Massera's theorems for various types of equations with discontinuous solutions

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Abstract

We present new Massera-type theorems for various types of equations with periodic right-hand sides. We deal with generalized ordinary differential equations, measure differential equations, impulsive equations (all of which might have discontinuous solutions), as well as dynamic equations on time scales. For scalar nonlinear equations, we find sufficient conditions guaranteeing that each bounded solution is asymptotic to a periodic solution. For linear systems, we show that the existence of a bounded solution implies the existence of a periodic solution. We include some examples to illustrate our results.

Keywords: Massera theorem; periodic solution; generalized ordinary differential equations; measure differential equations; impulsive equations; dynamic equations on time scales

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1 Introduction

In 1950, J. L. Massera published a remarkable paper [14] on the existence of periodic solutions to ordinary differential equations with periodic right-hand sides. He considered n-dimensional systems of the form $x'(t) = f(x(t), t)$, where f is 1-periodic in the second variable and smooth, and investigated the existence of periodic solutions of period 1. Let us mention three of his conclusions: 1) For scalar equations (i.e., $n = 1$), the existence of a bounded solution implies the existence of a 1-periodic solution. 2) In general, the previous theorem no longer holds for systems of equations (i.e., $n > 1$). 3) For linear systems, the existence of a bounded solution again implies the existence of a 1-periodic solution. In fact, assertion 1) could be strengthened: For a scalar equation, each bounded solution is asymptotic to a 1-periodic solution. This version is no longer true for linear systems: For example, the right-hand sides of the system $x'(t) = -y(t)$, $y'(t) = x(t)$ are 1-periodic with respect to t (since they do not depend explicitly on t), but all nonzero solutions of the system have the form $(x(t), y(t)) = (x_0 \cos t - y_0 \sin t, x_0 \sin t + y_0 \cos t)$.

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Thus, they are 2π-periodic, and not asymptotic to a 1-periodic solution. A good elementary overview of Massera-type theorems can be found in [4].

Since the publication of Massera's paper, numerous authors have investigated the validity of similar results for other types of equations. For example, the paper [5] deals with functional differential equations, [8] is concerned with systems of Volterra equations, [12] is devoted to partial differential equations, and [1, 13] discuss dynamic equations on time scales.

In the present paper, we obtain new Massera-type results for generalized ordinary differential equations in the sense of J. Kurzweil (see [10, 11, 24] for an overview of this concept). A characteristic feature of generalized ODEs is that their solutions are in general discontinuous. As a consequence, it turns out that a bounded solution of a T-periodic scalar generalized ODE need not be asymptotic to a T-periodic solution (see Example 3.3). However, in Section 3, we present a simple condition that guarantees the monotonicity of the flow at all discontinuity points, as well as the fact that bounded solutions of scalar nonlinear generalized ODEs are asymptotic to periodic solutions. In Section 4, we provide a direct analogue of Massera's theorem for systems of linear generalized ODEs.

One reason why generalized ODEs are attractive stems from the fact that they encompass other types of equations – classical ODEs, impulsive ODEs, measure differential equations, or dynamic equations on time scales. Thus, once we have Massera-type theorems for generalized ODEs, it is an easy task to obtain their counterparts for measure differential equations (Section 5), dynamic equations on time scales (Section 6), and impulsive ODEs (Section 7). As far as we are aware, the results for generalized ODEs, measure DEs, and impulsive ODEs are completely new, and the results for dynamic equations improve those from the existing literature (more details are given in Section 6). In particular, we do not require the right-hand side to be continuous in the time variable, and consider all equations in the integral form. Hence, even in the case of ODEs, the assumptions are weaker when compared to [4] or [14]. Finally, let us note that the results from Section 5 are also applicable to Stieltjes differential equations and distributional differential equations, both of which are closely related to measure differential equations; see [16].

2 Preliminaries

In this section, we recall some basic concepts from the theory of generalized ordinary differential equations. For more details, see [10, 11, 24].

Generalized ODEs are integral equations involving the Kurzweil integral, which was introduced in [10] and is based on the concept of δ-fine partitions: Given a function $\delta : [a,b] \to \mathbb{R}^+$, a tagged partition of $[a,b]$ with division points $a = s_0 \le s_1 \le \cdots \le s_k = b$ and tags $\tau_i \in [s_{i-1}, s_i]$, $i \in \{1,\dots,k\}$, is called δ-fine if $[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i))$ for each $i \in \{1,\dots,k\}$.

Definition 2.1. A function $U : [a,b] \times [a,b] \to \mathbb{R}^n$ is called Kurzweil integrable if there is an element $I \in \mathbb{R}^n$ having the following property: For every $\varepsilon > 0$, there is a function $\delta : [a,b] \to \mathbb{R}^+$ such that the inequality

$$\left\| \sum_{i=1}^{k} [U(\tau_i, s_i) - U(\tau_i, s_{i-1})] - I \right\| < \varepsilon$$

holds for every δ-fine tagged partition of $[a,b]$. In this case, I is called the Kurzweil integral of U over $[a,b]$ and is denoted by $\int_a^b \mathrm{D}_t U(\tau,t)$. Moreover, we define $\int_b^a \mathrm{D}_t U(\tau,t) = -\int_a^b \mathrm{D}_t U(\tau,t)$.

The Kurzweil integral has the usual properties of linearity, additivity with respect to adjacent intervals, integrability on subintervals, etc. More information can be found in [11, 24].

A function $f:[a,b]\to\mathbb{R}^n$ is called regulated if the limit $f(t-)=\lim_{s\mapsto t-} f(s)$ exists for every $t\in(a,b]$, and the limit $f(t+)=\lim_{s\mapsto t+} f(s)$ exists for every $t\in[a,b)$.

For the following statement, see [11, Corollary 14.18] or [24, Theorem 1.16].

Theorem 2.2. *Assume that $U:[a,b]\times[a,b]\to\mathbb{R}^n$ is Kurzweil integrable and $u:[a,b]\to\mathbb{R}^n$ is given by*

$$u(s)=u(a)+\int_a^s \mathrm{D}_t U(\tau,t),\quad s\in[a,b].$$

If U is regulated in the second variable, then u is regulated and satisfies

$$\begin{aligned} u(t+)&=u(t)+U(t,t+)-U(t,t),\quad t\in[a,b)\\ u(t-)&=u(t)+U(t,t-)-U(t,t),\quad t\in(a,b]. \end{aligned}$$

Given a pair of functions $f:[a,b]\to\mathbb{R}^n$, $g:[a,b]\to\mathbb{R}$, let $U(\tau,t)=f(\tau)g(t)$ for all $\tau,t\in[a,b]$. Then the integral $\int_a^b \mathrm{D}_t U(\tau,t)$ is usually denoted by $\int_a^b f(t)\,\mathrm{d}g(t)$, and is called the Perron-Stieltjes or Kurzweil-Stieltjes integral of f with respect to g. This integral appears in the next result, which is a Bihari-type inequality (i.e., a nonlinear version of the Gronwall inequality); its proof can be found in [24, Theorem 1.40].

Theorem 2.3. *Let $h:[a,b]\to\mathbb{R}$ be left-continuous and nondecreasing, $\omega:[0,\infty)\to\mathbb{R}$ continuous and increasing with $\omega(0)=0$, $\psi:[a,b]\to[0,\infty)$ bounded, and $k>0$ such that*

$$\psi(t)\le k+\int_a^t \omega(\psi(s))\,\mathrm{d}h(s),\quad t\in[a,b].$$

Given an arbitrary $u_0>0$, let

$$\Omega(u)=\int_{u_0}^u \frac{\mathrm{d}r}{\omega(r)},\quad u\in(0,\infty),$$

$\alpha=\lim_{u\to0+}\Omega(u)\ge-\infty$, and $\beta=\lim_{u\to\infty}\Omega(u)\le\infty$. Also, let $\Omega^{-1}:(\alpha,\beta)\to\mathbb{R}$ be the inverse function to Ω. If $\Omega(k)+h(b)-h(a)<\beta$, then

$$\psi(t)\le\Omega^{-1}(\Omega(k)+h(t)-h(a)),\quad t\in[a,b].$$

Definition 2.4. Consider an interval $I\subset\mathbb{R}$ and a function $F:\mathbb{R}^n\times I\to\mathbb{R}^n$. A function $x:I\to\mathbb{R}^n$ is called a solution of the generalized ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x=\mathrm{D}_t F(x,t), \tag{2.1}$$

if the relation

$$x(v)=x(u)+\int_u^v \mathrm{D}_t F(x(\tau),t) \tag{2.2}$$

holds for all $u,v\in I$.

If x is a solution of Eq. (2.1) and F is regulated in the second variable, then it follows from Theorem 2.2 that the one-sided limits of x at an arbitrary point t are

$$\begin{aligned} x(t+) &= x(t) + F(x(t), t+) - F(x(t), t), \\ x(t-) &= x(t) + F(x(t), t-) - F(x(t), t). \end{aligned} \tag{2.3}$$

This observation is used in the proof of the next lemma.

Lemma 2.5. *Assume that $F : \mathbb{R}^n \times [a, b] \to \mathbb{R}^n$ is regulated in the second variable, and $x : [a, b] \to \mathbb{R}^n$, $y : [a, b] \to \mathbb{R}^n$ are solutions of Eq. (2.1).*

1. *If $t_0 \in [a, b)$ and $x(t_0+) = y(t_0+)$, then the function $z : [a, b] \to \mathbb{R}^n$ given by*

$$z(t) = \begin{cases} y(t) & \text{if } t \in [a, t_0], \\ x(t) & \text{if } t \in (t_0, b], \end{cases}$$

is a solution of Eq. (2.1).

2. *If $t_0 \in (a, b]$ and $x(t_0-) = y(t_0-)$, then the function $w : [a, b] \to \mathbb{R}^n$ given by*

$$w(t) = \begin{cases} y(t) & \text{if } t \in [a, t_0), \\ x(t) & \text{if } t \in [t_0, b], \end{cases}$$

is a solution of Eq. (2.1).

Proof. We prove only the first statement; the second one is demonstrated in a similar way. Obviously, the relation

$$z(v) - z(u) = \int_u^v \mathrm{D}_t F(z(\tau), t)$$

holds for $u, v \in [a, t_0]$ as well as for $u, v \in (t_0, b]$; the only nontrivial case is when one of the points u, v lies in $[a, t_0]$ and the other in $(t_0, b]$. Without loss of generality, suppose that $a \le u \le t_0 < v \le b$. By Theorem 2.2, the limit $\lim_{s \to t_0+} \int_s^v \mathrm{D}_t F(x(\tau), t)$ exists. Hence, by Hake's theorem for the Kurzweil integral (see [24, Remark 1.15]), the integral $\int_{t_0}^v \mathrm{D}_t F(z(\tau), t)$ exists, and

$$\int_{t_0}^v \mathrm{D}_t F(z(\tau), t) = F(z(t_0), t_0+) - F(z(t_0), t_0) + \lim_{s \to t_0+} \int_s^v \mathrm{D}_t F(x(\tau), t).$$

Consequently, we get

$$\begin{aligned} \int_u^v \mathrm{D}_t F(z(\tau), t) &= \int_u^{t_0} \mathrm{D}_t F(y(\tau), t) + F(z(t_0), t_0+) - F(z(t_0), t_0) + \lim_{s \to t_0+} \int_s^v \mathrm{D}_t F(x(\tau), t) \\ &= y(t_0) - y(u) + F(y(t_0), t_0+) - F(y(t_0), t_0) + x(v) - x(t_0+) \\ &= y(t_0) - z(u) + F(y(t_0), t_0+) - F(y(t_0), t_0) + z(v) - y(t_0+) \\ &= y(t_0) - z(u) + z(v) - y(t_0) = z(v) - z(u), \end{aligned}$$

where the first inequality on the previous line follows from Theorem 2.2. This confirms that z is a solution of Eq. (2.1). $\square$

Let us introduce the following conditions:

(N1) For every bounded set $O \subset \mathbb{R}^n$, there exists a nondecreasing left-continuous function $h : \mathbb{R} \to \mathbb{R}$ such that

$$\|F(x, s_2) - F(x, s_1)\| \le |h(s_2) - h(s_1)|$$

for all $x \in O$, $s_1, s_2 \in \mathbb{R}$.

(N2) For every bounded set $O \subset \mathbb{R}^n$, there exist a nondecreasing left-continuous function $h : \mathbb{R} \to \mathbb{R}$ and a continuous increasing function $\omega : [0, \infty) \to \mathbb{R}$ such that $\omega(0) = 0$, $\lim_{u\to 0+} \int_u^v \frac{\mathrm{d}r}{\omega(r)} = \infty$ for a certain $v > 0$, and

$$\|F(x, s_2) - F(x, s_1) - F(y, s_2) + F(y, s_1)\| \le |h(s_2) - h(s_1)|\omega(\|x - y\|)$$

for all $x, y \in O$, $s_1, s_2 \in \mathbb{R}$.

(N3) There exist a number $T > 0$ and a function $M : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$F(x, t + T) - F(x, t) = M(x)$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Without loss of generality, we may assume that the functions h in (N1) and (N2) coincide. (If (N1) holds with $h = h_1$ and (N2) holds with $h = h_2$, then both (N1) and (N2) hold with $h = h_1 + h_2$.) Conditions (N1) and (N2) are standard assumptions in the theory of generalized ODEs (cf. [22, 24]). Since the function h in (N1) is nondecreasing, it is also regulated, and (N1) implies that F is regulated in the second variable. Moreover, left-continuity of h implies left-continuity of F in the second variable. Consequently, the relations (2.3) imply that each solution of Eq. (2.1) is regulated and left-continuous.

(N2) is an Osgood-type condition; we can often choose $\omega(r) = Lr$ for all $r \ge 0$, which corresponds to a Lipschitz-type condition. For our purposes, it is important that (N1) and (N2) imply that forward solutions of initial-value problems are unique; this result is presented in [24, Theorem 4.8].

Condition (N3) was already utilized in the earlier paper [15], and its role is the same as the role of the condition $f(x, t + T) = f(x, t)$ for a classical differential equation of the form $x'(t) = f(x(t), t)$. Indeed, the next lemma shows that if (N3) holds, then each solution of Eq. (2.1) can be shifted backward or forward in time by the period T.

Lemma 2.6. *Condition* (N3) *implies the following assertions:*

(i) *A function $x : [a + T, b + T] \to \mathbb{R}^n$ is a solution of Eq.* (2.1) *if and only if the function $y : [a, b] \to \mathbb{R}^n$ given by $y(t) = x(t + T)$ is a solution of Eq.* (2.1).

(ii) *Eq.* (2.1) *has a T-periodic solution if and only if there is a solution $x : [t_0, t_0 + T] \to \mathbb{R}^n$ such that $x(t_0 + T) = x(t_0)$.*

Proof. We start by proving (i). Note that if $U : [u+T, v+T] \times [u+T, v+T] \to \mathbb{R}^n$ is an arbitrary function, then the integral $\int_{u+T}^{v+T} \mathrm{D}_t U(\tau, t)$ exists if and only if the integral $\int_u^v \mathrm{D}_t U(\tau+T, t+T)$ exists; in this case, they have the same value. This follows immediately from the definition of the integral, since a δ-fine partition of $[u, v]$ is easily transformed into a δ-fine partition of $[u+T, v+T]$.

Hence, taking $U(\tau, t) = F(x(\tau), t)$, we see that if $x : [a+T, b+T] \to \mathbb{R}^n$ is a solution of Eq. (2.1), then $y : [a, b] \to \mathbb{R}^n$ given by $y(t) = x(t+T)$ satisfies

$$y(v) - y(u) = x(v+T) - x(u+T) = \int_{u+T}^{v+T} \mathrm{D}_t F(x(\tau), t) = \int_u^v \mathrm{D}_t F(x(\tau+T), t+T).$$

The last integral is defined using sums of the form

$$\sum_{i=1}^{k} [F(x(\tau_i+T), s_i+T) - F(x(\tau_i+T), s_{i-1}+T)],$$

and condition (N3) implies they coincide with

$$\sum_{i=1}^{k} [F(x(\tau_i+T), s_i) - F(x(\tau_i+T), s_{i-1})].$$

Consequently, the integral $\int_u^v \mathrm{D}_t F(x(\tau+T), t)$ exists if and only if $\int_u^v \mathrm{D}_t F(x(\tau+T), t+T)$ exists, and they have the same value. Therefore, if x is a solution of Eq. (2.1), we conclude that

$$y(v) - y(u) = \int_u^v \mathrm{D}_t F(x(\tau+T), t) = \int_u^v \mathrm{D}_t F(y(\tau), t),$$

proving that y is a solution of Eq. (2.1). Conversely, if $y : [a, b] \to \mathbb{R}^n$ is a solution of Eq. (2.1), a similar calculation shows that $x : [a+T, b+T] \to \mathbb{R}^n$ given by $x(t) = y(t-T)$ is a solution of Eq. (2.1).

Now, let us prove (ii). Clearly, if $x : \mathbb{R} \to \mathbb{R}^n$ is T-periodic, then $x(t_0+T) = x(t_0)$ for each $t_0 \in \mathbb{R}$. Conversely, suppose there is a solution $x : [t_0, t_0+T] \to \mathbb{R}^n$ such that $x(t_0+T) = x(t_0)$. Then we can extend x to a T-periodic function defined on the whole real line. Statement (i) guarantees that x will be a solution of Eq. (2.1) on each interval of the form $[t_0+nT, t_0+(n+1)T]$ with $n \in \mathbb{Z}$, and therefore a solution on the whole real line. □

Lemma 2.7. *If* (N1)–(N3) *hold, then the function* h *in* (N1) *and* (N2) *can be chosen in such a way that* $t \mapsto h(t+T) - h(t)$ *is a constant function on* $\mathbb{R}$.

Proof. Suppose that (N1)–(N3) hold, choose a bounded set $O \subset \mathbb{R}^n$, and let h, ω be the functions from (N1) and (N2). Denote $C = h(T) - h(0)$, and let $\tilde{h} : \mathbb{R} \to \mathbb{R}$ be given by

$$\tilde{h}(t) = \begin{cases} h(t) & \text{if } t \in [0, T), \\ h(t-nT) + nC & \text{if } t \in [nT, (n+1)T) \text{ and } n \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Obviously, $\tilde{h}$ is nondecreasing and left-continuous on each interval $[nT,(n+1)T)$ with $n \in \mathbb{Z}$. Since

$$\lim_{t\to(n+1)T-} \tilde{h}(t) = h(T-) + nC = h(T) + nC = h(0) + (n+1)C = \tilde{h}((n+1)T),$$

it follows that $\tilde{h}$ is nondecreasing and left-continuous on the whole real line. It is also clear that $\tilde{h}(t+T) - \tilde{h}(t) = C$ for each $t \in \mathbb{R}$. To finish the proof, let us show that (N1) and (N2) hold if h is replaced by $\tilde{h}$. To verify these conditions, it is enough to consider the case when $s_1 < s_2$.

For each pair of real numbers $s_1 < s_2$, there exist unique integers $k \le l$ such that $kT \le s_1 < (k+1)T$ and $lT \le s_2 < (l+1)T$.

Note that (N3) implies $F(x, t + kT) = F(x, t) + kM(x)$ for each $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Hence, if $k = l$, then for all $x, y \in O$ we get

$$\begin{gathered}\|F(x,s_2) - F(x,s_1)\| = \|F(x,s_2 - kT) - F(x,s_1 - kT)\| \\ \le h(s_2 - kT) - h(s_1 - kT) = h(s_2 - kT) + kC - h(s_1 - kT) - kC = \tilde{h}(s_2) - \tilde{h}(s_1),\end{gathered}$$

and

$$\begin{gathered}\|F(x,s_2) - F(x,s_1) - F(y,s_2) + F(y,s_1)\| \\ = \|F(x,s_2 - kT) - F(x,s_1 - kT) - F(y,s_2 - kT) + F(y,s_1 - kT)\| \\ \le \omega(\|x-y\|)(h(s_2 - kT) - h(s_1 - kT)) = \omega(\|x-y\|)(\tilde{h}(s_2) - \tilde{h}(s_1)).\end{gathered}$$

It remains to consider the case $k < l$. Note that for every $x \in O$, we have $\|M(x)\| = \|F(x,T) - F(x,0)\| \le h(T) - h(0) = C$ and for every $x, y \in O$ and $\tau \in \mathbb{R}$, we have $\|M(x) - M(y)\| = \|F(x,T) - F(x,0) - F(y,T) + F(y,0)\| \le \omega(\|x-y\|)(h(T) - h(0)) = \omega(\|x-y\|)C$. Therefore,

$$\begin{gathered}\|F(x,s_2) - F(x,s_1)\| \le \|F(x,s_2) - F(x,lT)\| \\ + \|F(x,lT) - F(x,(k+1)T)\| + \|F(x,(k+1)T) - F(x,s_1)\| \\ \le \|F(x,s_2 - lT) - F(x,0)\| + (l-(k+1))\|M(x)\| + \|F(x,T) - F(x,s_1 - kT)\| \\ \le h(s_2 - lT) - h(0) + (l-(k+1))C + h(T) - h(s_1 - kT) \\ = h(s_2 - lT) + lC - h(s_1 - kT) - kC = \tilde{h}(s_2) - \tilde{h}(s_1),\end{gathered}$$

and

$$\begin{gathered}\|F(x,s_2) - F(x,s_1) - F(y,s_2) + F(y,s_1)\| \\ \le \|F(x,s_2) - F(x,lT) - F(y,s_2) + F(y,lT)\| \\ + \|F(x,lT) - F(x,(k+1)T) - F(y,lT) + F(y,(k+1)T)\| \\ + \|F(x,(k+1)T) - F(x,s_1) - F(y,(k+1)T) + F(y,s_1)\| \\ = \|F(x,s_2 - lT) - F(x,0) - F(y,s_2 - lT) + F(y,0)\| \\ + \|(l-(k+1))M(x) - (l-(k+1))M(y)\| \\ + \|F(x,T) - F(x,s_1 - kT) - F(y,T) + F(y,s_1 - kT)\| \\ \le \omega(\|x-y\|)(h(s_2 - lT) - h(0)) + \omega(\|x-y\|)(h(T) - h(s_1 - kT)) + (l-(k+1))\omega(\|x-y\|)C\end{gathered}$$

$$= \omega(\|x-y\|)(h(s_2-lT)-h(s_1-kT)+C+lC-(k+1)C)$$
$$= \omega(\|x-y\|)(h(s_2-lT)+lC-h(s_1-kT)-kC) = \omega(\|x-y\|)(\tilde{h}(s_2)-\tilde{h}(s_1)).$$

This proves that (N1) and (N2) hold with h replaced by $\tilde{h}$. □

3 Massera's theorem for nonlinear generalized ODEs

In this section, we prove a Massera-type theorem for the scalar generalized ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \mathrm{D}_t F(x,t), \tag{3.1}$$

where $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We need the following additional condition:

(N4) If $u, v \in \mathbb{R}$ are such that $u < v$, then

$$u + F(u,t+) - F(u,t) \le v + F(v,t+) - F(v,t)$$

for all $t \in \mathbb{R}$.

Condition (N4) can be expressed in an equivalent way by requiring that for every $t \in \mathbb{R}$, the mapping $u \mapsto u + F(u,t+) - F(u,t)$ is nondecreasing on $\mathbb{R}$. If (N4) holds and x, y is a pair of solutions such that $x(t) \le y(t)$, it follows from the relations (2.3) that $x(t+) \le y(t+)$. (N4) generalizes a condition introduced in [17] in the context of measure differential equations, and later used in [20, 21] in the study of Stieltjes differential equations.

The next lemma shows that (N4) together with uniqueness of forward solutions implies that the flow generated by Eq. (3.1) is monotonic.

Lemma 3.1. *If* (N1), (N2) *and* (N4) *hold and* $x, y : [a,b] \to \mathbb{R}$ *are solutions of Eq.* (3.1) *satisfying* $x(a) \le y(a)$*, then* $x(t) \le y(t)$ *for all* $t \in (a,b]$.

Proof. For contradiction, assume there exists a $t \in (a,b]$ with $x(t) > y(t)$. Let

$$t_0 = \inf\{s \in (a,b] : x(s) > y(s)\}.$$

By the definition of infimum, we have either $t_0 = a$, or $t_0 > a$ and $x(t_0-) \le y(t_0-)$; in both cases, we can conclude (using left–continuity of solutions) that $x(t_0) \le y(t_0)$ and $t_0 < b$.

Recall that (N1) and (N2) imply uniqueness of forward solutions. If $x(t_0) = y(t_0)$, then the restrictions of x and y to $[t_0, b]$ are two different solutions of Eq. (3.1) with the same initial condition, which is a contradiction.

If $x(t_0) < y(t_0)$, then condition (N4) implies $x(t_0+) \le y(t_0+)$. If $x(t_0+) < y(t_0+)$, then $x < y$ on a right neighborhood of t_0, which is impossible by the definition of infimum. Thus, we necessarily have $x(t_0+) = y(t_0+)$. By Lemma 2.5, the function $z : [a,b] \to \mathbb{R}$ given by

$$z(t) = \begin{cases} y(t) & \text{if } t \in [a, t_0], \\ x(t) & \text{if } t \in (t_0, b], \end{cases}$$

is a solution of Eq. (3.1). Thus, y and z are two distinct solutions with $y(a) = z(a)$, a contradiction with the uniqueness of forward solutions. □

The next result is a version of Massera's theorem for scalar nonlinear generalized ODEs. The first part of the proof (the existence of a T-periodic solution) is similar to the classical proof presented in [14, Theorem 1] or [4, Theorem 4.1.10]. We say that a solution x of Eq. (3.1) is asymptotic to another solution y whenever $\lim_{t\to\infty}(x(t)-y(t))=0$.

Theorem 3.2 (Massera's Theorem for Nonlinear Generalized ODEs)**.** *Let $F:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be such that conditions* (N1)–(N4) *hold. Then each bounded solution $x:[t_0,\infty)\to\mathbb{R}$ of Eq.* (3.1) *is asymptotic to a T-periodic solution. In particular, the existence of a bounded solution implies the existence of a T-periodic solution.*

Proof. Let $x:[t_0,\infty)\to\mathbb{R}$ be a bounded solution and choose $M>0$ such that $|x(t)|\le M$ for all $t\in[t_0,\infty)$. For each $n\in\mathbb{N}$, let $x_n:[t_0,\infty)\to\mathbb{R}$ be given by

$$x_n(t)=x(t+nT),\qquad t\in[t_0,\infty).$$

By Lemma 2.6, x_n is a solution of Eq. (3.1), and satisfies $|x_n(t)|\le M$ for every $t\in[t_0,\infty)$.

Assume that $x(t_0)\le x_1(t_0)$; the case $x(t_0)\ge x_1(t_0)$ can be handled in a similar way. Using Lemma 3.1 and the definition of x_n, we see that $x_n(t)=x(t+nT)\le x_1(t+nT)=x_{n+1}(t)$ for every $n\in\mathbb{N}$ and $t\ge t_0$. Hence, $\{x_n\}_{n=1}^{\infty}$ is a bounded nondecreasing sequence of functions on $[t_0,\infty)$; let $y:[t_0,\infty)\to\mathbb{R}$ be the pointwise limit of this sequence. We claim that y is a solution of Eq. (3.1). Recalling that

$$x_n(t)=x_n(t_0)+\int_{t_0}^{t}\mathrm{D}_sF(x_n(\tau),s),\quad t\in[t_0,\infty),$$

it suffices to check that $\lim_{n\to\infty}\int_{t_0}^{t}\mathrm{D}_sF(x_n(\tau),s)=\int_{t_0}^{t}\mathrm{D}_sF(y(\tau),s)$ for every $t\in[t_0,\infty)$. This fact follows from a convergence theorem for the Kurzweil integral presented in [24, Theorem 1.29]), whose assumptions are satisfied thanks to conditions (N1) and (N2), in which we choose $O=[-M,M]$.

Observe that

$$y(t_0+T)=\lim_{n\to\infty}x_n(t_0+T)=\lim_{n\to\infty}x_{n+1}(t_0)=y(t_0).\tag{3.2}$$

By Lemma 2.6 and its proof, Eq. (3.1) has a T-periodic solution, which coincides with y on $[t_0,\infty)$. Let us show that x is asymptotic to y.

By Lemma 2.7, we can assume that the function h in (N1) and (N2) is such that $t\mapsto h(t+T)-h(t)$ is a constant function.

According to (N2), there exists a $v>0$ such that the function

$$\Omega(u)=\int_u^v\frac{\mathrm{d}r}{\omega(r)},\quad u\in(0,\infty),$$

is continuous, increasing, $\lim_{u\to0+}\Omega(u)=-\infty$, and $\beta=\lim_{u\to+\infty}\Omega(u)\le\infty$. Hence, the inverse function Ω^{-1} is increasing on its domain $(-\infty,\beta)$.

Given an arbitrary $\varepsilon>0$, find an $\eta>0$ such that $\Omega(\eta)+h(t_0+T)-h(t_0)<\beta$ and $\Omega^{-1}(\Omega(\eta)+h(t_0+T)-h(t_0))<\varepsilon$.

Since

$$\lim_{m\to\infty}x(t_0+mT)=\lim_{m\to\infty}x_m(t_0)=y(t_0)=y(t_0+mT),$$

there exists an $m_0 \in \mathbb{N}$ such that $|x(t_0 + mT) - y(t_0 + mT)| < \eta$ for all $m \geq m_0$.

Denote $\psi(t) = |x(t) - y(t)|$, $t \in [t_0, \infty)$. For each $t \geq t_0 + m_0 T$, there is a unique integer $m \geq m_0$ such that $t_0 + mT \leq t < t_0 + (m+1)T$. Using the definition of a solution, we get

$$\begin{aligned}\psi(t) &= \left| x(t_0 + mT) + \int_{t_0+mT}^{t} \mathrm{D}_s F(x(\tau), s) - y(t_0 + mT) - \int_{t_0+mT}^{t} \mathrm{D}_s F(y(\tau), s) \right| \\ &\leq |x(t_0 + mT) - y(t_0 + mT)| + \left| \int_{t_0+mT}^{t} \mathrm{D}_s [F(x(\tau), s) - F(y(\tau), s)] \right| \\ &\leq \eta + \int_{t_0+mT}^{t} \omega(|x(s) - y(s)|) \, \mathrm{d}h(s),\end{aligned}$$

where the last inequality is a consequence of (N2) and the definition of the integral.

The Bihari-type inequality from Theorem 2.3 gives the estimate

$$\begin{aligned}\psi(t) \leq \Omega^{-1}(\Omega(\eta) + h(t) - h(t_0 + mT)) &\leq \Omega^{-1}(\Omega(\eta) + h(t_0 + (m+1)T) - h(t_0 + mT)) \\ &= \Omega^{-1}(\Omega(\eta) + h(t_0 + T) - h(t_0)) < \varepsilon,\end{aligned}$$

which holds for each $t \geq t_0 + m_0 T$. This proves that $\lim_{t \to \infty} \psi(t) = 0$. □

The next example shows that condition (N4) cannot be omitted.

Example 3.3. Consider the generalized ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} x = \mathrm{D}_t F(x, t), \tag{3.3}$$

where $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by $F(x, t) = -2nx$ for all $t \in (n, n+1]$ and $n \in \mathbb{Z}$. This function F obviously satisfies (N3) with $T = 1$, since

$$F(x, t + 1) - F(x, t) = -2x$$

for all $x, t \in \mathbb{R}$. To see that F satisfies also (N1) and (N2), choose a bounded set $O \subset \mathbb{R}$, and find a $K \geq 1$ such that $O \subset [-K, K]$. Let $h(t) = 2nK$ for all $t \in (n, n+1]$ and $n \in \mathbb{Z}$, and $\omega(r) = r$ for each $r \in [0, \infty)$. Take an arbitrary $x \in O$ and $s_1, s_2 \in \mathbb{R}$. There exist unique integers k, l such that $s_1 \in (k, k+1]$ and $s_2 \in (l, l+1]$. Then we have the estimates

$$|F(x, s_2) - F(x, s_1)| = |-2kx + 2lx| = 2|l - k| \cdot |x| \leq 2K|l - k| = |h(s_2) - h(s_1)|,$$

and

$$\begin{aligned}|F(x, s_2) - F(x, s_1) - F(y, s_2) + F(y, s_1)| &= |-2kx + 2lx + 2ky - 2ly| \\ = 2|l - k| \cdot |x - y| \leq 2K|l - k| \cdot |x - y| &= |h(s_2) - h(s_1)| \omega(|x - y|).\end{aligned}$$

Each solution of Eq. (3.3) is regulated, left-continuous, and has to be constant on each interval $(n-1, n]$ with $n \in \mathbb{Z}$, because h has the same property. For each $n \in \mathbb{Z}$, (2.3) implies

$$x(n+) = x(n) + F(x(n), n+) - F(x(n), n) = x(n) - 2nx(n) + 2(n-1)x(n) = -x(n),$$

i.e., x changes sign at each integer.

Thus, we see that the unique solution of Eq. (3.3) corresponding to the initial condition $x(0) = x_0$ satisfies $x(t) = x_0$ for all $t \in (n-1, n]$ with $n \in \mathbb{Z}$ being even, and $x(t) = -x_0$ for all $t \in (n-1, n]$ with $n \in \mathbb{Z}$ being odd. Hence, there is a unique 1-periodic solution (the zero solution), all remaining solutions are 2-periodic, and they are not asymptotic to the 1-periodic solution.

The reason why Theorem 3.2 does not apply is that condition (N4) is violated, and the flow is not monotonic. To see why (N4) fails, observe that for every $n \in \mathbb{Z}$ we have $u + F(u, n+) - F(u, n) = -u$, so $u < v$ does not imply $u + F(u, n+) - F(u, n) \leq v + F(v, n+) - F(v, n)$.

Let us now present an example of a generalized ODE with discontinuous solutions where Theorem 3.2 is applicable.

Example 3.4. Consider the generalized ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} x = \mathrm{D}_t F(x, t),$$

where $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by $F(x, t) = (n + 1 - t)x$ for all $t \in (n, n+1]$, $n \in \mathbb{Z}$. Since $F(x, t+1) = F(x, t)$ for all $x, t \in \mathbb{R}$, condition (N3) is satisfied with $T = 1$.

To check that conditions (N1) and (N2) also hold, consider a bounded set $O \subset [-K, K] \subset \mathbb{R}$ with $K \geq 1$. Define $\omega : [0, \infty) \to \mathbb{R}$ by $\omega(r) = r$, and $h : \mathbb{R} \to \mathbb{R}$ by $h(s) = K(s + n)$ for all $s \in (n, n+1]$, $n \in \mathbb{Z}$. Choose arbitrary $s_1, s_2 \in \mathbb{R}$ with $s_1 \leq s_2$. There exist unique $k, l \in \mathbb{Z}$ such that $k \leq l$ and $s_1 \in (k, k+1]$, $s_2 \in (l, l+1]$. For all $x, y \in O$, we have

$$\begin{aligned} |F(x, s_2) - F(x, s_1)| &= |(l + 1 - s_2)x - (k + 1 - s_1)x| \leq K|(l - k) - (s_2 - s_1)| \\ &\leq K|(l - k) + (s_2 - s_1)| = |h(s_2) - h(s_1)|, \end{aligned}$$

and also

$$\begin{aligned} &|F(x, s_2) - F(x, s_1) - F(y, s_2) + F(y, s_1)| \\ &= |(l + 1 - s_2)x - (k + 1 - s_1)x - (l + 1 - s_2)y + (k + 1 - s_1)y| \\ &= |(l - k) - (s_2 - s_1)| \cdot |x - y| \leq K|(l - k) + (s_2 - s_1)| \cdot |x - y| = |h(s_2) - h(s_1)|\omega(|x - y|). \end{aligned}$$

Lastly, condition (N4) is always true at each t where the map $t \mapsto F(x, t)$ is continuous. So, we only need to check (N4) when $t \in \mathbb{Z}$ and in this case, we have

$$v + F(v, t+) - F(v, t) - (u + F(u, t+) - F(u, t)) = 2v - 2u \geq 0,$$

whenever $v > u$. Thus, according to Theorem 3.2, each bounded solution is asymptotic to a 1-periodic solution. We can verify this fact by the following explicit calculation:

Notice first that $F(x(\tau), t) = x(\tau)g(t)$, where $g(t) = n + 1 - t$ for $t \in (n, n+1]$ and $n \in \mathbb{Z}$. Thus, for each $x : [a, b] \to \mathbb{R}$, the integral $\int_a^b \mathrm{D}_s F(x(\tau), s)$ reduces to $\int_a^b x(s)\,\mathrm{d}g(s)$. Moreover, if $[a, b] \subset (n, n+1]$ with $n \in \mathbb{Z}$, then $g(t_2) - g(t_1) = t_1 - t_2$ for all $t_1, t_2 \in [a, b]$, and therefore $\int_a^b x(s)\,\mathrm{d}g(s) = -\int_a^b x(s)\,\mathrm{d}s$. Hence, in each interval $(n, n+1]$, the generalized ODE reduces to the ODE $x'(t) = -x(t)$. If we know the value $x(n+)$, we get the explicit solution $x(t) = x(n+)e^{-(t-n)}$ for all $t \in (n, n+1]$.

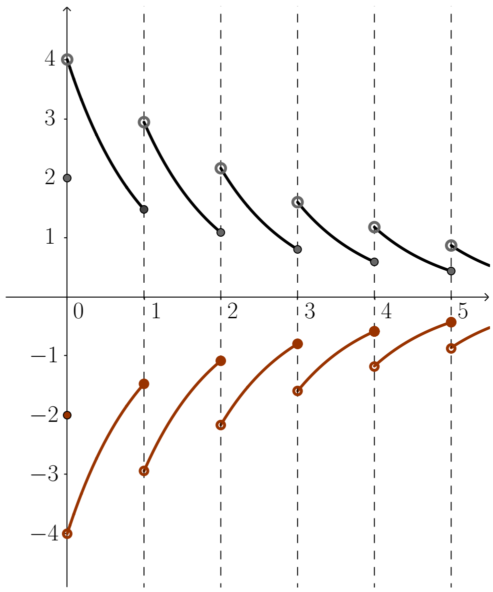

Figure 1: Solutions with $x(0) = 2$ (positive) and $x(0) = -2$ (negative)

It remains to determine what happens to the solution at an arbitrary integer n. From Theorem 2.2, we deduce that

$$x(n+) = x(n) + F(x(n), n+) - F(x(n), n) = 2x(n).$$

Combining the previous information, we get $x(t) = 2x(n)e^{-(t-n)}$ for all $t \in (n, n+1]$. Thus, given an arbitrary $x_0 \in \mathbb{R}$, it is easy to construct the corresponding solution $x : [0, \infty) \to \mathbb{R}$ satisfying $x(0) = x_0$; Figure 1 illustrates the solutions with $x(0) = 2$ and $x(0) = -2$.

Now, notice that for each $n \in \mathbb{N}$, we have $x(n+1) = 2x(n)e^{-1} < x(n)$, because $2/e < 1$. Consequently, it is easy to see that the only 1-periodic solution is the zero solution, any other solution is bounded, and converges to the zero solution, as predicted by Theorem 3.2.

4 Massera's theorem for linear generalized ODEs

In the previous section, we proved a version of Massera's theorem for scalar nonlinear generalized ODEs. Here we consider n-dimensional systems of linear generalized ODEs of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \mathrm{D}_t[A(t)x + f(t)], \tag{4.1}$$

where x and f take values in $\mathbb{R}^n$, and A takes values in $L(\mathbb{R}^n)$, the space of all real $n \times n$ matrices. Eq. (4.1) is a special case of Eq. (2.1) with

$$F(x, t) = A(t)x + f(t), \tag{4.2}$$

and is equivalent to the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} \mathrm{d}[A(s)]x(s) + f(t) - f(t_0).$$

We introduce the following conditions:

(L1) $A : \mathbb{R} \to L(\mathbb{R}^n)$ has locally bounded variation and $I - \Delta^- A(t)$ is invertible for each $t \in \mathbb{R}$.

(L2) $f : \mathbb{R} \to \mathbb{R}^n$ is regulated.

(L3) There exist $T > 0$, $C \in L(\mathbb{R}^n)$ and $D \in \mathbb{R}^n$ such that

$$A(t+T) - A(t) = C \quad \text{and} \quad f(t+T) - f(t) = D \quad \text{for every } t \in \mathbb{R}.$$

It is well known (see e.g. [18, Theorem 7.4.5]) that if conditions (L1) and (L2) hold, then for every $x_0 \in \mathbb{R}^n$, Eq. (4.1) has a unique solution defined on $[t_0, \infty)$ and satisfying $x(t_0) = x_0$; let us denote it by $x(\cdot, t_0, x_0)$. Clearly, we have

$$x(t, t_0, x_0) = x(t, t_0, 0) + x_h(t, t_0, x_0),$$

where x_h is the solution of the corresponding homogeneous equation (i.e., Eq. (4.1) with $f \equiv 0$). The set of all solutions of the homogeneous equation is a vector space, and the mapping $x_0 \mapsto x_h(t, t_0, x_0)$ is a linear mapping on $\mathbb{R}^n$. As a consequence, for every fixed $t \geq t_0$, the mapping

$$x_0 \mapsto x(t, t_0, x_0)$$

is continuous on $\mathbb{R}^n$.

Finally, note that if (L3) holds, then the function F given by (4.2) satisfies (N3).

Theorem 4.1 (Massera's Theorem for Linear Generalized ODEs)**.** *Let $A : \mathbb{R} \to L(\mathbb{R}^n)$ and $f : \mathbb{R} \to \mathbb{R}^n$ be such that* (L1)–(L3) *hold. If Eq.* (4.1) *has a bounded solution on* $[t_0, \infty)$*, then it has a T-periodic solution.*

Proof. Let $x(\cdot, t_0, x_0)$ be a bounded solution of Eq. (4.1) defined on $[t_0, \infty)$. Hence, there exists a constant $M > 0$ such that $\|x(t, t_0, x_0)\| \leq M$ for every $t \in [t_0, \infty)$. We know that the Poincaré map $P : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$P(y) = x(t_0 + T, t_0, y)$$

is continuous on $\mathbb{R}^n$. Let us consider its restriction to the set

$$\Lambda = \{y \in \mathbb{R}^n : \|x(t, t_0, y)\| \leq M \ \text{ for every } t \geq t_0\}.$$

The set Λ is nonempty because $x_0 \in \Lambda$. It is also easy to see that Λ is convex. In fact, if $x_0, y_0 \in \Lambda$ and $\alpha \in (0, 1)$, then

$$\begin{gathered} \alpha x(t, t_0, x_0) + (1-\alpha) x(t, t_0, y_0) = \\ \alpha \left(x_0 + \int_{t_0}^{t} \mathrm{d}[A(s)] x(s, t_0, x_0) + f(t) - f(t_0) \right) \\ +(1-\alpha) \left(y_0 + \int_{t_0}^{t} \mathrm{d}[A(s)] x(s, t_0, y_0) + f(t) - f(t_0) \right) \\ = \alpha x_0 + (1-\alpha) y_0 + \int_{t_0}^{t} \mathrm{d}[A(s)] (\alpha x(s, t_0, x_0) + (1-\alpha) x(s, t_0, y_0)) + f(t) - f(t_0). \end{gathered}$$

Hence, the function $\alpha x(\cdot, t_0, x_0) + (1-\alpha)x(\cdot, t_0, y_0)$ is a solution of Eq. (4.1). It coincides with the solution $x(\cdot, t_0, \alpha x_0 + (1-\alpha)y_0)$, which has the same value at t_0. Therefore,

$$\|x(t, t_0, \alpha x_0 + (1-\alpha)y_0)\| = \|\alpha x(t, t_0, x_0) + (1-\alpha)x(t, t_0, y_0)\| \leq \alpha M + (1-\alpha)M = M$$

for each $t \geq t_0$, implying that $\alpha x_0 + (1-\alpha)y_0 \in \Lambda$.

Notice that Λ is bounded, because $y \in \Lambda$ implies $\|y\| \leq M$. It is also closed: Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in Λ with $y_n \to y$. Then $\|x(t, t_0, y_n)\| \leq M$ for every $t \geq t_0$, and therefore $\|x(t, t_0, y)\| \leq M$ for every $t \geq t_0$, implying that $y \in \Lambda$.

For every $y \in \mathbb{R}^n$, it follows from Lemma 2.6 that the function $t \mapsto x(t+T, t_0, y)$ is a solution of Eq. (4.1). It has to coincide with the solution $t \mapsto x(t, t_0, P(y))$, which has the same value at t_0. Thus, if $y \in \Lambda$, we get

$$\|x(t, t_0, P(y))\| = \|x(t+T, t_0, y)\| \leq M.$$

This implies that $P(y) \in \Lambda$ for every $y \in \Lambda$. By the Brouwer fixed point theorem, we conclude that P has a fixed point $\tilde{x}_0 \in \Lambda$, i.e., $x(t_0, t_0, \tilde{x}_0) = \tilde{x}_0 = P(\tilde{x}_0) = x(t_0 + T, t_0, \tilde{x}_0)$. Lemma 2.6 implies that Eq. (4.1) has a T-periodic solution. $\square$

Example 4.2. Let us return to the generalized ODE presented in Example 3.3:

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \mathrm{D}_t F(x, t), \tag{4.3}$$

where $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by $F(x, t) = -2mx$ for all $t \in (m, m+1]$ and $m \in \mathbb{Z}$. This is a linear equation of the form (4.1) with $n = 1$, $f \equiv 0$ and $A(t) = -2m$ for all $t \in (m, m+1]$ and $m \in \mathbb{Z}$. It is clear that (L1) and (L2) are satisfied (note that A is left-continuous, and therefore $I - \Delta^- A(t) = I$ for each $t \in \mathbb{R}$), and (L3) holds with $T = 1$ and $C = -2$. We have seen that Theorem 3.2 is not applicable, but the assumptions of Theorem 4.1 are satisfied. We also know that all solutions are bounded; hence, Theorem 3.2 predicts the existence of a 1-periodic solution, which is the zero solution (recall that all nonzero solutions are 2-periodic).

5 Massera's theorems for measure differential equations

Once we have Massera's theorems for generalized ordinary differential equations, we can obtain the corresponding statements for measure differential equations, i.e., integral equations of the form

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s), s)\,\mathrm{d}g(s). \tag{5.1}$$

We begin with scalar nonlinear equations.

Theorem 5.1 (Massera's Theorem for Nonlinear Measure DEs)**.** *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be such that the following conditions hold:*

- *g is nondecreasing, left-continuous, and there exist $K, T > 0$ such that $g(t+T) - g(t) = K$ for each $t \in \mathbb{R}$.*

- *f is continuous in the first variable and T-periodic in the second variable.*
- *The integral $\int_{s_1}^{s_2} f(x,s)\,\mathrm{d}g(s)$ exists for each interval $[s_1,s_2]\subset\mathbb{R}$ and each $x\in\mathbb{R}$.*
- *For every bounded set $O\subset\mathbb{R}$, there exist a locally integrable function $m:\mathbb{R}\to[0,\infty)$ and a continuous increasing function $\omega:[0,\infty)\to\mathbb{R}$ such that $\omega(0)=0$, $\lim_{u\to 0+}\int_u^v \frac{\mathrm{d}r}{\omega(r)}=\infty$ for a certain $v>0$, and*

$$\left|\int_{s_1}^{s_2} f(x,s)\,\mathrm{d}g(s)\right| \le \int_{s_1}^{s_2} m(s)\,\mathrm{d}g(s), \tag{5.2}$$

$$\left|\int_{s_1}^{s_2} (f(x,s)-f(y,s))\,\mathrm{d}g(s)\right| \le \omega(|x-y|)\int_{s_1}^{s_2} m(s)\,\mathrm{d}g(s) \tag{5.3}$$

for each interval $[s_1,s_2]\subset\mathbb{R}$ and all $x,y\in O$.

- *If $u,v\in\mathbb{R}$ with $u<v$, then $u+f(u,t)\Delta^+g(t)\le v+f(v,t)\Delta^+g(t)$ for every $t\in\mathbb{R}$.*

Then each bounded solution $x:[t_0,\infty)\to\mathbb{R}$ of Eq. (5.1) is asymptotic to a T-periodic solution. In particular, the existence of a bounded solution implies the existence of a T-periodic solution.

Proof. Let

$$F(x,t)=\int_{t_0}^{t} f(x,s)\,\mathrm{d}g(s)$$

for all $x,t\in\mathbb{R}$. The assumptions (5.2) and (5.3) imply that F satisfies conditions (N1) and (N2) with $h(t)=\int_{t_0}^{t} m(s)\,\mathrm{d}g(s)$, $t\in\mathbb{R}$ (note that g is left-continuous, and Theorem 2.2 implies that h has the same property). Next, note that

$$F(x,t+T)-F(x,t)=\int_{t}^{t+T} f(x,s)\,\mathrm{d}g(s)=\int_{t_0}^{t_0+T} f(x,s)\,\mathrm{d}g(s),$$

where the last equality follows from the assumptions on f and g. Thus, F satisfies (N3) with $M(x)=\int_{t_0}^{t_0+T} f(x,s)\,\mathrm{d}g(s)$. As concerns (N4), it follows from the assumptions and Theorem 2.2 that if $u<v$, then

$$u+F(u,t+)-F(u,t)=u+f(u,t)\Delta^+g(t)\le v+f(v,t)\Delta^+g(t)=v+F(v,t+)-F(v,t)$$

for all $t\in\mathbb{R}$.

Finally, [24, Proposition 5.12][1] implies $\int_{t_0}^{t} \mathrm{D}_s F(x(\tau),s)=\int_{t_0}^{t} f(x(s),s)\,\mathrm{d}g(s)$ for each regulated function $x:[t_0,\infty)\to\mathbb{R}$. Hence, the measure differential equation (5.1) is equivalent to the generalized ODE

$$x(t)=x(t_0)+\int_{t_0}^{t} \mathrm{D}_s F(x(\tau),s)$$

(note that the solutions of both equations are necessarily regulated), and the proof is finished by applying Theorem 3.2. □

[1] The statement given in [24] assumes that $|f(x,s)|\le m(s)$ for all $x,s\in\mathbb{R}$. However, our weaker assumptions on f are sufficient if one avoids the dominated convergence theorem and instead uses the convergence theorem given in [18, Theorem 6.8.10].

A linear measure differential equation has the form (5.1) with $f(x,t) = p(t)x + q(t)$, i.e.,

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s) + q(s))\,\mathrm{d}g(s). \tag{5.4}$$

Massera's theorem for this type of equations reads as follows.

Theorem 5.2 (Massera's Theorem for Linear Measure DEs). *Let $g : \mathbb{R} \to \mathbb{R}$, $p : \mathbb{R} \to L(\mathbb{R}^n)$ and $q : \mathbb{R} \to \mathbb{R}^n$ be such that the following conditions hold:*

- *g is nondecreasing, left-continuous, and there exist $K, T > 0$ such that $g(t+T) - g(t) = K$ for each $t \in \mathbb{R}$.*
- *p, q are T-periodic and locally integrable with respect to g.*
- *There exists a locally integrable function $m : \mathbb{R} \to [0,\infty)$ such that*

  $$\left\| \int_{s_1}^{s_2} p(s)\,\mathrm{d}g(s) \right\| \le \int_{s_1}^{s_2} m(s)\,\mathrm{d}g(s) \tag{5.5}$$

  for each interval $[s_1, s_2] \subset \mathbb{R}$.

If Eq. (5.4) has a bounded solution on $[t_0,\infty)$, then it has a T-periodic solution.

Proof. Let

$$A(t) = \int_{t_0}^{t} p(s)\,\mathrm{d}g(s), \quad f(t) = \int_{t_0}^{t} q(s)\,\mathrm{d}g(s)$$

for all $t \in \mathbb{R}$. Since g is regulated and left-continuous, it follows from Theorem 2.2 that A and f have the same properties. Moreover, Eq. (5.5) implies that

$$\operatorname{var}_u^v A \le \int_u^v m(s)\,\mathrm{d}g(s)$$

for each interval $[u,v] \subset \mathbb{R}$, i.e., A has locally bounded variation. The periodicity of p, q and the assumptions on g imply that for every $t \in \mathbb{R}$, we have

$$\begin{aligned} A(t+T) - A(t) &= \int_t^{t+T} p(s)\,\mathrm{d}g(s) = \int_{t_0}^{t_0+T} p(s)\,\mathrm{d}g(s) = A(t_0+T) - A(t_0), \\ f(t+T) - f(t) &= \int_t^{t+T} q(s)\,\mathrm{d}g(s) = \int_{t_0}^{t_0+T} q(s)\,\mathrm{d}g(s) = f(t_0+T) - f(t_0). \end{aligned}$$

It follows that A, f satisfy conditions (L1)–(L3) with $C = A(t_0+T) - A(t_0)$ and $D = f(t_0 + T) - f(t_0)$. Finally, the substitution theorem for the Kurzweil-Stieltjes integral (see [18, Theorem 6.6.1]) implies $\int_{t_0}^{t} \mathrm{d}[A(s)]x(s) = \int_{t_0}^{t} p(s)x(s)\,\mathrm{d}g(s)$. Hence, the measure differential equation (5.4) is equivalent to the generalized linear ODE

$$x(t) = x(t_0) + \int_{t_0}^{t} \mathrm{d}[A(s)]x(s) + f(t) - f(t_0),$$

and the proof is finished by applying Theorem 4.1. $\square$

6 Massera's theorems for dynamic equations on time scales

We now proceed to Massera-type theorems for dynamic equations on time scales, which are known to be a special case of measure differential equations (see e.g. [18, 23]). Let $\mathbb{T}$ be a time scale, i.e., a closed nonempty subset of $\mathbb{R}$. The dynamic equation

$$x^{\Delta}(t) = f(x(t), t), \quad t \in \mathbb{T}, \tag{6.1}$$

has the integral form

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s), s)\Delta s, \quad t \in \mathbb{T}, \tag{6.2}$$

where the Δ-integral on the right-hand side can be understood in the sense of Riemann, Lebesgue, or Henstock-Kurzweil. More details about time scales, Δ-derivatives, Δ-integrals, and dynamic equations are available in [2, 3, 7].

We will focus on Eq. (6.2) with the Henstock-Kurzweil Δ-integral, which is the most general one; its definition can be found in [18] or [19]. Since we are interested in the existence of periodic solutions, we will deal with periodic time scales; the following definition may be found e.g. in [9].

Definition 6.1. Given a $T > 0$, a time scale $\mathbb{T}$ is called T-periodic if $t \in \mathbb{T}$ implies $t \pm T \in \mathbb{T}$.

Clearly, if $\mathbb{T}$ is T-periodic, then $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = \infty$. Throughout the rest of this section, we will always assume that $\mathbb{T}$ is a periodic time scale.

In order to distinguish between intervals on the real line and time scale intervals, we will use the notation $[\alpha, \beta]_{\mathbb{T}} = [\alpha, \beta] \cap \mathbb{T}$ whenever $\alpha, \beta \in \mathbb{T}$. An arbitrary function $h : \mathbb{T} \to \mathbb{R}^n$ can be extended to a function $h^* : \mathbb{R} \to \mathbb{R}^n$ as follows. For each $t \in \mathbb{R}$, we define $t^* = \inf\{s \in \mathbb{T}\colon\ t \le s\}$, and $h^*(t) = h(t^*)$; note that h^* coincides with h on $\mathbb{T}$.

The next result, whose proof is available in [6, Theorem 4.2] or [18, Theorem 8.6.8], describes the relation between the Δ-integral of a function defined on a time scale interval, and the Kurzweil-Stieltjes integral of its extension to the corresponding interval on the real line.

Theorem 6.2. *Let $f : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$ be an arbitrary function. Define $g(t) = t^*$ for every $t \in [a, b]$. Then the Kurzweil-Henstock Δ-integral $\int_a^b f(t)\Delta t$ exists if and only if the Kurzweil-Stieltjes integral $\int_a^b f^*(t)\,\mathrm{d}g(t)$ exists; in this case, both integrals have the same value.*

We also need the following observation, which was proved in [6, Lemma 4.4].

Lemma 6.3. *Let $a, b \in \mathbb{T}$, $a < b$, $g(t) = t^*$ for every $t \in [a, b]$. If $f : [a, b] \to \mathbb{R}^n$ is such that the integral $\int_a^b f(t)\,\mathrm{d}g(t)$ exists, then*

$$\int_c^d f(t)\,\mathrm{d}g(t) = \int_{c^*}^{d^*} f(t)\,\mathrm{d}g(t)$$

for every $c, d \in [a, b]$.

The next result, which describes the relation between dynamic equations and measure differential equations, follows from [18, Theorem 8.7.1].

Theorem 6.4. *Suppose that $a, b, t_0 \in \mathbb{T}$, $a \le t_0 \le b$, and consider a function $f : \mathbb{R}^n \times [a,b]_{\mathbb{T}} \to \mathbb{R}^n$. Let $g(t) = t^*$ and $f^*(x,t) = f(x,t^*)$ for all $t \in [a,b]$, $x \in \mathbb{R}^n$.*

If a function $x : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ satisfies

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s), s)\,\Delta s, \quad t \in [a,b]_{\mathbb{T}}, \tag{6.3}$$

then the function $y : [a,b] \to \mathbb{R}^n$ given by $y = x^*$ satisfies

$$y(t) = y(t_0) + \int_{t_0}^{t} f^*(y(s), s)\,\mathrm{d}g(s), \quad t \in [a,b]. \tag{6.4}$$

Conversely, each function $y : [a,b] \to \mathbb{R}^n$ satisfying Eq. (6.4) has the form $y = x^*$, where $x : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ satisfies Eq. (6.3).

Although the previous theorem was formulated for solutions defined on compact intervals, it is clear that a similar result holds also for unbounded intervals.

We now prove Massera's theorem for scalar nonlinear dynamic equations on time scales.

Theorem 6.5 (Massera's Theorem for Nonlinear Dynamic Equations)**.** *Suppose that $\mathbb{T}$ is a T-periodic time scale and $f : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ satisfies the following conditions:*

- *f is continuous in the first variable and T-periodic with respect to the second variable.*
- *The integral $\int_{s_1}^{s_2} f(x,s)\,\Delta s$ exists whenever $[s_1, s_2]_{\mathbb{T}} \subset \mathbb{T}$ and $x \in \mathbb{R}$.*
- *For every bounded set $O \subset \mathbb{R}$, there exist a locally Δ-integrable function $m : \mathbb{T} \to [0,\infty)$ and a continuous increasing function $\omega : [0,\infty) \to \mathbb{R}$ such that $\omega(0) = 0$, $\lim_{u\to 0+} \int_u^v \frac{\mathrm{d}r}{\omega(r)} = \infty$ for a certain $v > 0$, and*

  $$\left|\int_{s_1}^{s_2} f(x,s)\,\Delta s\right| \le \int_{s_1}^{s_2} m(s)\,\Delta s,$$
  $$\left|\int_{s_1}^{s_2} (f(x,s) - f(y,s))\,\Delta s\right| \le \omega(|x-y|) \int_{s_1}^{s_2} m(s)\,\Delta s$$

  whenever $[s_1, s_2]_{\mathbb{T}} \subset \mathbb{T}$ and $x, y \in O$.
- *If $u, v \in \mathbb{R}$ with $u < v$, then $u + f(u,t)\mu(t) \le v + f(v,t)\mu(t)$ for every $t \in \mathbb{T}$.*

Then each bounded solution $x : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ of Eq. (6.2) is asymptotic to a T-periodic solution on $\mathbb{T}$. In particular, the existence of a bounded solution implies the existence of a T-periodic solution.

Proof. Let $f^*(x,t) = f(x,t^*)$ and $g(t) = t^*$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}$. Then g is nondecreasing and left-continuous (see [23, Lemma 4]). Since $\mathbb{T}$ is T–periodic, we have $g(t+T) - g(t) = T$ for every $t \in \mathbb{R}$. Also, using the fact that $\Delta^+ g(t) = \mu(t)$ for $t \in \mathbb{T}$ and $\Delta^+ g(t) = 0$ otherwise, we see that if $u, v \in \mathbb{R}$ satisfy $u < v$, then

$$u + f(u,t)\Delta^+ g(t) \le v + f(v,t)\Delta^+ g(t) \quad \text{for every } t \in \mathbb{R}.$$

By Theorem 6.2 and Lemma 6.3, we have $\int_{s_1^*}^{s_2^*} f(x,s)\Delta s = \int_{s_1^*}^{s_2^*} f^*(x,s)\,\mathrm{d}g(s) = \int_{s_1}^{s_2} f^*(x,s)\,\mathrm{d}g(s)$ whenever $[s_1,s_2] \subset \mathbb{R}$ and $x \in \mathbb{R}$, i.e., the last integral exists. For a bounded set $O \subset \mathbb{R}$, we get

$$\left|\int_{s_1}^{s_2} f^*(x,s)\,\mathrm{d}g(s)\right| = \left|\int_{s_1^*}^{s_2^*} f(x,s)\,\Delta s\right| \le \int_{s_1^*}^{s_2^*} m(s)\,\Delta s = \int_{s_1}^{s_2} m^*(s)\,\mathrm{d}g(s)$$

whenever $[s_1,s_2] \subset \mathbb{R}$ and $x \in O$, and similarly

$$\left|\int_{s_1}^{s_2} (f^*(x,s) - f^*(y,s))\,\mathrm{d}g(s)\right| \le \omega(|x-y|)\int_{s_1}^{s_2} m^*(s)\,\mathrm{d}g(s)$$

whenever $[s_1,s_2] \subset \mathbb{R}$ and $x,y \in O$. This shows that the functions f^*, g satisfy the assumptions of Theorem 5.1.

If $x : [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}$ is a bounded solution of Eq. (6.2), then the function $y : [t_0,\infty) \to \mathbb{R}$ given by $y = x^*$ is bounded and (by Theorem 6.4) satisfies $y(t) = y(t_0) + \int_{t_0}^{t} f^*(y(s),s)\,\mathrm{d}g(s)$, $t \in [t_0,\infty)$. By Theorem 5.1, y is asymptotic to a T-periodic solution of the same equation defined on $\mathbb{R}$. By the second part of Theorem 6.4, this solution is the extension of a certain T-periodic solution of Eq. (6.2). □

Next, we consider a system of linear dynamic equations in the vector form

$$x^{\Delta}(t) = p(t)x + q(t), \quad t \in \mathbb{T},$$

whose integral form is

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s) + q(s))\,\Delta s, \quad t \in \mathbb{T}. \tag{6.5}$$

A version of Massera's theorem for this type of equations is presented below.

Theorem 6.6 (Massera's Theorem for Linear Dynamic Equations)**.** *Suppose that $\mathbb{T}$ is a T-periodic time scale, and $p : \mathbb{T} \to L(\mathbb{R}^n)$ and $q : \mathbb{T} \to \mathbb{R}^n$ satisfy the following conditions:*

- *p and q are T-periodic and locally Δ-integrable.*
- *There exists a locally Δ-integrable function $m : \mathbb{T} \to [0,\infty)$ such that*

$$\left\|\int_{s_1}^{s_2} p(s)\,\Delta s\right\| \le \int_{s_1}^{s_2} m(s)\,\Delta s$$

  whenever $[s_1,s_2]_{\mathbb{T}} \subset \mathbb{T}$.

If Eq. (6.5) has a bounded solution $x : [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}^n$, then it has a T-periodic solution on $\mathbb{T}$.

Proof. Let $g(t) = t^*$ for every $t \in \mathbb{R}$. Then g is nondecreasing and left-continuous (see [23, Lemma 4]). Since $\mathbb{T}$ is T-periodic, we have $g(t+T) - g(t) = T$ for every $t \in \mathbb{R}$. The functions p^* and q^* are T-periodic, because p, q are T-periodic and the time scale is T-periodic. The fact that they are locally integrable with respect to g follows from Theorem 6.2.

By Theorem 6.2 and Lemma 6.3, we get

$$\left\| \int_{s_1}^{s_2} p^*(s)\,\mathrm{d}g(s) \right\| = \left\| \int_{s_1^*}^{s_2^*} p(s)\,\Delta s \right\| \leq \int_{s_1^*}^{s_2^*} m(s)\,\Delta s = \int_{s_1}^{s_2} m^*(s)\,\mathrm{d}g(s).$$

whenever $[s_1, s_2] \subset \mathbb{R}$. Hence, the functions g, p^*, q^* satisfy the assumptions of Theorem 5.2.

The function $y : [t_0, \infty) \to \mathbb{R}^n$ given by $y = x^*$ is bounded, and (by Theorem 6.4) satisfies $y(t) = y(t_0) + \int_{t_0}^{t} (p^*(s)y(s) + q^*(s))\,\mathrm{d}g(s)$, $t \in [t_0, \infty)$. By Theorem 5.2, the last equation has a T-periodic solution on $\mathbb{R}$. By the second part of Theorem 6.4, this solution is the extension of a T–periodic solution of Eq. (6.5). □

Remark 6.7. In the existing literature, one can find several Massera-type theorems for dynamic equations on time scales; see [1, 13]. In [1], the results are restricted to the special case when $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, and do not cover periodic time scales. The paper [13] contains Massera's theorems for dynamic equations on periodic time scales, but the corresponding result for linear equations has stronger assumptions than our Theorem 6.6 (in [13, Theorem 2.1], the functions on the right-hand side are required to be rd-continuous), and there is no direct analogue of our Theorem 6.5 for nonlinear equations (in [13, Theorem 3.1], there is no information on the asymptotic behavior of bounded solutions). Let us note that the authors of [13] work with all time scales $\mathbb{T}$ such that $t \in \mathbb{T}$ implies $t + T \in \mathbb{T}$ and $\mu(t + T) = \mu(T)$. This definition is more general than our Definition 6.1, since it requires only periodicity in "forward time", and includes also time scales with $\inf \mathbb{T} > -\infty$. We have opted for Definition 6.1 only to ensure that if $g(t) = t^*$, then $t \mapsto g(t + T) - g(t)$ is constant on $\mathbb{R}$, as required by the results in Section 5. However, it is clear that Theorems 6.5 and 6.6 might be reformulated to be applicable to all time scales considered in [13], since each such time scale with $\inf \mathbb{T} = t_0 > -\infty$ can be enlarged by adding points smaller than t_0 in such a way that the new time scale is T-periodic according to Definition 6.1.

7 Massera's theorems for impulsive differential equations

In this section, our goal is to obtain Massera-type theorems for impulsive differential equations. Consider an impulsive ODE of the form

$$x'(t) = f(x(t), t), \tag{7.1}$$

$$\Delta^+ x(\tau_j) = I_j(x(\tau_j)), \quad j \in \mathbb{Z}, \tag{7.2}$$

with $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $I_j : \mathbb{R}^n \to \mathbb{R}^n$ for each $j \in \mathbb{Z}$, $\{\tau_j\}_{j \in \mathbb{Z}}$ being an increasing real sequence, and $\Delta^+ x(\tau_j) = x(\tau_j+) - x(\tau_j)$. It is common to assume that the equality in (7.1) holds almost everywhere, the solution x is left-continuous, and absolutely continuous on each interval $(\tau_{j-1}, \tau_j]$, $j \in \mathbb{Z}$. Then the corresponding integral form is

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s), s)\,\mathrm{d}s + \sum_{\substack{j \in \mathbb{Z},\\ t_0 \leq \tau_j < t}} I_j(x(\tau_j)), \tag{7.3}$$

where the integral on the right-hand side is the Lebesgue integral. In the following text, we consider the more general situation where the integral is understood in the sense of Kurzweil-Henstock. Since we are interested in the existence of periodic solutions, we will deal with periodic impulses described by the following condition:

(P) There exist $T > 0$ and $m \in \mathbb{N}$ such that $\tau_j = \tau_{j-m} + T$ and $I_j = I_{j-m}$ for each $j \in \mathbb{Z}$.

The next result, which follows from [6, Lemma 2.4] (see also [17, Lemma 5.2]), makes it possible to rewrite the right-hand side of Eq. (7.3) using a Stieltjes integral.

Theorem 7.1. *Let $a \le \eta_1 < \eta_2 < \cdots < \eta_k < b$. Consider a function $f : [a, b] \to \mathbb{R}^n$ and let $\tilde{f} : [a, b] \to \mathbb{R}^n$ be such that $\tilde{f}(s) = f(s)$ for every $s \in [a, b)\backslash\{\eta_1, \dots, \eta_k\}$. Let $g : [a, b] \to \mathbb{R}$ be a left-continuous function such that $\Delta^+ g(\eta_j) = 1$ for each $j \in \{1, \dots, k\}$, and $g(t) - g(u) = t - u$ whenever $[u, t] \cap \{\eta_1, \dots, \eta_k\} = \emptyset$. Then the Kurzweil-Stieltjes integral $\int_a^b \tilde{f}(s) \,\mathrm{d}g(s)$ exists if and only if the Kurzweil-Henstock integral $\int_a^b f(s) \,\mathrm{d}s$ exists. In this case, we have*

$$\int_a^b \tilde{f}(s) \,\mathrm{d}g(s) = \int_a^b f(s) \,\mathrm{d}s + \sum_{j=1}^{k} \tilde{f}(\eta_j).$$

We remark that each function g satisfying the assumptions of the previous theorem has the form $g(t) = g(a) + (t - a) + \sum_{j=1}^{k} \chi_{(\eta_j,\infty)}(t)$, $t \in [a, b]$, and is therefore unique up to an additive constant. The previous result implies the next theorem, which allows us to rewrite Eq. (7.3) as a measure differential equation. (A similar result may be found in [17, Theorem 5.3].)

Theorem 7.2. *Let $\{\tau_j\}_{j\in\mathbb{Z}}$ be an increasing real sequence with $\lim_{j\to\pm\infty} \tau_j = \pm\infty$. Consider functions $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $I_j : \mathbb{R}^n \to \mathbb{R}^n$ for each $j \in \mathbb{Z}$. A function $x : [a, b] \to \mathbb{R}^n$ is a solution of Eq. (7.3) on $[a, b]$ if and only if it is a solution of the measure differential equation*

$$x(t) = x(t_0) + \int_{t_0}^{t} \tilde{f}(x(s), s) \,\mathrm{d}g(s), \quad t \in [a, b],$$

where $\tilde{f} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is given by

$$\tilde{f}(z, t) = \begin{cases} f(z, t), & \text{if } t \in \mathbb{R} \setminus \{\tau_j : j \in \mathbb{Z}\}, \\ I_j(z), & \text{if } t = \tau_j \text{ for some } j \in \mathbb{Z}, \end{cases}$$

and $g : \mathbb{R} \to \mathbb{R}$ is given by $g(t) = t + j$ for each $t \in (\tau_j, \tau_{j+1}]$, $j \in \mathbb{Z}$.

Note that the function g described in Theorem 7.2 satisfies the conditions of Theorem 7.1. Moreover, the condition $\lim_{j\to\pm\infty} \tau_j = \pm\infty$ ensures that each bounded interval contains at most finitely many points of impulses, and therefore the sum in Eq. (7.3) is always finite.

Although Theorem 7.2 was formulated for solutions defined on compact intervals, it is clear that a similar result holds also for unbounded intervals (note that the definitions of $\tilde{f}$ and g do not depend on the choice of the interval).

We are now ready for Massera's theorem for nonlinear scalar impulsive differential equations.

Theorem 7.3 (Massera's Theorem for Nonlinear Impulsive Equations)**.** *Consider an increasing real sequence* $\{\tau_j\}_{j\in\mathbb{Z}}$ *and functions* $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *and* $I_j : \mathbb{R} \to \mathbb{R}$*,* $j \in \mathbb{Z}$*, such that condition* (P) *holds. Moreover, suppose that:*

- f *is continuous in the first variable and* T*-periodic in the second variable.*
- *The integral* $\int_{s_1}^{s_2} f(x, s)\,\mathrm{d}s$ *exists for each interval* $[s_1, s_2] \subset \mathbb{R}$ *and each* $x \in \mathbb{R}$*.*
- *For every bounded set* $O \subset \mathbb{R}$*, there exist a locally integrable function* $m : \mathbb{R} \to [0,\infty)$ *and a continuous increasing function* $\omega : [0,\infty) \to \mathbb{R}$ *such that* $\omega(0) = 0$*,* $\lim_{u\to 0+} \int_u^v \frac{\mathrm{d}r}{\omega(r)} = \infty$ *for a certain* $v > 0$*, and*
$$\left|\int_{s_1}^{s_2} f(x, s)\,\mathrm{d}s\right| \le \int_{s_1}^{s_2} m(s)\,\mathrm{d}s,$$
$$\left|\int_{s_1}^{s_2} (f(x, s) - f(y, s))\,\mathrm{d}s\right| \le \omega(|x - y|) \int_{s_1}^{s_2} m(s)\,\mathrm{d}s$$
for each interval $[s_1, s_2] \subset \mathbb{R}$ *and all* $x, y \in O$*. Moreover, for each* $j \in \{1, \dots, m\}$*, there exists a constant* $m_j \ge 0$ *such that*
$$|I_j(x)| \le m_j \quad \text{and} \quad |I_j(x) - I_j(y)| \le \omega(|x - y|)m_j$$
for all $x, y \in O$*.*
- *If* $u, v \in \mathbb{R}$ *with* $u < v$*, then* $u + I_j(u) \le v + I_j(v)$ *for each* $j \in \{1, \dots, m\}$*.*

Then each bounded solution $x : [t_0,\infty) \to \mathbb{R}$ *of Eq.* (7.3) *is asymptotic to a* T*-periodic solution. In particular, the existence of a bounded solution implies the existence of a* T*-periodic solution.*

Proof. Define $\tilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ as in Theorem 7.2. Then Eq. (7.3) is equivalent to the measure differential equation
$$x(t) = x(t_0) + \int_{t_0}^{t} \tilde{f}(x(s), s)\,\mathrm{d}g(s),$$
and it suffices to show that $\tilde{f}$, g satisfy the assumptions of Theorem 5.1.

It is clear g that is nondecreasing, left-continuous, and condition (P) ensures that $t \mapsto g(t + T) - g(t)$ is a constant function on $\mathbb{R}$. If $u < v$ and $t \in \mathbb{R} \setminus \{\tau_j : j \in \mathbb{Z}\}$, then
$$u + \tilde{f}(u, t)\Delta^+ g(t) = u < v = v + \tilde{f}(v, t)\Delta^+ g(t).$$
Otherwise, if $t = \tau_j$ for some $j \in \{1, \dots, m\}$, then
$$u + \tilde{f}(u, t)\Delta^+ g(t) = u + I_j(u) \le v + I_j(v) = v + \tilde{f}(v, t)\Delta^+ g(t).$$
By condition (P), it is clear that the same result holds for each $j \in \mathbb{Z}$.

From the definition of $\tilde{f}$, we see that it is continuous in the first variable and T-periodic in the second variable. By hypothesis, the integral $\int_{s_1}^{s_2} f(x, s)\,\mathrm{d}s$ exists for each interval $[s_1, s_2] \subset \mathbb{R}$ and each $x \in \mathbb{R}$. Applying Theorem 7.1, we see that the integral $\int_{s_1}^{s_2} \tilde{f}(x, s)\,\mathrm{d}g(s)$ also exists.

Given a bounded set $O \subset \mathbb{R}$, consider the functions $m : \mathbb{R} \to [0,\infty)$, $\omega : [0,\infty) \to \mathbb{R}$, and constants $m_1, \dots, m_m$, whose existence is guaranteed by the assumptions. We extend $m_1, \dots, m_m$ into a m-periodic sequence $\{m_j\}_{j\in\mathbb{Z}}$, and define $\tilde{m} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{m}(t) = \begin{cases} m(t), & \text{if } t \in \mathbb{R} \setminus \{\tau_j : j \in \mathbb{Z}\}, \\ m_j, & \text{if } t = \tau_j \text{ for some } j \in \mathbb{Z}. \end{cases} \tag{7.4}$$

By Theorem 7.1, for each interval $[s_1, s_2] \subset \mathbb{R}$ and all $x, y \in O$, we get

$$\begin{aligned} \left| \int_{s_1}^{s_2} \tilde{f}(x,s)\,\mathrm{d}g(s) \right| &= \left| \int_{s_1}^{s_2} f(x,s)\,\mathrm{d}s + \sum_{\substack{j\in\mathbb{Z},\\ s_1\le\tau_j<s_2}} I_j(x) \right| \\ &\le \int_{s_1}^{s_2} m(s)\,\mathrm{d}s + \sum_{\substack{j\in\mathbb{Z},\\ s_1\le\tau_j<s_2}} m_j = \int_{s_1}^{s_2} \tilde{m}(s)\,\mathrm{d}g(s), \end{aligned}$$

and similarly

$$\begin{aligned} \left| \int_{s_1}^{s_2} (\tilde{f}(x,s) - \tilde{f}(y,s))\,\mathrm{d}g(s) \right| &= \left| \int_{s_1}^{s_2} (f(x,s) - f(y,s))\,\mathrm{d}s + \sum_{\substack{j\in\mathbb{Z},\\ s_1\le\tau_j<s_2}} (I_j(x) - I_j(y)) \right| \\ &\le \omega(|x-y|) \int_{s_1}^{s_2} m(s)\,\mathrm{d}s + \sum_{\substack{j\in\mathbb{Z},\\ s_1\le\tau_j<s_2}} \omega(|x-y|)m_j = \omega(|x-y|) \int_{s_1}^{s_2} \tilde{m}(s)\,\mathrm{d}g(s). \end{aligned}$$

Thus, all assumptions of Theorem 5.1 are satisfied, and the proof is complete. □

Next, we turn our attention to linear impulsive differential equations in the integral form

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s) + q(s))\,\mathrm{d}s + \sum_{\substack{j\in\mathbb{Z},\\ t_0\le\tau_j<t}} (A_j x(\tau_j) + b_j), \tag{7.5}$$

with $p : \mathbb{R} \to L(\mathbb{R}^n)$, $q : \mathbb{R} \to \mathbb{R}^n$, $A_j \in L(\mathbb{R}^n)$, and $b_j \in \mathbb{R}^n$ for every $j \in \mathbb{Z}$. Eq. (7.5) represents a special case of Eq. (7.3) with $f(x,t) = p(t)x + q(t)$ and $I_j(x) = A_j x + b_j$ for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $j \in \mathbb{Z}$. In this case, we can reformulate condition (P) as follows:

(P') There exist $T > 0$ and $m \in \mathbb{N}$ such that $\tau_j = \tau_{j-m} + T$, $A_j = A_{j-m}$ and $b_j = b_{j-m}$ for each $j \in \mathbb{Z}$.

Theorem 7.4 (Massera's Theorem for Linear Impulsive Equations)**.** *Consider an increasing real sequence $\{\tau_j\}_{j\in\mathbb{Z}}$ and $A_j \in L(\mathbb{R}^n)$, $b_j \in \mathbb{R}^n$, $j \in \mathbb{Z}$, such that condition* (P') *holds. Moreover, consider functions $p : \mathbb{R} \to L(\mathbb{R}^n)$ and $q : \mathbb{R} \to \mathbb{R}^n$ satisfying the following conditions:*

- *p, q are T-periodic and locally integrable.*

- *There exists a locally integrable function $m : \mathbb{R} \to [0, \infty)$ such that*
$$\left\| \int_{s_1}^{s_2} p(s)\,\mathrm{d}s \right\| \le \int_{s_1}^{s_2} m(s)\,\mathrm{d}s$$
for each interval $[s_1, s_2] \subset \mathbb{R}$.

If Eq. (7.5) has a bounded solution on $[t_0, \infty)$, then it has a T-periodic solution.

Proof. Define $g : \mathbb{R} \to \mathbb{R}$ as in Theorem 7.2. Then Eq. (7.5) is equivalent to the linear measure differential equation
$$x(t) = x(t_0) + \int_{t_0}^{t} (\tilde{p}(s)x(s) + \tilde{q}(s))\,\mathrm{d}g(s),$$
where the functions $\tilde{p} : \mathbb{R} \to L(\mathbb{R}^n)$ and $\tilde{q} : \mathbb{R} \to \mathbb{R}^n$ are given by
$$\tilde{p}(t) = \begin{cases} p(t), & \text{if } t \in \mathbb{R} \setminus \{\tau_j : j \in \mathbb{Z}\}, \\ A_j, & \text{if } t = \tau_j \text{ for some } j \in \mathbb{Z}, \end{cases} \qquad \tilde{q}(t) = \begin{cases} q(t), & \text{if } t \in \mathbb{R} \setminus \{\tau_j : j \in \mathbb{Z}\}, \\ b_j, & \text{if } t = \tau_j \text{ for some } j \in \mathbb{Z}. \end{cases}$$
To finish the proof, it suffices to show that $\tilde{p}$, $\tilde{q}$, g satisfy the assumptions of Theorem 5.2. It is clear that g is nondecreasing and left-continuous. Condition (P') ensures that $t \mapsto g(t+T) - g(t)$ is a constant function on $\mathbb{R}$, and that $\tilde{p}$ and $\tilde{q}$ are T-periodic; the fact that they are locally integrable with respect to g follows from Theorem 7.1. Define $\tilde{m} : \mathbb{R} \to \mathbb{R}$ by
$$\tilde{m}(t) = \begin{cases} m(t), & \text{if } t \in \mathbb{R} \setminus \{\tau_j : j \in \mathbb{Z}\}, \\ \|A_j\|, & \text{if } t = \tau_j \text{ for some } j \in \mathbb{Z}. \end{cases}$$
By Theorem 7.1, for each interval $[s_1, s_2] \subset \mathbb{R}$, we get
$$\left\| \int_{s_1}^{s_2} \tilde{p}(s)\,\mathrm{d}g(s) \right\| = \left\| \int_{s_1}^{s_2} p(s)\,\mathrm{d}s + \sum_{\substack{j \in \mathbb{Z}, \\ s_1 \le \tau_j < s_2}} A_j \right\| \le \int_{s_1}^{s_2} m(s)\,\mathrm{d}s + \sum_{\substack{j \in \mathbb{Z}, \\ s_1 \le \tau_j < s_2}} \|A_j\| = \int_{s_1}^{s_2} \tilde{m}(s)\,\mathrm{d}g(s).$$
Hence, all assumptions of Theorem 5.2 are satisfied, and the proof is complete. □

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