Well-posedness results for abstract generalized differential equations and measure functional differential equations

Antonín Slavík
Charles University, Faculty of Mathematics and Physics,
Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail: slavik@karlin.mff.cuni.cz

Abstract

In the first part of the paper, we consider nonlinear generalized ordinary differential equations whose solutions take values in infinite-dimensional Banach spaces, and prove new theorems concerning the existence of solutions and continuous dependence on initial values and parameters. In the second part, we apply these results in the study of nonlinear measure functional differential equations and impulsive functional differential equations with infinite delay.

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1 Introduction

The concept of a generalized ordinary differential equation was originally introduced by J. Kurzweil in [20] as a tool in the study of continuous dependence of solutions to ordinary differential equations of the usual form $x'(t) = f(x(t), t)$. He observed that instead of dealing directly with the right-hand side f, it might be advantageous to work with the function $F(x,t) = \int_{t_0}^{t} f(x,s)\,\mathrm{d}s$, i.e., the primitive to f. In this connection, he also introduced an integral whose special case is the Kurzweil-Henstock integral. Gauge-type integrals are well known to specialists in integration theory, but they are also becoming more popular in the field of differential equations (see e.g. [5]).

Over the years, it became clear that the theory of generalized differential equations is not only useful in the study of classical nonautonomous differential equations (see e.g. [2]), but also represents a suitable tool for the investigation of equations with discontinuous solutions. (For other approaches to equations with discontinuous solutions or right-hand sides, such as measure differential equations, equations in Filippov's or Krasovskii's sense, distributional differential equations, or impulsive differential equations, see e.g. [6, 7, 14, 18, 19, 25, 22, 31].) In particular, generalized differential equations encompass other types of equations, such as equations with impulses, dynamic equations on time scales, functional differential equations with impulses, or measure functional differential equations (see e.g. [10, 11, 13, 27, 32, 37] and the references there). To deal with functional differential equations, it is necessary to consider generalized equations whose solutions take values in infinite-dimensional Banach spaces; this fact provides a motivation to the study of abstract generalized differential equations.

Unfortunately, the existing theory for abstract generalized equations is not as powerful as in the finite-dimensional case. The only exception is the class of linear equations, where the results are quite satisfactory (see [17, 27, 28]). Our goal is to rectify this situation and obtain new results concerning well-posedness of solutions to abstract nonlinear generalized differential equations under reasonably weak assumptions on the right-hand sides.

In the theory of classical ordinary differential equations, it is well known that Picard's theorem on the local existence and uniqueness of solutions of the problem

$$x'(t) = f(x(t), t), \quad t \in [a, b], \quad x(a) = x_0, \tag{1.1}$$

can be improved in the following way: Instead of working with a locally Lipschitz continuous right-hand side, it is enough to assume that f is a continuous function satisfying

$$\|f(x, t) - f(y, t)\| \le \omega(\|x - y\|),$$

where $\omega : [0, \infty) \to [0, \infty)$ is a continuous increasing function such that $\omega(0) = 0$ and

$$\lim_{v \to 0+} \int_v^u \frac{\mathrm{d}r}{\omega(r)} = \infty \tag{1.2}$$

for every $u > 0$. This existence-uniqueness result is known as Osgood's theorem (in [29], W. F. Osgood identified (1.2) to be a sufficient condition for uniqueness). The situation in finite-dimensional spaces is rather simple, since the existence of a local solution follows from Peano's theorem, and condition (1.2) together with Bihari's inequality guarantee uniqueness.

Remarkably, Peano's theorem is no longer valid in infinite-dimensional Banach spaces, but Osgood's theorem remains true (see e.g. [8, Theorem 3.2] and [35]). In this case, condition (1.2) is necessary to prove both existence and uniqueness. The proof uses the fact that a continuous right-hand side can be uniformly approximated by locally Lipschitz continuous right-hand sides; the corresponding initial-value problems have unique solutions, which are uniformly convergent to a solution of the original equation.

Things become more complicated when we switch to abstract generalized differential equations, whose right-hand sides are usually assumed to be continuous in x, but may be discontinuous in t. A local existence-uniqueness theorem, whose proof is based on the contraction mapping theorem, can be found in [13]; this is the analogue of Picard's theorem for generalized equations. In Section 3, we prove a more general Osgood-type existence and uniqueness theorem; again, the idea is to approximate the right-hand side by functions such that the theorem from [13] is applicable.

In Section 4, we discuss continuous dependence of solutions to generalized differential equations with respect to initial values and parameters. We obtain two new theorems, which generalize several results available in the literature. We also provide an example showing why we cannot expect the infinite-dimensional theorems to hold under exactly the same assumptions as in the finite-dimensional case.

In Section 5, we use the previous theory to study the well-posedness for nonlinear measure functional differential equations of the form

$$y(t) = y(t_0) + \int_{t_0}^t f(y_s, s)\, \mathrm{d}g(s).$$

These equations were introduced only relatively recently in [10], and were subsequently studied in [3, 11, 12, 26, 27, 37]. Fairly general results concerning the well-posedness for linear equations with finite delay have been obtained in [27], while the nonlinear case with finite delay was considered in [10]. Here we show that the theory from Sections 3 and 4 leads to new well-posedness theorems for nonlinear equations with infinite delay, which significantly improve the results from [10].

Functional differential equations with impulses, which represent an important special case of measure functional differential equations, are briefly discussed in Section 6.

2 Preliminaries

The theory of generalized ordinary differential equations is based on the concept of Kurzweil integral. We recall the definition here, and refer the reader to [21, 24, 32, 34] for more information about the properties of this integral and its special cases, the Kurzweil-Stieltjes and Kurzweil-Henstock integral.

Given a function $\delta : [a, b] \to \mathbb{R}^+$, a tagged partition of the interval $[a, b]$ with division points $a = s_0 \le s_1 \le \cdots \le s_k = b$ and tags $\tau_i \in [s_{i-1}, s_i]$, $i \in \{1, \dots, k\}$, is called δ-fine if

$$[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)), \ \ i \in \{1, \dots, k\}.$$

Let X be a Banach space. A function $U : [a, b] \times [a, b] \to X$ is called Kurzweil integrable on $[a, b]$, if there is an element $I \in X$ such that for every $\varepsilon > 0$, there is a function $\delta : [a, b] \to \mathbb{R}^+$ such that

$$\left\| \sum_{i=1}^{k} (U(\tau_i, s_i) - U(\tau_i, s_{i-1})) - I \right\| < \varepsilon$$

for every δ-fine tagged partition of $[a, b]$. In this case, we define $\int_a^b DU(\tau, t) = I$.

An important special case is the Kurzweil-Stieltjes integral (also known as the Perron-Stieltjes integral) of a function $f : [a, b] \to X$ with respect to a function $g : [a, b] \to \mathbb{R}$, which corresponds to the choice $U(\tau, t) = f(\tau)g(t)$. This integral will be denoted by $\int_a^b f(t)\,\mathrm{d}g(t)$. There is a useful simple criterion for the existence of the Kurzweil-Stieltjes integral: If $f : [a, b] \to X$ is regulated and $g : [a, b] \to \mathbb{R}$ has bounded variation, then $\int_a^b f(t)\,\mathrm{d}g(t)$ exists (see [33, Proposition 15]). We will often use this criterion in situations where g is a nondecreasing function.

We can now proceed to the definition of a generalized ordinary differential equation. Consider a set $B \subset X$, an interval $I \subset \mathbb{R}$, and a function $F : B \times I \to X$. A generalized ordinary differential equation with the right-hand side F has the form

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t), \quad t \in I, \tag{2.1}$$

which is a shorthand notation for the integral equation

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau), t), \quad [s_1, s_2] \subset I. \tag{2.2}$$

In other words, a function $x : I \to B$ is a solution of (2.1) if and only if (2.2) is satisfied. The reader should keep in mind that (2.1) is a symbolic notation only and does not necessarily mean that x is differentiable.

The next definition introduces classes of functions which occur frequently in the theory of generalized differential equations.

Definition 2.1. Assume that $B \subset X$, $G = B \times I$, $h_1 : I \to \mathbb{R}$, $h_2 : I \to \mathbb{R}$ are nondecreasing functions, and $\omega : [0, \infty) \to \mathbb{R}$ is a continuous increasing function with $\omega(0) = 0$. The class $\mathcal{F}(G, h_1, h_2, \omega)$ consists of all functions $F : G \to X$ satisfying the following conditions:

$$\|F(x, t_2) - F(x, t_1)\| \le h_1(t_2) - h_1(t_1), \quad x \in B, \quad [t_1, t_2] \subset I,$$

$$\|F(x, t_2) - F(x, t_1) - F(y, t_2) + F(y, t_1)\| \le \omega(\|x - y\|)(h_2(t_2) - h_2(t_1)), \quad x, y \in B, \quad [t_1, t_2] \subset I.$$

For the rest of the paper, let us make the following agreement: Whenever we write $F \in \mathcal{F}(G, h_1, h_2, \omega)$, we assume that h_1, h_2, ω satisfy the assumptions listed in Definition 2.1, without mentioning all of them explicitly.

In the special case when h_1 and h_2 are equal to the same function h, we write $\mathcal{F}(G, h, \omega)$ instead of $\mathcal{F}(G, h_1, h_2, \omega)$. Many authors focus solely on this special case. Indeed, in many situations, the important thing is the existence of a pair of functions such that $F \in \mathcal{F}(G, h_1, h_2, \omega)$, while the particular values of h_1 and h_2 play no role. In this case, the assumption $h_1 = h_2$ presents no loss of generality, because we always have

$$\mathcal{F}(G, h_1, h_2, \omega) \subset \mathcal{F}(G, h_1 + h_2, h_1 + h_2, \omega) = \mathcal{F}(G, h_1 + h_2, \omega).$$

Still, for our purposes, it is useful to distinguish h_1 and h_2. In Theorem 3.2, this will enable us to provide a more reasonable lower bound for the length of the interval where a local solution is guaranteed to exist.

Under certain assumptions on the right-hand side (see [32, Theorem 5.14]), a classical ordinary differential equation of the form

$$x'(t) = f(x(t), t), \quad t \in I,$$

is equivalent to the generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t), \quad t \in I,$$

where

$$F(x,t) = \int_{t_0}^{t} f(x,s)\,\mathrm{d}s, \quad t \in I,$$

and t_0 is an arbitrary point in I. In this situation, the two conditions from Definition 2.1 reduce to

$$\left\| \int_{t_1}^{t_2} f(x,s)\,\mathrm{d}s \right\| \le h_1(t_2) - h_1(t_1), \quad x \in B, \quad [t_1,t_2] \subset I, \tag{2.3}$$

$$\left\| \int_{t_1}^{t_2} (f(x,s) - f(y,s))\,\mathrm{d}s \right\| \le \omega(\|x-y\|)(h_2(t_2) - h_2(t_1)), \quad x,y \in B, \quad [t_1,t_2] \subset I. \tag{2.4}$$

However, we emphasize that not every generalized differential equation is equivalent to a classical one; different choices of F might lead to other types of equations (cf. Section 5).

In the rest of this section, we collect some basic facts about regulated functions, Kurzweil integration, and generalized ordinary differential equations.

The following proposition, which characterizes relatively compact sets in the space of regulated functions, was proved in [16, Theorem 2.18]. For an arbitrary interval $I \subset \mathbb{R}$, we use the symbol $G(I,X)$ to denote the space of all bounded regulated functions $f : I \to X$ (if I is a compact interval, then every regulated function $f : I \to X$ is necessarily bounded). The space $G(I,X)$ is equipped with the supremum norm

$$\|f\|_\infty = \sup_{t \in I} \|f(t)\|, \quad f \in G(I,X).$$

For an arbitrary set $B \subset X$, let $G(I,B)$ denote the set of all bounded regulated functions $f : I \to B$.

Theorem 2.2. *For every set $\mathcal{A} \subset G([a,b],\mathbb{R}^n)$, the following conditions are equivalent:*

1. *$\mathcal{A}$ is relatively compact.*
2. *The set $\{x(a);\, x \in \mathcal{A}\}$ is bounded, there are an increasing continuous function $\eta : [0,\infty) \to [0,\infty)$ with $\eta(0) = 0$, and an increasing function $K : [a,b] \to \mathbb{R}$ such that*

$$\|x(t_2) - x(t_1)\| \le \eta(K(t_2) - K(t_1))$$

whenever $x \in \mathcal{A}$ and $[t_1,t_2] \subset [a,b]$.

We now present two inequalities for the Kurzweil integral. Both follow easily from the definition of the integral, and are special cases of [32, Corollary 1.36].

Lemma 2.3. *Let $B \subset X$, $G = B \times [a,b]$. Assume that $F : G \to X$ satisfies*

$$\|F(x,t_2) - F(x,t_1)\| \le h_1(t_2) - h_1(t_1), \quad x \in B, \quad [t_1,t_2] \subset [a,b],$$

where $h_1 : [a,b] \to \mathbb{R}$ is a nondecreasing function. If $x : [a,b] \to B$ and the integral $\int_a^b DF(x(\tau),t)$ exists, then

$$\left\| \int_a^b DF(x(\tau),t) \right\| \le h_1(b) - h_1(a).$$

Consequently, if $x : [a,b] \to X$ is a solution of a generalized equation $\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t)$ whose right-hand side is an element of $\mathcal{F}(G,h_1,h_2,\omega)$, we have the estimate

$$\|x(t_2) - x(t_1)\| \le h_1(t_2) - h_1(t_1), \quad [t_1,t_2] \in [a,b].$$

Lemma 2.4. *Let $B \subset X$, $G = B \times [a,b]$. Assume that $F : G \to X$ belongs to the class $\mathcal{F}(G,h_1,h_2,\omega)$. If $x,\, y : [a,b] \to B$ are arbitrary functions, then*

$$\left\| \int_a^b D[F(x(\tau),t) - F(y(\tau),t)] \right\| \le \int_a^b \omega(\|x(t) - y(t)\|)\,\mathrm{d}h_2(t),$$

provided that the integrals on both sides exist.

The next three lemmas provide sufficient conditions for the existence of the Kurzweil integral $\int_a^b DF(x(\tau),t)$. The proof of the first statement can be found in the proof of [32, Corollary 3.15].

Lemma 2.5. *Let $B \subset X$, $G = B \times [a,b]$. Assume that $F : G \to X$ belongs to the class $\mathcal{F}(G,h,\omega)$. If $x : [a,b] \to B$ is a step function, i.e., if there exists a partition*

$$a = s_0 < s_1 < \cdots < s_m = b$$

and elements $c_1, \ldots, c_m \in X$ such that

$$x(s) = c_i, \quad s \in (s_{i-1}, s_i), \quad i \in \{1, \ldots, m\},$$

then the integral $\int_a^b DF(x(\tau),t)$ exists and equals

$$\sum_{j=1}^{m} \Bigg(F(c_j, s_j-) - F(c_j, s_{j-1}+) + F(x(s_{j-1}), s_{j-1}+) - F(x(s_{j-1}), s_{j-1}) + F(x(s_j), s_j) - F(x(s_j), s_j-) \Bigg).$$

The next lemma generalizes [1, Proposition 2.13], which is a special case of our statement corresponding to $\omega(r) = r$. (A finite-dimensional version can be found in [32, Corollary 3.15]; unfortunately, the proof no longer works in infinite dimension.)

Lemma 2.6. *Let $B \subset X$, $G = B \times [a,b]$. Assume that $F : G \to X$ belongs to the class $\mathcal{F}(G,h,\omega)$. If $x : [a,b] \to B$ is the uniform limit of a sequence of step functions $x_k : [a,b] \to B$, $k \in \mathbb{N}$, then the integral $\int_a^b DF(x(\tau),t)$ exists and equals $\lim_{k\to\infty} \int_a^b DF(x_k(\tau),t)$.*

Proof. By Lemma 2.5, the integral $\int_a^b DF(x_k(\tau),t)$ exists for every $k \in \mathbb{N}$. Let us prove the existence of $\lim_{k\to\infty} \int_a^b DF(x_k(\tau),t)$. For each pair $i, j \in \mathbb{N}$, Lemma 2.4 implies

$$\left\| \int_a^b D[F(x_i(\tau),t) - F(x_j(\tau),t)] \right\| \le \int_a^b \omega(\|x_i(t) - x_j(t)\|) \,\mathrm{d}h(t) \le \omega\left(\|x_i - x_j\|_\infty\right)(h(b) - h(a)).$$

The right-hand side approaches zero as $i, j \to \infty$, and therefore the Cauchy condition for the existence of $\lim_{k\to\infty} \int_a^b DF(x_k(\tau),t)$ is satisfied.

Choose an arbitrary $\varepsilon > 0$. There exists a $k \in \mathbb{N}$ such that

$$\left\| \int_a^b DF(x_k(\tau),t) - \lim_{l\to\infty} \int_a^b DF(x_l(\tau),t) \right\| < \frac{\varepsilon}{3},$$

$$\omega\left(\|x - x_k\|_\infty\right)(h(b) - h(a)) < \frac{\varepsilon}{3}.$$

Also, there exists a gauge δ on $[a,b]$ such that

$$\left\| \sum_{i=1}^{m} (F(x_k(\tau_i), s_i) - F(x_k(\tau_i), s_{i-1})) - \int_a^b DF(x_k(\tau),t) \right\| < \frac{\varepsilon}{3}$$

for every δ-fine tagged partition of $[a,b]$ with division points $s_0, s_1, \ldots, s_m$ and tags $\tau_1, \ldots, \tau_m$. For these partitions, we get

$$\left\| \sum_{i=1}^{m} (F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})) - \lim_{l\to\infty} \int_a^b DF(x_l(\tau),t) \right\|$$

$$\le \left\| \sum_{i=1}^{m} (F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) - F(x_k(\tau_i), s_i) + F(x_k(\tau_i), s_{i-1})) \right\|$$

$$+ \left\| \sum_{i=1}^{m} (F(x_k(\tau_i), s_i) - F(x_k(\tau_i), s_{i-1})) - \int_a^b DF(x_k(\tau),t) \right\| + \left\| \int_a^b DF(x_k(\tau),t) - \lim_{l\to\infty} \int_a^b DF(x_l(\tau),t) \right\|$$

$$\leq \sum_{i=1}^{m} \omega(\|x(\tau_i) - x_k(\tau_i)\|)(h(s_i) - h(s_{i-1})) + \frac{2\varepsilon}{3} \leq \omega\left(\|x - x_k\|_\infty\right)\left(h(b) - h(a)\right) + \frac{2\varepsilon}{3} < \varepsilon,$$

which proves that $\int_a^b DF(x(\tau), t)$ exists and equals $\lim_{l\to\infty} \int_a^b DF(x_l(\tau), t)$. $\square$

Since every regulated function is the uniform limit of step functions, we get the following corollary.

Lemma 2.7. *Let $B \subset X$, $G = B \times [a, b]$. Assume that $F : G \to X$ belongs to the class $\mathcal{F}(G, h_1, h_2, \omega)$. If $x : [a, b] \to B$ is a regulated function, then the integral $\int_a^b DF(x(\tau), t)$ exists.*

The proof of the next lemma is almost identical to the proof of [28, Proposition 3.8] (which is concerned with the special case when $F_k(t, x) = A_k(t)$, $k \in \mathbb{N}_0$), but we include it here for reader's convenience.

Lemma 2.8. *Let $B \subset X$, $G = B \times [a, b]$, and consider a sequence of functions $F_k : B \times [a, b] \to X$, $k \in \mathbb{N}_0$, such that*

$$\lim_{k\to\infty} F_k(x, t) = F_0(x, t), \quad (x, t) \in G.$$

Assume that $F_k \in \mathcal{F}(G, h, \omega)$ for every $k \in \mathbb{N}_0$. Then

$$\begin{aligned} \lim_{k\to\infty} F_k(x, t+) &= F_0(x, t+), \quad (x, t) \in B \times [a, b), \\ \lim_{k\to\infty} F_k(x, t-) &= F_0(x, t-), \quad (x, t) \in B \times (a, b]. \end{aligned}$$

Moreover, for every fixed $x \in B$, the sequence of functions $t \mapsto F_k(x, t)$, $k \in \mathbb{N}$, is uniformly convergent to the function $t \mapsto F_0(x, t)$ on $[a, b]$.

Proof. For every $x \in B$ and $t \in [a, b)$, we have $F_k(x, t+) \to F_0(x, t+)$ for $k \to \infty$, because

$$\|F_k(x, t+) - F_0(x, t+)\| \leq \|F_k(x, t+) - F_k(x, t + \delta)\| + \|F_k(x, t + \delta) - F_0(x, t + \delta)\|$$

$$+\|F_0(x, t + \delta) - F_0(x, t+)\| \leq 2(h(t + \delta) - h(t+)) + \|F_k(x, t + \delta) - F_0(x, t + \delta)\|,$$

and the right-hand side can be made arbitrarily small by choosing $\delta > 0$ sufficiently small and $k \in \mathbb{N}$ sufficiently large. Similarly, one can prove that $F_k(x, t-) \to F_0(x, t-)$ for $k \to \infty$.

Now, assume there is an $x \in B$ such that the sequence $t \mapsto F_k(x, t)$, $k \in \mathbb{N}$, is not uniformly convergent to $t \mapsto F_0(x, t)$. Then, there exist an $\varepsilon > 0$, a subsequence $\{F_{k_l}\}_{l=1}^{\infty}$, and a sequence $\{t_l\}_{l=1}^{\infty}$ such that

$$\|F_{k_l}(x, t_l) - F_0(x, t_l)\| \geq \varepsilon, \quad l \in \mathbb{N}. \tag{2.5}$$

Moreover, without loss of generality, we can assume that $\lim_{l\to\infty} t_l = t_0 \in [a, b]$. Then, at least one of the following statements has to be true:

a) The sequence $\{t_l\}_{l=1}^{\infty}$ has a subsequence $\{t_{l_m}\}_{m=1}^{\infty}$ whose terms are all smaller than t_0.

b) The sequence $\{t_l\}_{l=1}^{\infty}$ has a subsequence $\{t_{l_m}\}_{m=1}^{\infty}$ whose terms are all greater than t_0.

(If neither a) nor b) was true, then $t_l = t_0$ for infinitely many values of l; together with (2.5), this would contradict the fact that $F_k(x, t_0) \to F_0(x, t_0)$.) We show that a) leads to a contradiction, and leave the other case up to the reader. We have

$$\varepsilon \leq \|F_{k_{l_m}}(x, t_{l_m}) - F_0(x, t_{l_m})\| \leq \|F_{k_{l_m}}(x, t_{l_m}) - F_{k_{l_m}}(x, t_0-)\| + \|F_{k_{l_m}}(x, t_0-) - F_0(x, t_0-)\|$$

$$+\|F_0(x, t_0-) - F_0(x, t_{l_m})\| \leq 2(h(t_0-) - h(t_{l_m})) + \|F_{k_{l_m}}(x, t_0-) - F_0(x, t_0-)\|.$$

However, the expression on the right-hand side approaches zero for $m \to \infty$, which is a contradiction. $\square$

The following lemma represents a stronger version of [32, Lemma 8.1]; instead of pointwise convergence, we prove uniform convergence of the indefinite integrals. Also, the original condition that x has bounded variation is replaced by the weaker assumption of regulatedness.

Lemma 2.9. *Let $B \subset X$, $G = B \times [a,b]$, and consider a sequence of functions $F_k : G \to X$, $k \in \mathbb{N}_0$, such that*

$$\lim_{k\to\infty} F_k(x,t) = F_0(x,t), \quad (x,t) \in G.$$

Assume that $F_k \in \mathcal{F}(G,h,\omega)$ for every $k \in \mathbb{N}_0$. If $x : [a,b] \to B$ is regulated, then

$$\lim_{k\to\infty} \int_a^s DF_k(x(\tau),t) = \int_a^s DF_0(x(\tau),t)$$

uniformly with respect to $s \in [a,b]$.

Proof. First, let us verify the statement in the case when $x : [a,b] \to B$ is a step function: There exist a partition $a = s_0 < s_1 < \cdots < s_m = b$ and elements $c_1,\ldots,c_m \in X$ such that $x(t) = c_j$ for every $t \in (s_{j-1},s_j)$. Choose an arbitrary $\varepsilon > 0$. According to Lemma 2.8, there exists a $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\|F_k(c_j,s_j-) - F_0(c_j,s_j-)\| &< \varepsilon/(6m), \quad j \in \{1,\ldots,m\},\\
\|F_k(c_j,s_{j-1}+) - F_0(c_j,s_{j-1}+)\| &< \varepsilon/(6m), \quad j \in \{1,\ldots,m\},\\
\|F_k(x(s_{j-1}),s_{j-1}+) - F_0(x(s_{j-1}),s_{j-1}+)\| &< \varepsilon/(6m), \quad j \in \{1,\ldots,m\},\\
\|F_k(x(s_j),s_j-) - F_0(x(s_j),s_j-)\| &< \varepsilon/(6m), \quad j \in \{1,\ldots,m\},\\
\|F_k(x(s_j),s) - F_0(x(s_j),s)\| &< \varepsilon/(6m), \quad j \in \{0,\ldots,m\}, \quad s \in [a,b],\\
\|F_k(c_j,s) - F_0(c_j,s)\| &< \varepsilon/(6m), \quad j \in \{1,\ldots,m\}, \quad s \in [a,b],
\end{aligned}$$

for all $k \geq k_0$. By Lemma 2.5, we have

$$\begin{aligned}
\int_{s_{j-1}}^{s_j} DF_k(x(\tau),t) = {} & F_k(c_j,s_j-) - F_k(c_j,s_{j-1}+) + F_k(x(s_{j-1}),s_{j-1}+)\\
& - F_k(x(s_{j-1}),s_{j-1}) + F_k(x(s_j),s_j) - F_k(x(s_j),s_j-),
\end{aligned}$$

for all $k \in \mathbb{N}_0$, $j \in \{1,\ldots,m\}$, and therefore

$$\left\| \int_{s_{j-1}}^{s_j} DF_k(x(\tau),t) - \int_{s_{j-1}}^{s_j} DF_0(x(\tau),t) \right\| < \frac{\varepsilon}{m}, \quad k \geq k_0.$$

When $s \in (s_{j-1},s_j)$ for some $j \in \{1,\ldots,m\}$, we have $x(s) = c_j$,

$$\int_{s_{j-1}}^{s} DF_k(x(\tau),t) = F_k(c_j,s) - F_k(c_j,s_{j-1}+) + F_k(x(s_{j-1}),s_{j-1}+) - F_k(x(s_{j-1}),s_{j-1}),$$

and therefore

$$\left\| \int_{s_{j-1}}^{s} DF_k(x(\tau),t) - \int_{s_{j-1}}^{s} DF_0(x(\tau),t) \right\| < \frac{4\varepsilon}{6m} < \frac{\varepsilon}{m}, \quad k \geq k_0.$$

It follows that for every $s \in [a,b]$, we have

$$\left\| \int_a^s DF_k(x(\tau),t) - \int_a^s DF_0(x(\tau),t) \right\| < \varepsilon, \quad k \geq k_0.$$

Now, consider the general situation when $x : [a,b] \to B$ is regulated. Choose an arbitrary $\varepsilon > 0$. There exists a step function $\varphi : [a,b] \to B$ such that $\omega(\|x-\varphi\|_\infty) < \frac{\varepsilon}{2(h(b)-h(a)+1)}$. Also, there exists a $k_0 \in \mathbb{N}$ such that $\left\|\int_a^s DF_k(\varphi(\tau),t) - \int_a^s DF_0(\varphi(\tau),t)\right\| < \varepsilon/2$ for all $k \geq k_0$ and $s \in [a,b]$. Then

$$\begin{aligned}
\left\| \int_a^s DF_k(x(\tau),t) - \int_a^s DF_0(x(\tau),t) \right\| &\leq \left\| \int_a^s DF_k(x(\tau),t) - \int_a^s DF_k(\varphi(\tau),t) \right\|\\
+ \left\| \int_a^s DF_k(\varphi(\tau),t) - \int_a^s DF_0(\varphi(\tau),t) \right\| &+ \left\| \int_a^s DF_0(\varphi(\tau),t) - \int_a^s DF_0(x(\tau),t) \right\|\\
\leq 2\int_a^s \omega(\|x(\tau)-\varphi(\tau)\|)\,\mathrm{d}h(\tau) + \frac{\varepsilon}{2} &< 2\omega(\|x-\varphi\|_\infty)(h(s)-h(a)) + \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

holds for all $k \geq k_0$ and $s \in [a,b]$, and the proof is complete. □

The following lemma summarizes some properties of solutions to generalized differential equations; it is a consequence of [32, Corollary 3.11] and [32, Lemma 3.12].

Lemma 2.10. *Let $B \subset X$, $G = B \times [a,b]$. Consider a function $F : G \to X$ satisfying*

$$\|F(x,t_2) - F(x,t_1)\| \le h_1(t_2) - h_1(t_1), \quad x \in B, \quad [t_1,t_2] \subset [a,b], \tag{2.6}$$

where $h_1 : [a,b] \to \mathbb{R}$ is a nondecreasing function. If $x : [a,b] \to X$ is a solution of the generalized differential equation $\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t)$, then x is a regulated function with bounded variation. Moreover,

$$\begin{aligned} x(s+) &= x(s) + F(x(s), s+) - F(x(s), s), \quad s \in [a,b), \\ x(s-) &= x(s) + F(x(s), s-) - F(x(s), s), \quad s \in (a,b]. \end{aligned}$$

If h_1 is left-continuous, then x is left-continuous as well.

Consider an equation $\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t)$, where F satisfies (2.6) with a left-continuous function h_1. Assume that we have a solution of this equation on $[a,b)$, and would like to extend it to $[a,b]$. According to Lemma 2.10, the extension obtained by letting $x(b) = x(b-)$ is the only candidate for such a solution. In the next lemma, we verify that the limit always exists, and that the extension indeed provides a solution on $[a,b]$.

Lemma 2.11. *Assume that $B \subset X$, $G = B \times [a,b]$, and $F : G \to X$ satisfies (2.6) with a left-continuous function h_1. If $x : [a,b) \to X$ is a solution of $\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t)$, then the limit $x(b-)$ exists. If $x(b-) \in B$ and we extend x to $[a,b]$ by letting $x(b) = x(b-)$, we obtain a solution of $\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t)$ on $[a,b]$.*

Proof. The Cauchy condition for the existence of the left-sided limit $x(b-)$ is satisfied, because

$$\|x(s_1) - x(s_2)\| \le |h_1(s_1) - h_1(s_2)|, \quad s_1, s_2 \in [a, a+b),$$

and the left-sided limit of h_1 does exist. Assume that $x(b-) \in B$. Since

$$\left\| \int_a^b DF(x(\tau),t) - \int_a^s DF(x(\tau),t) \right\| = \left\| \int_s^b DF(x(\tau),t) \right\| \le h_1(b) - h_1(s), \quad s \in [a,b],$$

it follows that $\lim_{s\to b-} \int_a^s DF(x(\tau),t) = \int_a^b DF(x(\tau),t)$, and therefore

$$x(b) = x(b-) = x(a) + \lim_{s\to b-} \int_a^s DF(x(\tau),t) = \int_a^b DF(x(\tau),t). \qquad \square$$

The following theorem, which describes the properties of the indefinite Kurzweil-Stieltjes integral, is a special case of [32, Theorem 1.16].

Theorem 2.12. *Let $f : [a,b] \to \mathbb{R}^n$ and $g : [a,b] \to \mathbb{R}$ be a pair of functions such that g is regulated and $\int_a^b f \,\mathrm{d}g$ exists. Then the function*

$$h(t) = \int_a^t f(s) \,\mathrm{d}g(s), \quad t \in [a,b],$$

is regulated and satisfies

$$\begin{aligned} h(t+) &= h(t) + f(t)\Delta^+ g(t), \quad t \in [a,b), \\ h(t-) &= h(t) - f(t)\Delta^- g(t), \quad t \in (a,b], \end{aligned}$$

where $\Delta^+ g(t) = g(t+) - g(t)$ and $\Delta^- g(t) = g(t) - g(t-)$.

The next result is a Bihari-type inequality (i.e., a nonlinear version of the Gronwall inequality) for the Kurzweil-Stieltjes integral, which can be found in [32, Theorem 1.40].

Theorem 2.13. *Consider functions $\psi : [a,b] \to [0,+\infty)$, $h : [a,b] \to \mathbb{R}$, $\omega : [0,+\infty) \to \mathbb{R}$ such that ψ is bounded, h is nondecreasing and left-continuous, ω is continuous, increasing and $\omega(0) = 0$. Suppose there exists a $k > 0$ such that*

$$\psi(t) \leq k + \int_a^t \omega(\psi(s))\,\mathrm{d}h(s), \quad t \in [a,b].$$

For an arbitrary $u_0 > 0$, let

$$\Omega(u) = \int_{u_0}^u \frac{1}{\omega(r)}\,\mathrm{d}r, \quad u \in (0,\infty),$$

$\alpha = \lim_{u\to 0+} \Omega(u) \geq -\infty$, and $\beta = \lim_{u\to+\infty} \Omega(u) \leq \infty$. Also, let $\Omega^{-1} : (\alpha,\beta) \to \mathbb{R}$ be the inverse function to Ω. If $\Omega(k) + h(b) - h(a) < \beta$, then

$$\psi(t) \leq \Omega^{-1}(\Omega(k) + h(t) - h(a)), \quad t \in [a,b].$$

The next proposition is an Osgood-type uniqueness theorem for generalized differential equations. The finite-dimensional version can be found in [32, Theorem 4.11], and its proof remains valid even in the infinite-dimensional case (in fact, it is a fairly straightforward consequence of Theorem 2.13).

Theorem 2.14. *Let $B \subset X$, $G = B \times [a,b]$. Consider a function $F : G \to X$ such that $F \in \mathcal{F}(G,h,\omega)$, where $h : [a,b] \to \mathbb{R}$ is left-continuous, and*

$$\lim_{v\to 0+} \int_v^u \frac{\mathrm{d}r}{\omega(r)} = \infty$$

for every $u > 0$. If $\tilde{x} \in X$, $[a,b_1]$, $[a,b_2] \subset [a,b]$, and $x_1 : [a,b_1] \to B$, $x_2 : [a,b_2] \to B$ are solutions of the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t), \quad x(a) = \tilde{x},$$

then $x_1(t) = x_2(t)$ for every $t \in [a,b_1] \cap [a,b_2]$.

Finally, let us recall the following existence-uniqueness theorem for generalized ordinary differential equations whose right-hand sides are elements of $\mathcal{F}(G,h,\omega_1)$ with $\omega_1(r) = r$, $r \geq 0$. Its proof, which is based on the contraction mapping theorem, can be found in [13, Theorem 2.15].

Theorem 2.15. *Assume that $B \subset X$ is an open set, $G = B \times [a,b]$, $F : G \to X$ belongs to the class $\mathcal{F}(G,h,\omega_1)$ with a left-continuous function h and $\omega_1(r) = r$, $r \geq 0$.*

If $x_0 \in B$ is such that $x_0 + F(x_0,a+) - F(x_0,a) \in B$, then the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t), \quad x(a) = x_0, \tag{2.7}$$

has a unique local solution defined on a right neighborhood of a.

If x is a local solution of the initial-value problem (2.7), it follows from Lemma 2.10 that $x(a+) = x_0 + F(x_0,a+) - F(x_0,a)$; this explains the meaning of the condition $x_0 + F(x_0,a+) - F(x_0,a) \in B$.

3 An Osgood-type existence theorem

In this section, we prove an Osgood-type existence theorem for abstract generalized differential equations. Unfortunately, the finite-dimensional proof presented in [32, Theorem 4.2], which makes use of Helly's choice theorem, is no longer applicable. Our proof is based on the following lemma, which says that a right-hand side $F \in \mathcal{F}(B\times[a,b],h_1,h_2,\omega)$ can be approximated by a function F_ε which is in a certain sense close to F. Moreover, F_ε has the property that for every $x \in B$, there is a neighborhood $U(x)$ and a constant $L(x) > 0$ such that the restriction of F_ε to $U(x) \times [a,b]$ is an element of $\mathcal{F}(U(x)\times[a,b],h_1,L(x)h_1,\omega_1)$, where $\omega_1(r) = r$, $r \geq 0$.

Lemma 3.1. *Let B be an open subset of X and $G = B \times [a,b]$. Consider a function $F : G \to X$ such that $F \in \mathcal{F}(G, h_1, h_2, \omega)$. Then, for every $\varepsilon > 0$, there exists a function $F_\varepsilon : G \to X$ with the following properties:*

1. $\|(F - F_\varepsilon)(x,t_2) - (F - F_\varepsilon)(x,t_1)\| \le \varepsilon(h_2(t_2) - h_2(t_1))$ *for all* $x \in B$, $[t_1,t_2] \subset [a,b]$.
2. $\|F_\varepsilon(x,t_2) - F_\varepsilon(x,t_1)\| \le h_1(t_2) - h_1(t_1)$ *for all* $x \in B$, $[t_1,t_2] \subset [a,b]$.
3. *For every $x \in B$, there exist a neighborhood $U(x)$ and a constant $L(x) > 0$ such that*

$$\|F_\varepsilon(y,t_2) - F_\varepsilon(y,t_1) - F_\varepsilon(z,t_2) + F_\varepsilon(z,t_1)\| \le \|y - z\| L(x)(h_1(t_2) - h_1(t_1))$$

for all $y, z \in U(x)$, $[t_1,t_2] \subset [a,b]$.

Proof. There exists a $\delta > 0$ such that $\omega(\delta) < \varepsilon$. For every $x \in B$, let $U_\delta(x) = \{y \in B;\ \|x - y\| < \delta/2\}$. Clearly, the system $\{U_\delta(x);\ x \in B\}$ is an open cover of B. Using the fact that every metric space is paracompact (see e.g. [30] or [4, Corollary 2.2]), we conclude that $\{U_\delta(x);\ x \in B\}$ has a locally finite open refinement $\{W_j;\ j \in J\}$; that is, the system $\{W_j;\ j \in J\}$ is an open cover of B, every W_j is contained in $U_\delta(x)$ for a certain $x \in B$, and every $x \in B$ has a neighborhood $V(x)$ that intersects only finitely many sets W_j.

Let $\{\varphi_j;\ j \in J\}$ be a partition of unity subordinated to $\{W_j;\ j \in J\}$, i.e., a collection of functions such that for every $j \in J$, the support of φ_j is contained in W_j, and $\sum_{j \in J} \varphi_j(x) = 1$ for every $x \in B$. The existence of such a partition of unity follows from paracompactness again (see e.g. [4, Proposition 2.3]). Moreover, it is known that the functions φ_j can be chosen to be locally Lipschitz continuous; the standard way of achieving this goal (cf. [23, Lemma 1]) is to let

$$\psi_j(x) = \begin{cases} d(x, \partial W_j), & x \in W_j \\ 0, & x \notin W_j \end{cases}$$

(where d denotes the distance between a point and a set), $\psi(x) = \sum_{j \in J} \psi_j(x)$, and finally $\varphi_j(x) = \frac{\psi_j(x)}{\psi(x)}$. It is not difficult to verify that the functions ψ_j are Lipschitz continuous with the Lipschitz constant equal to 1, ψ is locally Lipschitz continuous (because $\{W_j;\ j \in J\}$ is locally finite), and φ_j are locally Lipschitz continuous.

Now, for every $j \in J$, choose an arbitrary $w_j \in W_j$, and let

$$F_\varepsilon(x,t) = \sum_{j \in J} \varphi_j(x) F(w_j, t), \quad x \in B, \quad t \in [a,b].$$

We claim that F_ε possesses the three properties listed in the statement of the theorem. Indeed, for all $x \in B$ and $[t_1,t_2] \subset [a,b]$, we have

$$\|(F - F_\varepsilon)(x,t_2) - (F - F_\varepsilon)(x,t_1)\| = \left\| \sum_{j \in J} \varphi_j(x)(F(x,t_2) - F(w_j,t_2) - F(x,t_1) + F(w_j,t_1)) \right\|$$

$$\le \sum_{j \in J} \varphi_j(x) \|F(x,t_2) - F(w_j,t_2) - F(x,t_1) + F(w_j,t_1)\| \le \sum_{j \in J} \varphi_j(x) \omega(\|x - w_j\|)(h_2(t_2) - h_2(t_1)).$$

Note that $\varphi_j(x)$ is nonzero only for those $j \in J$ such that $x \in W_j$; in this case, both x and w_j are elements of $U_\delta(y)$. Thus $\|x - w_j\| < \delta$, $\omega(\|x - w_j\|) < \varepsilon$, and we obtain the estimate

$$\sum_{j \in J} \varphi_j(x) \omega(\|x - w_j\|)(h_2(t_2) - h_2(t_1)) \le \varepsilon(h_2(t_2) - h_2(t_1)) \sum_{j \in J} \varphi_j(x) = \varepsilon(h_2(t_2) - h_2(t_1)).$$

Next, note that

$$\|F_\varepsilon(x,t_2) - F_\varepsilon(x,t_1)\| = \left\| \sum_{j \in J} \varphi_j(x)(F(w_j,t_2) - F(w_j,t_1)) \right\| \le \sum_{j \in J} \varphi_j(x)(h_1(t_2) - h_1(t_1)) = h_1(t_2) - h_1(t_1).$$

Finally, for every $x \in B$, there exists a neighborhood $U(x)$ that intersects only a finite number of sets from the open cover $\{W_j;\, j \in J\}$, say $k(x)$ of them. Without loss of generality, we can assume that $U(x)$ is so small that all functions φ_j whose support intersects $U(x)$ are Lipschitz-continuous in $U(x)$, i.e., satisfy

$$|\varphi_j(y) - \varphi_j(z)| \le C(x)\|y - z\|, \quad y, z \in U(x)$$

for a certain constant $C(x) \ge 0$. Then

$$\|F_\varepsilon(y, t_2) - F_\varepsilon(y, t_1) - F_\varepsilon(z, t_2) + F_\varepsilon(z, t_1)\| = \left\|\sum_{j \in J} (\varphi_j(y) - \varphi_j(z))(F(w_j, t_2) - F(w_j, t_1))\right\|$$

$$\le k(x)C(x)\|y - z\|(h_1(t_2) - h_1(t_1)), \quad y, z \in U(x), \quad [t_1, t_2] \subset [a, b]. \qquad \square$$

We are now ready to prove the promised existence-uniqueness theorem for abstract generalized differential equations. It generalizes Theorem 2.15, which corresponds to the special case when $\omega(r) = r$. Even in that case, our theorem provides more information since it specifies a lower bound for the length of the interval where the solution is guaranteed to exist.

Theorem 3.2. *Assume that $B \subset X$ is an open set, $G = B \times [a, b]$, $F : G \to X$ belongs to the class $\mathcal{F}(G, h_1, h_2, \omega)$, where h_1, h_2 are left-continuous, and*

$$\lim_{v \to 0+} \int_v^u \frac{\mathrm{d}r}{\omega(r)} = \infty \tag{3.1}$$

for every $u > 0$.

If $x_0 \in B$ is such that $x_0 + F(x_0, a+) - F(x_0, a) \in B$, then the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t), \quad x(a) = x_0, \tag{3.2}$$

has a unique local solution defined on a right neighborhood of a.

Moreover, if $\Delta > 0$ is such that the closed ball

$$\{x \in X;\, \|x - (x_0 + F(x_0, a+) - F(x_0, a))\| \le h_1(a + \Delta) - h_1(a+)\} \tag{3.3}$$

is contained in B, the solution is guaranteed to exist on $[a, a + \Delta]$.

Proof. Let $\Delta > 0$ be an arbitrary number with the property described in the statement of the theorem (since B is open, such a Δ always exists). Clearly, it is enough to find a unique solution of Eq. (3.2) on the interval $[a, a + \Delta_0]$, where

$$\Delta_0 = \inf\{t \in [a, a + \Delta];\, h_1(t) = h_1(a + \Delta)\}.$$

Indeed, if $\Delta_0 < \Delta$ and we have a solution x defined on $[a, a + \Delta_0]$, we can extend it to $[a, a + \Delta]$ by letting

$$x(s) = x(a + \Delta_0) + F(x(a + \Delta_0), (a + \Delta_0)+) - F(x(a + \Delta_0), a + \Delta_0), \quad s \in (a + \Delta_0, a + \Delta].$$

One can easily check that $\|x(s) - x(a+)\| \le h_1((a + \Delta_0)+) - h_1(a+)$, i.e., $x(s)$ lies in the closed ball (3.3). Also, for every $x \in B$, the function $t \mapsto F(x, t)$ is constant on $(a + \Delta_0, a + \Delta]$ (because h_1 is constant there). According to Lemma 2.5, we have $\int_{a+\Delta_0}^s DF(x(\tau), t) = F(x(a + \Delta_0), (a + \Delta_0)+) - F(x(a + \Delta_0), a + \Delta_0)$ for all $s \in (a + \Delta_0, a + \Delta]$. It follows that

$$x(s) = x(a + \Delta_0) + \int_{a+\Delta_0}^s DF(x(\tau), t) = x_0 + \int_a^s DF(x(\tau), t), \quad s \in (a + \Delta_0, a + \Delta],$$

i.e., x is a solution of Eq. (3.2) on $[a, a + \Delta]$; uniqueness follows from Theorem 2.14.

To show the existence of a unique solution on $[a, a + \Delta_0]$, it is sufficient to prove the existence of a unique solution on $[a, a + \Delta_1]$ for every $\Delta_1 \in (0, \Delta_0)$. Then we have a unique solution on $[a, a + \Delta_0)$, which can be extended to $[a, a+\Delta_0]$ using Lemma 2.11. The assumption $x((a+\Delta_0)-) \in B$ from the lemma will be satisfied, because $\|x(t) - x(a+)\| \leq h_1(t) - h_1(a+)$ for all $t \in [a, a+\Delta_0)$, and thus the values $x(t)$, $t \in [a, a+\Delta_0)$ lie in the closed ball (3.3).

Let $\Delta_1 \in (0, \Delta_0)$ and note that $h_1(a+\Delta) - h_1(a+\Delta_1) > 0$. We claim that for every $\varepsilon > 0$ satisfying

$$\varepsilon(h_2(a+) - h_2(a)) < h_1(a+\Delta) - h_1(a+\Delta_1), \tag{3.4}$$

there exists a unique solution $x : [a, a+\Delta_1] \to X$ of the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF_\varepsilon(x, t), \quad x(a) = x_0, \tag{3.5}$$

where F_ε is the function obtained from Lemma 3.1. To prove this, let C be the set of all $c \in [a, a+\Delta_1]$ such that (3.5) has a unique solution x on $[a, c]$. Clearly, C is nonempty (because $a \in C$) and closed (this is a consequence of Lemma 2.11). Our goal is to show that $C = [a, a+\Delta_1]$, which can be accomplished by proving that C is open in $[a, a+\Delta_1]$. To this end, it is necessary to show that for every $c \in [a, a+\Delta_1) \cap C$, the unique solution x of (3.5) defined on $[a, c]$ can be always extended to a larger interval. Let

$$\tilde{x}_c = x(c) + F_\varepsilon(x(c), c+) - F_\varepsilon(x(c), c).$$

Using the properties of F_ε listed in Lemma 3.1 and (3.4), we get

$$\begin{aligned}
&\|\tilde{x}_c - (x_0 + F(x_0, a+) - F(x_0, a))\| \\
&\leq \|\tilde{x}_c - (x_0 + F_\varepsilon(x_0, a+) - F_\varepsilon(x_0, a))\| + \|F_\varepsilon(x_0, a+) - F_\varepsilon(x_0, a) - F(x_0, a+) + F(x_0, a)\| \\
&\leq \|x(c) - x(a+)\| + \|F_\varepsilon(x(c), c+) - F_\varepsilon(x(c), c)\| + \varepsilon(h_2(a+) - h_2(a)) \\
&\leq h_1(c) - h_1(a+) + h_1(c+) - h_1(c) + h_1(a+\Delta) - h_1(a+\Delta_1) \leq h_1(a+\Delta) - h_1(a+),
\end{aligned}$$

i.e., $\tilde{x}_c \in B$. By Lemma 3.1, there are a neighborhood $U(\tilde{x}_c)$ of the point $\tilde{x}_c$ and a number $L(\tilde{x}_c) > 0$ such that $F_\varepsilon \in \mathcal{F}(U(\tilde{x}_c) \times [a, b], h_1, L(x)h_1, \omega_1) \subset \mathcal{F}(U(\tilde{x}_c) \times [a, b], (1 + L(x))h_1, \omega_1)$. We would like to use Theorem 2.15 find a unique local solution of $\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF_\varepsilon(x, t)$ defined on a right neighborhood of c, whose value at c is $x(c)$. If $x(c) \in U(\tilde{x}_c)$, all assumptions of the theorem are satisfied. If $x(c) \notin U(\tilde{x}_c)$, it is enough to find a local solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = D\tilde{F}_\varepsilon(y, t), \quad y(c) = \tilde{x}_c,$$

where $\tilde{F}_\varepsilon(y, c) = F_\varepsilon(y, c+)$ and $\tilde{F}_\varepsilon(y, t) = F_\varepsilon(y, t)$ for all $y \in U(\tilde{x}_c)$, $t \in [c, b]$; note that $F_\varepsilon \in \mathcal{F}(U(\tilde{x}_c) \times [c, b], (1 + L(x))h_1, \omega_1)$. Then, extend the solution x of (3.5) from $[a, c]$ to a larger interval by letting $x(t) = y(t)$ for $t > c$. This extended function x has the correct jump at c, and is a solution of (3.5) on a right neighborhood of c.

Let $n_0 \in \mathbb{N}$ be such that $\frac{1}{n_0}(h_2(a+) - h_2(a)) < h_1(a+\Delta) - h_1(a+\Delta_1)$. We have proved that for every integer $n \geq n_0$, there exists a function $x_n : [a, a+\Delta_1] \to X$ satisfying

$$x_n(s) = x_0 + \int_a^s DF_{1/n}(x_n(\tau), t), \quad s \in [a, a+\Delta_1]. \tag{3.6}$$

For each pair of integers $m, n \geq n_0$, we obtain

$$x_n(s) - x_m(s) = \int_a^s D[(F_{1/n} - F)(x_n(\tau), t)] + \int_a^s D[F(x_n(\tau), t) - F(x_m(\tau), t)] + \int_a^s D[(F - F_{1/m})(x_m(\tau), t)].$$

Lemma 2.4 implies

$$\left\| \int_a^s D[F(x_n(\tau), t) - F(x_m(\tau), t)] \right\| \leq \int_a^s \omega(\|x_n(t) - x_m(t)\|) \, \mathrm{d}h_2(t), \quad s \in [a, a+\Delta_1],$$

and Lemma 2.3 gives the estimate

$$\left\| \int_a^s D[(F_{1/k} - F)(x_k(\tau), t)] \right\| \leq \frac{h_2(s) - h_2(a)}{k}, \quad s \in [a, a + \Delta_1], \quad k \geq n_0.$$

Consequently,

$$\|x_n(s) - x_m(s)\| \leq \int_a^s \omega(\|x_n(t) - x_m(t)\|)\,\mathrm{d}h_2(t) + (h_2(b) - h_2(a)) \left(\frac{1}{n} + \frac{1}{m} \right), \quad s \in [a, a + \Delta_1], \quad m, n \geq n_0.$$

For an arbitrary $u_0 > 0$, the function

$$\Omega(u) = \int_{u_0}^u \frac{1}{\omega(r)}\,\mathrm{d}r, \quad u \in (0, \infty),$$

is continuous, increasing, $\alpha = \lim_{u \to 0+} \Omega(u) = -\infty$, and $\beta = \lim_{u \to +\infty} \Omega(u) \leq \infty$. Hence, the inverse function Ω^{-1} is increasing on its domain $(-\infty, \beta)$. For m, n sufficiently large, we have

$$\Omega \left((h_2(b) - h_2(a)) \left(\frac{1}{n} + \frac{1}{m} \right) \right) + h_2(b) - h_2(a) < \beta,$$

and it follows from Theorem 2.13 that

$$\|x_n(s) - x_m(s)\| \leq \Omega^{-1} \left(\Omega \left((h_2(b) - h_2(a)) \left(\frac{1}{n} + \frac{1}{m} \right) \right) + h_2(b) - h_2(a) \right), \quad s \in [a, a + \Delta_1].$$

As m, n increase, the argument of Ω^{-1} tends to $-\infty$, and therefore the whole right-hand side approaches zero. Thus $\{x_n\}_{n \geq n_0}$ is a Cauchy sequence in $G([a, a + \Delta_1], X)$, and has a limit $x : [a, a + \Delta_1] \to X$.

Observe that $\lim_{n \to \infty} \int_a^s DF_{1/n}(x_n(\tau), t) = \int_a^s DF(x(\tau), t)$ for every $s \in [a, a + \Delta_1]$, because

$$\left\| \int_a^s D[F_{1/n}(x_n(\tau), t) - F(x(\tau), t)] \right\| \leq \left\| \int_a^s D[(F_{1/n} - F)(x_n(\tau), t)] \right\| + \left\| \int_a^s D[F(x_n(\tau), t) - F(x(\tau), t)] \right\|$$

$$\leq \frac{h_2(s) - h_2(a)}{n} + \int_a^s \omega(\|x_n(t) - x(t)\|)\,\mathrm{d}h_2(t) \leq \frac{h_2(s) - h_2(a)}{n} + \omega(\|x_n - x\|_\infty)(h_2(s) - h_2(a)),$$

and the last expression approaches zero as $n \to \infty$.

By letting $n \to \infty$ in Eq. (3.6), we see that x is a solution of $\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t)$ on $[a, a + \Delta_1]$; uniqueness follows from Theorem 2.14. □

Remark 3.3. We have just proved that the condition

$$\{x \in X;\ \|x - (x_0 + F(x_0, a+) - F(x_0, a))\| \leq h_1(a + \Delta) - h_1(a+)\} \subset B$$

guarantees that the local solution exists on $[a, a + \Delta]$. However, assume that we a priori know that the solution of $\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t)$, $x(a) = x_0$, never leaves a certain set $Y \subset X$. Then, an inspection of the previous proof shows that it is enough to verify the following two conditions:

- If $x \in Y$ and $\|x - (x_0 + F(x_0, a+) - F(x_0, a))\| \leq h_1(a + \Delta) - h_1(a+)$, then $x \in B$.
- If $x \in Y$ and $t \in [a, a + \Delta)$, then $x + F(x, t+) - F(x, t) \in Y$.

We will use this observation in the proof of Osgood theorem for measure functional differential equations.

4 Continuous dependence

Following the usual convention, we state our continuous dependence theorems for sequences of initial-value problems of the following form:

$$\frac{\mathrm{d}x_k}{\mathrm{d}\tau} = DF_k(x_k, t), \quad t \in [a, b], \quad x_k(a) = \tilde{x}_k, \quad k \in \mathbb{N}_0, \tag{4.1}$$

where $F_k \to F_0$ and $\tilde{x}_k \to \tilde{x}_0$ for $k \to \infty$. However, the results may be easily adapted to initial-value problems where the right-hand side as well as the initial condition depends on a parameter $\lambda \in \Lambda$ (for example, Λ can be a metric space):

$$\frac{\mathrm{d}x_\lambda}{\mathrm{d}\tau} = DF(x_\lambda, t, \lambda), \quad t \in [a, b], \quad x_\lambda(a) = \tilde{x}(\lambda), \quad \lambda \in \Lambda,$$

where $F(x, t, \lambda) \to F(x, t, \lambda_0)$ and $\tilde{x}(\lambda) \to \tilde{x}(\lambda_0)$ for $\lambda \to \lambda_0$.

In this paper, we are interested in sequences of problems of the form (4.1), where all the right-hand sides F_k are elements of $\mathcal{F}(G, h, \omega)$ for a fixed pair of functions h, ω. In particular, our goal is to investigate infinite-dimensional counterparts of the following two finite-dimensional theorems, which can be found in [32] (see Theorems 8.2 and 8.6 there).

Theorem 4.1. *Let $B \subset \mathbb{R}^n$, $G = B \times [a, b]$, and consider a sequence of functions $F_k : G \to \mathbb{R}^n$, $k \in \mathbb{N}_0$, such that*

$$\lim_{k\to\infty} F_k(x, t) = F_0(x, t), \quad (x, t) \in G.$$

Assume that $F_k \in \mathcal{F}(G, h, \omega)$ for every $k \in \mathbb{N}_0$. Finally, suppose there exists a sequence of functions $x_k : [a, b] \to B$, $k \in \mathbb{N}_0$, such that

$$\frac{\mathrm{d}x_k}{\mathrm{d}\tau} = DF_k(x_k, t), \quad t \in [a, b], \quad k \in \mathbb{N},$$

and $\lim_{k\to\infty} x_k(s) = x_0(s)$ for every $s \in [a, b]$. Then

$$\frac{\mathrm{d}x_0}{\mathrm{d}\tau} = DF_0(x_0, t), \quad t \in [a, b].$$

Theorem 4.2. *Let $B \subset \mathbb{R}^n$, $G = B \times [a, b]$, and consider a sequence of functions $F_k : G \to \mathbb{R}^n$, $k \in \mathbb{N}_0$, such that*

$$\lim_{k\to\infty} F_k(x, t) = F_0(x, t), \quad (x, t) \in G.$$

Assume that $F_k \in \mathcal{F}(G, h, \omega)$ for every $k \in \mathbb{N}_0$, where h is left-continuous. Suppose that $\tilde{x}_0 \in B$ and $x_0 : [a, b] \to B$ is a unique solution of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF_0(x, t), \quad x(a) = \tilde{x}_0.$$

Finally, assume there exists a $\rho > 0$ such that $\|y - x_0(s)\| < \rho$ implies $y \in B$ whenever $s \in [a, b]$ (i.e., the ρ-neighborhood of x_0 is contained in B). Then, given an arbitrary sequence $\tilde{x}_k \in \mathbb{R}^n$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} \tilde{x}_k = \tilde{x}_0$, there is a $k_0 \in \mathbb{N}$ and a sequence of functions $x_k : [a, b] \to B$, $k \geq k_0$, which satisfy

$$\frac{\mathrm{d}x_k}{\mathrm{d}\tau} = DF_k(x_k, t), \quad t \in [a, b], \quad x_k(a) = \tilde{x}_k. \tag{4.2}$$

Moreover, $\lim_{k\to\infty} x_k(s) = x_0(s)$ for every $s \in [a, b]$.

We start by deriving an infinite-dimensional version of Theorem 4.1. The finite-dimensional proof given in [32] makes use of Helly's selection theorem (which is no longer valid in infinite dimension), but only to conclude that the limit function has bounded variation. Fortunately, this is easy to prove without Helly's theorem. Moreover, our proof does not depend on this fact because we have generalized Lemma 2.9 to regulated functions. Otherwise, the main idea of the proof is similar to the proof from [32].

Note the remarkable fact that (thanks to Lemma 2.9 and Theorem 2.2) the conclusion of our theorem is stronger than in the original finite-dimensional version: we prove the uniform convergence of the sequence $\{x_k\}_{k=1}^{\infty}$. The theorem also generalizes Theorem A.3 from [1], since we do not assume that h is left-continuous and has only finitely many discontinuities (although these assumptions are not mentioned explicitly in the statement of Theorem A.3, they can be found at the beginning of Appendix A in [1], and are needed in the proof of Theorem A.3).

Theorem 4.3. *Let $B \subset X$, $G = B \times [a,b]$, and consider a sequence of functions $F_k : G \to X$, $k \in \mathbb{N}_0$, such that*

$$\lim_{k\to\infty} F_k(x,t) = F_0(x,t), \quad (x,t) \in G.$$

Assume that $F_k \in \mathcal{F}(G,h,\omega)$ for every $k \in \mathbb{N}_0$. Finally, suppose there exists a sequence of functions $x_k : [a,b] \to B$, $k \in \mathbb{N}_0$, such that

$$\frac{\mathrm{d}x_k}{\mathrm{d}\tau} = DF_k(x_k,t), \quad t \in [a,b], \quad k \in \mathbb{N},$$

and $\lim_{k\to\infty} x_k(s) = x_0(s)$ for every $s \in [a,b]$. Then

$$\frac{\mathrm{d}x_0}{\mathrm{d}\tau} = DF_0(x_0,t), \quad t \in [a,b].$$

Moreover, the sequence $\{x_k\}_{k=1}^{\infty}$ is uniformly convergent to x_0.

Proof. We know that

$$x_k(s) = x_k(a) + \int_a^s DF_k(x_k(\tau),t), \quad s \in [a,b], \quad k \in \mathbb{N},$$

and our goal is to prove that

$$x_0(s) = x_0(a) + \int_a^s DF_0(x_0(\tau),t), \quad s \in [a,b]. \tag{4.3}$$

Clearly, it is enough to show that $\lim_{k\to\infty} \int_a^s DF_k(x_k(\tau),t) = \int_a^s DF_0(x_0(\tau),t)$ uniformly with respect to $s \in [a,b]$. The assumption $F_k \in \mathcal{F}(G,h,\omega)$ together with Lemma 2.3 imply

$$\|x_k(\beta) - x_k(\alpha)\| \le h(\beta) - h(\alpha), \quad k \in \mathbb{N}, \quad [\alpha,\beta] \subset [a,b].$$

Passing to the limit $k \to \infty$, we obtain

$$\|x_0(\beta) - x_0(\alpha)\| \le h(\beta) - h(\alpha), \quad [\alpha,\beta] \subset [a,b].$$

Since h is regulated, it follows that x_0 is regulated as well. (In fact, it is clear that x_0 has bounded variation.) By Lemma 2.7, the integrals $\int_a^b DF_0(x_0(\tau),t)$ and $\int_a^b DF_k(x_0(\tau),t)$ exist. We have

$$\int_a^s DF_k(x_k(\tau),t) - \int_a^s DF_0(x_0(\tau),t) = \int_a^s D[F_k(x_k(\tau),t) - F_k(x_0(\tau),t)] + \int_a^s D[F_k(x_0(\tau),t) - F_0(x_0(\tau),t)].$$

We need to show that both integrals on the right-hand side are convergent to zero for $k \to \infty$, and that the convergence is uniform with respect to $s \in [a,b]$. For the second integral, this is a consequence of Lemma 2.9. The first integral can be estimated using Lemma 2.4:

$$\left\|\int_a^s D[F_k(x_k(\tau),t) - F_k(x_0(\tau),t)]\right\| \le \int_a^s \omega(\|x_k(\tau) - x_0(\tau)\|)\,\mathrm{d}h(\tau)$$

Observe that the sequence of functions $\{x_k - x_0\}_{k=1}^{\infty}$ is uniformly bounded, because

$$\|x_k(s) - x_0(s)\| \le \|x_k(s) - x_k(a)\| + \|x_k(a) - x_0(a)\| + \|x_0(a) - x_0(s)\| \le M, \quad k \in \mathbb{N}, \quad s \in [a,b],$$

where $M = 2(h(b) - h(a)) + \sup_{k \in \mathbb{N}} \|x_k(a) - x_0(a)\|$. Since ω is continuous, the assumptions of the dominated convergence theorem for the Kurzweil-Stieltjes integral (see [32, Corollary 1.32]) are satisfied, and we get

$$\lim_{k\to\infty} \int_a^s \omega(\|x_k(\tau) - x_0(\tau)\|)\,\mathrm{d}h(\tau) = \int_a^s \lim_{k\to\infty} \omega(\|x_k(\tau) - x_0(\tau)\|)\,\mathrm{d}h(\tau) = 0, \quad s \in [a,b].$$

Let $\varphi_k(s) = \int_a^s \omega(\|x_k(\tau) - x_0(\tau)\|)\,\mathrm{d}h(\tau)$, $s \in [a,b]$, $k \in \mathbb{N}$. We know that $\varphi_k \to 0$ for $k \to \infty$, and it remains to check that the convergence is uniform on $[a,b]$. To this end, it is enough to verify that every subsequence of $\{\varphi_k\}_{k=1}^\infty$ has a subsequence which is uniformly convergent to zero. (Then it follows easily that $\{\varphi_k\}_{k=1}^\infty$ itself is convergent to zero; otherwise, we could find an $\varepsilon > 0$ and a subsequence $\{\varphi_{k_l}\}_{l=1}^\infty$ such that $\|\varphi_{k_l}\|_\infty \geq \varepsilon$, which is a contradiction.) For every $k \in \mathbb{N}$, we have $\varphi_k(a) = 0$, and

$$|\varphi_k(\beta) - \varphi_k(\alpha)| = \int_\alpha^\beta \omega(\|x_k(\tau) - x_0(\tau)\|)\,\mathrm{d}h(\tau) \leq \omega(M)(h(\beta) - h(\alpha))$$

whenever $[\alpha,\beta] \subset [a,b]$. Therefore, the second condition of Theorem 2.2 is satisfied with $\eta(r) = \omega(M)r$ and $K(t) = h(t) + t$. By this theorem, every subsequence of $\{\varphi_k\}_{k=1}^\infty$ has a subsequence which is uniformly convergent to zero, and the proof is complete. □

Our next goal is to obtain an infinite-dimensional version of Theorem 4.2. The proof given in [32] is again based on Helly's selection theorem. This time, there seems to be no simple way to avoid it, and we have to follow a different approach. Moreover, one cannot expect to prove an infinite-dimensional version of Theorem 4.2 under the same assumptions as in the finite-dimensional case. The reason is that in Theorem 4.2, the assumption $F_k \in \mathcal{F}(G,h,\omega)$ guarantees the local existence of solutions to Eq. (4.2) for all sufficiently large $k \in \mathbb{N}$ (see [32, Chapter 4]). Unfortunately, this is no longer true in a general Banach space. Example 4.5 shows that Theorem 4.2 fails in infinite dimension; the construction is based on an example from J. Dieudonné's paper [9], where he demonstrated that Peano's existence theorem need not hold in infinite-dimensional spaces.

First, we need the following inequality.

Lemma 4.4. *If $x, y \in \mathbb{R}$, then $|\sqrt{|x|} - \sqrt{|y|}| \leq \sqrt{|x-y|}$.*

Proof. The function $f(x) = \sqrt{x}$ is concave and $f(0) = 0$; hence, f is subadditive on $[0,\infty)$.

Given an arbitrary pair $x, y \in \mathbb{R}$, we have

$$\sqrt{|x|} = \sqrt{|x-y+y|} \leq \sqrt{|x-y|+|y|} \leq \sqrt{|x-y|} + \sqrt{|y|},$$

and therefore $\sqrt{|x|} - \sqrt{|y|} \leq \sqrt{|x-y|}$. A similar reasoning leads to the inequality $\sqrt{|y|} - \sqrt{|x|} \leq \sqrt{|y-x|}$, which completes the proof. □

Example 4.5. Let c_0 be the space of all real sequences $\{x_n\}_{n=0}^\infty$ such that $\lim_{n\to\infty} x_n = 0$; this space is equipped with the supremum norm. For an arbitrary $\lambda \in \mathbb{R}$, define the mapping $f_\lambda : c_0 \to c_0$ by

$$f_\lambda(\{x_n\}_{n=0}^\infty) = \lambda \left\{ \sqrt{|x_n|} + \frac{1}{n+1} \right\}_{n=0}^\infty.$$

Consider the abstract differential equation

$$x'(t) = f_\lambda(x(t)), \quad t \geq 0, \quad x(0) = 0. \tag{4.4}$$

For $\lambda = 0$, the equation has the unique solution $x(t) = 0$ for $t \geq 0$. On the other hand, for $\lambda > 0$, the equation is not even locally solvable. For contradiction, assume that $x = \{x_n\}_{n=0}^\infty$ is a solution defined on $[0,b]$ for some $b > 0$. Then

$$x_n'(t) = \lambda \left(\sqrt{|x_n(t)|} + \frac{1}{n+1} \right), \quad t \in [0,b], \quad x_n(0) = 0, \quad n \in \mathbb{N}_0.$$

By the comparison theorem for initial-value problems, we get $x_n(t) \geq y_n(t)$, where y_n is any solution of

$$y_n'(t) = \lambda\sqrt{|y_n(t)|}, \quad t \in [0,b], \quad y_n(0) = 0.$$

One solution of the last equation is $y_n(t) = (\lambda t)^2/4$, and therefore $x_n(t) \geq (\lambda t)^2/4$. For $t > 0$, this is in contradiction with the fact that $\lim_{n\to\infty} x_n(t) = 0$.

For any $\lambda \geq 0$ and $x, y \in c_0$, we have

$$\|f_\lambda(x)\| = \sup_{n\in\mathbb{N}_0} \lambda \left| \sqrt{|x_n|} + \frac{1}{n+1} \right| \leq \lambda(\sqrt{\|x\|} + 1), \tag{4.5}$$

$$\|f_\lambda(x) - f_\lambda(y)\| = \sup_{n\in\mathbb{N}_0} \lambda|\sqrt{|x_n|} - \sqrt{|y_n|}| \leq \sup_{n\in\mathbb{N}_0} \lambda\sqrt{|x_n - y_n|} = \lambda\sqrt{\|x-y\|}. \tag{4.6}$$

These inequalities guarantee that the initial-value problem (4.4) is equivalent to

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF_\lambda(x,t), \quad t \geq 0, \quad x(0) = 0, \tag{4.7}$$

where $F_\lambda(x,t) = \int_0^t f_\lambda(x)\,\mathrm{d}s = tf_\lambda(x)$; see [32, Theorem 5.14] (the proof in infinite dimension is the same as the finite-dimensional one, the only exception is that one has to rely on our Lemma 2.6 instead of its finite-dimensional counterpart). Hence, Eq. (4.7) has a unique global solution for $\lambda = 0$, and no local solutions for $\lambda > 0$.

Now, let $\lambda_0 = 0$ and let $\{\lambda_k\}_{k=1}^\infty$ be any sequence of positive real numbers such that $\lim_{k\to\infty} \lambda_k = 0$. Also, suppose that $b > 0$ and $B \subset c_0$ is an arbitrary bounded set containing the zero element. Clearly,

$$\lim_{k\to\infty} F_{\lambda_k}(x,t) = F_{\lambda_0}(x,t), \quad (x,t) \in B \times [0,b].$$

Our final goal is to verify that $F_{\lambda_k} \in \mathcal{F}(B \times [0,b], h, \omega)$ for all $k \in \mathbb{N}_0$. It follows from (4.5) and (4.6) that

$$\|F_{\lambda_k}(x,t_2) - F_{\lambda_k}(x,t_1)\| = (t_2 - t_1)\|f_{\lambda_k}(x)\| \leq (t_2 - t_1)\left(\sup_{k\in\mathbb{N}} |\lambda_k|\right)\left(\sup_{x\in B} \sqrt{\|x\|} + 1\right),$$

$$\begin{aligned} &\|F_{\lambda_k}(x,t_2) - F_{\lambda_k}(x,t_1) - F_{\lambda_k}(y,t_2) + F_{\lambda_k}(y,t_1)\| \\ &= (t_2 - t_1)\|f_{\lambda_k}(x) - f_{\lambda_k}(y)\| \leq (t_2 - t_1)\left(\sup_{k\in\mathbb{N}} |\lambda_k|\right)\sqrt{\|x-y\|} \end{aligned}$$

whenever $x, y \in B$ and $[t_1, t_2] \subset [0,b]$. This proves that $F_{\lambda_k} \in \mathcal{F}(B \times [0,b], h, \omega)$ with

$$h(t) = t\left(\sup_{k\in\mathbb{N}} |\lambda_k|\right)\left(\sup_{x\in B} \sqrt{\|x\|} + 1\right)$$

and $\omega(r) = \sqrt{r}$, and shows that Theorem 4.2 is no longer true in infinite dimension.

The following lemma will be needed in the proof of our continuous dependence theorem.

Lemma 4.6. *Let $B \subset X$, $G = B \times [a,b]$, and consider a sequence of functions $F_k : G \to X$, $k \in \mathbb{N}_0$, such that*

$$\lim_{k\to\infty} F_k(x,t) = F_0(x,t), \quad (x,t) \in G.$$

Assume that $F_k \in \mathcal{F}(G,h,\omega)$ for every $k \in \mathbb{N}_0$, where h is left-continuous and $\lim_{v\to 0+} \int_v^u \frac{\mathrm{d}r}{\omega(r)} = \infty$ for every $u > 0$. Suppose that for every $k \in \mathbb{N}_0$, $x_k : [a,b] \to B$ satisfies

$$\frac{\mathrm{d}x_k}{\mathrm{d}\tau} = DF_k(x_k,t), \quad t \in [a,b], \quad x_k(a) = \tilde{x}_k,$$

where $\lim_{k\to\infty} \tilde{x}_k = \tilde{x}_0$. Then $\{x_k\}_{k=1}^\infty$ is uniformly convergent to x_0 on $[a,b]$.

Proof. For every $s \in [a,b]$ and $k \in \mathbb{N}$, the integral $\int_a^b DF_k(x_0(\tau),t)$ exists by Lemma 2.7, and we obtain

$$\|x_0(s) - x_k(s)\| \leq \|\tilde{x}_0 - \tilde{x}_k\| + \left\| \int_a^s DF_0(x_0(\tau),t) - \int_a^s DF_k(x_k(\tau),t) \right\|$$

$$\leq \|\tilde{x}_0 - \tilde{x}_k\| + \left\| \int_a^s DF_0(x_0(\tau),t) - \int_a^s DF_k(x_0(\tau),t) \right\| + \left\| \int_a^s DF_k(x_0(\tau),t) - \int_a^s DF_k(x_k(\tau),t) \right\|.$$

Choose an arbitrary $\varepsilon > 0$. There exists a $k_0 \in \mathbb{N}$ such that $\|\tilde{x}_0 - \tilde{x}_k\| < \varepsilon/2$ and

$$\left\| \int_a^s DF_0(x_0(\tau),t) - \int_a^s DF_k(x_0(\tau),t) \right\| < \varepsilon/2$$

for every $k \geq k_0$ (the second statement follows from Lemma 2.9). These facts together with Lemma 2.4 imply that

$$\|x_0(s) - x_k(s)\| \leq \varepsilon + \int_a^s \omega(\|x_0(t) - x_k(t)\|)\,\mathrm{d}h(t), \quad k \geq k_0, \quad s \in [a,b].$$

By letting $\psi_k(s) = \|x_0(s) - x_k(s)\|$, $s \in [a,b]$, the last inequality can be rewritten as

$$\psi_k(s) \leq \varepsilon + \int_a^s \omega(\psi_k(t))\,\mathrm{d}h(t), \quad k \geq k_0, \quad s \in [a,b].$$

For an arbitrary $u_0 > 0$, the function

$$\Omega(u) = \int_{u_0}^u \frac{1}{\omega(r)}\,\mathrm{d}r, \quad u \in (0,\infty),$$

is continuous, increasing, $\alpha = \lim_{u\to 0+} \Omega(u) = -\infty$, and $\beta = \lim_{u\to+\infty} \Omega(u) \leq \infty$. Hence, the inverse function Ω^{-1} is increasing on its domain $(-\infty,\beta)$. Without loss of generality, we can assume that ε is so small that $\Omega(\varepsilon) + h(b) - h(a) < \beta$. It follows from Theorem 2.13 that

$$\psi_k(s) \leq \Omega^{-1}(\Omega(\varepsilon) + h(s) - h(a)) \leq \Omega^{-1}(\Omega(\varepsilon) + h(b) - h(a)), \quad s \in [a,b], \quad k \geq k_0.$$

For $\varepsilon \to 0+$, we have $\Omega(\varepsilon) + h(b) - h(a) \to -\infty$, and therefore $\Omega^{-1}(\Omega(\varepsilon) + h(b) - h(a)) \to 0$; this completes the proof. $\square$

We now proceed to an infinite-dimensional counterpart to Theorem 4.2. In comparison with that theorem, we restrict ourselves to the case $\lim_{v\to 0+} \int_v^u \frac{\mathrm{d}r}{\omega(r)} = \infty$ for every $u > 0$; by Theorem 2.14, this guarantees uniqueness of solutions. The proof is similar to the proof of Theorem 8.6 in [32], but the part which was originally based on Helly's selection theorem is now different and uses Lemma 4.6 instead. Also, the new Theorem 3.2 is needed in the proof. Again, the conclusion is stronger than in the original finite-dimensional version (we get uniform convergence of solutions instead of pointwise convergence).

Theorem 4.7. *Let $B \subset X$, $G = B \times [a,b]$, and consider a sequence of functions $F_k : G \to X$, $k \in \mathbb{N}_0$, such that*

$$\lim_{k\to\infty} F_k(x,t) = F_0(x,t), \quad (x,t) \in G.$$

Assume that $F_k \in \mathcal{F}(G,h,\omega)$ for every $k \in \mathbb{N}_0$, where h is left-continuous and

$$\lim_{v\to 0+} \int_v^u \frac{\mathrm{d}r}{\omega(r)} = \infty \tag{4.8}$$

for every $u > 0$. Let $x_0 : [a,b] \to B$ satisfy

$$\frac{\mathrm{d}x_0}{\mathrm{d}\tau} = DF_0(x_0,t), \quad x_0(a) = \tilde{x}_0.$$

Finally, assume there exists a $\rho > 0$ such that $\|y - x_0(s)\| < \rho$ implies $y \in B$ whenever $s \in [a,b]$.

Then, given an arbitrary sequence $\tilde{x}_k \in X$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} \tilde{x}_k = \tilde{x}_0$, there is a $k_0 \in \mathbb{N}$ and a sequence of functions $x_k : [a,b] \to B$, $k \geq k_0$, which satisfy

$$\frac{\mathrm{d}x_k}{\mathrm{d}\tau} = DF_k(x_k, t), \quad t \in [a,b], \quad x_k(a) = \tilde{x}_k.$$

Moreover, the sequence $\{x_k\}_{k=k_0}^{\infty}$ is uniformly convergent to x_0 on $[a,b]$.

Proof. It follows from the assumptions that

$$\|F_k(\tilde{x}_k, a+) - F_k(\tilde{x}_k, a) - F_k(\tilde{x}_0, a+) + F_k(\tilde{x}_0, a)\| \leq \omega(\|\tilde{x}_k - \tilde{x}_0\|)(h(a+) - h(a)) \to 0$$

for $k \to \infty$. According to Lemma 2.8, we have $F_k(\tilde{x}_0, a+) \to F_0(\tilde{x}_0, a+)$ for $k \to \infty$. Therefore, we get

$$F_k(\tilde{x}_0, a+) - F_k(\tilde{x}_0, a) \to F_0(\tilde{x}_0, a+) - F_0(\tilde{x}_0, a),$$

and consequently

$$\lim_{k\to\infty} (\tilde{x}_k + F_k(\tilde{x}_k, a+) - F_k(\tilde{x}_k, a)) = \tilde{x}_0 + F_0(\tilde{x}_0, a+) - F_0(\tilde{x}_0, a) = x_0(a+) \tag{4.9}$$

(the last equality follows from Lemma 2.10).

Choose a $\delta > 0$ such that $h(a+\delta) - h(a+) < \rho/2$. If $y \in X$ satisfies $\|y - x_0(a+)\| < \rho/2$, then

$$\|y - x_0(a+\delta)\| \leq \|y - x_0(a+)\| + \|x_0(a+) - x_0(a+\delta)\| < \rho/2 + h(a+\delta) - h(a+) < \rho,$$

and therefore $y \in B$, i.e., the $\rho/2$-neighborhood of $x_0(a+)$ is contained in B. This observation together with (4.9) imply the existence of a $k_0 \in \mathbb{N}$ such that the values

$$\tilde{x}_k + F_k(\tilde{x}_k, a+) - F_k(\tilde{x}_k, a), \quad k \geq k_0,$$

together with their $\rho/4$-neighborhoods, are contained in B. Moreover, we can assume that $\tilde{x}_k \in B$ for all $k \geq k_0$. By Theorem 3.2, there exist a $\Delta > 0$ and a sequence of functions $x_k : [a, a+\Delta] \to B$, $k \geq k_0$, such that

$$\frac{\mathrm{d}x_k}{\mathrm{d}\tau} = DF_k(x_k, t), \quad t \in [a, a+\Delta], \quad x_k(a) = \tilde{x}_k, \quad k \geq k_0 \tag{4.10}$$

(note that Theorem 3.2 requires the set B to be open; if necessary, we can replace our B by the open ρ-neighborhood of x_0). According to Lemma 4.6, the sequence $\{x_k\}_{k=k_0}^{\infty}$ is uniformly convergent to x_0.

Up to this point, we have verified the statement of the theorem on the interval $[a, a+\Delta]$. For contradiction, assume that the theorem does not hold on the whole interval $[a,b]$, i.e., there exists a $c \in (a,b)$ such that the theorem holds on $[a,d]$ for every $d < c$, but not on $[a,d]$ with $d > c$. For every $k \geq k_0$, it follows from Lemma 2.3 that

$$\|x_k(v) - x_k(u)\| \leq h(v) - h(u), \quad [u,v] \subset [a,c), \quad k \in \mathbb{N}.$$

Now, the existence of $\lim_{s\to c-} h(s)$ implies the existence of $\lim_{s\to c-} x_k(s)$ for every $k \in \mathbb{N}$. By letting $x_k(c) = x_k(c-)$, we see that for all $k \geq k_0$, Eq. (4.10) has a unique solution defined on the closed interval $[a,c]$. According to the Moore-Osgood theorem, we also have $\lim_{k\to\infty} x_k(c) = \lim_{s\to c-} x_0(s) = x_0(c)$. We can now follow the argumentation from the first part of the present proof with a replaced by c to conclude that the theorem holds on an interval $[a,d]$ with $d > c$, which is a contradiction. □

We conclude our discussion of continuous dependence theorems for generalized differential equations with two remarks:

- In this section, we were interested in continuous dependence theorems for sequences of equations where all the right-hand sides $F_k : G \to X$, $k \in \mathbb{N}_0$, are elements of the same class $\mathcal{F}(G, h, \omega)$. In the finite-dimensional case, there exist theorems applicable in the situation when $F_k \in \mathcal{F}(G, h_k, \omega)$ and the sequence $\{h_k\}_{k=0}^{\infty}$ satisfies some additional conditions; see [32, Theorem 8.5], [32, Theorem 8.8], [15, Theorem 2.4], and [15, Theorem 2.6]. Therefore, it is natural to ask whether these results remain valid in infinite dimension. It is not difficult to check that the answer is affirmative in case of [32, Theorem 8.5] and [15, Theorem 2.4], which are similar to Theorem 4.1, and their original proofs are still applicable without any changes. On the other hand, [32, Theorem 8.8] and [15, Theorem 2.6], which are similar to Theorem 4.2, are not true in infinite dimension; again, the reason is that the assumption $F_k \in \mathcal{F}(G, h_k, \omega)$ no longer guarantees local existence of solutions when $\omega : [0, \infty) \to [0, \infty)$ is an arbitrary increasing continuous function (cf. Example 4.5). We leave it as an open problem to find out whether the infinite-dimensional counterparts of these two theorems are valid under the additional assumption $\lim_{v\to 0+} \int_v^u \frac{\mathrm{d}r}{\omega(r)} = \infty$.

- In his recent book [21], J. Kurzweil considered abstract nonlinear generalized equations of the form

$$x(s) = x(a) + \int_a^s DF(x(\tau), \tau, t), \quad s \in [a, b], \tag{4.11}$$

where the integral on the right-hand side is the strong Kurzweil integral (see [21, Chapter 14]). On one hand, his equations are more general because F can depend explicitly on τ. On the other hand, strong Kurzweil integrability is a more restrictive property that ordinary Kurzweil integrability.

In [21, Lemma 23.8], we find a continuous dependence theorem for equations of the form (4.11). In the special case when the right-hand side F does not depend on τ, J. Kurzweil's four conditions (23.2)–(23.5) reduce to the two inequalities

$$\|F(x, t_2) - F(x, t_1)\| \le (1 + \|x\|)(\Phi(t_2) - \Phi(t_1)), \quad x \in B, \quad [t_1, t_2] \subset [a, b], \tag{4.12}$$

$$\|F(x, t_2) - F(x, t_1) - F(y, t_2) + F(y, t_1)\| \le \|x - y\|(\Phi(t_2) - \Phi(t_1)), \quad x, y \in B, \quad [t_1, t_2] \subset [a, b], \tag{4.13}$$

where $\Phi : [a, b] \to \mathbb{R}$ is a nondecreasing left-continuous function. Note that the theorem in [21] is stated for $B = X$, but remains valid for any $B \subset X$. For example, an ordinary differential equation with a locally Lipschitz-continuous right-hand side is equivalent to a generalized differential equation whose right-hand side F satisfies (4.13) only locally. Also, solutions of generalized differential equations are regulated, and therefore bounded on compact intervals. Thus, the situation when B is bounded is quite natural. In this case, it is not difficult to check that the conditions (4.12), (4.13) hold if and only if $F \in \mathcal{F}(B \times [a, b], h, \omega_1)$, where $\omega_1(r) = r$ for all $r \ge 0$. Therefore, J. Kurzweil's theorem is seemingly similar to our Theorem 4.7 with $\omega = \omega_1$, but its precise formulation is different: The theorem starts with two right-hand sides F_1, $F_2 : B \times [a, b] \to X$ satisfying the inequality

$$\|F_1(x, t_2) - F_1(x, t_1) - F_2(x, t_2) + F_2(x, t_1)\| \le (1 + \|x\|)(\Phi^*(t_2) - \Phi^*(t_1)),$$

for all $x \in B$ and $[t_1, t_2] \subset [a, b]$, where $\Phi^* : [a, b] \to \mathbb{R}$ is a nondecreasing left-continuous function. Then, assuming that x_1, $x_2 : [a, b] \to X$ satisfy $\frac{\mathrm{d}x_i}{\mathrm{d}\tau} = DF_i(x_i, t)$ and $x_1(a) = x_2(a)$, the theorem provides an estimate of the form $\|x_1 - x_2\|_\infty \le C(\Phi^*(b) - \Phi^*(a))$.

5 Measure functional differential equations

As an application of our results for abstract differential equations, let us study the well-posedness for measure functional differential equations of the form

$$y(t) = y(t_0) + \int_{t_0}^t f(y_s, s)\, \mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \tag{5.1}$$

where y and f take values in $\mathbb{R}^n$, and the integral on the right-hand side is the Kurzweil-Stieltjes integral with respect to a nondecreasing function $g : [t_0, t_0 + \sigma] \to \mathbb{R}$. These equations generalize other types of functional equations, such as classical functional differential equations, impulsive functional differential equations, or functional dynamic equations on time scales (see [10, 11]). In [10], it was shown that there is a one-to-one correspondence between measure functional differential equations with finite delay and generalized ordinary differential equations whose solutions take values in certain infinite-dimensional spaces. In [37], this correspondence was extended to equations with infinite delay and axiomatically described phase space. For simplicity, we do not discuss general phase spaces as in [27, 37], but restrict ourselves to the phase space $G((-\infty,0],\mathbb{R}^n)$.

The correspondence between measure functional differential equations and generalized ordinary differential equations is constructed as follows: We take $B\subset\mathbb{R}^n$, $O=G((-\infty,t_0+\sigma],B)$, $P=G((-\infty,0],B)$, and consider a function $f:P\times[t_0,t_0+\sigma]\to\mathbb{R}^n$ (note that $x_t\in P$ whenever $x\in O$ and $t\in[t_0,t_0+\sigma]$). Now, under certain assumptions, Eq. (5.1) is equivalent (in a sense described below) to the abstract generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}=DF(x,t),\quad t\in[t_0,t_0+\sigma],\tag{5.2}$$

in the Banach space $X=G((-\infty,t_0+\sigma],\mathbb{R}^n)$, where the solution x takes values in $O\subset X$, and the function $F:O\times[t_0,t_0+\sigma]\to X$ is given by

$$F(x,t)(\vartheta)=\begin{cases}0, & -\infty<\vartheta\le t_0,\\ \int_{t_0}^{\vartheta}f(x_s,s)\,\mathrm{d}g(s), & t_0\le\vartheta\le t\le t_0+\sigma,\\ \int_{t_0}^{t}f(x_s,s)\,\mathrm{d}g(s), & t\le\vartheta\le t_0+\sigma\end{cases}\tag{5.3}$$

for every $x\in O$ and $t\in[t_0,t_0+\sigma]$.

At this moment, we need the following conditions concerning the function f:

(A) The integral $\int_{t_0}^{t_0+\sigma}f(y_t,t)\,\mathrm{d}g(t)$ exists for every $y\in O$.

(B) There exists a function $M:[t_0,t_0+\sigma]\to[0,\infty)$, which is Kurzweil-Stieltjes integrable with respect to g, such that

$$\left\|\int_a^b f(y_t,t)\,\mathrm{d}g(t)\right\|\le\int_a^b M(t)\,\mathrm{d}g(t),\quad y\in O,\quad [a,b]\subseteq[t_0,t_0+\sigma].$$

(C) There exists a function $L:[t_0,t_0+\sigma]\to[0,\infty)$, which is Kurzweil-Stieltjes integrable with respect to g, and a continuous increasing function $\omega:[0,\infty)\to[0,\infty)$ such that $\omega(0)=0$,

$$\left\|\int_a^b (f(y_t,t)-f(z_t,t))\,\mathrm{d}g(t)\right\|\le\int_a^b L(t)\omega(\|y_t-z_t\|_\infty)\,\mathrm{d}g(t),\quad y,z\in O,\quad [a,b]\subseteq[t_0,t_0+\sigma].$$

The next lemma is a straightforward generalization of [37, Lemma 3.4], which corresponds to the special case $\omega(r)=r$.

Lemma 5.1. *Assume that $B\subset\mathbb{R}^n$, $O=G((-\infty,t_0+\sigma],B)$, $P=G((-\infty,0],B)$, $g:[t_0,t_0+\sigma]\to\mathbb{R}$ is a nondecreasing function, $f:P\times[t_0,t_0+\sigma]\to\mathbb{R}^n$ satisfies conditions* (A), (B), (C). *Then, the function $F:O\times[t_0,t_0+\sigma]\to G((-\infty,t_0+\sigma],\mathbb{R}^n)$ given by* (5.3) *is an element of $\mathcal{F}(O\times[t_0,t_0+\sigma],h_1,h_2,\omega)$, where $h_1(t)=\int_{t_0}^t M(s)\,\mathrm{d}g(s)$ and $h_2(t)=\int_{t_0}^t L(s)\,\mathrm{d}g(s)$ for all $s\in[t_0,t_0+\sigma]$.*

Proof. Condition (A) guarantees that the integrals in the definition of F exist. When $[s_1,s_2]\subset[t_0,t_0+\sigma]$, we have

$$F(y,s_2)(\tau)-F(y,s_1)(\tau)=\begin{cases}0, & -\infty<\tau\le s_1,\\ \int_{s_1}^{\tau}f(y_s,s)\,\mathrm{d}g(s), & s_1\le\tau\le s_2,\\ \int_{s_1}^{s_2}f(y_s,s)\,\mathrm{d}g(s), & s_2\le\tau\le t_0+\sigma\end{cases}$$

for every $y\in O$. Using condition (B), we get

$$\|F(y,s_2)-F(y,s_1)\|_\infty=\sup_{\tau\in[s_1,s_2]}\|F(y,s_2)(\tau)-F(y,s_1)(\tau)\|=$$

$$=\sup_{\tau\in[s_1,s_2]}\left\|\int_{s_1}^{\tau}f(y_s,s)\,\mathrm{d}g(s)\right\|\le\int_{s_1}^{s_2}M(s)\,\mathrm{d}g(s)=h_1(s_2)-h_1(s_1).$$

Similarly, condition (C) implies that for every $y, z \in O$, we have

$$\|F(y,s_2) - F(y,s_1) - F(z,s_2) + F(z,s_1)\|_\infty$$

$$= \sup_{\tau\in[s_1,s_2]} \|F(y,s_2)(\tau) - F(y,s_1)(\tau) - F(z,s_2)(\tau) + F(z,s_1)(\tau)\| = \sup_{\tau\in[s_1,s_2]} \left\| \int_{s_1}^{\tau} (f(y_s,s) - f(z_s,s))\,\mathrm{d}g(s) \right\|$$

$$\le \int_{s_1}^{s_2} L(s)\omega(\|y_s - z_s\|_\infty)\,\mathrm{d}g(s) \le \omega(\|y-z\|_\infty) \left(\int_{s_1}^{s_2} L(s)\,\mathrm{d}g(s) \right) = \omega(\|y-z\|_\infty)(h_2(s_2) - h_2(s_1)). \qquad \square$$

The next two theorems describe the precise relationship between solutions of the measure functional differential equation (5.1) and solutions of the generalized differential equation (5.2).

The proofs for the special case $\omega(r) = r$ can be found in [37]. In the general case, the proofs require some small modifications that are described below.

Theorem 5.2. *Assume that $B \subset \mathbb{R}^n$ is open, $O = G((-\infty, t_0+\sigma], B)$, $P = G((-\infty, 0], B)$, $\phi \in P$, $g : [t_0, t_0+\sigma] \to \mathbb{R}$ is a nondecreasing function, $f : P \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ satisfies conditions* (A), (B), (C), *and $F : O \times [t_0, t_0+\sigma] \to G((-\infty, t_0+\sigma], \mathbb{R}^n)$ is given by* (5.3).

If $y \in O$ is a solution of the measure functional differential equation

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} f(y_s,s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\ y_{t_0} &= \phi, \end{aligned}$$

then the function $x : [t_0, t_0+\sigma] \to O$ given by

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), & \vartheta \in (-\infty, t], \\ y(t), & \vartheta \in [t, t_0+\sigma] \end{cases}$$

is a solution of the generalized ordinary differential equation (5.2).

Proof. A proof for the special case $\omega(r) = r$ can be found in [37, Theorem 3.6]. In the general case, it is enough to modify the proof from [37] as follows:

Given an $\varepsilon > 0$, there exists an $r_0 > 0$ such that $\omega(r) \le \varepsilon$ for all $r \in [0, r_0]$. The function $h(t) = \int_{t_0}^{t} M(s)\,\mathrm{d}g(s)$ has only finitely many points $t \in [t_0, t_0+\sigma]$ such that $\Delta^+ h(t) \ge r_0$; denote these points by $t_1, \dots, t_m$. Find a gauge $\delta : [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

$$\delta(\tau) < \min\left\{ \frac{t_k - t_{k-1}}{2},\ k = 2, \dots, m \right\}, \quad \tau \in [t_0, t_0+\sigma],$$

$$\delta(\tau) < \min\{|\tau - t_k|;\ k = 1, \dots, m\}, \quad \tau \in [t_0, t_0+\sigma] \backslash \{t_1, \dots, t_m\},$$

$$\int_{t_k}^{t_k+\delta(t_k)} L(s)\omega(\|y_s - x(t_k)_s\|_\infty)\,\mathrm{d}g(s) < \frac{\varepsilon}{2m+1}, \quad k \in \{1, \dots, m\},$$

$$\|y(\rho) - y(\tau)\| \le r_0, \quad \tau \in [t_0, t_0+\sigma] \backslash \{t_1, \dots, t_m\}, \quad \rho \in [\tau, \tau + \delta(\tau)).$$

Then proceed as in [37] with obvious modifications. $\square$

Theorem 5.3. *Assume that $B \subset \mathbb{R}^n$ is open, $O = G((-\infty, t_0+\sigma], B)$, $P = G((-\infty, 0], B)$, $\phi \in P$, $g : [t_0, t_0+\sigma] \to \mathbb{R}$ is a nondecreasing function, $f : P \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ satisfies conditions* (A), (B), (C), *and $F : O \times [t_0, t_0+\sigma] \to G((-\infty, t_0+\sigma], \mathbb{R}^n)$ is given by* (5.3).

If $x : [t_0, t_0+\sigma] \to O$ is a solution of the generalized ordinary differential equation (5.2) *with the initial condition*

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & \vartheta \in (-\infty, t_0], \\ \phi(0), & \vartheta \in [t_0, t_0+\sigma], \end{cases}$$

then the function $y \in O$ defined by

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & \vartheta \in (-\infty, t_0], \\ x(\vartheta)(\vartheta), & \vartheta \in [t_0, t_0 + \sigma] \end{cases}$$

is a solution of the measure functional differential equation

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} f(y_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \\ y_{t_0} &= \phi. \end{aligned}$$

Proof. A proof for the special case $\omega(r) = r$ can be found in [37, Theorem 3.7]. In the general case, it is enough to modify the proof from [37] in the same way as described in the proof of Theorem 5.2, except that the last condition on the gauge δ should be replaced by

$$|h(\rho) - h(\tau)| \le r_0, \quad \tau \in [t_0, t_0 + \sigma] \backslash \{t_1, \ldots, t_m\}, \quad \rho \in [\tau, \tau + \delta(\tau)).$$

Then proceed as in [37] with obvious modifications. □

We now present an Osgood-type existence theorem for measure functional differential equations with infinite delay based on Theorem 3.2. Our result generalizes [10, Theorem 5.3] and [37, Theorem 3.12], which corresponds to the special case $\omega(r) = r$. Even in that case, our theorem provides more information since it specifies a lower bound for the length of the interval where the solution is guaranteed to exist.

Theorem 5.4. *Assume that $B \subset \mathbb{R}^n$ is open, $O = G((-\infty, t_0+\sigma], B)$, $P = G((-\infty, 0], B)$, $g : [t_0, t_0+\sigma] \to \mathbb{R}$ is nondecreasing and left-continuous function, and $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies conditions (A), (B), (C), where the function $\omega : [0, \infty) \to [0, \infty)$ is such that*

$$\lim_{v \to 0+} \int_{v}^{u} \frac{\mathrm{d}r}{\omega(r)} = \infty$$

for every $u > 0$.

If $\phi \in P$ is such that $\phi(0) + f(\phi, t_0)\Delta^+ g(t_0) \in B$, then the initial-value problem

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \qquad y_{t_0} = \phi, \tag{5.4}$$

has a unique local solution defined on a right neighborhood of t_0.

Moreover, if $\Delta > 0$ is such that the closed ball

$$\{x \in \mathbb{R}^n;\ \|x - (\phi(0) + f(\phi, t_0)\Delta^+ g(t_0))\| \le \int_{t_0+}^{t_0+\Delta} M(s)\,\mathrm{d}g(s)\} \tag{5.5}$$

is contained in B, the solution is guaranteed to exist on $[t_0, t_0 + \Delta]$.

Proof. According to Lemma 5.1, the function F given by (5.3) is an element of $\mathcal{F}(O \times [t_0, t_0+\sigma], h_1, h_2, \omega)$, where $h_1(t) = \int_{t_0}^{t} M(s)\,\mathrm{d}g(s)$ and $h_2(t) = \int_{t_0}^{t} L(s)\,\mathrm{d}g(s)$. Since g is left-continuous, h_1 and h_2 have the same property.

By Theorems 5.2 and 5.3, the initial-value problem (5.4) is equivalent to

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t), \quad t \in [t_0, t_0 + \sigma], \quad x(t_0) = x_0, \tag{5.6}$$

where x_0 equals

$$x_0(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & \vartheta \in (-\infty, t_0], \\ \phi(0), & \vartheta \in [t_0, t_0 + \sigma]. \end{cases}$$

The function F is regulated with respect to the second variable (this follows from the fact that $\mathcal{F}(O \times [t_0, t_0+\sigma], h_1, h_2, \omega)$, where h_1 is regulated). Thus for every $t \in [t_0, t_0+\sigma)$, the right-sided limit $F(x, t+)$ exists and $F(x, t+)(\vartheta) = \lim_{\delta \to 0+} F(x, t+\delta)(\vartheta)$. Using Theorem 2.12 and the definition of F given in (5.3), we obtain

$$(F(x,t+) - F(x,t))(\vartheta) = \begin{cases} 0, & \vartheta \in (-\infty, t], \\ f(x_t, t)\Delta^+ g(t), & \vartheta \in (t, t_0+\sigma]. \end{cases}$$

By Theorem 3.2, Eq. (5.6) has a unique local solution if $x_0 + F(x_0, t_0+) - F(x_0, t_0) \in O$. Since

$$(x_0 + F(x_0, t_0+) - F(x_0, t_0))(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & \vartheta \in (-\infty, t_0], \\ \phi(0) + f(\phi, t_0)\Delta^+ g(t_0), & \vartheta \in (t_0, t_0+\sigma], \end{cases}$$

the condition is satisfied if and only if $\phi(0) + f(\phi, t_0)\Delta^+ g(t_0) \in B$.

For every $x \in O$ and $t \in [t_0, t_0+\sigma]$, $F(x,t)$ vanishes on $(-\infty, t_0]$. Thus, the solution of (5.6) can never leave the set

$$Y = \{x \in G((-\infty, t_0+\sigma], \mathbb{R}^n);\ x(\vartheta) = \phi(\vartheta - t_0) \text{ for } \vartheta \in (-\infty, t_0]\}.$$

By Remark 3.3, the solution is guaranteed to exist on $[a, a+\Delta]$ if the following statements hold:

- If $x \in Y$ and $t \in [t_0, t_0+\Delta)$, then $x + F(x, t+) - F(x,t) \in Y$.
- If $x \in Y$ and $\|x - (x_0 + F(x_0, t_0+) - F(x_0, t_0))\| \le h_1(t_0+\Delta) - h_1(t_0+)$, then $x \in O$.

The first condition is clearly satisfied, because $F(x, t+) - F(x,t)$ equals zero on $(-\infty, t_0]$. Also, we have $h_1(t_0+\Delta) - h_1(t_0+) = \int_{t_0+}^{t_0+\Delta} M(s)\,\mathrm{d}g(s)$ and

$$(x - (x_0 + F(x_0, t_0+) - F(x_0, t_0)))(\vartheta) = \begin{cases} 0, & \vartheta \in (-\infty, t_0], \\ x(\vartheta) - (\phi(0) + f(\phi, t_0)\Delta^+ g(t_0)), & \vartheta \in (t_0, t_0+\sigma], \end{cases}$$

for every $x \in Y$. Hence, the second condition is satisfied if the closed ball (5.5) is contained in B. $\square$

Let us proceed to continuous dependence of solutions to nonlinear measure functional differential equations. The only theorem available in the literature is [10, Theorem 6.3], which applies to equations with finite delay and is similar in spirit to Theorem 4.3 (i.e., it states that under certain assumptions, the limit of solutions is a solution again). The next result for measure functional differential equations with infinite delay is based on Theorem 4.7. Therefore, it is new even in the special case when the delay is finite and $\omega(r) = r$.

Theorem 5.5. *Assume that $B \subset \mathbb{R}^n$, $O = G((-\infty, t_0+\sigma], B)$, $P = G((-\infty, 0], B)$, $g : [t_0, t_0+\sigma] \to \mathbb{R}$ is a nondecreasing left-continuous function, and $f_k : P \times [t_0, t_0+\sigma] \to \mathbb{R}^n$, $k \in \mathbb{N}_0$, is a sequence of functions satisfying the following conditions:*

1. *The integral $\int_{t_0}^{t_0+\sigma} f_k(y_t, t)\,\mathrm{d}g(t)$ exists for every $k \in \mathbb{N}_0$, $y \in O$.*
2. *There exists a function $M : [t_0, t_0+\sigma] \to [0, \infty)$, which is Kurzweil-Stieltjes integrable with respect to g, such that*
$$\left\| \int_a^b f_k(y_t, t)\,\mathrm{d}g(t) \right\| \le \int_a^b M(t)\,\mathrm{d}g(t), \quad k \in \mathbb{N}_0, \quad y \in O, \quad [a,b] \subseteq [t_0, t_0+\sigma].$$
3. *There exists a function $L : [t_0, t_0+\sigma] \to [0, \infty)$, which is Kurzweil-Stieltjes integrable with respect to g, and a continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ such that $\omega(0) = 0$, $\lim_{v \to 0+} \int_v^u \frac{\mathrm{d}r}{\omega(r)} = \infty$ for every $u > 0$, and*
$$\left\| \int_a^b (f_k(y_t, t) - f_k(z_t, t))\,\mathrm{d}g(t) \right\| \le \int_a^b L(t)\omega(\|y_t - z_t\|_\infty)\,\mathrm{d}g(t), \quad k \in \mathbb{N}_0, \quad y, z \in O, \quad [a,b] \subseteq [t_0, t_0+\sigma].$$

4. For every $y \in O$,

$$\lim_{k\to\infty}\int_{t_0}^t f_k(y_s,s)\,\mathrm{d}g(s)=\int_{t_0}^t f_0(y_s,s)\,\mathrm{d}g(s)$$

uniformly with respect to $t\in[t_0,t_0+\sigma]$.

Suppose that $\phi_0\in P$ and $y_0:(-\infty,t_0+\sigma]\to B$ satisfies

$$y_0(t)=y_0(t_0)+\int_{t_0}^t f_0((y_0)_s,s)\,\mathrm{d}g(s),\quad t\in[t_0,t_0+\sigma],\qquad (y_0)_{t_0}=\phi_0.$$

Also, assume there exists a $\rho>0$ such that $\|y-y_0(s)\|<\rho$ implies $y\in B$ whenever $s\in(-\infty,t_0+\sigma]$. Then, given an arbitrary sequence $\phi_k\in P$, $k\in\mathbb{N}$, such that $\lim_{k\to\infty}\|\phi_k-\phi_0\|_\infty=0$, there are a $k_0\in\mathbb{N}$ and a sequence of functions $y_k:(-\infty,t_0+\sigma]\to B$, $k\ge k_0$, which satisfy

$$y_k(t)=y_k(t_0)+\int_{t_0}^t f_k((y_k)_s,s)\,\mathrm{d}g(s),\quad t\in[t_0,t_0+\sigma],\qquad (y_k)_{t_0}=\phi_k. \tag{5.7}$$

Moreover, the sequence $\{y_k\}_{k=k_0}^\infty$ is uniformly convergent to y_0 on $(-\infty,t_0+\sigma]$.

Proof. For every $k\in\mathbb{N}_0$, let the function $F_k:O\times[t_0,t_0+\sigma]\to G((-\infty,t_0+\sigma],\mathbb{R}^n)$ be given by

$$F_k(x,t)(\vartheta)=\begin{cases}0, & -\infty<\vartheta\le t_0,\\ \int_{t_0}^{\vartheta} f_k(x_s,s)\,\mathrm{d}g(s), & t_0\le\vartheta\le t\le t_0+\sigma,\\ \int_{t_0}^{t} f_k(x_s,s)\,\mathrm{d}g(s), & t\le\vartheta\le t_0+\sigma\end{cases}$$

for every $x\in O$ and $t\in[t_0,t_0+\sigma]$. According to Lemma 5.1, we have $F_k\in\mathcal{F}(O\times[t_0,t_0+\sigma],h_1,h_2,\omega)$, where h_1, h_2 are left-continuous functions. Moreover, it follows from assumption 4 that

$$\lim_{k\to\infty}F_k(x,t)=F_0(x,t),\quad (x,t)\in O\times[t_0,t_0+\sigma].$$

For every $k\in\mathbb{N}_0$, let $\tilde{x}_k\in O$ be defined as

$$\tilde{x}_k(\vartheta)=\begin{cases}\phi_k(\vartheta-t_0), & \vartheta\in(-\infty,t_0],\\ \phi_k(0), & \vartheta\in[t_0,t_0+\sigma].\end{cases}$$

By Theorem 5.2, the function $x_0:[t_0,t_0+\sigma]\to O$ given by

$$x_0(t)(\vartheta)=\begin{cases}y_0(\vartheta), & \vartheta\in(-\infty,t],\\ y_0(t), & \vartheta\in[t,t_0+\sigma]\end{cases}$$

satisfies

$$\frac{\mathrm{d}x_0}{\mathrm{d}\tau}=DF_0(x_0,t),\quad t\in[t_0,t_0+\sigma],\quad x_0(t_0)=\tilde{x}_0.$$

Also, note that $\lim_{k\to\infty}\tilde{x}_k=\tilde{x}_0$. Hence, by Theorem 4.7, there is a $k_0\in\mathbb{N}$ and a sequence of functions $x_k:[t_0,t_0+\sigma]\to O$, $k\ge k_0$, which is uniformly convergent to x_0 on $[t_0,t_0+\sigma]$, and

$$\frac{\mathrm{d}x_k}{\mathrm{d}\tau}=DF_k(x_k,t),\quad t\in[t_0,t_0+\sigma],\quad x_k(t_0)=\tilde{x}_k,\quad k\ge k_0.$$

For every $k\in\mathbb{N}$, let $y_k\in O$ be given by

$$y_k(\vartheta)=\begin{cases}x_k(t_0)(\vartheta), & \vartheta\in(-\infty,t_0],\\ x_k(\vartheta)(\vartheta), & \vartheta\in[t_0,t_0+\sigma].\end{cases}$$

By Theorem 5.3, the function y_k is a solution of the initial-value problem (5.7). Also, since the sequence $\{x_k\}_{k=k_0}^\infty$ is uniformly convergent to x_0, it follows that $\{y_k\}_{k=k_0}^\infty$ is uniformly convergent to y_0. □

6 Functional differential equations with impulses

In this section, we consider the well-posedness for impulsive functional differential equations of the form

$$\begin{aligned} y'(t) &= f(y_t, t), \quad \text{almost everywhere in } [t_0, t_0+\sigma], \\ \Delta^+ y(t_i) &= I_i(y(t_i)), \quad i \in \{1, \dots, m\}, \end{aligned}$$

where the impulses take place at preassigned times $t_1, \dots, t_m \in [t_0, t_0+\sigma)$, and their action is described by the operators $I_i : \mathbb{R}^n \to \mathbb{R}^n$, $i \in \{1, \dots, m\}$; the solution is assumed to be left-continuous at every point t_i, and absolutely continuous on every interval whose intersection with $\{t_1, \dots, t_m\}$ is empty. The equivalent integral form of the problem is

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s)\,\mathrm{d}s + \sum_{\substack{i \in \{1,\dots,m\}, \\ t_i < t}} I_i(y(t_i)), \quad t \in [t_0, t_0+\sigma]. \tag{6.1}$$

Using the following lemma, it is possible to convert impulsive functional differential equations into measure functional differential equations. The lemma is a consequence of [11, Lemma 2.4].

Lemma 6.1. *Let $m \in \mathbb{N}$, $a \le t_1 < t_2 < \dots < t_m < b$, and*

$$g(s) = s + \sum_{i=1}^{m} \chi_{(t_i,\infty)}(s), \quad s \in [a, b]$$

(where χ_A denotes the characteristic function of a set $A \subset \mathbb{R}$). Consider an arbitrary pair of functions $f : [a, b] \to \mathbb{R}$ and $\tilde{f} : [a, b] \to \mathbb{R}$ such that $\tilde{f}(s) = f(s)$ for every $s \in [a, b] \backslash \{t_1, \dots, t_m\}$.

Then the integral $\int_a^b \tilde{f}(s)\,\mathrm{d}g(s)$ exists if and only if the integral $\int_a^b f(s)\,\mathrm{d}s$ exists; in that case, we have

$$\int_a^t \tilde{f}(s)\,\mathrm{d}g(s) = \int_a^t f(s)\,\mathrm{d}s + \sum_{\substack{i \in \{1,\dots,m\}, \\ t_i < t}} \tilde{f}(t_i), \quad t \in [a, b].$$

According to the last lemma, the impulsive equation (6.1) is equivalent to the measure functional differential equation

$$y(t) = y(t_0) + \int_{t_0}^{t} \tilde{f}(y_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \tag{6.2}$$

where $g(s) = s + \sum_{i=1}^{m} \chi_{(t_i,\infty)}(s)$ and

$$\tilde{f}(x, s) = \begin{cases} f(x, s) & \text{for } s \in [t_0, t_0+\sigma] \backslash \{t_1, \dots, t_m\}, \\ I_i(x(0)) & \text{for } s = t_i,\ i \in \{1, \dots, m\}. \end{cases}$$

Now, it is a simple task to obtain an Osgood-type existence theorem for impulsive functional differential equations with infinite delay.

Theorem 6.2. *Let $B \subset \mathbb{R}^n$ be open, $O = G((-\infty, t_0+\sigma], B)$, $P = G((-\infty, 0], B)$, $m \in \mathbb{N}$, $t_0 \le t_1 < t_2 < \dots < t_m < t_0+\sigma$. Consider functions $f : P \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ and $I_1, \dots, I_m : B \to \mathbb{R}^n$ such that the following conditions are satisfied:*

1. *The integral $\int_{t_0}^{t_0+\sigma} f(y_t, t)\,\mathrm{d}t$ exists for every $y \in O$.*
2. *There exists an integrable function $M : [t_0, t_0+\sigma] \to [0, \infty)$ such that*

$$\left\| \int_a^b f(y_t, t)\,\mathrm{d}t \right\| \le \int_a^b M(t)\,\mathrm{d}t, \quad y \in O, \quad [a, b] \subseteq [t_0, t_0+\sigma].$$

3. There exists an integrable function $L : [t_0, t_0 + \sigma] \to [0, \infty)$ *and a continuous increasing function* $\omega : [0, \infty) \to [0, \infty)$ *such that* $\omega(0) = 0$, $\lim_{v\to 0+} \int_v^u \frac{\mathrm{d}r}{\omega(r)} = \infty$ *for every* $u > 0$, *and*

$$\left\| \int_a^b (f(y_t, t) - f(z_t, t))\,\mathrm{d}t \right\| \le \int_a^b L(t)\omega(\|y_t - z_t\|_\infty)\,\mathrm{d}t, \quad y, z \in O, \quad [a, b] \subseteq [t_0, t_0 + \sigma].$$

4. There exist constants $\alpha_1, \dots, \alpha_m \ge 0$ *such that* $\|I_i(x)\| \le \alpha_i$ *for* $i \in \{1, \dots, m\}$, $x \in B$.

5. There exists a constant $\beta > 0$ *such that* $\|I_i(x) - I_i(y)\| \le \beta\omega(\|x - y\|)$ *for* $i \in \{1, \dots, m\}$, x, $y \in B$.

Let $\phi \in P$ *and assume that either* $t_0 < t_1$, *or* $t_0 = t_1$ *and* $\phi(0) + I_1(\phi(0)) \in B$. *Then, the initial-value problem*

$$y(t) = y(t_0) + \int_{t_0}^t f(y_s, s)\,\mathrm{d}s + \sum_{\substack{i\in\{1,\dots,m\},\\ t_i<t}} I_i(y(t_i)), \quad t \in [t_0, t_0 + \sigma], \qquad y_{t_0} = \phi. \tag{6.3}$$

has a unique local solution defined on a right neighborhood of t_0.

Moreover, if $\Delta > 0$ *is such that either* $t_0 < t_1$ *and*

$$\{x \in \mathbb{R}^n;\ \|x - \phi(0)\| \le \int_{t_0}^{t_0+\Delta} M(s)\,\mathrm{d}s + \sum_{i=1}^m \alpha_i\} \subset B,$$

or $t_0 = t_1$ *and*

$$\{x \in \mathbb{R}^n;\ \|x - (\phi(0) + I_1(\phi(0)))\| \le \int_{t_0}^{t_0+\Delta} M(s)\,\mathrm{d}s + \sum_{i=2}^m \alpha_i\} \subset B,$$

then the solution is guaranteed to exist on $[t_0, t_0 + \Delta]$.

Proof. It is enough to prove that the equivalent measure functional differential equation (6.2) has a unique local solution satisfying $y_{t_0} = \phi$. Let us verify that all assumptions of Theorem 5.5 are satisfied.

Assume that $[a, b] \subset [t_0, t_0 + \sigma]$ and $y, z \in O$. From Lemma 6.1, we obtain

$$\left\| \int_a^b \tilde{f}(y_t, t)\,\mathrm{d}g(t) \right\| = \left\| \int_a^b f(y_t, t)\,\mathrm{d}t + \sum_{\substack{i\in\{1,\dots,m\},\\ a\le t_i<b}} I_i(y(t_i)) \right\| \le \int_a^b M(t)\,\mathrm{d}t + \sum_{\substack{i\in\{1,\dots,m\},\\ a\le t_i<b}} \alpha_i = \int_a^b \tilde{M}(t)\,\mathrm{d}g(t),$$

where

$$\tilde{M}(t) = \begin{cases} M(t) & \text{for } t \in [t_0, t_0 + \sigma]\backslash\{t_1, \dots, t_m\}, \\ \alpha_i & \text{for } t = t_i,\ i \in \{1, \dots, m\}. \end{cases}$$

Similarly,

$$\left\| \int_a^b \left(\tilde{f}(y_t, t) - \tilde{f}(z_t, t)\right) \mathrm{d}g(t) \right\| = \left\| \int_a^b (f(y_t, t) - f(z_t, t))\,\mathrm{d}t + \sum_{\substack{i\in\{1,\dots,m\},\\ a\le t_i<b}} (I_i(y(t_i)) - I_i(z(t_i))) \right\|$$

$$\le \int_a^b L(t)\omega(\|y_t - z_t\|_\infty)\,\mathrm{d}t + \sum_{\substack{i\in\{1,\dots,m\},\\ a\le t_i<b}} \beta\omega(\|y(t_i) - z(t_i)\|)$$

$$\le \int_a^b L(t)\omega(\|y_t - z_t\|_\infty)\,\mathrm{d}t + \sum_{\substack{i\in\{1,\dots,m\},\\ a\le t_i<b}} \beta\omega(\|y_{t_i} - z_{t_i}\|_\infty) = \int_a^b \tilde{L}(t)\omega(\|y_t - z_t\|_\infty)\,\mathrm{d}g(t),$$

where

$$\tilde{L}(t) = \begin{cases} L(t) & \text{for } t \in [t_0, t_0+\sigma]\backslash\{t_1,\dots,t_m\}, \\ \beta & \text{for } t = t_i,\ i \in \{1,\dots,m\}. \end{cases}$$

We have either $t_0 < t_1$ and $\phi(0) + \tilde{f}(\phi, t_0)\Delta^+ g(t_0) = \phi(0) \in B$, or $t_0 = t_1$ and $\phi(0) + \tilde{f}(\phi, t_0)\Delta^+ g(t_0) = \phi(0) + I_1(\phi(0)) \in B$. Hence, all assumptions of Theorem 5.5 are satisfied.

Finally, the two conditions on the length Δ follow from the identity

$$\int_{t_0+}^{t_0+\Delta} \tilde{M}(s)\,\mathrm{d}g(s) = \int_{t_0}^{t_0+\Delta} M(s)\,\mathrm{d}s + \sum_{\substack{i\in\{1,\dots,m\},\\ t_0<t_i<t_0+\sigma}} \alpha_i. \qquad \square$$

Our last result is a continuous dependence theorem for impulsive functional differential equations with infinite delay.

Theorem 6.3. *Let $B \subset \mathbb{R}^n$ be open, $O = G((-\infty, t_0+\sigma], B)$, $P = G((-\infty, 0], B)$, $m \in \mathbb{N}$, $t_0 \le t_1 < t_2 < \dots < t_m < t_0 + \sigma$. Consider sequences of functions $f_k : P \times [t_0, t_0+\sigma] \to \mathbb{R}^n$, $k \in \mathbb{N}_0$, and $I_1^k, \dots, I_m^k : B \to \mathbb{R}^n$, $k \in \mathbb{N}_0$, which satisfy the following conditions:*

1. *The integral $\int_{t_0}^{t_0+\sigma} f_k(y_t, t)\,\mathrm{d}t$ exists for every $k \in \mathbb{N}_0$, $y \in O$.*
2. *There exists an integrable function $M : [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that*
$$\left\| \int_a^b f_k(y_t, t)\,\mathrm{d}t \right\| \le \int_a^b M(t)\,\mathrm{d}t, \quad k \in \mathbb{N}_0, \quad y \in O, \quad [a,b] \subseteq [t_0, t_0+\sigma].$$
3. *There exist a continuous increasing function $\omega : [0,\infty) \to [0,\infty)$ and an integrable function $L : [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that*
$$\left\| \int_a^b (f_k(y_t, t) - f_k(z_t, t))\,\mathrm{d}t \right\| \le \int_a^b L(t)\omega(\|y_t - z_t\|_\infty)\,\mathrm{d}t, \quad k \in \mathbb{N}_0, \quad y, z \in O, \quad [a,b] \subseteq [t_0, t_0+\sigma]$$
and $\lim_{v\to 0+} \int_v^u \frac{\mathrm{d}r}{\omega(r)} = \infty$ for every $u > 0$.
4. *For every $y \in O$,*
$$\lim_{k\to\infty} \int_{t_0}^t f_k(y_s, s)\,\mathrm{d}s = \int_{t_0}^t f_0(y_s, s)\,\mathrm{d}s$$
uniformly with respect to $t \in [t_0, t_0+\sigma]$.
5. *There exists a constant $\alpha > 0$ such that $\|I_i^k(x)\| \le \alpha$ for every $i \in \{1,\dots,m\}$, $k \in \mathbb{N}_0$ and $x \in B$.*
6. *There exists a constant $\beta > 0$ such that $\|I_i^k(x) - I_i^k(y)\| \le \beta\omega(\|x-y\|)$ for every $i \in \{1,\dots,m\}$, $k \in \mathbb{N}_0$ and $x, y \in B$.*
7. *For every $x \in B$ and $k \in \{1,\dots,m\}$, $\lim_{k\to\infty} I_i^k(x) = I_i^0(x)$.*

Suppose that $\phi_0 \in P$ and $y_0 : (-\infty, t_0+\sigma] \to B$ satisfies

$$y_0(t) = y_0(t_0) + \int_{t_0}^t f_0((y_0)_s, s)\,\mathrm{d}s + \sum_{\substack{i\in\{1,\dots,m\},\\ t_i<t}} I_i^0(y_0(t_i)), \quad t \in [t_0, t_0+\sigma], \qquad (y_0)_{t_0} = \phi_0.$$

Also, assume there exists a $\rho > 0$ such that $\|y - y_0(s)\| < \rho$ implies $y \in B$ whenever $s \in (-\infty, t_0+\sigma]$. Then, given an arbitrary sequence $\phi_k \in P$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} \|\phi_k - \phi_0\|_\infty = 0$, there are a $k_0 \in \mathbb{N}$ and a sequence of functions $y_k : (-\infty, t_0+\sigma] \to B$, $k \ge k_0$, which satisfy

$$y_k(t) = y_k(t_0) + \int_{t_0}^t f_k((y_k)_s, s)\,\mathrm{d}s + \sum_{\substack{i\in\{1,\dots,m\},\\ t_i<t}} I_i^k(y_k(t_i)), \quad t \in [t_0, t_0+\sigma], \qquad (y_k)_{t_0} = \phi_k.$$

Moreover, the sequence $\{y_k\}_{k=k_0}^\infty$ is uniformly convergent to y_0 on $(-\infty, t_0+\sigma]$.

Proof. Let $g(s) = s + \sum_{i=1}^{m} \chi_{(t_i,\infty)}(s)$ and

$$\tilde{f}_k(x,s) = \begin{cases} f_k(x,s), & s \in [t_0, t_0+\sigma]\backslash\{t_1,\dots,t_m\}, \\ I_i^k(x(0)), & s = t_i,\ i \in \{1,\dots,m\} \end{cases}$$

for every $k \in \mathbb{N}_0$. Then

$$y_k(t) = y_k(t_0) + \int_{t_0}^{t} f_k((y_k)_s, s)\,\mathrm{d}s + \sum_{\substack{i\in\{1,\dots,m\},\\ t_i<t}} I_i^k(y_k(t_i)), \quad t \in [t_0, t_0+\sigma]$$

holds if and only if

$$y_k(t) = y_k(t_0) + \int_{t_0}^{t} \tilde{f}_k((y_k)_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma].$$

Hence, the statement of our theorem follows from Theorem 5.5 once we verify that all its assumptions are satisfied. The validity of assumptions 1 and 4 is a straightforward consequence of Lemma 6.1.

As in the proof of Theorem 6.2, one can verify that

$$\left\|\int_a^b \tilde{f}_k(y_t,t)\,\mathrm{d}g(t)\right\| \le \int_a^b \tilde{M}(t)\,\mathrm{d}g(t), \quad [a,b] \subset [t_0, t_0+\sigma],$$

$$\left\|\int_a^b \Big(\tilde{f}_k(y_t,t) - \tilde{f}_k(z_t,t)\Big)\,\mathrm{d}g(t)\right\| \le \int_a^b \tilde{L}(t)\omega(\|y_t - z_t\|_\infty)\,\mathrm{d}g(t),$$

which means that assumptions 2 and 3 are satisfied, too. □

7 Conclusion

The results obtained in this paper are also applicable to abstract dynamic equations on time scales, as well as (impulsive) functional dynamic equations on time scales. As shown in [10, 11, 36], these types of equations can be transformed to measure (functional) differential equations. Thus, one can easily obtain Osgood-type theorems and continuous dependence theorems for these equations.

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