

EXTREMAL SOLUTIONS OF MEASURE DIFFERENTIAL EQUATIONS

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Abstract

We investigate the existence of the least and greatest solutions to measure differential equations, as well as the relation between the extremal solutions and lower or upper solutions. Along the way, we prove a fairly general local existence theorem and an analogue of Peano's uniqueness theorem for measure differential equations. Finally, we show that the general theory is applicable in the study of differential equations with impulses or dynamic equations on time scales.

Keywords: measure differential equations, extremal solution, lower solution, upper solution, equations with impulses, dynamic equations on time scales

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1 Introduction

The study of extremal solutions to ordinary differential equations goes back to G. Peano's 1886 paper [31] dealing with local solvability of scalar differential equations with continuous right-hand sides. Given the initial-value problem

$$y'(t) = f(y(t), t), \quad y(a) = y_0, \quad (1.1)$$

Peano proved the existence of a pair of solutions $y_{\min}, y_{\max} : [a, a + \delta] \rightarrow \mathbb{R}$ that are extremal in the sense that every other solution $y : [a, a + \delta] \rightarrow \mathbb{R}$ satisfies $y_{\min} \leq y \leq y_{\max}$. The functions y_{\min} and y_{\max} were obtained as the supremum of all strict lower solutions and the infimum of all strict upper solutions, respectively. Although Peano's proof was not completely satisfactory, the basic idea is correct, and similar arguments can still be found in modern textbooks; see e.g. [14, Lemma 1.3], [24, Theorem 8.53], or the nice expository papers [33, 34] describing other possible approaches to extremal solutions. Questions connected with the existence of extremal solutions and their relation to lower and upper solutions still continue to attract researchers' attention. One possible direction is to study extremal solutions to equations of the form (1.1) with discontinuous right-hand sides; see e.g. [2, 3, 6, 13, 15, 19, 35, 36, 37, 38] and the references there. In this case, it is common to deal with solutions of Eq. (1.1) in Carathéodory's sense, i.e., they are absolutely continuous and satisfy the differential equations almost everywhere. Extremal solutions of various types of equations with discontinuous solutions, such as equations with impulses or distributional differential equations, were considered e.g. in [16, 17, 18, 20, 29, 39].

In the present paper, we are concerned with extremal solutions of the so-called measure differential equations, i.e., integral equations of the form

$$y(t) = y_0 + \int_a^t f(y(s), s) dg(s), \quad (1.2)$$

where the integral on the right-hand side is the Kurzweil-Stieltjes integral (also known as the Perron-Stieltjes integral), and g is a nondecreasing function. Measure differential equations generalize other types of equations, such as classical differential equations (corresponding to $g(s) = s$), equations with impulses, or dynamic equations on time scales (see Sections 5 and 6). Their solutions need not be differentiable or continuous; a solution can have up to countably many discontinuities located at the points where g is discontinuous (cf. Theorem 2.3).

Although it is not difficult to find examples where Eq. (1.2) does not possess extremal solutions, we show that a simple condition guarantees the existence of the greatest and the least solution. The condition is trivially satisfied at all points where g is continuous, and therefore our result generalizes the classical theorem due to Peano. Our approach to extremal solutions is inspired by [33, 34] and does not rely on iterative techniques or on lower/upper solutions. We are interested not only in local extremal solutions, but also in the global existence of noncontinuable extremal solutions. Along the way, we obtain an analogue of Peano's uniqueness theorem for measure differential equations whose right-hand sides are nonincreasing in y . Then we introduce the concepts of lower and upper solutions to Eq. (1.2), and discuss their relation with extremal solutions. Finally, we present several new theorems on the extremal solutions of equations with impulses and dynamic equations on time scales.

To make the paper self-contained, Section 2 provides a summary of all basic properties of Kurzweil-Stieltjes integrals that will be needed in the rest of the paper. In Section 3, we prove a fairly general local existence theorem for measure differential equations, as well as several auxiliary results concerned with continuation of solutions. Section 4 contains the main results on extremal solutions and lower/upper solutions to scalar measure differential equations. In Section 5, we recall that equations with impulses represent a special case of measure differential equations, and use this fact to obtain new results about extremal solutions of impulsive equations. A similar approach is followed in Section 6, where we study extremal solutions of dynamic equations on time scales.

2 Preliminaries

We begin with a brief overview of some properties of the Kurzweil-Stieltjes (or Perron-Stieltjes) integral, whose definition can be found in many sources; see e.g. [9, 10, 41, 42, 43, 44, 45]. Nevertheless, for readers who are not familiar with this concept, it is sufficient to know that the integral has the usual properties of linearity, additivity with respect to adjacent subintervals, as well as the properties described in this section.

In what follows, given functions $f : [a, b] \rightarrow \mathbb{R}^n$ and $g : [a, b] \rightarrow \mathbb{R}$, the Kurzweil-Stieltjes integral of f with respect to g on $[a, b]$ will be denoted by $\int_a^b f(s) dg(s)$, or simply $\int_a^b f dg$. If this integral exists, we also define $\int_b^a f dg = -\int_a^b f dg$.

The following result is a particular case of [41, Theorem 1.14].

Theorem 2.1. *Let $f : [a, b] \rightarrow \mathbb{R}^n$ and $g : [a, b] \rightarrow \mathbb{R}$. Then the following statements hold:*

1. *If the integral $\int_t^b f dg$ exists for every $t \in (a, b]$ and if $\lim_{t \rightarrow a^+} \left(\int_t^b f dg + f(a)(g(t) - g(a)) \right) = I$ for a certain $I \in \mathbb{R}^n$, then $\int_a^b f dg$ exists and equals I .*

2. If the integral $\int_a^t f dg$ exists for every $t \in [a, b)$ and if $\lim_{t \rightarrow b^-} \left(\int_a^t f dg + f(b)(g(b) - g(t)) \right) = I$ for a certain $I \in \mathbb{R}^n$, then $\int_a^b f dg$ exists and equals I .

Remark 2.2. For simplicity, we will denote by $\int_{a+}^b f dg$ the limit $\lim_{t \rightarrow a+} \int_t^b f dg$, when it exists. Similarly, the symbol $\int_a^{b-} f dg$ will stand for the limit $\lim_{t \rightarrow b-} \int_a^t f dg$.

The following result, which describes the properties of the indefinite Kurzweil-Stieltjes integral, is a special case of [41, Theorem 1.16]. Given a regulated function g , the symbols $\Delta^+g(t)$ and $\Delta^-g(t)$ will be used throughout this paper to denote

$$\Delta^+g(t) = g(t+) - g(t), \quad t \in [a, b), \quad \Delta^-g(t) = g(t) - g(t-), \quad t \in (a, b].$$

We also make the convention that $\Delta^+g(b) = 0$ and $\Delta^-g(a) = 0$.

Theorem 2.3. Let $f : [a, b] \rightarrow \mathbb{R}^n$ and $g : [a, b] \rightarrow \mathbb{R}$ be such that g is regulated and the integral $\int_a^b f dg$ exists. Then the function

$$u(t) = \int_a^t f dg, \quad t \in [a, b],$$

is regulated and satisfies

$$\begin{aligned} u(t+) &= u(t) + f(t)\Delta^+g(t), \quad t \in [a, b), \\ u(t-) &= u(t) - f(t)\Delta^-g(t), \quad t \in (a, b]. \end{aligned}$$

The next convergence result for the Kurzweil-Stieltjes integral is based on Lemma 5.4 and Theorem 5.5 from [26] (see also [41, Theorem 1.28] and [41, Remark 1.30]). Although the original theorem in [26] is formulated for a sequence of real-valued functions, the result still holds for functions taking values in \mathbb{R}^n (to see this, it is enough to apply the original statement to all components of the vector-valued functions).

Theorem 2.4. Let $g : [a, b] \rightarrow \mathbb{R}$ be a function of bounded variation. Assume that $f, f_k : [a, b] \rightarrow \mathbb{R}^n$, $k \in \mathbb{N}$, are functions satisfying the following conditions:

1. The integral $\int_a^b f_k dg$ exists for every $k \in \mathbb{N}$.
2. For each $\tau \in [a, b]$, $\lim_{k \rightarrow \infty} f_k(\tau) = f(\tau)$.
3. There exists a constant $K > 0$ such that for every division $a = \sigma_0 < \sigma_1 < \dots < \sigma_l = b$ of $[a, b]$ and every finite sequence $m_1, \dots, m_l \in \mathbb{N}$, we have

$$\left\| \sum_{j=1}^l \int_{\sigma_{j-1}}^{\sigma_j} f_{m_j} dg \right\| \leq K.$$

Then the integral $\int_a^b f dg$ exists and

$$\int_a^b f dg = \lim_{k \rightarrow \infty} \int_a^b f_k dg.$$

Finally, we recall a characterization of relatively compact sets in the space of regulated functions, which is a consequence of [11, Theorem 2.18]. As usual, $G([a, b], \mathbb{R}^n)$ denotes the space of all regulated functions $x : [a, b] \rightarrow \mathbb{R}^n$, equipped with the supremum norm $\|x\|_\infty = \sup_{t \in [a, b]} \|x(t)\|$.

We remark that, though the theorem in [11] requires h to be an increasing function, it is enough to assume that h is nondecreasing and let $\tilde{h}(t) = h(t) + t$, $t \in [a, b]$, to see that the original assumption is satisfied.

Theorem 2.5. Let $\mathcal{A} \subset G([a, b], \mathbb{R}^n)$. Assume that the set $\{x(a); x \in \mathcal{A}\}$ is bounded, and there is a nondecreasing function $h : [a, b] \rightarrow \mathbb{R}$ such that

$$\|x(t_2) - x(t_1)\| \leq h(t_2) - h(t_1)$$

whenever $x \in \mathcal{A}$ and $[t_1, t_2] \subset [a, b]$. Then \mathcal{A} is relatively compact.

3 Existence and continuation of solutions

In this section, we consider measure differential equations of the form

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s) dg(s), \quad t \in [a, b], \quad (3.1)$$

where $g : [a, b] \rightarrow \mathbb{R}$ is a function of bounded variation, $f : B \times [a, b] \rightarrow \mathbb{R}^n$, $t_0 \in [a, b]$, and $y_0 \in B$, where $B \subseteq \mathbb{R}^n$. Using the method of successive approximations, we will prove a Peano-type (or Carathéodory-type) existence theorem. Results on continuation of solution will be also investigated.

We introduce the following system of conditions, which will be useful for our purposes:

(C1) For every $y \in B$, the integral $\int_a^b f(y, t) dg(t)$ exists.

(C2) There exists a function $M : [a, b] \rightarrow \mathbb{R}$, which is Kurzweil-Stieltjes integrable with respect to g , such that

$$\left\| \int_u^v f(y, t) dg(t) \right\| \leq \int_u^v M(t) dg(t)$$

for every $y \in B$ and $[u, v] \subseteq [a, b]$.

(C3) For each $t \in [a, b]$, the mapping $y \mapsto f(y, t)$ is continuous in B .

The next lemma presents a consequence of conditions (C1), (C2), (C3) that is crucial for the forthcoming results.

Lemma 3.1. Assume that $g : [a, b] \rightarrow \mathbb{R}$ is a function of bounded variation and $f : B \times [a, b] \rightarrow \mathbb{R}^n$ satisfies conditions (C1), (C2), (C3). Then for each regulated function $y : [a, b] \rightarrow B$, the integral $\int_a^b f(y(t), t) dg(t)$ exists, and we have

$$\left\| \int_u^v f(y(t), t) dg(t) \right\| \leq \int_u^v M(t) dg(t), \quad [u, v] \subseteq [a, b]. \quad (3.2)$$

Proof. We begin by proving the statement in the case when $y : [a, b] \rightarrow B$ is a step function. Then there exists a division $a = t_0 < t_1 < \dots < t_m = b$ such that y is constant on each interval (t_{i-1}, t_i) , $i \in \{1, \dots, m\}$. Consider a fixed $i \in \{1, \dots, m\}$. Choose an arbitrary $\tau \in (t_{i-1}, t_i)$ and let

$$\begin{aligned} h(s) &= \int_{\tau}^s M(t) dg(t), \quad s \in [t_{i-1}, t_i], \\ F(s) &= \int_{\tau}^s f(y(t), t) dg(t), \quad s \in (t_{i-1}, t_i), \end{aligned}$$

where M is the function specified in condition (C2); note that the integral appearing in the definition of F is guaranteed to exist by condition (C1). It follows from Theorem 2.3 that h is regulated, and therefore the Cauchy conditions for the existence of the limits $h(t_{i-1}+)$ and $h(t_i-)$ are satisfied. Since

$$\|F(v) - F(u)\| = \left\| \int_u^v f(y(t), t) dg(t) \right\| \leq \int_u^v M(t) dg(t) = h(v) - h(u), \quad [u, v] \subseteq (t_{i-1}, t_i), \quad (3.3)$$

it follows that the Cauchy conditions for the existence of $F(t_{i-1}+)$ and $F(t_i-)$ are also satisfied. By Theorem 2.1, the integrals $\int_{t_{i-1}}^{\tau} f(y(t), t) dg(t)$ and $\int_{\tau}^{t_i} f(y(t), t) dg(t)$ exist, and we have

$$\begin{aligned}\int_{t_{i-1}}^{\tau} f(y(t), t) dg(t) &= F(\tau) - F(t_{i-1}+) + f(y(t_{i-1}), t_{i-1})\Delta^+g(t_{i-1}), \\ \int_{\tau}^{t_i} f(y(t), t) dg(t) &= F(t_i-) - F(\tau) + f(y(t_i), t_i)\Delta^-g(t_i).\end{aligned}$$

Consequently, the integral $\int_{t_{i-1}}^{t_i} f(y(t), t) dg(t)$ exists as well. Note that

$$\begin{aligned}\|f(y(t_{i-1}), t_{i-1})\Delta^+g(t_{i-1})\| &= \left\| \int_{t_{i-1}}^{t_{i-1}^+} f(y(t_{i-1}), t) dg(t) \right\| \leq \int_{t_{i-1}}^{t_{i-1}^+} M(t) dg(t), \\ \|f(y(t_i), t_i)\Delta^-g(t_i)\| &= \left\| \int_{t_i^-}^{t_i} f(y(t_i), t) dg(t) \right\| \leq \int_{t_i^-}^{t_i} M(t) dg(t).\end{aligned}$$

These observations together with (3.3) imply that the estimate $\left\| \int_u^v f(y(t), t) dg(t) \right\| \leq \int_u^v M(t) dg(t)$ holds also if $u = t_{i-1}$ or $v = t_i$.

By the additivity of the integral with respect to adjacent intervals, we conclude that $\int_a^b f(y(t), t) dg(t)$ exists and (3.2) holds.

To finish the proof, consider the case when $y : [a, b] \rightarrow B$ is a regulated function. There exists a sequence of step functions $\{y_k\}_{k=1}^{\infty}$ which is convergent to y on $[a, b]$. Without loss of generality, we can assume that all functions y_k take values in B (see the explanation before Lemma 10 in [43]). By the first part of the proof, we know that the integral $\int_a^b f(y_k(t), t) dg(t)$ exists for each $k \in \mathbb{N}$, and

$$\left\| \int_u^v f(y_k(t), t) dg(t) \right\| \leq \int_u^v M(t) dg(t), \quad [u, v] \subseteq [a, b]. \quad (3.4)$$

Hence, for every division $a = \sigma_0 < \sigma_1 < \dots < \sigma_l = b$ and every finite sequence $m_1, \dots, m_l \in \mathbb{N}$, we have

$$\left\| \sum_{j=1}^l \int_{\sigma_{j-1}}^{\sigma_j} f(y_{m_j}(s), s) dg(s) \right\| \leq \int_a^b M(t) dg(t).$$

Finally, condition (C3) implies that $\lim_{k \rightarrow \infty} f(y_k(s), s) = f(y(s), s)$, $s \in [a, b]$. Theorem 2.4 guarantees that the integral $\int_a^b f(y(s), s) dg(s)$ exists and equals $\lim_{k \rightarrow \infty} \int_a^b f(y_k(s), s) dg(s)$. This fact together with (3.4) imply that (3.2) holds. \square

If f satisfies conditions (C1), (C2), (C3) and y is a solution of Eq. (3.1), the estimate (3.2) implies that

$$\|y(v) - y(u)\| \leq h(v) - h(u), \quad [u, v] \subseteq [a, b],$$

where $h(s) = \int_a^s M(t) dg(t)$, $s \in [a, b]$. An immediate consequence is that y has bounded variation.

We now present the main result of this section, a local existence theorem for measure differential equations.

Theorem 3.2. *Assume that $g : [a, b] \rightarrow \mathbb{R}$ is a function of bounded variation, $f : B \times [a, b] \rightarrow \mathbb{R}^n$ satisfies conditions (C1), (C2), (C3), $t_0 \in [a, b]$ and $y_0 \in B$. If $y_+ = y_0 + f(y_0, t_0)\Delta^+g(t_0)$ and $y_- = y_0 - f(y_0, t_0)\Delta^-g(t_0)$ are interior points of B , then there exist $\delta_-, \delta_+ > 0$ such that Eq. (3.1) has a solution*

on $[t_0 - \delta_-, t_0 + \delta_+] \cap [a, b]$.

Moreover, δ_- and δ_+ can be chosen to be any numbers such that the balls

$$\{x \in \mathbb{R}^n; \|x - y_+\| \leq \int_{t_0+}^{t_0+\delta_+} M(s) dg(s)\}, \quad \{x \in \mathbb{R}^n; \|x - y_-\| \leq \int_{t_0-\delta_-}^{t_0-} M(s) dg(s)\}$$

are contained in B .

Proof. We start by proving the existence of a solution on a right neighborhood of t_0 . Assume that $t_0 < b$; otherwise, there is nothing to prove. Let $h(t) = \int_{t_0}^t M(s) dg(s)$, $t \in [t_0, b]$. Condition (C2) implies that $\int_u^v M(s) dg(s) \geq 0$ for each $[u, v] \subseteq [a, b]$. Hence, h is a nondecreasing function. Noting that y_+ is an interior point of B , there exists an $R_+ > 0$ such that the ball

$$\{x \in \mathbb{R}^n; \|x - y_+\| \leq R_+\}$$

is contained in B . Let $\delta_+ > 0$ be any number such that

$$h(t_0 + \delta_+) - h(t_0+) = \int_{t_0+}^{t_0+\delta_+} M(s) dg(s) \leq R_+. \quad (3.5)$$

We will show that there exists a solution of Eq. (3.1) defined on $[t_0, t_0 + \delta_+]$.

Let $y_1 : [t_0, t_0 + \delta_+] \rightarrow \mathbb{R}^n$ be given by $y_1(t) = y_0$, $t \in [t_0, t_0 + \delta_+]$. For each $k \in \mathbb{N}$, $k \geq 2$, define $y_k : [t_0, t_0 + \delta_+] \rightarrow \mathbb{R}^n$ recursively by

$$y_k(t) = y_0 + \int_{t_0}^t f(y_{k-1}(s), s) dg(s), \quad t \in [t_0, t_0 + \delta_+].$$

We claim that the functions y_k are well defined, regulated, and take values in B . The statement is obvious for $k = 1$. By induction, if we assume the statement holds for a certain k , then Lemma 3.1 implies that y_{k+1} is well defined, and is regulated by Theorem 2.3. Using Theorem 2.1, Lemma 3.1, and the estimate (3.5), for $t \in (t_0, t_0 + \delta_+]$ we have

$$\begin{aligned} \|y_{k+1}(t) - y_+\| &= \left\| \int_{t_0}^t f(y_k(s), s) dg(s) - f(y_0, t_0)\Delta^+g(t_0) \right\| \\ &= \left\| \int_{t_0}^t f(y_k(s), s) dg(s) - f(y_k(t_0), t_0)\Delta^+g(t_0) \right\| \\ &= \left\| \int_{t_0+}^t f(y_k(s), s) dg(s) \right\| \leq \int_{t_0+}^t M(s) dg(s) \\ &= h(t) - h(t_0+) \leq h(t_0 + \delta_+) - h(t_0+) \leq R_+. \end{aligned} \quad (3.6)$$

Consequently, $y_{k+1}(t) \in B$ for every $t \in [t_0, t_0 + \delta_+]$.

For each $k \in \mathbb{N}$, we have $y_k(t_0) = y_0$ and

$$\|y_{k+1}(t) - y_{k+1}(s)\| = \left\| \int_s^t f(y_k(\tau), \tau) dg(\tau) \right\| \leq h(t) - h(s)$$

whenever $[s, t] \subseteq [t_0, t_0 + \delta_+]$. By Theorem 2.5, the set $\{y_k; k \in \mathbb{N}\}$ is a relatively compact subset of $G([t_0, t_0 + \delta_+], \mathbb{R}^n)$, and thus contains a subsequence which converges to a function $y \in G([t_0, t_0 + \delta_+], \mathbb{R}^n)$. Relabeling, we may assume that $\lim_{k \rightarrow \infty} \|y_k - y\|_\infty = 0$. It follows from (3.6) that

$$\|y(t) - y_+\| \leq R_+, \quad t \in [t_0, t_0 + \delta_+],$$

and hence y takes values in B . Note that, by Lemma 3.1, the integral $\int_a^b f(y_k(t), t) dg(t)$ exists for each $k \in \mathbb{N}$, while condition (C3) implies that $\lim_{k \rightarrow \infty} f(y_k(s), s) = f(y(s), s)$, $s \in [t_0, t_0 + \delta_+]$. Moreover, for every division $t_0 = \sigma_0 < \sigma_1 < \dots < \sigma_l = t_0 + \delta_+$ of $[t_0, t_0 + \delta_+]$ and every finite sequence $m_1, \dots, m_l \in \mathbb{N}$, we have

$$\left\| \sum_{j=1}^l \int_{\sigma_{j-1}}^{\sigma_j} f(y_{m_j}(s), s) dg(s) \right\| \leq \int_{t_0}^{t_0 + \delta_+} M(s) dg(s).$$

Thus, applying Theorem 2.4, we see that for each $t \in [t_0, t_0 + \delta_+]$

$$y(t) = \lim_{k \rightarrow \infty} y_{k+1}(t) = y_0 + \lim_{k \rightarrow \infty} \int_{t_0}^t f(y_k(s), s) dg(s) = y_0 + \int_{t_0}^t f(y(s), s) dg(s),$$

wherefrom we conclude that y is a solution of the given measure differential equation on $[t_0, t_0 + \delta_+]$. To prove the existence of a solution on a left neighborhood of t_0 , it is enough to reverse time and consider the measure differential equation

$$z(t) = y_0 + \int_{-t_0}^t \hat{f}(z(s), s) d\hat{g}(s), \quad t \in [-t_0, -a],$$

where $\hat{g}(t) = g(-t)$ and $\hat{f}(z, t) = f(z, -t)$. Using the first part of the proof, one can conclude that the equation has a solution z defined on $[-t_0, -t_0 + \delta_-]$, where $\delta_- > 0$ is any number such that

$$\int_{t_0 - \delta_-}^{t_0^-} M(s) dg(s) \leq R_-,$$

R_- being a number such that the ball $\{x \in \mathbb{R}^n; \|x - y_-\| \leq R_-\}$ is contained in B . Consequently, $x(t) = z(-t)$, $t \in [t_0 - \delta_-, t_0]$, is a solution of Eq. (3.1) on $[t_0 - \delta_-, t_0]$. \square

Remark 3.3. Peano's classical theorem on the local existence of solutions to differential equations with continuous right-hand sides is a special case of Theorem 3.2. In that case, we have $g(s) = s$, $y_+ = y_- = y_0$, and f is continuous. Hence, if we take $B \subset \mathbb{R}^n$ to be a closed ball containing y_0 in its interior, then f is bounded on $B \times [a, b]$, and conditions (C1), (C2), (C3) are satisfied.

The more general Carathéodory's local existence theorem also follows from Theorem 3.2: Assume that $B \subset \mathbb{R}^n$ is a closed ball containing y_0 in its interior and $f : B \times [a, b] \rightarrow \mathbb{R}^n$ is a Carathéodory function, i.e., $y \mapsto f(y, t)$ is continuous for each $t \in [a, b]$, $t \mapsto f(y, t)$ is measurable for each $y \in B$, and there exists a Lebesgue integrable function $M : [a, b] \rightarrow \mathbb{R}$ such that $\|f(y, t)\| \leq M(t)$ for all $(y, t) \in B \times [a, b]$. It follows that for each $y \in B$, $t \mapsto f(y, t)$ is Lebesgue integrable, and therefore also Kurzweil integrable. Hence, conditions (C1), (C2), (C3) are satisfied.

Remark 3.4. With a simple adaptation of the proof of Theorem 3.2, it is possible to obtain an existence theorem for measure functional differential equations of the form

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^t f(y_s, s) dg(s), \quad t \in [t_0, t_0 + \sigma], \\ y_{t_0} &= \phi. \end{aligned} \tag{3.7}$$

Equations of this type were studied in several papers; see e.g. [9, 44, 45] and the references there. Picard-type existence and uniqueness theorems for measure functional differential equations can be found in [9, Theorem 5.3] and [44, Theorem 3.12], while a more general Osgood-type existence and uniqueness

theorem was obtained in [45, Theorem 5.2]. Using the method of successive approximations as in the proof of Theorem 3.2, we arrive at the following Peano-type (or Carathéodory-type) existence theorem: *Assume that $g : [t_0, t_0 + \sigma] \rightarrow \mathbb{R}$ is a function of bounded variation, $B \subseteq \mathbb{R}^n$, and $f : G([-r, 0], B) \times [t_0, t_0 + \sigma] \rightarrow \mathbb{R}^n$ has the following properties:*

1. *The integral $\int_{t_0}^{t_0 + \sigma} f(y_t, t) dg(t)$ exists whenever $y : [t_0 - r, t_0 + \sigma] \rightarrow B$ is regulated.*
2. *There exists a function $M : [a, b] \rightarrow \mathbb{R}$, which is Kurzweil-Stieltjes integrable with respect to g , such that $\|\int_u^v f(y_t, t) dg(t)\| \leq \int_u^v M(t) dg(t)$ for every regulated function $y : [t_0 - r, t_0 + \sigma] \rightarrow B$ and $[u, v] \subseteq [t_0, t_0 + \sigma]$.*
3. *f is continuous in the first variable.*

If $\phi \in G([-r, 0], B)$ is such that $\phi(0) + f(\phi, t_0)\Delta^+g(t_0)$ is an interior point of B , then there exists a $\delta > 0$ such that Eq. (3.7) has a solution on $[t_0, t_0 + \delta]$.

The rest of this section will be devoted to continuation of solutions.

Lemma 3.5. *Assume that $g : [a, b] \rightarrow \mathbb{R}$ is a function of bounded variation, $f : B \times [a, b] \rightarrow \mathbb{R}^n$ satisfies conditions (C1), (C2), (C3), and $B \subseteq \mathbb{R}^n$ is open. Suppose that $y_0 \in B$, $y_+ = y_0 + f(y_0, t_0)\Delta^+g(t_0) \in B$ and $y_- = y_0 - f(y_0, t_0)\Delta^-g(t_0) \in B$.*

If δ_- and δ_+ are any numbers such that the balls

$$\{x \in \mathbb{R}^n; \|x - y_+\| \leq \int_{t_0+}^{t_0+\delta_+} M(s) dg(s)\}, \quad \{x \in \mathbb{R}^n; \|x - y_-\| \leq \int_{t_0-\delta_-}^{t_0-} M(s) dg(s)\}$$

are contained in B , then each solution $y : I \rightarrow B$ of Eq. (3.1), where I is a closed subinterval of $[t_0 - \delta_-, t_0 + \delta_+] \cap [a, b]$ containing t_0 , can be extended to $[t_0 - \delta_-, t_0 + \delta_+] \cap [a, b]$.

Proof. Let us prove that y can be continued forward in time up to $\min(t_0 + \delta_+, b)$; the existence of a backward continuation can be obtained in a similar way. Assume that $t_1 = \sup I < \min(t_0 + \delta_+, b)$; otherwise, there is nothing to prove. Denote $y_1 = y(t_1)$ and $\widehat{y}_+ = y_1 + f(y_1, t_1)\Delta^+g(t_1)$. Let $z : [a, b] \rightarrow B$ be given by

$$z(t) = \begin{cases} y(t_0), & t \in [a, t_0], \\ y(t), & t \in [t_0, t_1], \\ y(t_1), & t \in (t_1, b]. \end{cases}$$

Since z is regulated, Lemma 3.1 and Theorem 2.3 imply

$$\begin{aligned} \int_{t_0+}^{t_1+} f(z(s), s) dg(s) &= \int_{t_0}^{t_1} f(z(s), s) dg(s) + f(z(t_1), t_1)\Delta^+g(t_1) - f(z(t_0), t_0)\Delta^+g(t_0) \\ &= y_0 + \int_{t_0}^{t_1} f(y(s), s) dg(s) + f(y(t_1), t_1)\Delta^+g(t_1) - y_0 - f(y(t_0), t_0)\Delta^+g(t_0) = \widehat{y}_+ - y_+. \end{aligned}$$

Now, if $x \in \mathbb{R}^n$ satisfies

$$\|x - \widehat{y}_+\| \leq \int_{t_1+}^{t_0+\delta_+} M(s) dg(s), \tag{3.8}$$

it follows that

$$\|x - y_+\| \leq \|x - \widehat{y}_+\| + \|\widehat{y}_+ - y_+\| \leq \int_{t_1+}^{t_0+\delta_+} M(s) dg(s) + \int_{t_0+}^{t_1+} M(s) dg(s) = \int_{t_0+}^{t_0+\delta_+} M(s) dg(s).$$

Hence, the ball consisting of all $x \in \mathbb{R}^n$ satisfying (3.8) is contained in B . By Theorem 3.2, the equation

$$y(t) = y_1 + \int_{t_1}^t f(y(s), s) dg(s)$$

has a solution defined on $[t_1, t_0 + \delta_+] \cap [a, b]$, which completes the proof. \square

The next example shows that a solution defined on $[t_0, T)$ need not be extendable to $[t_0, T]$ even if the limit $y(T-)$ exists. Moreover, if the extension exists, it need not be unique.

Example 3.6. Assume that $f : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ is given by $f(y, t) = y$, and $g : [0, 1] \rightarrow \mathbb{R}$ is given by $g(t) = t$ for $t \in [0, 1)$, and $g(1) = 2$. Consider Eq. (3.1) with an arbitrary $y_0 \in \mathbb{R}$. On the interval $[0, 1)$, the equation reduces to

$$y(t) = y_0 + \int_0^t y(s) ds, \quad t \in [0, 1),$$

whose solution is $y(t) = y_0 e^t$, $t \in [0, 1)$. If the solution can be extended to $[0, 1]$, then Theorem 2.3 implies $y(1-) = y(1) - f(y(1), 1)\Delta^-g(1) = y(1) - y(1) = 0$. This condition is violated if $y_0 \neq 0$, which means that the solution cannot be extended to $[0, 1]$. On the other hand, if $y_0 = 0$, we can choose $y(1)$ to be an arbitrary number. We have

$$\lim_{t \rightarrow 1-} \left(\int_0^t y(s) dg(s) + y(1)\Delta^-g(1) \right) = y(1).$$

By Theorem 2.1, $\int_0^1 y(s) dg(s)$ exists and equals $y(1)$, i.e., the extended function $y : [0, 1] \rightarrow \mathbb{R}$ is indeed a solution of our equation on $[0, 1]$.

Generalizing the ideas from the previous example, we arrive at the following result.

Lemma 3.7. *If $g : [a, b] \rightarrow \mathbb{R}$ is a function of bounded variation, $B \subseteq \mathbb{R}^n$, $y_0 \in B$ and $f : B \times [a, b] \rightarrow \mathbb{R}^n$, then the following statements hold:*

1. *If y is a solution of Eq. (3.1) on $[t_0, T)$, then it can be extended to a solution on $[t_0, T]$ if and only if $y(T-)$ exists and there is a vector $\hat{y} \in B$ such that $y(T-) = \hat{y} - f(\hat{y}, T)\Delta^-g(T)$.*
2. *If y is a solution of Eq. (3.1) on $(T, t_0]$, then it can be extended to a solution on $[T, t_0]$ if and only if $y(T+)$ exists and there is a vector $\hat{y} \in B$ such that $y(T+) = \hat{y} + f(\hat{y}, T)\Delta^+g(T)$.*

In both cases, the extension of the solution can be obtained by letting $y(T) = \hat{y}$.

Proof. It is enough to prove the first statement; the second one can be obtained by ‘‘reversing time’’ as in the proof of Theorem 3.2.

If the solution y can be extended to $[t_0, T]$, then Theorem 2.3 implies $y(T-) = y(T) - f(y(T), T)\Delta^-g(T)$. Hence, the existence of $y(T-)$ and of a vector $\hat{y} \in B$ such that $y(T-) = \hat{y} - f(\hat{y}, T)\Delta^-g(T)$ are necessary conditions. Let us show that they are also sufficient. Let $y(T) = \hat{y}$. We know that

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s) dg(s), \quad t \in [t_0, T).$$

Passing to the limit $t \rightarrow T-$, we see that $\lim_{t \rightarrow T-} \int_{t_0}^t f(y(s), s) dg(s) = y(T-) - y_0$. Thus,

$$\lim_{t \rightarrow T-} \left(\int_{t_0}^t f(y(s), s) dg(s) + f(y(T), T)\Delta^-g(T) \right) = y(T-) - y_0 + f(\hat{y}, T)\Delta^-g(T) = \hat{y} - y_0 = y(T) - y_0.$$

By Theorem 2.1, $\int_{t_0}^T f(y(s), s) dg(s)$ exists and equals $y(T) - y_0$. Consequently, y is a solution of Eq. (3.1) on $[t_0, T]$. \square

4 Extremal solutions, lower and upper solutions

In this section, we turn our attention to the scalar equation

$$y(t) = y_0 + \int_a^t f(y(s), s) dg(s), \quad t \in [a, b], \quad (4.1)$$

where $g : [a, b] \rightarrow \mathbb{R}$ is nondecreasing and left-continuous, $f : B \times [a, b] \rightarrow \mathbb{R}$, $B \subseteq \mathbb{R}$, and $y_0 \in B$. According to Theorem 2.3, left-continuity of g implies that each solution of Eq. (4.1) is left-continuous. One reason for focusing only on left-continuous functions g is that we want to avoid technical difficulties connected with extension of solutions from intervals of the form $[u, v)$ to $[u, v]$; cf. Example 3.6 and Lemma 3.7. From the viewpoint of applications, this restriction is not a serious one; see Sections 5 and 6. Note also that we deal only with solutions going forward in time, i.e., the initial condition is imposed at the left endpoint of the interval. Again, the reason is to avoid problems with extension of solutions from $(u, v]$ to $[u, v]$.

The least and greatest solutions to measure differential equations are defined in the standard way as we can see in the following definition.

Definition 4.1. Let $I \subseteq [a, b]$ be an interval with $a \in I$ and let $z : I \rightarrow \mathbb{R}$ be a solution of Eq. (4.1). We say that z is the greatest solution of Eq. (4.1) on I if any other solution $y : I \rightarrow \mathbb{R}$ satisfies

$$y(t) \leq z(t) \quad \text{for every } t \in I.$$

Symmetrically, we say that z is the least solution of Eq. (4.1) on I if any other solution $y : I \rightarrow \mathbb{R}$ satisfies

$$z(t) \leq y(t) \quad \text{for every } t \in I.$$

When we want to emphasize that we are considering solutions defined on I and taking values in B , we say that z is the greatest or the least solution of Eq. (4.1) in $B \times I$.

The greatest solution and the least solution are collectively referred to as the extremal solutions.

The following example illustrates that Eq. (4.1) with a discontinuous function g need not have extremal solutions. The function g introduced in the example is left-continuous, but discontinuous from the right at $t = 1$.

Example 4.2. Suppose that $g : [0, 2] \rightarrow \mathbb{R}$ and $f : [0, \infty) \times [0, 2] \rightarrow \mathbb{R}$ are given by

$$g(t) = \begin{cases} t, & t \in [0, 1], \\ t + 1, & t \in (1, 2], \end{cases}$$

$$f(y, t) = \begin{cases} 3y^{2/3}, & t \in [0, 1), \\ 2(1 - y), & t = 1, \\ 0, & t \in (1, 2]. \end{cases}$$

Consider Eq. (4.1) with $y_0 = 0$. On $[0, 1)$, the equation reduces to $y'(t) = 3y(t)^{2/3}$. Since g is left-continuous at 1, each solution has the same property. Next, note that

$$y(1+) = y(1) + f(y(1), 1)\Delta^+g(1) = y(1) + 2(1 - y(1)) = 2 - y(1).$$

Finally, each solution has to be constant on $(1, 2]$. It follows that all solutions have the form

$$z_\tau(t) = \begin{cases} 0, & t \in [0, \tau], \\ (t - \tau)^3, & t \in (\tau, 1], \\ 2 - (1 - \tau)^3, & t \in (1, 2], \end{cases}$$

where $\tau \in [0, 1]$ is a parameter. Note that z_0 is the greatest solution on $[0, 1]$, while $z_1 > z_0$ on $(1, 2]$. Hence, there is no greatest solution on $[0, 2]$. Similarly, z_1 is the least solution on $[0, 1]$, but there is no least solution on $[0, 2]$.

The previous example shows that additional assumptions are needed to ensure the existence of extremal solutions. Therefore, we introduce the following condition:

(C4) If $u, v \in B$, with $u < v$, then $u + f(u, t)\Delta^+g(t) \leq v + f(v, t)\Delta^+g(t)$ for every $t \in [a, b]$.

Note that in Example 4.2, the condition is violated at $t = 1$.

Remark 4.3. The importance of condition (C4) stems from the following observation: If $y_1, y_2 : I \rightarrow B$ are two solutions of Eq. (4.1), where $I \subseteq [a, b]$ is an interval with $a \in I$, and if $y_1(\tau) \leq y_2(\tau)$ for some $\tau \in I \setminus \{\sup I\}$, then condition (C4) together with Theorem 2.3 imply

$$y_1(\tau+) = y_1(\tau) + f(y_1(\tau), \tau)\Delta^+g(\tau) \leq y_2(\tau) + f(y_2(\tau), \tau)\Delta^+g(\tau) = y_2(\tau+).$$

This observation will be used in the proofs of subsequent theorems.

The next result shows that conditions (C1), (C2), (C3) together with (C4) are sufficient to guarantee the existence of extremal solutions. Inspired by R. L. Pouso's proof of [34, Theorem 3.1] (which is concerned with the existence of extremal solutions to classical ordinary differential equations), we find the greatest solution as the solution with the largest integral. (See also the paper [25], where the authors employ a similar method for finding extremal solutions of a second-order periodic boundary-value problem.)

Theorem 4.4. Assume that $g : [a, b] \rightarrow \mathbb{R}$ is nondecreasing and left-continuous, $B \subseteq \mathbb{R}$ is closed, $y_0 \in B$ and $f : B \times [a, b] \rightarrow \mathbb{R}$ satisfies conditions (C1), (C2), (C3), (C4). If Eq. (4.1) has a solution on $[a, b]$, then it has the greatest solution and the least solution on $[a, b]$.

Proof. Let us prove the existence of the greatest solution on $[a, b]$. Let S be the set of all solutions of Eq. (4.1) defined on $[a, b]$ and taking values in B . If $x \in S$ and $[s, t] \subseteq [a, b]$, Lemma 3.1 implies

$$|x(t) - x(s)| = \left| \int_s^t f(x(\tau), \tau) dg(\tau) \right| \leq \int_s^t M(\tau) dg(\tau) = h(t) - h(s),$$

where $h(t) = \int_a^t M(\tau) dg(\tau)$, $t \in [a, b]$, is nondecreasing (see the proof of Theorem 3.2). Hence, it follows from Theorem 2.5 that S is a relatively compact subset of $G([a, b], \mathbb{R})$. In order to prove that S is compact, it is enough to show that S is closed. To this aim, consider a sequence $\{x_k\}_{k=1}^\infty$ in S such that $\lim_{k \rightarrow \infty} \|x_k - x\|_\infty = 0$ for some function $x \in G([a, b], \mathbb{R})$. Since B is closed, it follows that x takes values in B . It is clear that $x(a) = y_0$. From Theorem 2.4 (the assumptions are satisfied because of (C1), (C2), (C3) and Lemma 3.1), we have

$$\lim_{k \rightarrow \infty} \int_a^t f(x_k(s), s) dg(s) = \int_a^t f(x(s), s) dg(s) \quad \text{for every } t \in [a, b],$$

which shows that x is a solution of (4.1) on $[a, b]$, i.e. $x \in S$, and therefore S is closed.

Note that each $x \in S$ is regulated, and therefore Riemann integrable. Let $F : S \rightarrow \mathbb{R}$ be given by $F(x) = \int_a^b x(s) ds$, $x \in S$. Because F is continuous on the compact set S , it attains a maximum for a certain $y_{\max} \in S$. That is,

$$\int_a^b x(s) ds \leq \int_a^b y_{\max}(s) ds, \quad x \in S.$$

Let us show that y_{\max} is the greatest solution of Eq. (4.1). For contradiction, assume there exist $y \in S$ and $\tau \in (a, b]$ such that $y_{\max}(\tau) < y(\tau)$. Then left-continuity of g implies $y_{\max}(\tau-) < y(\tau-)$, and Remark 4.3 implies that $y_{\max}(\tau+) \leq y(\tau+)$. Let

$$\alpha = \sup\{t \in [a, \tau); y(t) \leq y_{\max}(t)\}.$$

By the definition of supremum, we have either $y(\alpha) \leq y_{\max}(\alpha)$, or there is a sequence of points $\{t_k\}_{k=1}^{\infty}$ from $[a, \alpha)$ such that $t_k \rightarrow \alpha$ and $y(t_k) \leq y_{\max}(t_k)$ for each $k \in \mathbb{N}$. In the latter case, $y(\alpha-) \leq y_{\max}(\alpha-)$, and left-continuity implies $y(\alpha) \leq y_{\max}(\alpha)$. Thus, the inequality $y(\alpha) \leq y_{\max}(\alpha)$ is always true, which necessarily means that $\alpha \neq \tau$, i.e., $\alpha < \tau$.

It follows from Remark 4.3 that $y(\alpha+) \leq y_{\max}(\alpha+)$. On the other hand, it follows from the definition of α that $y > y_{\max}$ on $(\alpha, \tau]$ and, consequently, $y(\alpha+) \geq y_{\max}(\alpha+)$. Thus, the only possibility is that $y(\alpha+) = y_{\max}(\alpha+)$.

We now distinguish between two cases concerning the behavior of y_{\max} and y on the right of τ :

1. For all $t > \tau$, we have $y(t) \geq y_{\max}(t)$. In this case, consider the function $z : [a, b] \rightarrow B$ given by

$$z(t) = \begin{cases} y_{\max}(t), & t \in [a, \alpha], \\ y(t), & t \in (\alpha, b]. \end{cases}$$

The fact that $y(\alpha+) = y_{\max}(\alpha+)$ implies that z is a solution of Eq. (4.1) on $[a, b]$. Indeed, z is obviously a solution on $[a, \alpha]$, and for each $t \in (\alpha, b]$ we obtain

$$\begin{aligned} z(t) - z(\alpha) &= z(t) - z(\alpha+) + z(\alpha+) - z(\alpha) = y(t) - y(\alpha+) + y(\alpha+) - y_{\max}(\alpha) \\ &= \int_{\alpha+}^t f(y(s), s) dg(s) + y_{\max}(\alpha+) - y_{\max}(\alpha) = \int_{\alpha+}^t f(y(s), s) dg(s) + f(y_{\max}(\alpha), \alpha) \Delta^+ g(\alpha) \\ &= \int_{\alpha+}^t f(z(s), s) dg(s) + f(z(\alpha), \alpha) \Delta^+ g(\alpha) = \int_{\alpha}^t f(z(s), s) dg(s), \end{aligned}$$

which confirms that z is a solution of Eq. (4.1) on $[a, b]$.

Since $z \geq y_{\max}$ on $[a, b]$ and $y > y_{\max}$ on $(\alpha, \tau]$, we have $\int_a^b z(s) ds > \int_a^b y_{\max}(s) ds$, a contradiction with the definition of y_{\max} .

2. There exists a $t > \tau$ such that $y(t) < y_{\max}(t)$. In this case, let

$$\beta = \inf\{t \in (\tau, b]; y(t) \leq y_{\max}(t)\}.$$

Since $y(t) > y_{\max}(t)$ for $t \in [\tau, \beta)$, we see that $y(\beta-) \geq y_{\max}(\beta-)$; the latter fact and left-continuity imply $y(\beta) \geq y_{\max}(\beta)$, and consequently (by Remark 4.3) $y(\beta+) \geq y_{\max}(\beta+)$.

By the definition of infimum, we have either $y(\beta) \leq y_{\max}(\beta)$, or there is a sequence of points $\{u_k\}_{k=1}^{\infty}$ from $(\beta, b]$ such that $u_k \rightarrow \beta$ and $y(u_k) \leq y_{\max}(u_k)$ for each $k \in \mathbb{N}$. Obviously, both possibilities lead to the conclusion $y(\beta+) \leq y_{\max}(\beta+)$. Thus, it follows that $y(\beta+) = y_{\max}(\beta+)$.

Now, consider the function

$$z(t) = \begin{cases} y_{\max}(t), & t \in [a, \alpha] \cup (\beta, b], \\ y(t), & t \in (\alpha, \beta]. \end{cases}$$

As in the previous case, the facts that $y(\alpha+) = y_{\max}(\alpha+)$ and $y(\beta+) = y_{\max}(\beta+)$ imply that z is a solution on $[a, b]$.

Since $z \geq y_{\max}$ on $[a, b]$ and $y > y_{\max}$ on (α, β) , we have $\int_a^b z(s) ds > \int_a^b y_{\max}(s) ds$, a contradiction with the definition of y_{\max} .

The existence of the least solution can be proved similarly by finding the minimum of the mapping $F(x) = \int_a^b x(s) ds$, $x \in S$. \square

Example 4.5. In the previous theorem, the assumption that B is closed cannot be omitted. Indeed, consider the classical example

$$y'(t) = 3y(t)^{2/3}, \quad t \in [0, 1], \quad y(0) = 0. \quad (4.2)$$

All solutions have the form

$$z_\tau(t) = \begin{cases} 0, & t \in [0, \tau], \\ (t - \tau)^3, & t \in [\tau, 1], \end{cases}$$

where $\tau \in [0, 1]$ is a parameter. For $B = [0, 1]$, the greatest solution of Eq. (4.2) in $B \times [0, 1]$ is z_0 . However, for $B = [0, 1)$, the solution z_0 escapes from B at $t = 1$, and there is no greatest solution of Eq. (4.2) in $B \times [0, 1]$.

As an application of Theorem 4.4, we will now derive a generalization of Peano's uniqueness theorem for differential equations whose right-hand sides are nonincreasing in y (cf. [1, Theorem 1.3.1] or [33, Corollary 2.2]). We replace condition (C2) by the following weaker assumption:

(C2') For each compact set $C \subseteq B$, there exists a function $M : [a, b] \rightarrow \mathbb{R}$, which is Kurzweil-Stieltjes integrable with respect to g , such that

$$\left| \int_u^v f(y, t) dg(t) \right| \leq \int_u^v M(t) dg(t)$$

for every $y \in C$ and $[u, v] \subseteq [a, b]$.

(Note that in Section 3, we were interested only in the local existence theory, where it is always possible to choose B to be a compact set. Thus, there was no loss of generality when dealing with the apparently stronger condition (C2).)

Theorem 4.6. *Assume that $g : [a, b] \rightarrow \mathbb{R}$ is nondecreasing and left-continuous, $B \subseteq \mathbb{R}$ is closed, $y_0 \in B$ and $f : B \times [a, b] \rightarrow \mathbb{R}$ satisfies conditions (C1), (C2'), (C3), (C4). If the function f is nonincreasing in the first variable, then Eq. (4.1) has at most one solution on $[a, b]$.*

Proof. Consider a pair of solutions $y_1, y_2 : [a, b] \rightarrow B$. Both of them are regulated, and therefore bounded, i.e., there exists an $r \geq 0$ such that the values of y_1, y_2 are contained in the compact set $C = B \cap [-r, r]$. By the assumption, f satisfies condition (C2) on the set $C \times [a, b]$. Thus, Theorem 4.4 ensures the existence of the greatest solution $y_{\max} : [a, b] \rightarrow C$ and the least solution $y_{\min} : [a, b] \rightarrow C$. We have

$$y_{\min}(t) \leq y_{\max}(t) = y_0 + \int_a^t f(y_{\max}(s), s) dg(s) \leq y_0 + \int_a^t f(y_{\min}(s), s) dg(s) = y_{\min}(t), \quad t \in [a, b],$$

which means that $y_{\min} = y_{\max}$. Since both y_1 and y_2 lie between y_{\min} and y_{\max} , they have to coincide. \square

Theorem 4.7. *Assume that $g : [a, b] \rightarrow \mathbb{R}$ is nondecreasing and left-continuous, $B \subseteq \mathbb{R}$, and $f : B \times [a, b] \rightarrow \mathbb{R}$ satisfies conditions (C1), (C2'), (C3), (C4). Suppose also that $y_0 \in B$ and $y_+ = y_0 + f(y_0, a)\Delta^+g(a)$ is an interior point of B . Then there exists a $\delta > 0$ such that the following statements hold:*

1. *Eq. (4.1) has the greatest solution y_{\max} and the least solution y_{\min} in $B \times [a, a + \delta]$.*
2. *For any other solution $y : I \rightarrow B$ of Eq. (4.1), where $a \in I \subsetneq [a, a + \delta]$, we have $y_{\min}(t) \leq y(t) \leq y_{\max}(t)$ for all $t \in I$.*

Proof. Take an arbitrary compact set $C \subseteq B$ such that $y_0 \in C$ and y_+ is an interior point of C . Then the restriction of f to $C \times [a, b]$ satisfies condition (C2). Let M be the function from this condition and take an arbitrary $\delta > 0$ such that the interval $\{x \in \mathbb{R}; |x - y_+| \leq \int_{a+}^{a+\delta} M(s) dg(s)\}$ is contained in the interior of C . It follows from Theorem 3.2 that Eq. (4.1) has a solution on $[a, a + \delta]$.

The existence of the greatest solution y_{\max} and the least solution y_{\min} of Eq. (4.1) in $C \times [a, a + \delta]$ is guaranteed by Theorem 4.4. We claim that y_{\max} and y_{\min} are also extremal in the whole set $B \times [a, a + \delta]$. To see this, it is enough to show that no solution defined on $[a, a + \delta]$ can leave the set C . Assume for contradiction there exists a solution $y : [a, a + \delta] \rightarrow B$ that leaves C , and let

$$\alpha = \inf\{t \in [a, a + \delta]; y(t) \notin C\}.$$

Left-continuity of y and closedness of C imply that $y(\alpha) \in C$ (if $\alpha = a$, the statement is obvious). Consequently, $\alpha < a + \delta$. Observe that

$$\begin{aligned} |y(\alpha+) - y_+| &\leq |y(\alpha) - y(a+)| + |f(y(\alpha), \alpha)\Delta^+g(\alpha)| = \left| \int_{a+}^{\alpha} f(y(s), s) dg(s) \right| + \left| \int_{\alpha}^{\alpha+} f(y(\alpha), s) dg(s) \right| \\ &\leq \int_{a+}^{\alpha} M(s) dg(s) + \int_{\alpha}^{\alpha+} M(s) dg(s) \leq \int_{a+}^{a+\delta} M(s) dg(s), \end{aligned}$$

which implies that $y(\alpha+)$ is contained in the interior of C . Consequently, the values of y on a right neighborhood of α lie in C , which contradicts the definition of α .

We now proceed to the proof of the second statement. We already know it is enough to consider solutions $y : I \rightarrow C$ of Eq. (4.1), where $a \in I \subsetneq [a, a + \delta]$. Choose an arbitrary $t \in I$. According to Lemma 3.5, there exists a solution $z : [a, a + \delta] \rightarrow C$ of Eq. (4.1) such that $z = y$ on $[a, t]$. Since y_{\max} , y_{\min} are the extremal solutions on $[a, a + \delta]$, we conclude that $y_{\min} \leq z \leq y_{\max}$ on $[a, a + \delta]$, and in particular $y_{\min}(t) \leq z(t) = y(t) \leq y_{\max}(t)$. \square

Remark 4.8. In the classical case when $g(s) = s$, the condition (C4) is vacuous, and Theorem 4.7 generalizes the classical Peano's theorem on the local existence of extremal solutions (see e.g. [34, Theorem 3.1]).

The next result provides global information on the existence of noncontinuable extremal solutions. The proof is inspired by the proof of [33, Theorem 2.3].

Theorem 4.9. *Assume that $g : [a, b] \rightarrow \mathbb{R}$ is nondecreasing and left-continuous, $B \subseteq \mathbb{R}$ is open, and $f : B \times [a, b] \rightarrow \mathbb{R}$ satisfies conditions (C1), (C2'), (C3), (C4). Suppose also that $y_0 \in B$ and $y_0 + f(y_0, a)\Delta^+g(a) \in B$. Then there exist intervals $I, J \subset [a, b]$ containing a , and noncontinuable solutions $y_{\max} : I \rightarrow B$, $y_{\min} : J \rightarrow B$ of Eq. (4.1) such that the following statements hold:*

1. *For any solution $y : I' \rightarrow B$ of Eq. (4.1) with $a \in I' \subseteq I$, we have $y(t) \leq y_{\max}(t)$ for all $t \in I'$.*

2. For any solution $y : J' \rightarrow B$ of Eq. (4.1) with $a \in J' \subseteq J$, we have $y_{\min}(t) \leq y(t)$ for all $t \in J'$.

Proof. We prove only the first statement; the proof of the second one is similar. Consider the set

$$\mathcal{T} = \{\tau > a; \text{ there exists a solution } y_\tau : [a, \tau] \rightarrow B \text{ of Eq. (4.1) such that any solution } y : I' \rightarrow B \\ \text{with } a \in I' \subseteq [a, \tau] \text{ satisfies } y \leq y_\tau \text{ on } I'\}.$$

By Theorem 4.7, there exists a $\delta > 0$ such that $a + \delta \in \mathcal{T}$, i.e., \mathcal{T} is nonempty. Denote $T = \sup \mathcal{T}$, and

$$I = \begin{cases} [a, T), & T \notin \mathcal{T}, \\ [a, T], & T \in \mathcal{T}. \end{cases}$$

Define the function $y_{\max} : I \rightarrow B$ as follows: For an arbitrary $t \in I$, find $\tau \in \mathcal{T}$ with $\tau \geq t$, and let $y_{\max}(t) = y_\tau(t)$. This definition is meaningful, since for $\tau_1, \tau_2 \in \mathcal{T}$, it follows from the definition of \mathcal{T} that $y_{\tau_1} = y_{\tau_2}$ in the intersection of their domains.

Let us show that y_{\max} cannot be continued to the right. For contradiction, assume that y_{\max} can be extended to a larger subinterval of $[a, b]$, and denote the extended function again by y_{\max} . We distinguish two cases:

1. $I = [a, T)$, and the extended function y_{\max} is defined on an interval containing $[a, T]$. For an arbitrary solution $y : [a, T] \rightarrow B$ of Eq. (4.1), we have $y \leq y_{\max}$ on $[a, T)$, and therefore $y(T-) \leq y_{\max}(T-)$. It follows from left-continuity that $y(T) \leq y_{\max}(T)$. Thus $T \in \mathcal{T}$, a contradiction.
2. $I = [a, T]$, and the extended function y_{\max} is defined on a certain interval which contains $[a, T]$ as a proper subinterval. Then $y_{\max}(T+) = y_{\max}(T) + f(y_{\max}(T), T)\Delta^+g(T)$ must be an element of B . By Theorem 4.7, the equation

$$y(t) = y_{\max}(T) + \int_T^t f(y(s), s) dg(s), \quad t \geq T, \quad (4.3)$$

has the greatest solution w on a certain interval $[T, T + \varepsilon]$. We claim that $T + \varepsilon \in \mathcal{T}$. Indeed, consider the function $\tilde{y}_{\max} : [a, T + \varepsilon] \rightarrow B$ given by

$$\tilde{y}_{\max}(t) = \begin{cases} y_{\max}(t), & t \in [a, T], \\ w(t), & t \in [T, T + \varepsilon]. \end{cases}$$

Let $y : I' \rightarrow B$ be any solution of Eq. (4.1) with $a \in I' \subseteq [a, T + \varepsilon]$. Obviously, $y \leq \tilde{y}_{\max}$ on $[a, T]$. Suppose that $y(\tau) > \tilde{y}_{\max}(\tau)$ for a certain $\tau \in (T, T + \varepsilon]$, i.e., $y(\tau) > w(\tau)$. Let $\alpha = \sup\{t \in [T, \tau); y(t) \leq w(t)\}$. We can now argue as in the final part of proof of Theorem 4.4: We have $y > w$ either on $(\alpha, T + \varepsilon]$, or on an interval (α, β) , where $\beta \in (\alpha, T + \varepsilon]$, $y(\beta) \geq w(\beta)$, and $y(\beta+) = w(\beta+)$. In any case, the solution z of Eq. (4.3) obtained by pasting w on $[T, \alpha]$ and y on a right neighborhood of α is not majorized by w , which contradicts the fact that w was the greatest solution of (4.3) on $[T, T + \varepsilon]$. The previous considerations imply that $y \leq \tilde{y}_{\max}$ on I' , and therefore $T + \varepsilon \in \mathcal{T}$, which is a contradiction.

□

Our next task is to study the relation between extremal solutions and lower or upper solutions.

Definition 4.10. Let $I \subseteq [a, b]$ be an interval with $a \in I$. A regulated function $\alpha : I \rightarrow B$ is said to be a lower solution of Eq. (4.1) on I if $\alpha(a) \leq y_0$ and

$$\alpha(v) - \alpha(u) \leq \int_u^v f(\alpha(s), s) dg(s), \quad [u, v] \subseteq I. \quad (4.4)$$

Symmetrically, a regulated function $\beta : I \rightarrow B$ is an upper solution of Eq. (4.1) on I if $\beta(a) \geq y_0$ and

$$\beta(v) - \beta(u) \geq \int_u^v f(\beta(s), s) dg(s), \quad [u, v] \subseteq I. \quad (4.5)$$

Remark 4.11. If $\alpha : I \rightarrow B$ is a lower solution of Eq. (4.1), then (4.4) and Theorem 2.3 imply

$$\Delta^+ \alpha(t) = \alpha(t+) - \alpha(t) \leq f(\alpha(t), t) \Delta^+ g(t), \quad t \in I, \quad (4.6)$$

$$\Delta^- \alpha(t) = \alpha(t) - \alpha(t-) \leq f(\alpha(t), t) \Delta^- g(t), \quad t \in I. \quad (4.7)$$

Obviously, the reverse inequalities hold for upper solutions.

Theorem 4.12. Assume that $g : [a, b] \rightarrow \mathbb{R}$ is nondecreasing and left-continuous, $B \subseteq \mathbb{R}$ is open, and $f : B \times [a, b] \rightarrow \mathbb{R}$ satisfies conditions (C1), (C2'), (C3), (C4). Suppose also that $y_0 \in B$ and $y_0 + f(y_0, a) \Delta^+ g(a) \in B$. If $y_{\max} : I \rightarrow B$ and $y_{\min} : J \rightarrow B$, where $a \in I \subseteq [a, b]$ and $a \in J \subseteq [a, b]$, are the noncontinuable extremal solutions of Eq. (4.1) described in Theorem 4.9, then the following statements hold:

1. If $\alpha : I' \rightarrow B$, where $a \in I' \subseteq I$, is a lower solution of Eq. (4.1), then $\alpha \leq y_{\max}$ on I' .
2. If $\beta : J' \rightarrow B$, where $a \in J' \subseteq J$, is an upper solution of Eq. (4.1), then $\beta \geq y_{\min}$ on J' .

Consequently,

$$\begin{aligned} y_{\max}(t) &= \max\{\alpha(t); \alpha \text{ is a lower solution of Eq. (4.1) on } [a, t]\}, \quad t \in I, \\ y_{\min}(t) &= \min\{\beta(t); \beta \text{ is an upper solution of Eq. (4.1) on } [a, t]\}, \quad t \in J. \end{aligned}$$

Proof. Let us prove the first statement; the proof of the second one is symmetrical. For contradiction, assume there is a lower solution $\alpha : I' \rightarrow B$ and $t_1 \in I'$ such that $\alpha(t_1) > y_{\max}(t_1)$. Let

$$t_2 = \sup\{t \in [a, t_1]; \alpha(t) \leq y_{\max}(t)\}.$$

By the definition of supremum, we have either $\alpha(t_2) \leq y_{\max}(t_2)$, or there is a sequence of points $\{u_k\}_{k=1}^{\infty}$ from $[a, t_2)$ such that $u_k \rightarrow t_2$ and $\alpha(u_k) \leq y_{\max}(u_k)$ for each $k \in \mathbb{N}$. In the latter case, $\alpha(t_2-) \leq y_{\max}(t_2-)$. Using (4.7) and the fact that $\Delta^- g(t) = 0$, we get $\Delta^- \alpha(t_2) \leq 0$, and therefore

$$\alpha(t_2) = \alpha(t_2-) + \Delta^- \alpha(t_2) \leq \alpha(t_2-) \leq y_{\max}(t_2-) = y_{\max}(t_2).$$

Thus, the inequality $\alpha(t_2) \leq y_{\max}(t_2)$ is always true, and implies that $t_2 < t_1$.

Denote $\tilde{x} = y_{\max}(t_2)$ and consider the equation

$$z(t) = \tilde{x} + \int_{t_2}^t \tilde{f}(z(s), s) dg(s), \quad t \in [t_2, t_1], \quad (4.8)$$

where $\tilde{f} : B \times [t_2, t_1] \rightarrow \mathbb{R}$ is given by

$$\tilde{f}(u, t) = \begin{cases} f(u, t), & u \geq \alpha(t), \\ f(\alpha(t), t), & u < \alpha(t). \end{cases}$$

Clearly, \tilde{f} satisfies condition (C3). If $y \in B$, then the function $m(t) = \max\{y, \alpha(t)\}$, $t \in [t_1, t_2]$, is regulated and $\tilde{f}(y, t) = f(m(t), t)$. Lemma 3.1 then implies that \tilde{f} satisfies conditions (C1) and (C2'). Moreover, $\tilde{x} + \tilde{f}(\tilde{x}, t_2)\Delta^+g(t_2) = y_{\max}(t_2+) \in B$. Take an arbitrary compact set $C \subset B$ such that $\tilde{x} \in C$ and $y_{\max}(t_2+)$ is an interior point of C . Then the restriction of \tilde{f} to $C \times [a, b]$ satisfies conditions (C1), (C2), (C3). Therefore, by Theorem 3.2, Eq. (4.8) has a solution z on $[t_2, t_2 + \eta] \subset [t_2, t_1]$, for some $\eta > 0$. Let us show that

$$z(t) \geq \alpha(t) \quad \text{for all } t \in [t_2, t_2 + \eta]. \quad (4.9)$$

For contradiction, assume that there exists a $\tau \in (t_2, t_2 + \eta]$ such that $z(\tau) < \alpha(\tau)$. Let

$$\nu = \sup\{t \in [t_2, \tau); \alpha(t) \leq z(t)\}.$$

Using the same argument as in the beginning of this proof, we can show that $\alpha(\nu) \leq z(\nu)$, and thus $\nu < \tau$. By the definition of ν , we have $z(t) < \alpha(t)$ for all $t \in (\nu, \tau]$, and therefore $z(\nu+) \leq \alpha(\nu+)$. On the other hand, Eq. (4.6) and condition (C4) imply

$$\alpha(\nu+) \leq \alpha(\nu) + f(\alpha(\nu), \nu)\Delta^+g(\nu) \leq z(\nu) + f(z(\nu), \nu)\Delta^+g(\nu) = z(\nu+).$$

Hence, the only possibility is that $z(\nu+) = \alpha(\nu+)$. Using the definition of \tilde{f} , we obtain

$$z(\tau) - z(\nu+) = \int_{\nu+}^{\tau} \tilde{f}(z(s), s) dg(s) = \int_{\nu+}^{\tau} f(\alpha(s), s) dg(s) \geq \alpha(\tau) - \alpha(\nu+),$$

which implies that $z(\tau) \geq \alpha(\tau)$; this is a contradiction. Thus, we have proved that (4.9) holds. Since $y_{\max}(t_2) = z(t_2) = \tilde{x}$, the function $y : [a, t_2 + \eta] \rightarrow \mathbb{R}$ given by

$$y(t) = \begin{cases} y_{\max}(t), & t \in [a, t_2], \\ z(t), & t \in [t_2, t_2 + \eta]. \end{cases}$$

defines a solution of Eq. (4.1) on $[a, t_2 + \eta]$. It follows from Theorem 4.9 that $y(t) \leq y_{\max}(t)$ for $t \in [a, t_2 + \eta]$. On the other hand, (4.9) and the definition of t_2 imply that $y(t) = z(t) \geq \alpha(t) > y_{\max}(t)$ for all $t \in (t_2, t_2 + \eta]$, which leads to a contradiction. \square

5 Ordinary differential equations with impulses

In this section, we take advantage of the known relation between impulsive systems and measure differential equations (cf. [10, 12, 45]), and study the existence of extremal solutions for differential equations with impulses at preassigned times. More precisely, we are concerned with the initial-value problem

$$\begin{aligned} y'(t) &= f(y(t), t), \quad \text{a.e. in } [a, b], \\ \Delta^+y(t_k) &= I_k(y(t_k)), \quad k \in \{1, \dots, m\}, \\ y(a) &= y_0, \end{aligned} \quad (5.1)$$

where $m \in \mathbb{N}$, $a \leq t_1 < \dots < t_m < b$, $f : B \times [a, b] \rightarrow \mathbb{R}^n$, $I_1, \dots, I_m : B \rightarrow \mathbb{R}^n$, $y_0 \in B$ and $B \subseteq \mathbb{R}^n$. The solutions of Eq. (5.1) are assumed to be left-continuous on $[a, b]$, and absolutely continuous on $[a, t_1]$, $(t_1, t_2], \dots, (t_m, b]$. The three relations in Eq. (5.1) are equivalent with the single integral equation

$$y(t) = y_0 + \int_a^t f(y(s), s) ds + \sum_{k; t_k < t} I_k(y(t_k)), \quad t \in [a, b], \quad (5.2)$$

where the integral on the right-hand side is the Lebesgue integral.

The next example shows that in general, an impulsive equation need not have extremal solutions.

Example 5.1. Consider the impulsive differential equation

$$\begin{aligned} y'(t) &= f(y(t), t), \text{ a.e. in } [0, 2], \\ \Delta^+ y(1) &= I(y(1)) \\ y(0) &= 0, \end{aligned} \tag{5.3}$$

where $I : [0, \infty) \rightarrow \mathbb{R}$ and $f : [0, \infty) \times [0, 2] \rightarrow \mathbb{R}$ are given by

$$\begin{aligned} I(y) &= 2(1 - y), \quad y \in [0, \infty), \\ f(y, t) &= \begin{cases} 3y^{2/3}, & t \in [0, 1), \\ 0, & t \in [1, 2]. \end{cases} \end{aligned}$$

It is not hard to see that the solutions of Eq. (5.3) are exactly the same as the solutions of the measure differential equation from Example 4.2. Therefore, Eq. (5.3) has no greatest/least solution on $[0, 2]$.

The following lemma, which is a consequence of [10, Lemma 2.4], is important for our purposes.

Lemma 5.2. Let $m \in \mathbb{N}$, $a \leq t_1 < t_2 < \dots < t_m < b$, and

$$g(s) = s + \sum_{k=1}^m \chi_{(t_k, \infty)}(s), \quad s \in [a, b] \tag{5.4}$$

(the symbol χ_A denotes the characteristic function of a set $A \subset \mathbb{R}$). Consider an arbitrary function $f : [a, b] \rightarrow \mathbb{R}^n$ and let $\tilde{f} : [a, b] \rightarrow \mathbb{R}^n$ be such that $\tilde{f}(s) = f(s)$ for every $s \in [a, b] \setminus \{t_1, \dots, t_m\}$. Then, g is left-continuous on $(a, b]$, and the Kurzweil-Stieltjes integral $\int_a^b \tilde{f}(s) dg(s)$ exists if and only if the Kurzweil-Henstock integral $\int_a^b f(s) ds$ exists. In this case, we have

$$\int_a^t \tilde{f}(s) dg(s) = \int_a^t f(s) ds + \sum_{k; t_k < t} \tilde{f}(t_k), \quad t \in [a, b].$$

We now deduce the following relation between Eq. (5.2) and a measure differential equation; it is a straightforward modification of [10, Theorem 3.1].

Theorem 5.3. Let $m \in \mathbb{N}$, $a \leq t_1 < t_2 < \dots < t_m < b$, $B \subseteq \mathbb{R}^n$, $y_0 \in B$, $f : B \times [a, b] \rightarrow \mathbb{R}^n$, and $I_1, \dots, I_m : B \rightarrow \mathbb{R}^n$. If $y : [a, b] \rightarrow B$ is a solution of Eq. (5.2), then it is a solution of the measure differential equation

$$y(t) = y_0 + \int_a^t \tilde{f}(y(s), s) dg(s), \quad t \in [a, b], \tag{5.5}$$

where $g : [a, b] \rightarrow \mathbb{R}$ is given by (5.4) and for each $z \in B$,

$$\tilde{f}(z, t) = \begin{cases} f(z, t), & t \in [a, b] \setminus \{t_1, \dots, t_m\}, \\ I_k(z), & t = t_k \text{ for some } k \in \{1, \dots, m\}. \end{cases} \tag{5.6}$$

If there exists a Lebesgue integrable function $M : [a, b] \rightarrow \mathbb{R}$ such that $\|f(y, t)\| \leq M(t)$ for every $y \in B$ and $t \in [a, b]$, then each solution of Eq. (5.5) is a solution of Eq. (5.2).

Proof. According to Lemma 5.2, the measure differential equation (5.5) is equivalent to the equation

$$y(t) = y_0 + \int_a^t f(y(s), s) ds + \sum_{k; t_k < t} I_k(y(t_k)), \quad t \in [a, b],$$

where the integral on the right-hand side is the Kurzweil-Henstock integral. Thus, the first part of the theorem follows from the fact that Lebesgue integrability implies Kurzweil-Henstock integrability. Conversely, if $s \mapsto f(y(s), s)$ is Kurzweil-Henstock integrable, then it is measurable. Hence, if it has a Lebesgue-integrable majorant M , it is Lebesgue integrable. \square

Let $m \in \mathbb{N}$, $a \leq t_1 < t_2 < \dots < t_m < b$, $B \subseteq \mathbb{R}^n$. We introduce the following system of conditions concerning the functions $f : B \times [a, b] \rightarrow \mathbb{R}^n$ and $I_1, \dots, I_m : B \rightarrow \mathbb{R}^n$:

(P1) For every $y \in B$, the function $t \mapsto f(y, t)$ is measurable.

(P2) There exists a Lebesgue integrable function $M : [a, b] \rightarrow \mathbb{R}$ such that $\|f(y, t)\| \leq M(t)$ for every $y \in B$ and $t \in [a, b]$.

(P3) For each $t \in [a, b] \setminus \{t_1, \dots, t_m\}$, the mapping $y \mapsto f(y, t)$ is continuous in B .

(P4) For each $k \in \{1, \dots, m\}$, there exists a constant $m_k > 0$ such that

$$\|I_k(y)\| \leq m_k \text{ for every } y \in B.$$

(P5) The function $I_k : B \rightarrow \mathbb{R}^n$ is continuous for each $k \in \{1, \dots, m\}$.

Consider the function $\tilde{f} : B \times [a, b] \rightarrow \mathbb{R}^n$ defined by (5.6). In view of Lemma 5.2, if f satisfies conditions (P1) and (P2), then \tilde{f} satisfies condition (C1). Moreover, if conditions (P3) and (P5) are satisfied, then \tilde{f} satisfies condition (C3). The next lemma provides a similar relation between conditions (P2), (P4), and (C2).

Lemma 5.4. *Let $m \in \mathbb{N}$, $a \leq t_1 < t_2 < \dots < t_m < b$, $B \subseteq \mathbb{R}^n$. Assume that $f : B \times [a, b] \rightarrow \mathbb{R}^n$ satisfies condition (P2) and $I_1, \dots, I_m : B \rightarrow \mathbb{R}^n$ satisfy condition (P4). Then the function $\tilde{f} : B \times [a, b] \rightarrow \mathbb{R}^n$ given by (5.6) satisfies condition (C2).*

Proof. Let $M : [a, b] \rightarrow \mathbb{R}$ be the function from condition (P2). For $y \in B$ and $[u, v] \subseteq [a, b]$, by Lemma 5.2 and conditions (P2) and (P4), we have

$$\left\| \int_u^v \tilde{f}(y, t) dg(t) \right\| = \left\| \int_u^v f(y, t) dt + \sum_{k; u \leq t_k < v} I_k(y) \right\| \leq \int_u^v M(t) dt + \sum_{k; u \leq t_k < v} m_k. \quad (5.7)$$

Considering the function $\tilde{M} : [a, b] \rightarrow \mathbb{R}$ defined by

$$\tilde{M}(t) = \begin{cases} M(t), & t \in [a, b] \setminus \{t_1, \dots, t_m\}, \\ m_k, & t = t_k \text{ for some } k \in \{1, \dots, m\}, \end{cases}$$

by Lemma 5.2, we can see that the right hand side of the inequality (5.7) equals $\int_u^v \tilde{M}(t) dg(t)$. Therefore, we conclude that \tilde{f} satisfies condition (C2). \square

From now on, we focus on the scalar case of Eq. (5.2), and define extremal solutions in the obvious way.

Definition 5.5. Let $I \subseteq [a, b]$ be an interval with $a \in I$ and let $z : I \rightarrow \mathbb{R}$ be a solution of Eq. (5.2). We say that z is the greatest solution of Eq. (5.2) on I if any other solution $y : I \rightarrow \mathbb{R}$ satisfies

$$y(t) \leq z(t) \text{ for every } t \in I.$$

Symmetrically, we say that z is the least solution of Eq. (5.2) on I if any other solution $y : I \rightarrow \mathbb{R}$ satisfies

$$z(t) \leq y(t) \text{ for every } t \in I.$$

In order to derive results on the existence of extremal solutions for impulsive differential equations based on the theorems from previous sections, we need to discuss whether condition (C4) is fulfilled by the function \tilde{f} defined in (5.6).

Noting that the function g given by (5.4) is such that $\Delta^+g(t_k) = 1$ for each $k \in \{1, \dots, m\}$ and $\Delta^+g(t) = 0$ otherwise, we introduce the following condition for the impulse functions:

(P6) If $u, v \in B$, with $u < v$, then $u + I_k(u) \leq v + I_k(v)$ for every $k \in \{1, \dots, m\}$.

Clearly, \tilde{f} satisfies condition (C4) provided $I_1, \dots, I_m : B \rightarrow \mathbb{R}$ satisfy (P6). Therefore, as a direct consequence of Theorems 4.4 and 5.3 we obtain the following result.

Theorem 5.6. *Assume that $m \in \mathbb{N}$, $a \leq t_1 < t_2 < \dots < t_m < b$, $B \subseteq \mathbb{R}$ is closed, $y_0 \in B$, and the functions $f : B \times [a, b] \rightarrow \mathbb{R}$, $I_1, \dots, I_m : B \rightarrow \mathbb{R}$ satisfy conditions (P1), (P2), (P3), (P4), (P5), (P6). If Eq. (5.2) has a solution on $[a, b]$, then it has the greatest solution and the least solution on $[a, b]$.*

Before we proceed, we need the following weaker versions of conditions (P2) and (P4):

(P2') For each compact set $C \subseteq B$, there exists a Lebesgue integrable function $M : [a, b] \rightarrow \mathbb{R}$ such that $|f(y, t)| \leq M(t)$ for every $y \in C$ and $t \in [a, b]$.

(P4') For each compact set $C \subseteq B$ and each $k \in \{1, \dots, m\}$, there exists a constant $m_k > 0$ such that $|I_k(y)| \leq m_k$ for every $y \in C$.

If conditions (P2') and (P4') hold, then Lemma 5.4 implies that the function \tilde{f} given by (5.6) satisfies condition (C2').

By combining Theorems 4.6 and 5.3, we get the following analogue of Peano's uniqueness result for equations with impulses.

Theorem 5.7. *Assume that $m \in \mathbb{N}$, $a \leq t_1 < t_2 < \dots < t_m < b$, $B \subseteq \mathbb{R}$ is closed, $y_0 \in B$, and the functions $f : B \times [a, b] \rightarrow \mathbb{R}$, $I_1, \dots, I_m : B \rightarrow \mathbb{R}$ satisfy conditions (P1), (P2'), (P3), (P4'), (P5), (P6). If f is nonincreasing in the first variable and I_1, \dots, I_m are nonincreasing, then Eq. (5.2) has at most one solution on $[a, b]$.*

The next comparison result follows from Theorems 4.9 and 5.3.

Theorem 5.8. *Assume that $m \in \mathbb{N}$, $a \leq t_1 < t_2 < \dots < t_m < b$, $B \subseteq \mathbb{R}$ is open, $y_0 \in B$, and the functions $f : B \times [a, b] \rightarrow \mathbb{R}$, $I_1, \dots, I_m : B \rightarrow \mathbb{R}$ satisfy conditions (P1), (P2'), (P3), (P4'), (P5), (P6). If $t_1 = a$, suppose that $y_0 + I_1(y_0) \in B$. Then there exist intervals $I, J \subseteq [a, b]$ containing a , and noncontinuable solutions $y_{\max} : I \rightarrow B$, $y_{\min} : J \rightarrow B$ of Eq. (5.2) such that the following statements hold:*

1. For any solution $y : I' \rightarrow B$ of Eq. (5.2) with $a \in I' \subseteq I$, we have $y(t) \leq y_{\max}(t)$ for all $t \in I'$.
2. For any solution $y : J' \rightarrow B$ of Eq. (5.2) with $a \in J' \subseteq J$, we have $y_{\min}(t) \leq y(t)$ for all $t \in J'$.

In the forthcoming results we will consider the following definition of lower and upper solutions of impulsive differential equations.

Definition 5.9. Let $m \in \mathbb{N}$, $a \leq t_1 < t_2 < \dots < t_m < b$ and put $J_0 = [a, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k \in \{1, \dots, m-1\}$ and $J_m = (t_m, b]$. Consider an interval $I \subseteq [a, b]$ with $a \in I$. A regulated function $\alpha : I \rightarrow B$ is called a lower solution of Eq. (5.2) on I , if $\alpha(a) \leq y_0$,

$$\alpha(v) - \alpha(u) \leq \int_u^v f(\alpha(s), s) ds \quad \text{whenever } [u, v] \subseteq J_k \cap I, k \in \{0, \dots, m\},$$

$$\alpha(t_k+) \leq \alpha(t_k) + I_k(\alpha(t_k)) \quad \text{for each } k \in \{1, \dots, m\} \text{ such that } t_k < \sup I.$$

Symmetrically, a regulated function $\beta : I \rightarrow B$ is an upper solution of Eq. (5.2) on I , if $\beta(a) \geq y_0$,

$$\begin{aligned} \beta(v) - \beta(u) &\geq \int_u^v f(\beta(s), s) ds \quad \text{whenever } [u, v] \subseteq J_k \cap I, k \in \{0, \dots, m\}, \\ \beta(t_k+) &\geq \beta(t_k) + I_k(\beta(t_k)) \quad \text{for each } k \in \{1, \dots, m\} \text{ such that } t_k < \sup I. \end{aligned}$$

Lemma 5.10. *If $\alpha : I \rightarrow B$ is a lower/upper solution of Eq. (5.2), then it is a lower/upper solution of Eq. (5.5).*

Proof. We will prove the statement for lower solutions, the other one is symmetrical. Let $\alpha : I \rightarrow B$ be a lower solution of Eq. (5.2) and let $u, v \in I$, $u < v$ be given. Lemma 5.2 implies

$$\int_u^v \tilde{f}(\alpha(t), t) dg(t) = \int_u^v f(\alpha(t), t) dt + \sum_{k; u \leq t_k < v} I_k(\alpha(t_k)).$$

If $[u, v] \cap \{t_1, \dots, t_m\} = \emptyset$, then

$$\int_u^v \tilde{f}(\alpha(t), t) dg(t) = \int_u^v f(\alpha(t), t) dt \geq \alpha(v) - \alpha(u).$$

Otherwise, there exist indices $l, N \in \{1, \dots, m\}$ such that $u \leq t_l < \dots < t_N < v$. Note that, by the definition of lower solution, for each $k \in \{l, \dots, N-1\}$ we have

$$\int_{t_k}^{t_{k+1}} f(\alpha(t), t) dt = \lim_{r \rightarrow t_k^+} \int_r^{t_{k+1}} f(\alpha(t), t) dt \geq \lim_{r \rightarrow t_k^+} [\alpha(t_{k+1}) - \alpha(r)] = \alpha(t_{k+1}) - \alpha(t_k+),$$

and, similarly, $\int_{t_N}^v f(\alpha(t), t) dt \geq \alpha(v) - \alpha(t_N+)$. These inequalities together with Lemma 5.4 imply that

$$\begin{aligned} \int_u^v \tilde{f}(\alpha(t), t) dg(t) &= \int_u^v f(\alpha(t), t) dt + \sum_{k=l}^N I_k(\alpha(t_k)) \\ &= \int_u^{t_l} f(\alpha(t), t) dt + \sum_{k=l}^{N-1} \int_{t_k}^{t_{k+1}} f(\alpha(t), t) dt + \int_{t_N}^v f(\alpha(t), t) dt + \sum_{k=l}^N I_k(\alpha(t_k)) \\ &\geq \alpha(t_l) - \alpha(u) + \sum_{k=l}^{N-1} [\alpha(t_{k+1}) - \alpha(t_k+)] + \alpha(v) - \alpha(t_N+) + \sum_{k=l}^N I_k(\alpha(t_k)) \\ &= \alpha(v) - \alpha(u) + \sum_{k=l}^N [\alpha(t_k) - \alpha(t_k+) + I_k(\alpha(t_k))] \geq \alpha(v) - \alpha(u), \end{aligned}$$

(where the last inequality is due to the fact that $\alpha(t_j) - \alpha(t_j+) + I_j(\alpha(t_j)) \geq 0$ for $t_j \in I$). Therefore, we conclude that $\alpha : I \rightarrow B$ is a lower solution of Eq. (5.5). \square

By combining Lemma 5.10 and Theorems 4.12 and 5.3, we obtain the following result.

Theorem 5.11. *Assume that $m \in \mathbb{N}$, $a \leq t_1 < t_2 < \dots < t_m < b$, $B \subseteq \mathbb{R}$ is open, $y_0 \in B$, and the functions $f : B \times [a, b] \rightarrow \mathbb{R}$, $I_1, \dots, I_m : B \rightarrow \mathbb{R}$ satisfy conditions (P1), (P2'), (P3), (P4'), (P5), (P6). If $t_1 = a$, suppose that $y_0 + I_1(y_0) \in B$.*

If $y_{\max} : I \rightarrow B$ and $y_{\min} : J \rightarrow B$, where $a \in I \subseteq [a, b]$ and $a \in J \subseteq [a, b]$, are the noncontinuable extremal solutions of Eq. (5.2) described in Theorem 5.8, then the following statements hold:

1. If $\alpha : I' \rightarrow B$, where $a \in I' \subseteq I$, is a lower solution of Eq. (5.2), then $\alpha \leq y_{\max}$ on I' .
2. If $\beta : J' \rightarrow B$, where $a \in J' \subseteq J$, is an upper solution of Eq. (5.2), then $\beta \geq y_{\min}$ on J' .

Consequently,

$$\begin{aligned} y_{\max}(t) &= \max\{\alpha(t); \alpha \text{ is a lower solution of Eq. (5.2) on } [a, t]\}, \quad t \in I, \\ y_{\min}(t) &= \min\{\beta(t); \beta \text{ is an upper solution of Eq. (5.2) on } [a, t]\}, \quad t \in J. \end{aligned}$$

Remark 5.12. Extremal solutions and lower/upper solutions are often discussed in the literature related to impulsive differential equations. They appear not only in the context of first-order initial-value problems, but also in the study of periodic boundary value problems subject to impulses (see the monograph [27] by V. Lakshmikantham et al.), functional differential equations with impulses (see the article [39] by R. L. Pouso and J. Tomeček), or distributional differential equations (see the papers [16, 17, 20] by S. Heikkilä and E. Talvila), which include impulsive equations as a special case. In the following remarks, we point out some differences between the existing approaches and our results:

1. Lower/upper solutions of differential equations with or without impulses are usually defined by means of differential inequalities, and are supposed to be piecewise continuously differentiable, piecewise absolutely continuous, or to have bounded variation (see e.g. [2, 8, 13, 27, 29, 30, 36, 38, 39]). Our definitions of lower and upper solutions are based on integral inequalities, and we require them to be merely regulated, i.e., they may have up to countably many discontinuity points. A similarly general approach can be found in [16], where the lower/upper solutions are regulated and left-continuous, in [17], where they are bounded and left-continuous, or in [20], where they are locally Kurzweil-Henstock integrable and left-continuous.
2. Results on extremal solutions of impulsive differential equations are sometimes formulated for systems of equations (see e.g. [16, 17, 18, 27]), while we are concerned with the scalar case only.
3. The methods of obtaining extremal solutions often rely on monotone iterative techniques and assume that the right-hand side f and the impulse functions are nondecreasing in y (see e.g. [16, 17, 18, 20]). These assumptions lead not only to the existence of extremal solutions, but also to the fact that they are nondecreasing with respect to the right-hand side and initial condition. Our results do not require f to be nondecreasing in y , and the impulse functions are only assumed to satisfy condition (P6), which is weaker than monotonicity. On the other hand, unlike the above-mentioned papers, we require f to be continuous in y .

6 Dynamic equations on time scales

In this section, we use the known relation between dynamic equations on time scales and measure differential equations (cf. [12, 40, 43]) to obtain new theorems on extremal solutions and lower or upper solutions of dynamic equations. We suppose that the reader is familiar with the elements of the time scales calculus, including the notions of the Lebesgue and Kurzweil-Henstock Δ -integrals (see [4, 5, 22, 32]). Assume that $\mathbb{T} \subset \mathbb{R}$ is a time scale, $a, b, t_0 \in \mathbb{T}$, $a \leq t_0 \leq b$, $B \subseteq \mathbb{R}^n$, $f : B \times [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^n$, and $y_0 \in B$. Instead of dealing with the usual form of the dynamic equation

$$y^{\Delta}(t) = f(y(t), t), \quad t \in [a, b]_{\mathbb{T}}, \quad y(t_0) = y_0, \quad (6.1)$$

we work with the more general integral form

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s) \Delta s, \quad t \in [a, b]_{\mathbb{T}}, \quad (6.2)$$

where the integral on the right-hand side is the Lebesgue Δ -integral. Each solution of Eq. (6.2) is necessarily continuous. Hence, if f is rd-continuous, then the integral on the right-hand side of Eq. (6.2) is simply the Riemann Δ -integral, and Eq. (6.2) reduces back to the classical form (6.1).

We need to recall the relation between the Kurzweil-Henstock Δ -integral and Kurzweil-Stieltjes integral, which was described in [43] and later refined in [10]. For each $t \in [a, b]$, let $t^* = \inf\{s \in \mathbb{T}; s \geq t\}$. Since \mathbb{T} is a closed set, we have $t^* \in [a, b]_{\mathbb{T}}$. For each function $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^n$, we consider its extension $f^* : [a, b] \rightarrow \mathbb{R}^n$ given by $f^*(t) = f(t^*)$, $t \in [a, b]$. The next statement is taken over from [10, Theorem 4.2].

Theorem 6.1. *Let $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^n$ be an arbitrary function. Define $g(s) = s^*$ for every $s \in [a, b]$. Then the Kurzweil-Henstock Δ -integral $\int_a^b f(t) \Delta t$ exists if and only if the Kurzweil-Stieltjes integral $\int_a^b f^*(t) dg(t)$ exists; in this case, both integrals have the same value.*

Using the previous theorem and some ideas from [43] and [10], we obtain the following relation between dynamic equations of the form (6.2) and measure differential equations.

Theorem 6.2. *If $y : [a, b]_{\mathbb{T}} \rightarrow B$ is a solution of Eq. (6.2), then $y^* : [a, b] \rightarrow B$ is a solution of the measure differential equation*

$$z(t) = y_0 + \int_{t_0}^t f^*(z(s), s) dg(s), \quad t \in [a, b], \quad (6.3)$$

where $g(t) = t^*$, $t \in [a, b]$, and

$$f^*(z, t) = f(z, t^*), \quad z \in B, \quad t \in [a, b]. \quad (6.4)$$

If there exists a Lebesgue Δ -integrable function $M : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ such that $\|f(y, t)\| \leq M(t)$ for every $y \in B$ and $t \in [a, b]$, then each solution $z : [a, b] \rightarrow B$ of Eq. (6.3) has the form $z = y^*$, where $y : [a, b]_{\mathbb{T}} \rightarrow B$ is a solution of Eq. (6.2).

Proof. Assume that $y : [a, b]_{\mathbb{T}} \rightarrow B$ is a solution of Eq. (6.2). Note that Lebesgue Δ -integrability implies Kurzweil-Henstock Δ -integrability. Hence, for each $t \in [a, b]_{\mathbb{T}}$, Theorem 6.1 guarantees that the Kurzweil-Stieltjes integral $\int_{t_0}^t f(y(s^*), s^*) dg(s)$ exists and equals $\int_{t_0}^t f(y(s), s) \Delta s$. If $t \in [a, b] \setminus \mathbb{T}$, then g is constant on the interval $[t, t^*]$. Thus $\int_t^{t^*} f(y(s^*), s^*) dg(s) = 0$, and

$$\int_{t_0}^t f(y(s^*), s^*) dg(s) = \int_{t_0}^{t^*} f(y(s^*), s^*) dg(s) - \int_t^{t^*} f(y(s^*), s^*) dg(s) = \int_{t_0}^{t^*} f(y(s), s) \Delta s.$$

To sum up, the relation $\int_{t_0}^t f(y(s^*), s^*) dg(s) = \int_{t_0}^{t^*} f(y(s), s) \Delta s$ holds for each $t \in [a, b]$. Consequently,

$$y^*(t) = y(t^*) = y_0 + \int_{t_0}^{t^*} f(y(s), s) \Delta s = y_0 + \int_{t_0}^t f(y(s^*), s^*) dg(s) = y_0 + \int_{t_0}^t f^*(y^*(s), s) dg(s), \quad t \in [a, b],$$

which proves that y^* is a solution of Eq. (6.3).

Conversely, assume that $z : [a, b] \rightarrow B$ is a solution of Eq. (6.3). Since the function g is constant on each interval $[u, v]$ such that $[u, v] \cap \mathbb{T} = \emptyset$, it follows that z has the same property. Hence, $z = y^*$, where $y : [a, b]_{\mathbb{T}} \rightarrow B$ is such that $y(t) = z(t)$ for $t \in [a, b]_{\mathbb{T}}$. Using Theorem 6.1, we get

$$y(t) = y_0 + \int_{t_0}^t f^*(z(s), s) dg(s) = y_0 + \int_{t_0}^t f(y^*(s), s^*) dg(s) = y_0 + \int_{t_0}^t f(y(s), s) \Delta s, \quad t \in [a, b]_{\mathbb{T}},$$

where the last integral is the Kurzweil-Henstock Δ -integral. However, if f has the Lebesgue Δ -integrable majorant M , the integral $\int_{t_0}^t f(y(s), s) \Delta s$ also exists as a Lebesgue Δ -integral, i.e., y is a solution of Eq. (6.2). \square

Given a function $f : B \times [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^n$, we introduce the following set of conditions:

(T1) For every $y \in B$, the function $t \mapsto f(y, t)$ is Lebesgue Δ -measurable.

(T2) There exists a Lebesgue Δ -integrable function $M : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ such that $\|f(y, t)\| \leq M(t)$ for every $y \in B$ and $t \in [a, b]_{\mathbb{T}}$.

(T3) For each $t \in [a, b]_{\mathbb{T}}$, the mapping $y \mapsto f(y, t)$ is continuous in B .

Consider the extended function $f^* : B \times [a, b] \rightarrow \mathbb{R}^n$ defined by (6.4). Obviously, if f satisfies condition (T3), then f^* satisfies (C3). Similarly, by Theorem 6.1, if f satisfies conditions (T1) and (T2), then f^* satisfies (C1). Finally, if f satisfies condition (T2), then f^* satisfies (C2) with the function M replaced by M^* .

In the rest of this section, we focus on scalar equations having the form

$$y(t) = y_0 + \int_a^t f(y(s), s) \Delta s, \quad t \in [a, b]_{\mathbb{T}}, \quad (6.5)$$

where $B \subseteq \mathbb{R}$ and $f : B \times [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$; note that the initial condition is now imposed at the left endpoint.

Definition 6.3. Let $I \subseteq [a, b]_{\mathbb{T}}$ be a time scale interval with $a \in I$ and let $z : I \rightarrow \mathbb{R}$ be a solution of Eq. (6.5). We say that z is the greatest solution of (6.5) on I if any other solution $y : I \rightarrow \mathbb{R}$ satisfies

$$y(t) \leq z(t) \quad \text{for every } t \in I.$$

Symmetrically, we say that z is the least solution of (6.5) on I if any other solution $y : I \rightarrow \mathbb{R}$ satisfies

$$z(t) \leq y(t) \quad \text{for every } t \in I.$$

The next example, which is a simple adaptation of Example 4.2, shows that dynamic equations need not have extremal solutions.

Example 6.4. Let $\mathbb{T} = [0, 1] \cup [2, 3]$. Suppose that $f : [0, \infty) \times \mathbb{T} \rightarrow \mathbb{R}$ is given by

$$f(y, t) = \begin{cases} 3y^{2/3}, & t \in [0, 1), \\ 2(1 - y), & t = 1, \\ 0, & t \in [2, 3]. \end{cases}$$

Consider Eq. (6.2) with $a = t_0 = 0$, $b = 3$, and $y_0 = 0$. On $[0, 1)$, the equation reduces to $y'(t) = 3y(t)^{2/3}$. Each solution is left-continuous at $t = 1$, and satisfies

$$y(2) = y(1) + \int_1^2 f(y(t), t) \Delta t = y(1) + f(y(1), 1)\mu(1) = y(1) + 2(1 - y(1)) = 2 - y(1).$$

Finally, each solution has to be constant on $[2, 3]$. It follows that all solutions have the form

$$z_\tau(t) = \begin{cases} 0, & t \in [0, \tau], \\ (t - \tau)^3, & t \in (\tau, 1], \\ 2 - (1 - \tau)^3, & t \in [2, 3], \end{cases}$$

where $\tau \in [0, 1]$ is a parameter. Note that z_0 is the greatest solution on $[0, 1]$, while $z_1 > z_0$ on $[2, 3]$. Hence, there is no greatest solution on \mathbb{T} . Similarly, z_1 is the least solution on $[0, 1]$, but there is no least solution on \mathbb{T} .

Taking into account the relation between dynamic equations and measure differential equations, we expect that the existence of extremal solutions will be guaranteed if the functions f^* , g satisfy condition (C4). We have $\Delta^+g(t) = \mu(t)$ for each $t \in [a, b]_{\mathbb{T}}$, and $\Delta^+g(t) = 0$ otherwise. Thus, we set the following condition:

(T4) If $u, v \in B$, with $u < v$, then $u + f(u, t)\mu(t) \leq v + f(v, t)\mu(t)$ for every $t \in [a, b]_{\mathbb{T}}$.

Obviously, condition (C4) holds if (T4) is satisfied. By combining Theorems 4.4 and 6.2, we obtain the following result.

Theorem 6.5. *Assume that $B \subseteq \mathbb{R}$ is closed, $y_0 \in B$, and $f : B \times [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ satisfies conditions (T1), (T2), (T3), (T4). If Eq. (6.5) has a solution on $[a, b]_{\mathbb{T}}$, then it has the greatest solution and the least solution on $[a, b]_{\mathbb{T}}$.*

To formulate the next results, we introduce the following weaker version of condition (T2):

(T2') For each compact set $C \subseteq B$, there exists a Lebesgue Δ -integrable function $M : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ such that $\|f(y, t)\| \leq M(t)$ for every $y \in C$ and $t \in [a, b]_{\mathbb{T}}$.

Note that if $f : B \times [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ satisfies condition (T2'), then $f^* : B \times [a, b] \rightarrow \mathbb{R}$ satisfies (C2').

As a consequence of Theorems 4.6 and 6.2, we have the following analogue of Peano's uniqueness theorem for dynamic equations.

Theorem 6.6. *Assume that $B \subseteq \mathbb{R}$ is closed, $y_0 \in B$, and $f : B \times [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ satisfies conditions (T1), (T2'), (T3), (T4). If the function f is nonincreasing in the first variable, then Eq. (6.5) has at most one solution on $[a, b]_{\mathbb{T}}$.*

The combination of Theorems 4.9 and 6.2 leads to the following result.

Theorem 6.7. *Assume that $B \subseteq \mathbb{R}$ is open and $f : B \times [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ satisfies conditions (T1), (T2'), (T3), (T4). Moreover, suppose that $y_0 \in B$ and $y_0 + f(y_0, a)\mu(a) \in B$. Then there exist time scale intervals $I, J \subset [a, b]_{\mathbb{T}}$ containing a , and noncontinuable solutions $y_{\max} : I \rightarrow B$, $y_{\min} : J \rightarrow B$ of Eq. (6.5) such that the following statements hold:*

1. For any solution $y : I' \rightarrow B$ of Eq. (6.5) with $a \in I' \subseteq I$, we have $y(t) \leq y_{\max}(t)$ for all $t \in I'$.
2. For any solution $y : J' \rightarrow B$ of Eq. (6.5) with $a \in J' \subseteq J$, we have $y_{\min}(t) \leq y(t)$ for all $t \in J'$.

We now proceed to lower and upper solutions of dynamic equations.

Definition 6.8. Let $I \subseteq [a, b]_{\mathbb{T}}$ be a time scale interval with $a \in I$. A regulated function $\alpha : I \rightarrow B$ is called a lower solution of Eq. (6.5) on I if $\alpha(a) \leq y_0$ and

$$\alpha(v) - \alpha(u) \leq \int_u^v f(\alpha(s), s) \Delta s \quad \text{whenever } u, v \in I, u < v.$$

Symmetrically, a regulated function $\beta : I \rightarrow B$ is an upper solution of Eq. (6.5) on I if $\beta(a) \geq y_0$ and

$$\beta(v) - \beta(u) \geq \int_u^v f(\beta(s), s) \Delta s \quad \text{whenever } u, v \in I, u < v.$$

Lemma 6.9. *If $\alpha : I \rightarrow B$ is a lower/upper solution of Eq. (6.5) on I , then $\alpha^* : I^* \rightarrow B$ is a lower/upper solution of Eq. (6.3) on $I^* = \{t; t^* \in I\}$.*

Proof. Assume that $\alpha : I \rightarrow B$ is a lower solution of Eq. (6.5). Choose an arbitrary pair $u, v \in I^*$, $u < v$. Since g is constant on $[u, u^*]$ and $[v, v^*]$, the integrals $\int_u^{u^*} f(\alpha^*(s), s^*) dg(s)$ and $\int_v^{v^*} f(\alpha^*(s), s^*) dg(s)$ exist and are equal to zero. Thus,

$$\alpha^*(v) - \alpha^*(u) = \alpha(v^*) - \alpha(u^*) \leq \int_{u^*}^{v^*} f(\alpha(s), s) \Delta s = \int_{u^*}^{v^*} f(\alpha^*(s), s^*) dg(s) = \int_u^v f(\alpha^*(s), s^*) dg(s),$$

which shows that $\alpha^* : I^* \rightarrow B$ is a lower solution of Eq. (6.3). The statement about upper solution can be proved in the same way. \square

The following result is a consequence of Lemma 6.9 and Theorems 4.12 and 6.2.

Theorem 6.10. *Assume that $B \subseteq \mathbb{R}$ is open, $f : B \times [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ satisfies conditions (T1), (T2'), (T3), (T4), $y_0 \in B$, and $y_0 + f(y_0, a)\mu(a) \in B$.*

If $y_{\max} : I \rightarrow B$ and $y_{\min} : J \rightarrow B$, where $a \in I \subseteq [a, b]_{\mathbb{T}}$ and $a \in J \subseteq [a, b]_{\mathbb{T}}$, are the noncontinuable extremal solutions of Eq. (6.5) described in Theorem 6.7, then the following statements hold:

1. *If $\alpha : I' \rightarrow B$, where $a \in I' \subseteq I$, is a lower solution of Eq. (6.5), then $\alpha \leq y_{\max}$ on I' .*
2. *If $\beta : J' \rightarrow B$, where $a \in J' \subseteq J$, is an upper solution of Eq. (6.5), then $\beta \geq y_{\min}$ on J' .*

Consequently,

$$\begin{aligned} y_{\max}(t) &= \max\{\alpha(t); \alpha \text{ is a lower solution of Eq. (6.5) on } [a, t]_{\mathbb{T}}\}, \quad t \in I, \\ y_{\min}(t) &= \min\{\beta(t); \beta \text{ is an upper solution of Eq. (6.5) on } [a, t]_{\mathbb{T}}\}, \quad t \in J. \end{aligned}$$

Remark 6.11. Extremal solutions and lower/upper solutions of dynamic equations were already studied in B. Kaymakçalan's paper [23] (see Theorems 5.1 and 5.2 there), and later in the book [28] (see Theorems 2.3.1 and 2.4.1). For the sake of comparison of our results with the ones in [23] and [28], we mention the following facts:

1. Instead of dealing with the dynamic equation $y^\Delta(t) = f(y(t), t)$, we work with the more general integral equation (6.2). Similarly, our definitions of lower and upper solutions are based on integral rather than differential inequalities, and thus are less restrictive. Unlike [23] and [28], the right-hand side is not required to be rd-continuous.
2. The results in [28] are formulated for systems of equations, and the right-hand side is assumed to be quasimonotone nondecreasing. On the other hand, we have restricted our attention to scalar equations only.
3. Theorems 5.1 and 5.2 in [23] assume that for each t , the function $u \mapsto f(u, t)\mu(t)$ is nondecreasing. This assumption is stronger than our condition (T4), which coincides with the one given in [28].

7 Open problems

Interested readers are invited to think about the following open questions:

- Is it possible to extend the theory developed in Section 4 to the vector case where both f and y take values in \mathbb{R}^n ? For comparing vector-valued functions, we can use the componentwise partial ordering of \mathbb{R}^n , where $y \leq z$ if and only if $y_i \leq z_i$ for all $i \in \{1, \dots, n\}$. As in the classical case (see e.g. [2, 7, 15]), one can expect that to obtain the existence of extremal solutions, the right-hand side f should be assumed to be quasimonotone nondecreasing, i.e., such that for each $i \in \{1, \dots, n\}$ and $(y, t), (z, t) \in B \times [a, b]$, the relations $y \leq z$ and $y_i = z_i$ imply $f_i(y, t) \leq f_i(z, t)$.
- For classical ordinary differential equations, the existence of a lower solution $\alpha : [a, b] \rightarrow \mathbb{R}$ and an upper solution $\beta : [a, b] \rightarrow \mathbb{R}$, where $\alpha \leq \beta$, guarantees the existence of a solution lying between α and β (see e.g. [34, Theorem 4.1]). Is there an analogue of this statement for measure differential equations?
- In [21, Theorem 19.1], R. Henstock established a local existence theorem for integral equations of the form

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s) ds, \quad t \in [a, b],$$

where the integral on the right-hand side is the Kurzweil-Henstock integral. Henstock's approach was thoroughly analyzed in [41, Chapter 2]; it turned out that Henstock's conditions, which are denoted by (H1), (H2), (H3) in [41], cover a somewhat larger class of right-hand sides than the Carathéodory theory. Our conditions (C1), (C3) coincide with Henstock's conditions (H1), (H2), but condition (C2) is stronger than (H3). Would it be possible to replace (C2) by an analogue of (H3) in order to get a more general local existence result for Eq. (3.1)?

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