alpha, \beta]_{\mathbb{T}} \to B_c$ be a unique solution of

$$x^\Delta(t) = f_0(x(t), t).$$

Finally, assume there exists a $\rho > 0$ such that $\|y - x(s)\| < \rho$ implies $y \in B_c$ whenever $s \in [\alpha, \beta]_{\mathbb{T}}$. Then, given an arbitrary sequence of n-dimensional vectors $\{y_k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} y_k = x(\alpha)$, there is a $k_0 \in \mathbb{N}$ and a sequence of functions $x_k : [\alpha, \beta]_{\mathbb{T}} \to B_c$, $k \geq k_0$, which satisfy

$$x_k^\Delta(t) = f_k(x_k(t), t), \quad t \in [\alpha, \beta]_{\mathbb{T}}, \quad x_k(\alpha) = y_k,$$

$$\lim_{k\to\infty} x_k(s) = x(s), \quad s \in [\alpha, \beta]_{\mathbb{T}}.$$

Proof. Using the same reasoning as in the proof of Theorem 14, we construct a sequence of functions $\{F_k\}_{k=0}^\infty$ defined on $G = B_c \times [\alpha, \beta]$. All these functions satisfy conditions (F1)–(F2) with $h(t) = \int_{t_0}^t (l^*(s) + m^*(s))\,\mathrm{d}g(s)$; note that $g(t) = t^*$ is a left-continuous function, and thus h is left-continuous according to Theorem 2. It follows from Theorem 12 that x^* is a unique solution of

$$\frac{\mathrm{d}x^*}{\mathrm{d}\tau} = DF_0(x^*, t), \quad t \in [\alpha, \beta].$$

The proof is finished by applying Theorem 15. □

Let us note that Theorem 4.11 in [9] states that if the function F satisfies conditions (F1)–(F2) with ω such that

$$\lim_{v\to 0+} \int_v^u \frac{\mathrm{d}r}{\omega(r)} = \infty \tag{16}$$

for every $u > 0$, then every solution $x : [\alpha, \beta] \to B_c$ of the generalized equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t)$$

is unique. Now, it is often the case that the function f is Lipschitz-continuous with respect to x on $B_c \times [\alpha, \beta]$, and thus f satisfies conditions (C1)–(C3) with $l(t) = L$ and $\omega(r) = r$. In this case, we see that (16) is true and therefore every solution is unique.

6 Stability

The dynamic equation

$$x^{\Delta}(t) = f(x(t), t)$$

has the trivial solution $x \equiv 0$ if and only if $f(0,t) = 0$ for every $t \in \mathbb{T}$. The present section is devoted to the investigation of stability of this trivial solution. The problem of stability has already been considered in a number of papers, see e.g. [5], [6], [7]. However, the theorem which will be obtained in this section describes two slightly different types of stability.

Consider a set $I \subset \mathbb{R}$ and a function $f : I \to \mathbb{R}^n$. Given a finite set of points

$$D = \{t_0, t_1, \dots, t_k\} \subset I$$

such that $t_0 \le t_1 \le \dots \le t_k$, let

$$v(f, D) = \sum_{i=1}^{k} \|f(t_i) - f(t_{i-1})\|.$$

The variation of f over I is defined as

$$\operatorname*{var}_{t \in I} f(t) = \sup_{D} v(f, D),$$

where the supremum ranges over all finite subsets D of I.

Note that when I is an interval on the real line, then we obtain the usual variation of a function over an interval, but our slightly more general definition permits us to calculate the variation of a function defined on a time scale interval $[a, b]_{\mathbb{T}}$.

Lemma 17. *Given an arbitrary function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$, we have*

$$\operatorname*{var}_{t \in [a,b]_{\mathbb{T}}} f(t) = \operatorname*{var}_{t \in [a,b]} f^*(t).$$

Proof. The statement follows from the fact that if $D = \{t_0, t_1, \dots, t_k\} \subset [a, b]$, then $D^* = \{t_0^*, t_1^*, \dots, t_k^*\} \subset [a, b]_{\mathbb{T}}$ and $v(f, D) = v(f^*, D^*)$. □

We start with a Lyapunov-type theorem for the generalized equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t).$$

Note that this equation has the trivial solution $x \equiv 0$ on an interval $I \subset \mathbb{R}$ if and only if $F(0, t_1) = F(0, t_2)$ for each pair $t_1, t_2 \in I$.

Theorem 18. *Let $c > 0$, $t_0 \in \mathbb{R}$ and $G = B_c \times [t_0, \infty)$. Consider a function $F : G \to \mathbb{R}^n$ which satisfies conditions* (F1)–(F2) *and $F(0, t_1) = F(0, t_2)$ for every $t_1, t_2 \ge t_0$. Assume there exists a number $a \in (0, c)$ and a function $V : [t_0, \infty) \times \overline{B_a} \to \mathbb{R}$ with the following properties:*

(V1) *$t \mapsto V(t, x)$ is left-continuous for every $x \in \overline{B_a}$.*

(V2) *There exists a continuous increasing function $b : [0, \infty) \to \mathbb{R}$ such that $b(\rho) = 0$ if and only if $\rho = 0$ and $V(t, x) \ge b(\|x\|)$ for every $t \in [t_0, \infty)$ and $x \in \overline{B_a}$.*

(V3) $V(t,0)=0$ *for every* $t\in[t_0,\infty)$.

(V4) *There exists a constant* $K>0$ *such that* $\|V(t,x)-V(t,y)\|\le K\|x-y\|$ *for every* $t\in[t_0,\infty)$ *and* $x,y\in\overline{B_a}$.

(V5) $t\mapsto V(t,x(t))$ *is nonincreasing along every solution* x *of the generalized equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}=DF(x,t).$$

Then the following statements are true:

(1) *For every* $\varepsilon>0$*, there is a* $\delta>0$ *such that if* $\alpha\ge t_0$ *and* $y:[\alpha,\beta]\to B_c$ *is a left-continuous function with bounded variation which satisfies* $\|y(\alpha)\|<\delta$ *and*

$$\operatorname*{var}_{s\in[\alpha,\beta]}\left(y(s)-\int_\alpha^s DF(y(\tau),t)\right)<\delta,$$

then $\|y(t)\|<\varepsilon$ *for every* $t\in[\alpha,\beta]$.

(2) *For every* $\varepsilon>0$*, there is a* $\delta>0$ *such that if* $P:[\alpha,\beta]\to B_c$ *is a left-continuous function with*

$$\operatorname*{var}_{s\in[\alpha,\beta]}P(s)<\delta,$$

then an arbitrary function $y:[\alpha,\beta]\to B_c$ *which is a solution of*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}=D\big(F(x,t)+P(t)\big)$$

and $\|y(\alpha)\|<\delta$ *satisfies* $\|y(t)\|<\varepsilon$ *for every* $t\in[\alpha,\beta]$.

Proof. See Theorem 10.8 and Theorem 10.13 in [9]. □

The statement (1) is called *variational stability*; it says that functions which are initially small, and which are "almost solutions" of the given generalized equation, are close to zero in the whole interval. The statement (2) is called *stability with respect to perturbations*; it says that functions which are initially small, and which are solutions of a generalized equation with a small perturbation term, are again close to zero in the whole interval.

We now proceed to a similar theorem concerning dynamic equations on time scales.

Theorem 19. *Let* $c>0$ *and* $t_0\in\mathbb{T}$*. Consider a function* $f:B_c\times[t_0,\infty)_{\mathbb{T}}\to\mathbb{R}^n$ *which satisfies conditions* (C1)–(C3) *and* $f(0,t)=0$ *for every* $t\in[t_0,\infty)_{\mathbb{T}}$*. Assume there exists a number* $a\in(0,c)$ *and function* $V:[t_0,\infty)_{\mathbb{T}}\times\overline{B_a}\to\mathbb{R}$ *with the following properties:*

(V1) $t\mapsto V(t,x)$ *is left-continuous for every* $x\in\overline{B_a}$.

(V2) *There exists a continuous increasing function* $b:[0,\infty)\to\mathbb{R}$ *such that* $b(\rho)=0$ *if and only if* $\rho=0$ *and* $V(t,x)\ge b(\|x\|)$ *for every* $t\in[t_0,\infty)_{\mathbb{T}}$ *and* $x\in\overline{B_a}$.

(V3) $V(t,0)=0$ *for every* $t\in[t_0,\infty)_{\mathbb{T}}$.

(V4) *There exists a constant* $K > 0$ *such that* $\|V(t,x) - V(t,y)\| \le K\|x - y\|$ *for every* $t \in [t_0, \infty)_{\mathbb{T}}$ *and* $x, y \in \overline{B_a}$.

(V5) $t \mapsto V(t, x(t))$ *is nonincreasing along every solution* x *of the dynamic equation*

$$x^{\Delta}(t) = f(x(t), t).$$

Then the following statements are true:

(1) *For every* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that if* $\alpha \ge t_0$ *and* $y : [\alpha, \beta]_{\mathbb{T}} \to B_c$ *is a left-continuous function with bounded variation which satisfies* $\|y(\alpha)\| < \delta$ *and*

$$\operatorname*{var}_{s \in [\alpha,\beta]_{\mathbb{T}}} \left(y(s) - \int_{\alpha}^{s} f(y(t), t)\, \Delta t \right) < \delta,$$

then $\|y(t)\| < \varepsilon$ *for every* $t \in [\alpha, \beta]_{\mathbb{T}}$.

(2) *For every* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that if* $p : [\alpha, \beta]_{\mathbb{T}} \to B_c$ *is an rd-continuous function and*

$$\int_{\alpha}^{\beta} \|p(t)\|\, \Delta t < \delta,$$

then every function $y : [\alpha, \beta]_{\mathbb{T}} \to B_c$ *such that* $\|y(\alpha)\| < \delta$ *and*

$$y^{\Delta}(t) = f(y(t), t) + p(t), \quad t \in [\alpha, \beta]_{\mathbb{T}}$$

satisfies $\|y(t)\| < \varepsilon$ *for every* $t \in [\alpha, \beta]_{\mathbb{T}}$.

Proof. It is sufficient to apply Theorem 18 to the functions

$$F(x,t) = \int_{t_0}^{t} f^*(x, s)\, \mathrm{d}g(s),$$

$$V^*(t,x) = V(t^*, x).$$

To prove (1), note that

$$\operatorname*{var}_{s \in [\alpha,\beta]_{\mathbb{T}}} \left(y(s) - \int_{\alpha}^{s} f(y(t), t)\, \Delta t \right) < \delta$$

implies

$$\operatorname*{var}_{s \in [\alpha,\beta]} \left(y^*(s) - \int_{\alpha}^{s} DF(y^*(\tau), t) \right) < \delta$$

(this follows from Lemma 17, Theorem 5, and Lemma 10). To prove (2), note that

$$y^{\Delta}(t) = f(y(t), t) + p(t), \quad t \in [\alpha, \beta]_{\mathbb{T}}$$

implies

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = D\big(F(x,t) + P(t)\big)$$

with $P(t) = \int_{\alpha}^{t} p^*(s)\, \mathrm{d}g(s)$, and that

$$\int_{\alpha}^{\beta} \|p(t)\|\, \Delta t < \delta$$

implies

$$\operatorname*{var}_{s\in[\alpha,\beta]_{\mathbb{T}}}\left(\int_\alpha^s p(t)\,\Delta t\right) < \delta$$

(this is easy to see from the definition of variation) and consequently

$$\operatorname*{var}_{s\in[\alpha,\beta]} P(s) = \operatorname*{var}_{s\in[\alpha,\beta]}\left(\int_\alpha^s p^*(t)\,\mathrm{d}g(s)\right) < \delta$$

(this follows from Lemma 17 and Theorem 5). □

7 Conclusion

We have outlined a method which enables us to translate existing results concerning generalized ordinary differential equations into the language of dynamic equations on time scales.

The readers are invited to examine existing sources on generalized differential equations to find theorems which might be interesting in the time scale setting; the amount of literature devoted to generalized equations is still growing.

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