Lucky Cars: Expected Values and Generating Functions

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Abstract. We count lucky cars in parking functions of a given length, and provide elementary derivations for the generating functions, expected values and variances. We consider not only classical parking functions, but also problems with additional parking slots, and with cars of different sizes.

1. PARKING FUNCTIONS AND LUCKY CARS. The classical concept of a parking function arises from the following problem: Consider n parking slots in a one-way street. There are n cars that arrive sequentially and their preferred parking slots are $\pi_1, \ldots, \pi_n \in \{1, \ldots, n\}$. Car i will park in the first empty slot (if any) whose number is greater than or equal to π_i. If all cars are able to park, we say that $\pi = (\pi_1, \ldots, \pi_n)$ is a parking function of length n.

We are interested in counting lucky cars, where car i is called lucky if it succeeds in parking in its preferred spot π_i. Let $a_l(n)$ be the number of parking functions of length n with exactly l lucky cars. Note that $a_0(n) = 0$ for each $n \in \mathbb{N}$, because the first car is always lucky, and $a_n(n) = n!$, because all cars are lucky if and only if they have distinct preferences.

For a fixed $n \in \mathbb{N}$, the numbers $a_0(n), \ldots, a_n(n)$ can be calculated from the generating function

$$\sum_{l=0}^{n} a_l(n) z^l = z \prod_{i=1}^{n-1} (i + (n-i+1)z), \tag{1}$$

which was obtained by I. M. Gessel and S. Seo in [**3**, Section 10]. For example, if $n = 3$, the polynomial on the right-hand side is $6z^3 + 8z^2 + 2z$. Indeed, there are

- 6 parking functions with 3 lucky cars: $(1, 2, 3)$ and its permutations,
- 8 parking functions with 2 lucky cars: $(1, 1, 3)$, $(1, 2, 1)$, $(1, 2, 2)$, $(1, 3, 1)$, $(2, 1, 1)$, $(2, 1, 2)$, $(2, 2, 1)$, $(3, 1, 1)$,
- 2 parking functions with 1 lucky car: $(1, 1, 1)$, $(1, 1, 2)$.

We can also consider a more general problem with n cars, but $n + c - 1$ parking slots instead of n. The cars now have preferences $\pi_1, \ldots, \pi_n \in \{1, \ldots, n + c - 1\}$, and the parking process is the same as before. If all cars are able to park, we say that $\pi = (\pi_1, \ldots, \pi_n)$ is a c-parking function of length n. (Hence, a 1-parking function is the classical parking function introduced earlier.)

If $a_l(n, c)$ is the number of c-parking functions of length n, we have

$$\sum_{l=0}^{n} a_l(n, c) z^l = cz \prod_{i=1}^{n-1} (i + (n-i+c)z). \tag{2}$$

Gessel and Seo also proved this result in [**3**, Section 10].

A detailed survey of parking functions by C. H. Yan is available in [**6**]. Perhaps the best known result says that the number of all parking functions of length n is

$(n+1)^{n-1}$. An elegant proof based on adding an extra parking slot and arranging the resulting $n+1$ slots clockwise in a circle is due to H. O. Pollak; it was first described by J. Riordan in [**4**], see also [**6**, p. 836]. A straightforward modification of this proof shows that the number of c-parking functions of length n is $c(n+c)^{n-1}$. These formulas can be also obtained by letting $z = 1$ in (1) and (2), respectively.

Starting with (1) and dividing by $(n+1)^{n-1}$, P. Diaconis and A. Hicks have discovered in [**1**, p. 139] that the expected number of lucky cars and the variance of this number for a random parking function of length n are

$$\mu_n = n - \frac{n(n-1)}{2(n+1)} = \frac{n}{2}\left(1 + \frac{2}{n+1}\right), \quad \sigma_n^2 = \sum_{i=1}^{n-1}\left(1 - \frac{i}{n+1}\right)\frac{i}{n+1}. \quad (3)$$

Thus, for a random parking function, approximately half of the cars are lucky.

Unfortunately, the proof of (1) in [**3**] is not entirely elementary. It begins by showing that the double generating function $Q(x,z) = \sum_{n=0}^{\infty}\sum_{l=0}^{n} a_l(n) z^l \frac{x^n}{n!}$ satisfies the differential equation $\frac{\partial Q}{\partial x} = zQ^2 + xQ\frac{\partial Q}{\partial x}$, and then verifying that the exponential generating function of the right-hand side of (1) satisfies this equation.

Our aim is to present a completely elementary derivation of the relations (3) and of the more general formulas for lucky cars in a random c-parking function of length n, namely

$$\mu_{n,c} = \frac{n}{2}\left(1 + \frac{c+1}{n+c}\right), \quad \sigma_{n,c}^2 = \sum_{i=1}^{n-1}\left(1 - \frac{i}{n+c}\right)\frac{i}{n+c}. \quad (4)$$

At the same time, we will obtain an elementary proof of the identities (1) and (2).

Finally, we will show that essentially the same method is applicable to the problem with cars of different sizes introduced by R. Ehrenborg and A. Happ in [**2**]. The corresponding generating function as well as the formula for the expected number of lucky cars that we derive in the last section are new.

2. AN ELEMENTARY APPROACH TO LUCKY CARS. Instead of dealing with parking functions for n cars and $n+c-1$ parking slots in a row, we will follow Pollak's idea and consider n cars and $n+c$ parking slots arranged clockwise in a circle. As before, each car has a preferred slot. If it is occupied, the car cruises clockwise until it finds an empty slot. We make the following observations:

- For each choice of parking preferences $\pi_1, \dots, \pi_n \in \{1, \dots, n+c\}$, all n cars are able to park, and c slots remain empty.
- Let F be the mapping that takes a vector of preferences $\pi = (\pi_1, \dots, \pi_n)$ and yields a vector $F(\pi)$ whose components are the components of π increased by 1 modulo $n+c$ (i.e., we identify 0 with $n+c$). In other words, all preferences in π are shifted clockwise by one position. Consequently, the final parking positions of cars with preferences $F(\pi)$ are obtained by a clockwise shift of the positions of cars with preferences π.
- For each π, consider the sequence of vectors

$$\pi, F(\pi), \dots, F^{n+c-1}(\pi). \quad (5)$$

  Exactly c of them (those where the additional slot $n+c$ remains empty) correspond to c-parking functions in the original problem with $n+c-1$ slots in a row.

- The shift F does not influence whether a car is lucky or not. Thus, all the vectors in (5) yield the same number of lucky cars.

These observations imply that the probability of a car i being lucky in the circular problem is the same as in the original problem with slots in a row. However, the calculation for the circular problem is much simpler: Suppose that the parking preferences $\pi_1, \ldots, \pi_n \in \{1, \ldots, n+c\}$ are chosen independently and uniformly at random. For each $i \in \{1, \ldots, n\}$, let

$$X_i = \begin{cases} 1 & \text{if car } i \text{ is lucky,} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Thus, $X_1, \ldots, X_n$ are random variables depending on the choice of parking preferences. When car i arrives, $i-1$ slots in the circle are already occupied, and the remaining ones are empty. Hence,

$$P(X_i = 0) = \frac{i-1}{n+c}, \quad EX_i = P(X_i = 1) = 1 - \frac{i-1}{n+c}. \tag{7}$$

By linearity of the expected value, the expected number of lucky cars is

$$\begin{aligned} \mu_{n,c} = E(X_1 + \cdots + X_n) &= \sum_{i=1}^{n} EX_i = \sum_{i=1}^{n} \left(1 - \frac{i-1}{n+c}\right) \\ &= n - \frac{1}{n+c} \frac{(n-1)n}{2} = \frac{n(n+2c+1)}{2(n+c)}, \end{aligned}$$

which proves the first identity in (4).

What happens if the number of parking slots is k times greater than the number of cars, i.e., if $c = (k-1)n + 1$? For a large number of cars, the proportion of lucky cars will be approximately

$$\lim_{n\to\infty} \frac{\mu_{n,(k-1)n+1}}{n} = \lim_{n\to\infty} \frac{(2k-1)n+3}{2(kn+1)} = \frac{2k-1}{2k}.$$

The random variables $X_1, \ldots, X_n$ also lead to a simple proof of (2). The probability generating function for each random variable X_i is

$$G_i(z) = P(X_i = 0) + P(X_i = 1)z = \frac{i-1}{n+c} + \left(1 - \frac{i-1}{n+c}\right) z.$$

Since summation of independent random variables corresponds to multiplication of their probability generating functions, we see that

$$G(z) = G_1(z) \cdots G_n(z) = \prod_{i=1}^{n} \left(\frac{i-1}{n+c} + \left(1 - \frac{i-1}{n+c}\right) z \right)$$

is the probability generating function for the random variable $X_1 + \cdots + X_n$, i.e., for the number of lucky cars. Denote by $p_l(n,c)$ the probability that for a random c-parking function of length n, exactly l cars out of n are lucky. Then

$$G(z) = \sum_{l=0}^{n} p_l(n,c) z^l = \prod_{i=1}^{n} \left(\frac{i-1}{n+c} + \left(1 - \frac{i-1}{n+c}\right) z \right)$$

$$= z \prod_{i=1}^{n-1} \left(\frac{i}{n+c} + \left(1 - \frac{i}{n+c}\right) z \right). \tag{8}$$

Recall that the total number of c-parking functions is $c(n+c)^{n-1}$. Multiplying the previous equality by this constant, we get

$$\sum_{l=0}^{n} a_l(n,c) z^l = cz \prod_{i=1}^{n-1} (i + (n-i+c)z), \tag{9}$$

which is formula (2). For $c = 1$, we recover the special case (1).

The derivation of (3) in [**1**, p. 139] contains essentially the same discrete random variables $X_1, \ldots, X_n$ (with $c = 1$) as in (7). However, [**1**] does not provide the combinatorial meaning of these variables (although the authors might have been aware of it), and they are introduced only formally as the random variables whose probability generating functions correspond to the factors in (8). Our approach is different: We first introduce $X_1, \ldots, X_n$ by means of (6), and subsequently use them to deduce (8) and (9).

The generating function (9) is useful for calculating the values $a_k(n,c)$, but it can reveal more information. For example, note that the polynomial

$$z \mapsto c \prod_{i=1}^{n-1} (i + (n-i+c)z)$$

has only real negative roots. According to [**5**, Theorem 4.27], the sequence of its coefficients, $\{a_{l+1}(n,c)\}_{l=0}^{n-1}$, is logarithmically concave. By [**5**, Proposition 4.26], the sequence $\{a_l(n,c)\}_{l=0}^{n}$ is unimodal, i.e., it consists of a nondecreasing part followed by a nonincreasing part. Clearly, $\{p_l(n,c)\}_{l=0}^{n}$ has the same property.

As a different application, let us calculate the variance for the number of lucky cars. For a random variable whose probability generating function is G, the variance can be calculated as $(\log G)'(1) + (\log G)''(1)$; see [**5**, Section 4.1]. For the generating function from (8), we get

$$(\log G)'(z) = \frac{1}{z} + \sum_{i=1}^{n-1} \frac{1 - \frac{i}{n+c}}{\frac{i}{n+c} + \left(1 - \frac{i}{n+c}\right) z},$$

$$(\log G)''(z) = -\frac{1}{z^2} - \sum_{i=1}^{n-1} \frac{\left(1 - \frac{i}{n+c}\right)^2}{\left(\frac{i}{n+c} + \left(1 - \frac{i}{n+c}\right) z\right)^2},$$

and therefore the variance for the number of lucky cars in c-parking functions of length n is

$$\begin{aligned} \sigma_{n,c}^2 &= \sum_{i=1}^{n-1} \left(1 - \frac{i}{n+c}\right) - \sum_{i=1}^{n-1} \left(1 - \frac{i}{n+c}\right)^2 \\ &= \sum_{i=1}^{n-1} \left(1 - \frac{i}{n+c}\right) \frac{i}{n+c} = \frac{n(n-1)(n+1+3c)}{6(n+c)^2}, \end{aligned}$$

which proves the second identity in (4).

3. CARS OF DIFFERENT SIZES. One possible generalization of the original parking problem is to consider cars of different sizes $y_1, \dots, y_n \in \mathbb{N}$. The number of parking slots is now $k = \sum_{i=1}^n y_i$, and the cars have preferences $\pi_1, \dots, \pi_n \in \{1, \dots, k\}$. According to the scenario described in [**2**], car i searches for the first available slot whose number is greater than or equal to π_i; if the following $y_i - 1$ slots are empty as well, the car parks successfully; otherwise, the parking process fails. If all cars are able to park, then $\pi = (\pi_1, \dots, \pi_n)$ is called a parking function[1] for $(y_1, \dots, y_n)$. In this setting, car i is lucky if it parks in a block of consecutive slots starting at π_i.

We want to calculate the expected number of lucky cars, and find the generating function for the numbers $a_l(y_1, \dots, y_n)$ that count parking functions for $(y_1, \dots, y_n)$ with exactly l lucky cars.

Consider instead $k + 1$ parking slots arranged clockwise in a circle, n cars of sizes $y_1, \dots, y_n \in \mathbb{N}$, and preferences $\pi_1, \dots, \pi_n \in \{1, \dots, k+1\}$. If all cars are able to park (this is no longer guaranteed!), exactly one slot remains empty, and we say that $\pi = (\pi_1, \dots, \pi_n)$ is a circular parking function for $(y_1, \dots, y_n)$. As in Section 2, the probability of car i being lucky does not depend on whether we choose a random parking function, or a random circular parking function. (Use again the shift mapping F, which preserves lucky cars, and observe that if π is a circular parking function, then exactly one of π, $F(\pi)$, ..., $F^k(\pi)$ is a parking function.)

To calculate the expected number of lucky cars for a random circular parking function, we rely on the following observation from [**2**, Section 2]: After all cars have parked, the circle is divided into $n + 1$ blocks corresponding to the n cars and one empty slot. The position of the first car and the order of the blocks completely determine where each car parks.

Car 1 is always lucky. In general, car i will end in one of the $n + 2 - i$ remaining available blocks. It can happen that the beginning of the block coincides with the car's preference, making the car lucky. Alternatively, the car's preference might coincide with one of the $y_1 + \cdots + y_{i-1}$ already occupied slots, and it will need to cruise until it arrives in its destination. To sum up, we have $y_1 + \cdots + y_{i-1} + n + 2 - i$ choices for car i, with $n + 2 - i$ of them making it lucky. Therefore, for a random circular parking function for $(y_1, \dots, y_n)$, the expected number of lucky cars is

$$\sum_{i=1}^{n} \frac{n + 2 - i}{y_1 + \cdots + y_{i-1} + n + 2 - i}. \tag{10}$$

As in Section 2, we can now obtain the generating function for the sequence $\{p_l(y_1, \dots, y_n)\}_{l=0}^n$, whose terms are the probabilities that, in a random parking function for $(y_1, \dots, y_n)$, exactly l cars are lucky:

$$\begin{aligned}\sum_{l=0}^{n} p_l(y_1, \dots, y_n) z^l &= \prod_{i=1}^{n} \frac{y_1 + \cdots + y_{i-1} + (n + 2 - i)z}{y_1 + \cdots + y_{i-1} + n + 2 - i} \\ &= z \prod_{i=1}^{n-1} \frac{y_1 + \cdots + y_i + (n + 1 - i)z}{y_1 + \cdots + y_i + n + 1 - i}.\end{aligned}$$

[1]The authors of [**2**] prefer the term "parking sequence," which might be more appropriate, but we stick to "parking function" for the sake of consistency.

Since the total number of parking functions for $(y_1, \dots, y_n)$ is

$$\prod_{i=1}^{n-1} (y_1 + \cdots + y_i + n + 1 - i)$$

(see [**2**, Theorem 3]), the generating function for the numbers $a_l(y_1, \dots, y_n)$ is

$$\sum_{l=0}^{n} a_l(y_1, \dots, y_n) z^l = z \prod_{i=1}^{n-1} (y_1 + \cdots + y_i + (n+1-i)z). \tag{11}$$

If $y_1 = \cdots = y_n = 1$, the previous formula reduces to (1).

As a quick check, consider the case $n = 3$ and $(y_1, y_2, y_3) = (2, 2, 1)$. The polynomial on the right-hand side of (11) is $6z^3 + 16z^2 + 8z$. Indeed, there are

- 6 parking functions with 3 lucky cars:
  $(1, 3, 5)$, $(1, 4, 3)$, $(2, 4, 1)$, $(3, 1, 5)$, $(4, 1, 3)$, $(4, 2, 1)$,
- 16 parking functions with 2 lucky cars:
  $(1, 1, 5)$, $(1, 2, 5)$, $(1, 3, 1)$, $(1, 3, 2)$, $(1, 3, 3)$, $(1, 3, 4)$, $(1, 4, 1)$, $(1, 4, 2)$,
  $(2, 2, 1)$, $(2, 3, 1)$, $(3, 1, 1)$, $(3, 1, 2)$, $(3, 1, 3)$, $(3, 1, 4)$, $(4, 1, 1)$, $(4, 1, 2)$,
- 8 parking functions with 1 lucky car:
  $(1, 1, 1)$, $(1, 1, 2)$, $(1, 1, 3)$, $(1, 1, 4)$, $(1, 2, 1)$, $(1, 2, 2)$, $(1, 2, 3)$, $(1, 2, 4)$.

The expected number of lucky cars is $(6 \cdot 3 + 16 \cdot 2 + 8 \cdot 1)/30 = 29/15$, as predicted by (10).

Observe again that the product on the right-hand side of (11) has only negative real roots, and therefore $\{a_k(y_1, \dots, y_n)\}_{l=0}^{n}$ as well as $\{p_k(y_1, \dots, y_n)\}_{l=0}^{n}$ are unimodal sequences. The calculation of the variance for the number of lucky cars is left as an exercise to the reader.

REFERENCES

1. Diaconis, P., Hicks, A. (2017). Probabilizing parking functions. *Adv. Appl. Math.* 89: 125–155.
2. Ehrenborg, R., Happ, A. (2016). Parking cars of different sizes. *Amer. Math. Monthly*. 123(10): 1045–1048.
3. Gessel, I. M., Seo, S. (2006). A refinement of Cayley's formula for trees. *Electron. J. Comb.* 11(2), paper no. R27.
4. Riordan, J. (1969). Ballots and trees. *J. Comb. Theory*. 6: 408–411.
5. Wilf, H. S. (2006). *generatingfunctionology*, 3rd edition. Wellesley, MA: A K Peters.
6. Yan, C. H. (2015). Parking functions. In: Bóna, M., ed.: *Handbook of Enumerative Combinatorics*. CRC Press, pp. 835–893.

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