Mixing Problems with Many Tanks

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Abstract. We revisit the classical calculus problem of describing the flow of brine in a system of tanks connected by pipes. For various configurations involving an arbitrary number of tanks, we show that the corresponding linear system of differential equations can be solved analytically. Finally, we analyze the asymptotic behavior of solutions for a general closed system of tanks. It turns out that the problem is closely related to the study of Laplacian matrices for directed graphs.

1. INTRODUCTION. Consider two tanks filled with brine, which are connected by a pair of pipes. One pipe brings brine from the first tank to the second tank at a given rate, while the second pipe carries brine in the opposite direction at the same rate. This situation suggests the following problem. Assuming that the initial concentrations in both tanks are known and that we have a perfect mixing in both tanks, find the concentrations in both tanks after a given period of time. Does the problem sound familiar? Similar mixing problems appear in many differential equations textbooks (see, e.g., [**3**], [**10**], and especially [**5**], which has an impressive collection of mixing problems). Most authors restrict themselves to mixing problems involving two or three tanks arranged in various configurations (a cascade with brine flowing in a single direction only, a linear arrangement of tanks connected by pairs of pipes, a cyclic arrangement of tanks, etc.). The problem then leads to a linear system of differential equations for the unknown concentrations, which is solved by calculating the eigenvalues and eigenvectors of the corresponding matrix.

An exercise in [**5**, p. 380] asks the reader to consider a cascade of n tanks and to solve the corresponding differential equations numerically in the case when $n = 10$. In this article, we discuss a variety of mixing problems with n tanks (including the one from [**5**]) and show that they can be solved exactly. Not only is it satisfying to obtain analytic solutions of these problems, but we will also have the opportunity to review a number of other interesting topics such as the Gershgorin circle theorem, recurrence relations, the heat equation, circulant matrices, and Laplacian matrices.

2. STAR ARRANGEMENT OF TANKS. We start with the case where n tanks (with $n > 1$) of the same volume V are arranged in the shape of a star. The central tank T_1 is connected by a pair of pipes to each of the remaining tanks $T_2, \dots, T_n$ as shown in Figure 1.

Assume that the flow through each pipe is f gallons per unit of time. Consequently, the volume V in each tank remains constant. Let $x_i(t)$ be the amount of salt in tank T_i at time t. This yields the differential equations

$$x_1'(t) = -(n-1)f\frac{x_1(t)}{V} + \sum_{i=2}^{n} f\frac{x_i(t)}{V},$$

$$x_i'(t) = f\frac{x_1(t)}{V} - f\frac{x_i(t)}{V}, \quad \text{for } 2 \le i \le n.$$

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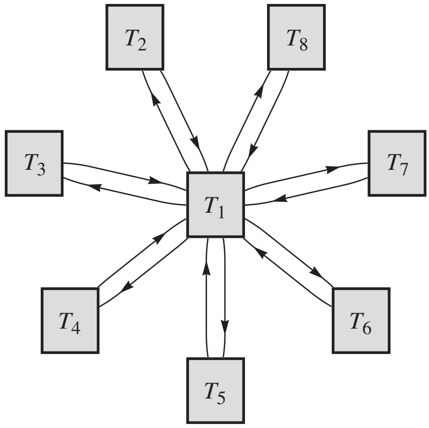

Figure 1. Tanks in a star arrangement

Without loss of generality (by taking suitable time units), we may assume that $f = V$. The system of differential equations can be written in the form $x'(t) = Ax(t)$, where $x(t) = (x_1(t), \dots, x_n(t))^T$ and

$$A = \begin{pmatrix} -(n-1) & 1 & \cdots & 1 \\ 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & -1 \end{pmatrix}.$$

The problem of obtaining a general solution of this system is equivalent to finding the eigenvalues and eigenvectors of A. The first task is to calculate the characteristic polynomial of A, i.e., the determinant of the $n \times n$ matrix

$$B = A - \lambda I = \begin{pmatrix} -(n-1) - \lambda & 1 & \cdots & 1 \\ 1 & -1 - \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & -1 - \lambda \end{pmatrix}.$$

By definition,

$$\det B = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) b_{1\pi(1)} \cdots b_{n\pi(n)},$$

where the summation runs over all permutations π of $\{1, \dots, n\}$. Clearly, the only permutations that contribute nonzero terms are the following.

- The identity permutation, which contributes $(-(n-1) - \lambda)(-1 - \lambda)^{n-1}$.
- Permutations where $\pi(1) = i$ for a certain $i \in \{2, \dots, n\}$, $\pi(i) = 1$, and $\pi(j) = j$ for $j \in \{2, \dots, n\} \backslash \{i\}$. Each of these permutations is a transposition, and contributes $-(-1 - \lambda)^{n-2}$ to the final sum.

It follows that

$$\begin{aligned} \det(A - \lambda I) &= (-(n-1) - \lambda)(-1 - \lambda)^{n-1} - (n-1)(-1 - \lambda)^{n-2} \\ &= (-1 - \lambda)^{n-2} \lambda (n + \lambda), \end{aligned}$$

i.e., A has two simple eigenvalues $\lambda = 0$ and $\lambda = -n$, as well as the eigenvalue $\lambda = -1$ of multiplicity $n-2$.

The next step is to calculate the eigenvectors. For $\lambda = 0$, all eigenvectors $v = (v_1, \ldots, v_n)$ have to satisfy $v_1 = v_2, v_1 = v_3, \ldots, v_1 = v_n$; the simplest choice is $v = (1, \ldots, 1)$. This corresponds to the physical situation where every tank contains the same amount of salt. For $\lambda = -n$, we obtain

$$A - \lambda I = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & -1+n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & -1+n \end{pmatrix}.$$

The eigenvector components must satisfy $v_i = v_1/(1-n)$, for $2 \le i \le n$, and we can take $v = (1-n, 1, \ldots, 1)$. Finally, if $\lambda = -1$, then

$$A - \lambda I = \begin{pmatrix} -n+2 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

All eigenvectors must be such that $v_1 = 0$ and $v_2 + \cdots + v_n = 0$. For example, we obtain a linearly independent set of $n-2$ eigenvectors $v^3, \ldots, v^n \in \mathbb{R}^n$ by taking $v_2^k = 1$, $v_k^k = -1$, and $v_l^k = 0$ for other values of l.

The general solution is a linear combination of functions of the form $e^{\lambda t} v$, where v is an eigenvector corresponding to λ. Since $e^{\lambda t} \to 0$ for $\lambda = -n$ and $\lambda = -1$, every solution approaches the state where all tanks contain the same amount of salt. In other words, this state corresponds to a globally asymptotically stable equilibrium.

3. TANKS IN A ROW. In our next example, we have a row of n tanks $T_1, \ldots, T_n$, with neighboring tanks connected by a pair of pipes (see Figure 2).

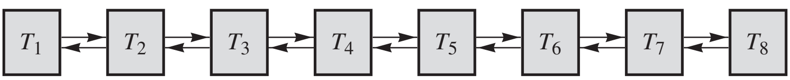

Figure 2. A linear arrangement of tanks

Again, the flow through each pipe is f gallons per unit of time. Using the same notation as before, we obtain the following system of differential equations:

$$x_1'(t) = -f\frac{x_1(t)}{V} + f\frac{x_2(t)}{V},$$

$$x_i'(t) = f\frac{x_{i-1}(t)}{V} - 2f\frac{x_i(t)}{V} + f\frac{x_{i+1}(t)}{V}, \quad \text{for } 2 \le i \le n-1,$$

and

$$x_n'(t) = -f\frac{x_{n-1}(t)}{V} - f\frac{x_n(t)}{V}.$$

Without loss of generality, we assume that $f = V$, and switch to the vector form $x'(t) = Ax(t)$, where

$$A = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}.$$

The matrix A is an example of a tridiagonal and "almost-Toeplitz" matrix (a Toeplitz matrix has constant values along lines parallel to the main diagonal). Before calculating its characteristic polynomial, it is useful to obtain some preliminary information about the location of the eigenvalues of A. Since A is a symmetric matrix, all eigenvalues must be real. According to the Gershgorin circle theorem, the eigenvalues of A are contained in the union of the intervals $[a_{ii} - r_i, a_{ii} + r_i]$, where

$$r_i = \sum_{j \in \{1,\ldots,n\} \setminus \{i\}} |a_{ij}|, \quad \text{for } 1 \le i \le n.$$

In our case, we have $a_{11} = a_{nn} = -1$ and $r_1 = r_n = 1$, while $a_{ii} = -2$ and $r_i = 2$ for $i \in \{2, \ldots, n-1\}$. Consequently, all eigenvalues of A are contained in the interval $[-4, 0]$.

Let us now return to the calculation of $\det(A - \lambda I)$. We start by expanding the determinant with respect to the first column:

$$\det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2-\lambda & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2-\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2-\lambda & 1 \\ 0 & 0 & 0 & \cdots & 1 & -1-\lambda \end{pmatrix}$$

$$= (-1-\lambda) \det \begin{pmatrix} -2-\lambda & 1 & \cdots & 0 & 0 \\ 1 & -2-\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -2-\lambda & 1 \\ 0 & 0 & \cdots & 1 & -1-\lambda \end{pmatrix}$$

$$- \det \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & -2-\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -2-\lambda & 1 \\ 0 & 0 & \cdots & 1 & -1-\lambda \end{pmatrix}.$$

The second determinant remains unchanged if we omit the first row, together with the first column. Consequently,

$$\det(A - \lambda I) = (-1-\lambda) D_{n-1} - D_{n-2},$$

where D_k (for every $k \in \mathbb{N}$) stands for the determinant of the $k \times k$ matrix

$$\begin{pmatrix} -2-\lambda & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2-\lambda & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2-\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2-\lambda & 1 \\ 0 & 0 & 0 & \cdots & 1 & -1-\lambda \end{pmatrix}.$$

The value of D_k can be calculated by the same method that works for tridiagonal Toeplitz matrices (see [**8**, p. 133]). Expanding this determinant with respect to the first column, we obtain the recurrence relation

$$D_k = (-2-\lambda)D_{k-1} - D_{k-2}, \quad \text{for } k \geq 2.$$

The initial values are $D_1 = -1-\lambda$ and $D_0 = 1$. We apply the standard procedure for solving second-order linear homogeneous recurrence relations (see [**6**]). The auxiliary equation

$$x^2 + (2+\lambda)x + 1 = 0$$

has discriminant $(2+\lambda)^2 - 4 = \lambda(\lambda+4)$, which is nonpositive for $\lambda \in [-4, 0]$. (We restrict our attention to $\lambda \in [-4, 0]$ because we are trying to find the roots of $\det(A - \lambda)$, and we already know they are contained in $[-4, 0]$.)

$$x_{1,2} = \frac{-\lambda-2}{2} \pm i\frac{\sqrt{-\lambda(\lambda+4)}}{2}.$$

Performing the substitution $\lambda + 2 = u$, we obtain

$$x_{1,2} = \frac{-u}{2} \pm i\frac{\sqrt{4-u^2}}{2} = e^{\pm i\gamma},$$

where $\gamma \in [0, \pi]$ satisfies $\cos\gamma = -u/2$, $\sin\gamma = \frac{\sqrt{4-u^2}}{2}$. Thus D_k is a linear combination of $x_1^k = e^{ik\gamma}$ and $x_2^k = e^{-ik\gamma}$, and $D_k = \alpha\cos(k\gamma) + \beta\sin(k\gamma)$ for some constants α, β and every $k \geq 0$. Using the initial conditions $D_0 = 1$ and $D_1 = -1-\lambda = 1-u = 1+2\cos\gamma$, we find $\alpha = 1$ and $\beta = (1+\cos\gamma)/\sin\gamma = \cot(\gamma/2)$, provided that $\gamma \neq 0$ (this assumption will be justified shortly). Therefore, $D_k = \cos(k\gamma) + \cot(\gamma/2)\sin(k\gamma)$ and

$$\begin{aligned} \det(A-\lambda I) &= (-1-\lambda)D_{n-1} - D_{n-2} = D_n + D_{n-1} \\ &= \cos(n\gamma) + \cos((n-1)\gamma) + \cot\left(\frac{\gamma}{2}\right)(\sin(n\gamma) + \sin((n-1)\gamma)) \\ &= 2\cos\left(\frac{\gamma}{2}\right)\cos\left((2n-1)\frac{\gamma}{2}\right) + 2\cot\left(\frac{\gamma}{2}\right)\left(\cos\left(\frac{\gamma}{2}\right)\sin\left((2n-1)\frac{\gamma}{2}\right)\right) \\ &\quad \times 2\cot\left(\frac{\gamma}{2}\right)\left(\sin\left(\frac{\gamma}{2}\right)\cos\left((2n-1)\frac{\gamma}{2}\right) + \cos\left(\frac{\gamma}{2}\right)\sin\left((2n-1)\frac{\gamma}{2}\right)\right) \\ &= 2\cot\left(\frac{\gamma}{2}\right)\sin(n\gamma). \end{aligned}$$

For $\gamma \in (0, \pi]$, the last expression vanishes when $\gamma = k\pi/n$, for $k \in \{1, \dots, n\}$. Returning back from γ to λ, we conclude that

$$\lambda_k = -2\cos\frac{k\pi}{n} - 2, \quad \text{for } k \in \{1, \dots, n\},$$

are the eigenvalues of A; since we found n distinct eigenvalues, there is no need to investigate the case $\gamma = 0$, which we excluded earlier.

The eigenvectors corresponding to λ_k are solutions of the linear homogeneous system with the matrix

$$\begin{pmatrix} 1+2\cos\frac{k\pi}{n} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2\cos\frac{k\pi}{n} & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2\cos\frac{k\pi}{n} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\cos\frac{k\pi}{n} & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1+2\cos\frac{k\pi}{n} \end{pmatrix}.$$

It follows that the components of an eigenvector $v = (v_1, \dots, v_n)$ corresponding to the eigenvalue λ_k are given by the relations

$$v_2 = -\left(1 + 2\cos\frac{k\pi}{n}\right) v_1, \tag{1}$$

and

$$v_i = -2\left(\cos\frac{k\pi}{n}\right) v_{i-1} - v_{i-2}, \quad \text{for } i \in \{3, \dots, n\}, \tag{2}$$

while v_1 can be an arbitrary nonzero number. In particular, $\lambda_n = 0$, and the corresponding eigenvectors have all components identical. Since the remaining eigenvalues are negative, we conclude again that a general solution always approaches the state with all tanks containing the same amount of salt.

Note that (2) is a second-order linear homogeneous recurrence relation with constant coefficients; by considering (1) as the initial condition, we can apply the standard algorithm (details are left to the reader) to conclude that

$$v_i = (-1)^i \cos\left(\frac{k\pi i}{n}\right) - (-1)^i \cot\left(\frac{k\pi}{2n}\right) \sin\left(\frac{k\pi i}{n}\right), \quad \text{for } i \in \{1, \dots, n\},$$

satisfies both (1) and (2); hence, $v = (v_1, \dots, v_n)$ is an eigenvector corresponding to λ_k.

The mixing problem we have just solved is closely related to the one-dimensional heat equation, i.e., the partial differential equation of the form

$$\frac{\partial f}{\partial t}(t, x) = k\frac{\partial^2 f}{\partial x^2}(t, x),$$

which models the conduction of heat in a one-dimensional rod (see [**3**], [**10**]); in particular, $f(t, x)$ is the temperature at the point x on the rod at time t. The same equation also describes the one-dimensional diffusion in a homogeneous medium; in this case, $f(t, x)$ corresponds to the concentration of a chemical at the point x at time t. Consider a line segment, represented by the interval $[a, b]$, whose endpoints are insulated—no heat transfer or diffusion takes place across the endpoints. Mathematically, this means that

$$\frac{\partial f}{\partial x}(t, a) = \frac{\partial f}{\partial x}(t, b) = 0$$

for every t. In view of these boundary conditions, we extend the spatial domain of f by letting $f(t, x) = f(t, a)$ for $x < a$ and $f(t, x) = f(t, b)$ for $x > b$. We now discretize the spatial domain of the heat equation by considering the grid of equally spaced points $x_i = a + i\Delta x$, where $\Delta x = (b - a)/n$ and n is an arbitrary positive integer. Let $y_i(t) = f(t, x_i)$ for every $i \in \{-1, \ldots, n + 1\}$. We approximate the second-order partial derivatives $\frac{\partial^2 f}{\partial x^2}$ at the grid points x_i by the second-order central differences:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2}(t, x_0) &\doteq \frac{1}{\Delta x}\left(\frac{f(t, x_1) - f(t, x_0)}{\Delta x} - \frac{f(t, x_0) - f(t, x_{-1})}{\Delta x}\right) \\
&= \frac{-f(t, x_0) + f(t, x_1)}{(\Delta x)^2}, \\
\frac{\partial^2 f}{\partial x^2}(t, x_i) &\doteq \frac{1}{\Delta x}\left(\frac{f(t, x_{i+1}) - f(t, x_i)}{\Delta x} - \frac{f(t, x_i) - f(t, x_{i-1})}{\Delta x}\right) \\
&= \frac{f(t, x_{i-1}) - 2f(t, x_i) + f(t, x_{i+1})}{(\Delta x)^2}, \quad \text{for } i \in \{1, \ldots, n - 1\}, \\
\frac{\partial^2 f}{\partial x^2}(t, x_n) &\doteq \frac{1}{\Delta x}\left(\frac{f(t, x_{n+1}) - f(t, x_n)}{\Delta x} - \frac{f(t, x_n) - f(t, x_{n-1})}{\Delta x}\right) \\
&= \frac{f(t, x_{n-1}) - f(t, x_n)}{(\Delta x)^2}.
\end{aligned}$$

(The symbol $\doteq$ means "is approximately equal to".) Since

$$y_i'(t) = \frac{\partial f}{\partial t}(t, x_i) = k\frac{\partial^2 f}{\partial x^2}(t, x_i),$$

it follows that

$$\begin{aligned}
y_0'(t) &\doteq \frac{k}{(\Delta x)^2}(-y_0(t) + y_1(t)), \\
y_i'(t) &\doteq \frac{k}{(\Delta x)^2}(y_{i-1}(t) - 2y_i(t) + y_{i+1}(t)), \quad \text{for } i \in \{1, \ldots, n - 1\},
\end{aligned}$$

and

$$y_n'(t) \doteq \frac{k}{(\Delta x)^2}(y_{n-1}(t) - y_n(t)).$$

The right-hand side of this system is the same as in our mixing problem for $n + 1$ tanks in a row. Thus, we have discovered that this particular mixing problem is equivalent to the spatial discretization of the one-dimensional heat equation. Conversely, we can start with the mixing problem as a toy model for heat conduction or diffusion, and derive the one-dimensional heat equation by letting $n \to \infty$ and $\Delta x \to 0$ (see also [**6**, p. 167] for a related model corresponding to both spatial and temporal discretization of the heat equation).

It should be mentioned that our matrix A appears also in situations where we discretize a second-order ordinary differential equation subject to suitable boundary conditions; many examples can be found in [**12**].

4. TANKS IN A CIRCLE. Another quite natural mixing problem is obtained by placing n tanks along a circle and connecting neighboring tanks by pairs of pipes (see Figure 3); note that this type of mixing problem can be interpreted as the spatial discretization of the one-dimensional heat equation on a circle.

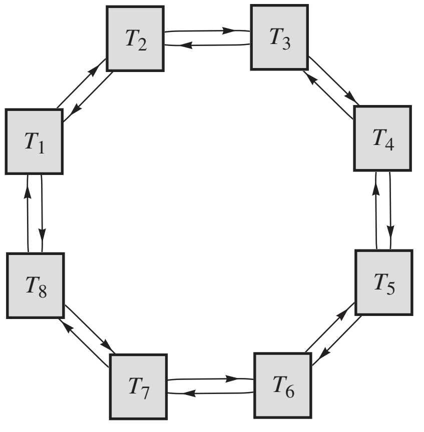

Figure 3. A cyclic arrangement of tanks

This configuration is quite similar to the linear arrangement and we leave it to the reader to verify that the matrix of the resulting linear system of differential equations is

$$A = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

This is an example of a circulant matrix, i.e., a matrix whose rows are obtained by cyclically shifting the first row. Since the problem of calculating the eigenvalues and eigenvectors of a circulant matrix is well known, we state only the results and refer the reader to appropriate sources (for the general theory of circulant matrices, see [**9**] and the references given there; for a direct calculation of the eigenvalues of A without circulant matrix theory, see Lemma 4.9 in [**2**]).

1. All circulant matrices of order n share the same set of eigenvectors, namely

$$v^j = (1, \omega_j, \omega_j^2, \ldots, \omega_j^{n-1}), \quad j \in \{0, \ldots, n-1\},$$

   where $\omega_j = \exp(2\pi ij/n) = \cos(2\pi j/n) + i\sin(2\pi j/n)$.
2. A circulant matrix whose first row is $(c_0, c_1, \ldots, c_{n-1})$ has eigenvalues

$$\lambda_j = c_0 + c_1\omega_j + c_2\omega_j^2 + \cdots + c_{n-1}\omega_j^{n-1}, \quad j \in \{0, \ldots, n-1\}.$$

In our case, we have $c_0 = -2$, $c_1 = c_{n-1} = 1$, and $c_i = 0$ otherwise. Consequently,

$$\begin{aligned}\lambda_j &= -2 + \exp(2\pi i j/n) + \exp(2\pi i(n-1)j/n) \\ &= -2 + \exp(2\pi i j/n) + \exp(-2\pi i j/n) = -2 + 2\cos(2\pi j/n).\end{aligned}$$

In particular, $\lambda_0 = 0$ and the corresponding eigenvector is $v^0 = (1, \dots, 1)$.

Note that

$$\lambda_j = -2 + 2\cos(2\pi j/n) = -2 + 2\cos(2\pi(n-j)/n) = \lambda_{n-j}$$

for $j \in \{1, \dots, n-1\}$. Thus, for odd values of n, the spectrum of A consists of the simple eigenvalue λ_0 and the eigenvalues $\lambda_1, \dots, \lambda_{(n-1)/2}$ having multiplicity 2. For even values of n, we have the simple eigenvalues $\lambda_0, \lambda_{n/2}$ (with $(-1, 1, \dots, -1, 1)$ being the eigenvector corresponding to $\lambda_{n/2}$) and the eigenvalues $\lambda_1, \dots, \lambda_{(n/2)-1}$ having multiplicity 2. In any case, all eigenvalues except λ_0 are negative, and thus all solutions of our system approach the equilibrium state with all tanks containing the same amount of salt.

Again, we remark that the matrix A occurs quite frequently in situations where we discretize second-order ordinary differential equations subject to suitable boundary conditions; see [**12**] for a detailed discussion.

5. ANALYSIS OF A GENERAL MIXING PROBLEM. Suppose we are not able to find the general solution of a certain mixing problem, i.e., we are unable to calculate the eigenvalues and eigenvectors of the corresponding matrix. Is it still possible to obtain some information about the general qualitative behavior of the solutions? According to physical intuition, it seems reasonable to expect that every solution approaches the state with all tanks containing the same amount of salt. Can we prove this fact rigorously?

We provide an affirmative answer for mixing problems satisfying the following conditions.

(1) All tanks hold the same constant volume V of brine; consequently, the total amount of brine flowing into a particular tank equals the total amount of brine flowing out of the tank.
(2) Each pipe connecting a pair of tanks transports the same volume f of brine per unit of time. By a suitable choice of time units, we can assume that $f/V = 1$.
(3) The mixing problem is irreducible in the following sense: The set of tanks $\{T_1, \dots, T_n\}$ cannot be partitioned into two disjoint nonempty groups such that every pipe connects tanks from the same group.

Such a mixing problem again leads to a linear system of differential equations. What does the corresponding matrix A look like? Note that A need not be symmetric, i.e., the fact that there is a pipe transporting brine from tank T_i to tank T_j need not imply the existence of a pipe transporting brine in the opposite direction. For example, the simple cyclic arrangement displayed in Figure 4 satisfies our assumptions.

By the first condition, the sum of each row of A is zero. By the second condition, $a_{ij} = 1$ if there is a pipe transporting brine from the ith tank to the jth tank, a_{ii} equals minus the number of pipes originating in the ith tank, and all remaining entries of A are zero.

Matrices of a similar form are well known in algebraic graph theory. Given a directed graph $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$, the matrix $L = \{l_{ij}\}_{i,j=1}^n$ given by

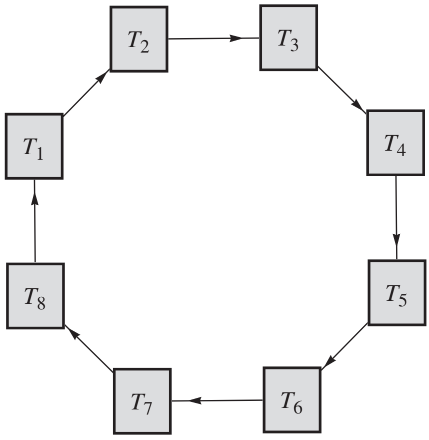

Figure 4. A simple cycle with neighbors connected by a single pipe only

$$l_{ij} = \begin{cases} \deg^+(v_i) & \text{if } i = j, \\ -1 & \text{if } i \neq j \text{ and } (v_i, v_j) \in E, \\ 0 & \text{otherwise} \end{cases}$$

(where $\deg^+(v)$ stands for the outdegree of the vertex v) is known as the Laplacian matrix of the directed graph G.

What is the relation between mixing problems and Laplacian matrices? A mixing problem can be represented by a directed graph G with vertices $T_1, \dots, T_n$ corresponding to tanks and edges representing the pipes. It follows from our previous discussion that such a mixing problem leads to the linear system $x'(t) = Ax(t)$, where $L = -A$ is the Laplacian matrix of G.

Clearly, λ is an eigenvalue of A if and only if $-\lambda$ is an eigenvalue of L (in this case, both eigenvalues share the same eigenvectors). For an arbitrary directed graph, the sum of each row in L is zero, and we see that $(1, \dots, 1)$ is an eigenvector corresponding to the zero eigenvalue. To prove that all solutions of our general mixing problem tend to the equilibrium state with all tanks containing the same amount of salt, it is sufficient to show that the zero eigenvalue is a simple one, and that all remaining eigenvalues of L have positive real parts. The second assertion is an immediate consequence of the Gershgorin circle theorem, which says that all eigenvalues are contained in the union of the n disks with centers l_{ii} and radii $\sum_{j \neq i} |l_{ij}| = -\sum_{j \neq i} l_{ij} = l_{ii}$, where $i \in \{1, \dots, n\}$. To prove that the zero eigenvalue is a simple one, note that the graphs arising from our mixing problems have two important properties: They are connected and balanced, i.e., the indegree $\deg^-(v)$ of an arbitrary vertex v equals its outdegree $\deg^+(v)$. However, a balanced connected graph is also strongly connected, i.e., there is a directed path from each vertex in the graph to every other vertex. (Assume that the graph has more than one strongly connected component. The number of edges leaving a certain component equals the number of edges entering that component, and we can find an oriented cycle passing through different components—a contradiction.)

We now proceed to show that every balanced connected directed graph has a simple zero eigenvalue. For undirected graphs, this result is well known (see [**2**], [**7**]); the usual proof is based on the observation that $L = MM^T$, where L is the Laplacian matrix and M is the incidence matrix. Such a decomposition is possible only when L is symmetric, and thus we have to use a different approach. Unfortunately, the literature on Laplacian matrices of directed graphs is rather scarce. The authors of [**11**] consider a connected balanced directed graph and its Laplacian matrix L and prove that the null space of L has dimension 1 (i.e., the geometric multiplicity of the zero eigenvalue is 1). Then they conclude that zero must be a simple eigenvalue (i.e., the algebraic multiplicity of the zero eigenvalue is 1). This result is correct, but a further explanation is necessary to show that the geometric multiplicity of the zero eigenvalue is equal to its algebraic multiplicity. Another paper dealing with Laplacian matrices of directed graphs is [**1**]. The authors consider weighted directed graphs (not necessarily balanced), and the results presented in their paper are far too general for our purposes. Here we present a short proof for balanced unweighted graphs based on the following lemma.

Lemma 1. *If L is the Laplacian matrix of a balanced directed graph $G = (V, E)$ on n vertices and $x \in \mathbb{R}^n$ is an arbitrary vector, then*

$$x^T Lx = \frac{1}{2} \sum_{(v_i,v_j)\in E} (x_i - x_j)^2.$$

Proof. We have

$$\begin{aligned} 2x^T Lx &= 2\sum_{i,j=1}^{n} l_{ij}x_i x_j = \sum_{i=1}^{n} l_{ii}x_i^2 + 2\sum_{\substack{i,j\in\{1,\dots,n\},\\ i\neq j}} l_{ij}x_i x_j + \sum_{j=1}^{n} l_{jj}x_j^2 \\ &= \sum_{i=1}^{n} \deg^+(v_i)x_i^2 - 2\sum_{(v_i,v_j)\in E} x_i x_j + \sum_{j=1}^{n} \deg^+(v_j)x_j^2. \end{aligned}$$

Observing that

$$\begin{aligned} \sum_{i=1}^{n} \deg^+(v_i)x_i^2 &= \sum_{(v_i,v_j)\in E} x_i^2, \\ \sum_{j=1}^{n} \deg^+(v_j)x_j^2 &= \sum_{j=1}^{n} \deg^-(v_j)x_j^2 \\ &= \sum_{(v_i,v_j)\in E} x_j^2, \end{aligned}$$

we obtain

$$2x^T Lx = \sum_{(v_i,v_j)\in E} (x_i^2 - 2x_i x_j + x_j^2) = \sum_{(v_i,v_j)\in E} (x_i - x_j)^2,$$

which proves the statement. ■

Theorem 2. *The null space of the Laplacian matrix of a connected balanced directed graph $G = (V, E)$ has dimension* 1.

Proof. Let L be the Laplacian matrix and consider an arbitrary vector $x \in \mathbb{R}^n$ such that $Lx = 0$. By Lemma 1,

$$0 = x^T L x = \frac{1}{2} \sum_{(v_i, v_j) \in E} (x_i - x_j)^2.$$

Consequently, all terms in the last sum must vanish and $x_i = x_j$ whenever $(v_i, v_j) \in E$. It follows that $x_i = x_j$ if there is a directed path between v_i and v_j. Since our graph is strongly connected, we conclude that $x_1 = \cdots = x_n$, i.e., the null space of L is spanned by the single vector $(1, \ldots, 1)$. ■

Theorem 2 was already proved in [**11**], but our proof is different and shorter. We now proceed to show that the algebraic multiplicity of the zero eigenvalue equals its geometric multiplicity.

Theorem 3. *For the Laplacian matrix of a connected balanced directed graph, the zero eigenvalue is a simple one.*

Proof. Let L be the Laplacian matrix. We have

$$\begin{aligned} \det(L - \lambda I) &= (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0 \\ &= (-1)^n (\lambda - \lambda_1) \cdots (\lambda - \lambda_n), \end{aligned}$$

where $a_0, \ldots, a_{n-1}$ are certain constants and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of L, including multiplicity. Without loss of generality, assume that λ_1 is the zero eigenvalue. Then

$$a_1 = (-1)^n \sum_{k=1}^{n} \left(\prod_{\substack{i \in \{1, \ldots, n\}, \\ i \neq k}} (-\lambda_i) \right) = -\lambda_2 \cdots \lambda_n,$$

and the proof will be complete once we show that $a_1 \neq 0$.

By Theorem 2, the dimension of the null space of L is 1, and it is thus possible to obtain an invertible matrix L_0 by deleting the ith row and jth column of L for suitable values of i and j. The value of the determinant

$$\det(L - \lambda I) = \det \begin{pmatrix} l_{11} - \lambda & \cdots & l_{1n} \\ \vdots & \ddots & \vdots \\ l_{n1} & \cdots & l_{nn} - \lambda \end{pmatrix}$$

remains unchanged if we first add all rows except the ith one to the ith row, and then add all columns except the jth one to the jth column. Using the fact that both the rows and columns of L have zero sums, we see that

$$\det(L-\lambda I) = \det\begin{pmatrix} l_{11}-\lambda & \cdots & -\lambda & \cdots & l_{1n} \\ \vdots & & \vdots & & \vdots \\ -\lambda & \cdots & -n\lambda & \cdots & -\lambda \\ \vdots & & \vdots & & \vdots \\ l_{n1} & \cdots & -\lambda & \cdots & l_{nn}-\lambda \end{pmatrix}$$

$$= \lambda \det\begin{pmatrix} l_{11}-\lambda & \cdots & -\lambda & \cdots & l_{1n} \\ \vdots & & \vdots & & \vdots \\ -1 & \cdots & -n & \cdots & -1 \\ \vdots & & \vdots & & \vdots \\ l_{n1} & \cdots & -\lambda & \cdots & l_{nn}-\lambda \end{pmatrix}.$$

Consequently, the value a_1, which is the coefficient of the linear term in the characteristic polynomial of L, equals the coefficient of the absolute term in the last determinant. This absolute term can be obtained by substituting $\lambda = 0$, which gives

$$a_1 = \det\begin{pmatrix} l_{11} & \cdots & 0 & \cdots & l_{1n} \\ \vdots & & \vdots & & \vdots \\ -1 & \cdots & -n & \cdots & -1 \\ \vdots & & \vdots & & \vdots \\ l_{n1} & \cdots & 0 & \cdots & l_{nn} \end{pmatrix} = -n(-1)^{i+j}\det L_0 \neq 0$$

(we have expanded the determinant with respect to the jth column). ■

Corollary 4. *Consider a general mixing problem with n tanks such that conditions* (1)–(3) *are satisfied. Then*

$$\lim_{t\to\infty} x_i(t) = \frac{x_1(0)+\cdots+x_n(0)}{n}, \quad \text{for } i \in \{1,\ldots,n\}.$$

Proof. The mixing problem leads to a linear system $x'(t) = Ax(t)$, where A has a simple zero eigenvalue and all remaining eigenvalues have negative real parts. The eigenvectors corresponding to the zero eigenvalue have all components identical. Consequently, for $t \to \infty$, any solution of the system tends toward the state where all tanks contain the same amount of salt. Since the total amount of salt is preserved, all components of the solution must approach the arithmetic mean of the initial conditions. ■

Note that for disconnected balanced graphs, we have the following result.

Theorem 5. *For the Laplacian matrix of a balanced directed graph, the algebraic multiplicity of the zero eigenvalue equals the number of connected components.*

Proof. After a suitable permutation of the vertices, the Laplacian matrix of the given graph is the block diagonal matrix

$$L = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_k \end{pmatrix},$$

where k is the number of connected components and L_i is the Laplacian matrix of the ith component. We have

$$\det(L - \lambda I) = \prod_{i=1}^{k} \det(L_i - \lambda I),$$

and the conclusion follows from the fact that each L_i has a simple zero eigenvalue. ∎

Returning to mixing problems, we see that the assumption of irreducibility cannot be omitted; for example, if we take two disjoint circular tank arrangements, it is true that each cycle approaches the equilibrium state with all tanks having the same amount of salt, but the equilibrium value can be different for each cycle.

6. CASCADE OF TANKS. So far, we have been interested in tank configurations that are closed in the sense that the total amount of salt in the system remains constant. The exercise from [**5**] mentioned in the introduction is concerned with a configuration of n tanks $T_1, \dots, T_n$ that is not closed, namely a cascade of n tanks (see Figure 5). For every $i \in \{1, \dots, n-1\}$, there is a pipe transporting brine from T_i to T_{i+1} at the rate of f gallons per unit of time. At the same rate, fresh water flows into tank T_1 and brine flows out of tank T_n, ensuring that the amount of liquid in each tank remains constant. This type of mixing problem can serve as a model describing the spread of pollution in a cascade of lakes.

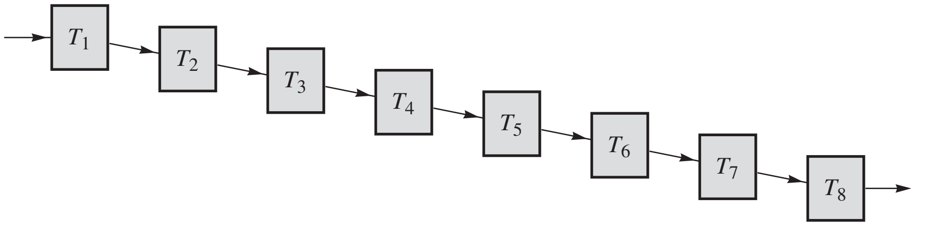

Figure 5. A cascade of tanks

Assuming that the tanks have volumes $V_1, \dots, V_n$, we obtain the system

$$x_1'(t) = -k_1 x_1(t),$$
$$x_i'(t) = k_{i-1} x_{i-1}(t) - k_i x_i(t), \quad \text{for } 2 \le i \le n,$$

where $k_i = f/V_i$. We have

$$A - \lambda I = \begin{pmatrix} -k_1 - \lambda & 0 & 0 & \cdots & 0 & 0 \\ k_1 & -k_2 - \lambda & 0 & \cdots & 0 & 0 \\ 0 & k_2 & -k_3 - \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -k_{n-1} - \lambda & 0 \\ 0 & 0 & 0 & \cdots & k_{n-1} & -k_n - \lambda \end{pmatrix},$$

and therefore

$$\det(A - \lambda I) = \prod_{i=1}^{n} (-k_i - \lambda).$$

We focus on the case when $k_1 = \cdots = k_n$, which corresponds to identical volumes $V_1 = \cdots = V_n$ (the reader can verify that, somewhat surprisingly, the case when all volumes are different is easier to solve). In this case, the number $\lambda = -k_1 = \cdots = -k_n$ is an eigenvalue of multiplicity n, and

$$A - \lambda I = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \lambda & 0 \end{pmatrix}$$

has the form of a nilpotent Jordan block. The nullspace of $A - \lambda I$ has dimension 1 and is spanned by the vector $v^1 = (0, \dots, 0, 1)$. Since we need n linearly independent solutions, we have to look at the nullspaces of higher powers of $A - \lambda I$. With each successive power, the only nonzero band containing the values λ shifts one position lower. It follows that the vector $v^k = (0, \dots, 0, 1, \dots, 1)$, whose first $n - k$ components are zero and the remaining equal to one, satisfies $(A - \lambda I)^k v^k = 0$. This gives a system of n linearly independent solutions of the form

$$\begin{aligned} e^{At} v^k &= e^{(A-\lambda I)t} e^{\lambda t} v^k \\ &= \left(I + (A - \lambda I)\frac{t}{1!} + (A - \lambda I)^2 \frac{t^2}{2!} + \cdots + (A - \lambda I)^{k-1} \frac{t^{k-1}}{(k-1)!} \right) e^{\lambda t} v^k, \end{aligned}$$

where $k \in \{1, \dots, n\}$. Since $\lambda < 0$, all solutions approach the state with no salt in the tanks.

7. CONCLUSION. The aim of this article was to present a topic that is accessible to undergraduate students and displays a nice interplay between differential equations, linear algebra, and graph theory. We remark that similar mathematical problems occur in a completely different setting, namely in the study of coordination of multiagent systems (see [**11**], [**4**] and the references there).

A possible project for the interested reader is to find other configurations (closed or not) of n tanks leading to systems of differential equations that can be solved analytically. A slightly more ambitious project is to consider mixing problems where we take into account the time necessary for the transport of brine between two tanks. The corresponding mathematical model then becomes a linear system of delay differential equations, which is naturally more difficult to analyze.

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Yet Another Generalization of a Celebrated Inequality of the Gamma Function

Let n be a positive integer and let $\overline{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n$. If $a \in \mathbb{R}$, we let $\overline{a}$ stand for $(a, \ldots, a) \in \mathbb{R}^n$. By $\overline{p} \succeq \overline{q}$ we mean that $p_i \geq q_i$ for $i = 1, 2, \ldots, n$. For a fixed $\overline{p} \succeq \overline{0}$, the generalized unit $\overline{p}$-ball is defined by $B_{\overline{p}} = \{\overline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_1|^{p_1} + \cdots + |x_n|^{p_n} \leq 1\}$, and its volume is given by (see [**2, 3**])

$$V_{\overline{p}} = V(B_{\overline{p}}) = 2^n \frac{\Gamma\left(1+\frac{1}{p_1}\right)\Gamma\left(1+\frac{1}{p_2}\right)\cdots\Gamma\left(1+\frac{1}{p_n}\right)}{\Gamma\left(1+\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_n}\right)}. \tag{1}$$

If $\overline{1} \preceq \overline{p} \preceq \overline{q}$, then $B_{\overline{1}} \subseteq B_{\overline{p}} \subseteq B_{\overline{q}} \subseteq B_{\overline{\infty}} = \{\overline{x} \in \mathbb{R}^n : \max_{1\leq i\leq n} |x_i| \leq 1\}$, as is easy to check (in fact, the proof goes as in the case of the classical unit p-balls). Therefore

$$V_{\overline{1}} \leq V_{\overline{p}} \leq V_{\overline{q}} \leq V_{\overline{\infty}} = 2^n. \tag{2}$$

Substituting (1) into (2), and denoting $\alpha_i = 1/p_i$ and $\beta_i = 1/q_i$, we obtain

$$\frac{1}{n!} \leq \frac{\prod_{i=1}^n \Gamma(1+\alpha_i)}{\Gamma\left(1+\sum_{i=1}^n \alpha_i\right)} \leq \frac{\prod_{i=1}^n \Gamma(1+\beta_i)}{\Gamma\left(1+\sum_{i=1}^n \beta_i\right)} \leq 1, \qquad 0 < \beta_i \leq \alpha_i \leq 1.$$

The case $\alpha_i = x \in [0,1]$ gives $1/n! \leq \left[\Gamma(1+x)\right]^n / \Gamma(1+nx) \leq 1$, which is an inequality that has sparked much interest in recent years (see [**1**]).

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