Dynamic diffusion-type equations on discrete-space domains

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Abstract

We consider a class of partial dynamic equations on discrete-space domains which includes, as a special case, the discrete-space versions of the diffusion (heat) equation. We focus on initial-value problems and study the existence and uniqueness of forward and backward solutions. Moreover, we discuss other topics such as sum and sign preservation, maximum and minimum principles, or symmetry of solutions.

Keywords: diffusion equation, heat equation, semidiscrete equations, partial dynamic equations, time scales, random walk

MSC 2010 subject classification: 34N05, 35F10, 39A14, 65M06

1 Introduction

The classical diffusion (heat) equation $u_t = k\nabla^2 u$ describes the particle (or heat) distribution as a function of time. In this paper, we consider a class of diffusion-type equations with discrete space and arbitrary (continuous, discrete or mixed) time, namely

$$u^{\Delta_t}(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t), \quad x \in \mathbb{Z},\ t \in \mathbb{T}, \tag{1.1}$$

where $\mathbb{T}$ is a time scale (arbitrary closed subset of $\mathbb{R}$). The symbol u^{Δ_t} denotes the partial Δ-derivative with respect to t, which becomes the standard partial derivative u_t when $\mathbb{T} = \mathbb{R}$, and the forward partial difference $\Delta_t u$ when $\mathbb{T} = \mathbb{Z}$. Since the differences with respect to x never appear in this paper, we omit the lower index t in u^{Δ_t} and write u^{Δ} instead. We employ the time scale calculus to be able to study equations with continuous, discrete or mixed time domains in a unified way. Readers who are not familiar with the basic principles and notations of this recent mathematical tool are kindly asked to consult Stefan Hilger's original paper [10] or the survey monograph [7].

Our motivation comes first and foremost from the absence of a systematic theory for discrete-space diffusion problems, although some more or less isolated facts about special cases of Eq. (1.1) can be found in the literature (see [2, 11, 12, 26]). Moreover, the lack of suitable mathematical tools is in contrast with the use of (1.1) in applications:

- When $a = c$ and $b = -2a$, Eq. (1.1) represents a discretized version of the classical diffusion equation. Depending on the time scale, we can obtain the semidiscrete diffusion equation ($\mathbb{T} = \mathbb{R}$), or the purely discrete diffusion equation ($\mathbb{T} = \mathbb{Z}$).
- The case $a = 0$ and $0 < c = -b$ corresponds to the discrete-space transport equation.
- For $\mathbb{T} = \mathbb{Z}$, $a = c = 1/2$ and $b = -1$, Eq. (1.1) reduces to

$$u(x,t+1) = \frac{1}{2}u(x+1,t) + \frac{1}{2}u(x-1,t), \tag{1.2}$$

which (together with the initial condition $u(0,0) = 1$ and $u(x,0) = 0$ for $x \neq 0$) describes the one-dimensional symmetric random walk on $\mathbb{Z}$ starting from the origin; the value $u(x,t)$ is the probability that the random walk visits point x at time t. More generally, consider a nonsymmetric random walk on $\mathbb{Z}$, where the probabilities of going left, remaining at the same position, or going right are $p, q, r \in [0,1]$, with $p + q + r = 1$. This random walk is described by Eq. (1.1), where $\mathbb{T} = \mathbb{Z}$, $a = p$, $b = q - 1$ and $c = r$. For $\mathbb{T} = \mathbb{R}$, we obtain a continuous-time Markov process which is similar to the well-known birth-death process (the difference is that in our situation, x can be positive as well as negative). Finally, for a general time scale $\mathbb{T}$, solutions of (1.1) can be regarded as heterogeneous stochastic processes.

- Applications of (1.1) go far beyond stochastic processes. For example, the semidiscrete diffusion equation appears in signal and image processing [15], while the discrete diffusion equation has been used to model mutations in biology [8].

From a theoretical point of view, our work could be perceived as a contribution to the study of partial dynamic equations (see, e.g., [3, 11, 12, 16]). Alternatively, since we consider discrete space, our dynamic diffusion equations can be viewed as infinite systems of ordinary dynamic equations (see, e.g., [18]).

The paper is organized as follows. In Section 2, we provide some auxiliary results regarding the time scale exponential function. In Section 3, we study the existence and uniqueness of both forward and backward solutions. Section 4 deals with topics related to stochastic processes, such as space sum preservation, sign preservation, or maximum and minimum principles. In Section 5, we show that equations with symmetric right-hand sides possess symmetric solutions and characterize their maxima. Finally, in Section 6, we conclude the paper with a summarizing table and a set of open problems.

2 Preliminaries

Before we start our investigations of dynamic diffusion equations, it is necessary to present some auxiliary results concerning the time scale exponential function.

We need the time scale polynomials $h_k : \mathbb{T}^2 \to \mathbb{R}$, which are defined as follows:

$$\begin{aligned} h_0(t,s) &= 1, \quad t,s \in \mathbb{T}, \\ h_{k+1}(t,s) &= \int_s^t h_k(\tau,s)\Delta\tau, \quad t,s \in \mathbb{T},\ k \in \mathbb{N}_0. \end{aligned}$$

The following estimate can be found in [6, Theorem 4.1]:

$$0 \le h_k(t,s) \le \frac{(t-s)^k}{k!}, \quad k \in \mathbb{N}_0,\ t \ge s. \tag{2.1}$$

Let X be a Banach space and $\mathcal{L}(X)$ the space of all bounded linear operators on X. Consider a point $t_0 \in \mathbb{T}$, and an operator $A \in \mathcal{L}(X)$ such that $I + A\mu(t)$ is invertible for every $t \in (-\infty, t_0)_{\mathbb{T}}$. Then, the initial-value problem

$$\begin{aligned} x^\Delta(t) &= Ax(t), \quad t \in \mathbb{T}, \\ x(t_0) &= I, \end{aligned} \tag{2.2}$$

has a unique solution on $\mathbb{T}$ (see [10, Theorem 5.7] or [7, Theorem 8.24]). Its value at a point $t \in \mathbb{T}$ is denoted by $e_A(t,t_0)$, and the function $t \mapsto e_A(t,t_0)$, whose values are elements of $\mathcal{L}(X)$, is called the exponential function corresponding to A.

For $t \ge t_0$, the exponential $e_A(t,t_0)$ can be expressed in the following way:

$$e_A(t,t_0) = \sum_{k=0}^{\infty} A^k h_k(t,t_0), \quad t \in [t_0,\infty)_{\mathbb{T}}. \tag{2.3}$$

Indeed, it follows from the estimate (2.1) that the series is absolutely and locally uniformly convergent. Moreover, using term by term differentiation, it is easy to verify that (2.2) is satisfied.

At this point, we emphasize that the series representation in (2.3) is no longer generally valid for $t < t_0$, because the series need not be convergent (this depends on the choice of the time scale, which influences the behavior of the time scale polynomials).

When $t \in \mathbb{T}$ is a right-scattered point, it is known that $e_A(\sigma(t), t_0) = (I + A\mu(t))e_A(t, t_0)$; this is an immediate consequence of (2.2) and the definition of the Δ-derivative. Hence, if $[t_0, t]_{\mathbb{T}}$ contains only a finite number of points, we obtain

$$e_A(t, t_0) = \prod_{\tau \in [t_0, t)_{\mathbb{T}}} (I + A\mu(\tau)). \tag{2.4}$$

We now show that the time scale exponential function can be obtained by means of a certain limit process, which resembles the definition of the Riemann integral with sums replaced by products (or, more precisely, compositions of operators). First, we need a few definitions.

Given a time scale interval $[a, b]_{\mathbb{T}}$, we use the symbol $\mathcal{P}(a, b)$ to denote the set of all partitions of $[a, b]_{\mathbb{T}}$. For every $D \in \mathcal{P}(a, b)$, let $m(D)$ be the number of subintervals in D. Therefore, every partition D has the form $a = t_0 < t_1 < \ldots < t_{m(D)} = b$, where $t_0, \ldots, t_{m(D)} \in \mathbb{T}$.

For every $\delta > 0$, let $P_\delta(a, b)$ denote the set of all partitions $D \in \mathcal{P}(a, b)$ such that for each $i \in \{1, \ldots, m(D)\}$, we have either $t_i - t_{i-1} \le \delta$, or $t_i - t_{i-1} > \delta$ and $t_i = \sigma(t_{i-1})$.

Consider a Banach space X, a function $F : \mathcal{P}(a, b) \to X$, and an element $L \in X$. We write $\lim_{\|D\| \to 0} F(D) = L$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\|F(D) - L\| < \varepsilon$ for every $D \in \mathcal{P}_\delta(a, b)$.

Recall that the exponential function $t \mapsto e_A(t, t_0)$ is the unique solution of the initial-value problem (2.2). Consider a point $t \in [t_0, \infty)_{\mathbb{T}}$ and imagine that we are trying to calculate the approximate value of $e_A(t, t_0)$ by Euler's method. To this end, we take a partition $D \in \mathcal{P}(t_0, t)$ and use the approximation

$$u(t_i) \doteq u(t_{i-1}) + u^\Delta(t_{i-1})(t_i - t_{i-1}) = (I + A(t_i - t_{i-1}))u(t_{i-1}).$$

Repeated application of this formula leads to the approximation (the operators on the right-hand side commute)

$$\begin{aligned} e_A(t, t_0) &\doteq (I + A(t_{m(D)} - t_{m(D)-1}))(I + A(t_{m(D)-1} - t_{m(D)-2})) \cdots (I + A(t_1 - t_0)) \\ &= (I + A(t_1 - t_0))(I + A(t_2 - t_1)) \cdots (I + A(t_{m(D)} - t_{m(D)-1})), \end{aligned}$$

which we expect to approach the true value of $e_A(t, t_0)$ for finer and finer partitions of $[t_0, t]_{\mathbb{T}}$. We confirm this fact in Theorem 2.2 below.

Lemma 2.1. *If $x_1, \ldots, x_m$ are nonnegative real numbers, then*

$$\sum_{1 \le i_1 < \cdots < i_k \le m} x_{i_1} \cdots x_{i_k} \le \frac{(x_1 + \cdots + x_m)^k}{k!}, \quad k \in \{1, \ldots, m\}.$$

Proof. By Maclaurin's inequality (see [25, Chapter 12]), we have

$$\left(\frac{\sum_{1 \le i_1 < \cdots < i_k \le m} x_{i_1} \cdots x_{i_k}}{\binom{m}{k}} \right)^{1/k} \le \frac{x_1 + \cdots + x_m}{m}, \quad k \in \{1, \ldots, m\}.$$

Therefore,

$$\sum_{1 \le i_1 < \cdots < i_k \le m} x_{i_1} \cdots x_{i_k} \le \binom{m}{k} \left(\frac{x_1 + \cdots + x_m}{m} \right)^k \le \frac{(x_1 + \cdots + x_m)^k}{k!}, \quad k \in \{1, \ldots, m\}. \qquad \square$$

Theorem 2.2. *Consider a Banach space X, a time scale interval $[t_0, t]_{\mathbb{T}}$, and an operator $A \in \mathcal{L}(X)$. For every $D \in \mathcal{P}(t_0, t)$, let*

$$P(D) = (I + A(t_1 - t_0))(I + A(t_2 - t_1)) \cdots (I + A(t_{m(D)} - t_{m(D)-1})). \tag{2.5}$$

Then $\lim_{\|D\| \to 0} P(D) = e_A(t, t_0)$.

Proof. The statement holds if $[t_0, t]_{\mathbb{T}}$ contains only a finite number of points. To see this, consider a positive number $\delta \leq \max_{\tau \in [t_0,t)_{\mathbb{T}}} \mu(\tau)$. Then, for every $D \in \mathcal{P}_\delta(a, b)$, we have

$$P(D) = \prod_{t \in [t_0,t)_{\mathbb{T}}} (I + A\mu(t)) = e_A(t, t_0),$$

where the last equality follows from (2.4), and it is clear that $\lim_{\|D\| \to 0} P(D) = e_A(t, t_0)$.

Now, assume that $[t_0, t]_{\mathbb{T}}$ has infinitely many points. In view of Eq. (2.3), it is enough to show that $\lim_{\|D\|\to 0} P(D) = \sum_{k=0}^{\infty} A^k h_k(t, t_0)$. From the definition of $P(D)$ in Eq. (2.5), we see that

$$P(D) = I + P_1(D) + \cdots P_{m(D)}(D),$$

where

$$P_k(D) = A^k \left(\sum_{1 \leq i_1 < \cdots < i_k \leq m(D)} (t_{i_1} - t_{i_1 - 1}) \cdots (t_{i_k} - t_{i_k - 1}) \right), \quad k \in \{1, \ldots, m(D)\}.$$

For every $k \in \mathbb{N}$, let $\chi_k : [t_0, t]^k_{\mathbb{T}} \to \mathbb{R}$ be given by

$$\chi_k(s_1, \ldots, s_k) = \begin{cases} 1 & \text{if } s_1 < s_2 < \cdots < s_k, \\ 0 & \text{otherwise.} \end{cases}$$

We obtain

$$\lim_{\|D\|\to 0} \sum_{1 \leq i_1 < \cdots < i_k \leq m(D)} (t_{i_1} - t_{i_1-1}) \cdots (t_{i_k} - t_{i_k-1}) = \lim_{\|D\|\to 0} \sum_{i_1,\ldots,i_k=1}^{m(D)} \chi_k(t_{i_1}, \ldots, t_{i_k})(t_{i_1} - t_{i_1-1}) \cdots (t_{i_k} - t_{i_k-1})$$

$$= \int_{t_0}^{t} \cdots \int_{t_0}^{t} \chi_k(s_1, \ldots, s_k) \Delta s_1 \cdots \Delta s_k = \int_{t_0}^{t} \left(\int_{t_0}^{s_1} \left(\cdots \left(\int_{t_0}^{s_{k-1}} \Delta s_k \right) \cdots \right) \Delta s_2 \right) \Delta s_1 = h_k(t, t_0).$$

On the last line, we first used the Riemann-type definition of the k-dimensional integral on time scales (see [4]), then Fubini's theorem, and finally the definition of the time scale polynomials. Note that a multidimensional integral on the product of time scales is just the ordinary Lebesgue integral with respect to a suitable measure (see [5]), and hence the use of Fubini's theorem is justified. To sum up, we have just proved that

$$\lim_{\|D\|\to 0} P_k(D) = A^k h_k(t, t_0).$$

Consider an arbitrary $\varepsilon > 0$. Since $\sum_{k=0}^{\infty} \|A\|^k \frac{(t-t_0)^k}{k!}$ is convergent, there exists a $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty} \|A\|^k \frac{(t - t_0)^k}{k!} < \varepsilon/3.$$

Now, we claim there is a $\delta > 0$ such that every partition $D \in \mathcal{P}_\delta(t_0, t)$ has at least k_0 subintervals. Indeed, consider any partition with k_0 subintervals $t_0 = s_0 < s_1 < \cdots < s_{k_0} = t$, and choose a positive δ smaller than $\min_{1 \leq i \leq k_0}(s_i - s_{i-1})$. Then every partition $D \in \mathcal{P}_\delta(t_0, t)$ has at least one division point in each of the intervals $(s_0, s_1]_{\mathbb{T}}, \ldots, (s_{k_0-1}, s_{k_0}]_{\mathbb{T}}$, and hence contains at least k_0 subintervals.

Moreover, assume that the number δ is such that

$$\|P_k(D) - A^k h_k(t, t_0)\| < \frac{\varepsilon}{3k_0}, \quad k \in \{1, \ldots, k_0\}, \quad D \in \mathcal{P}_\delta(t_0, t).$$

For every $D \in \mathcal{P}(t_0, t)$, Lemma 2.1 (with $x_j = t_j - t_{j-1}$, $j \in \{1, \ldots, m(D)\}$) gives

$$\|P_k(D)\| \leq \|A\|^k \sum_{1 \leq i_1 < \cdots < i_k \leq m(D)} (t_{i_1} - t_{i_1-1}) \cdots (t_{i_k} - t_{i_k-1}) \leq \|A\|^k \frac{(t - t_0)^k}{k!}, \quad k \in \{1, \ldots, m(D)\}.$$

These facts imply that for every partition $D \in \mathcal{P}_\delta(t_0, t)$, we have the estimate

$$\left\| P(D) - \sum_{k=0}^{\infty} A^k h_k(t,t_0) \right\| = \left\| \sum_{k=1}^{m(D)} P_k(D) - \sum_{k=1}^{\infty} A^k h_k(t,t_0) \right\| \le \sum_{k=1}^{k_0} \|P_k(D) - A^k h_k(t,t_0)\|$$

$$+ \sum_{k=k_0+1}^{m(D)} \|P_k(D)\| + \sum_{k=k_0+1}^{\infty} \|A\|^k h_k(t,t_0) < k_0 \frac{\varepsilon}{3k_0} + 2 \sum_{k=k_0+1}^{\infty} \|A\|^k \frac{(t-t_0)^k}{k!} < \varepsilon. \qquad \square$$

Let us mention that our previous result is closely connected with the theory of product integrals. The idea of obtaining the solution of a linear system of differential equations as a limit of products of the form $P(D)$ goes back to Vito Volterra [31], who considered matrix-valued functions A defined on a real interval (i.e., he worked with finite-dimensional spaces only, but with A dependent on t). Later, the theory was generalized to infinite-dimensional spaces by Pesi Rustom Masani [17]. For more information about product integrals and their history, see [24]. Product integrals of matrix-valued functions defined on time scales have been introduced in [23]. Our proof of Theorem 2.2 combines the ideas of both Volterra and Masani (cf. [24], Theorems 2.4.3 and 5.5.10).

3 General theory

In this section, we consider the equation

$$u^\Delta(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t), \tag{3.1}$$

where a, b, c are real numbers. We study the existence and uniqueness of solutions to initial-value problems, and prove a superposition principle.

When $t \in \mathbb{T}$ is a right-scattered point, Eq. (3.1) implies

$$u(x,\sigma(t)) = u(x,t) + \mu(t)u^\Delta(x,t) = a\mu(t)u(x+1,t) + (1+b\mu(t))u(x,t) + c\mu(t)u(x-1,t),$$

i.e., the sequence $\{u(x,t)\}_{x\in\mathbb{Z}}$ determines $\{u(x,\sigma(t))\}_{x\in\mathbb{Z}}$ uniquely. Also, when $t \in \mathbb{T}$ is a left-dense point, it follows from continuity that $u(x,t) = \lim_{\tau\to t-} u(x,\tau)$, i.e., the values $\{u(x,t)\}_{x\in\mathbb{Z}}$ are uniquely determined by the values $\{u(x,\tau)\}_{x\in\mathbb{Z}}$, where $\tau < t$.

To sum up, when we go forward in time, existence and uniqueness of solutions corresponding to an initial condition are straightforward for all time scales with no right-dense points. It remains to settle the matter for time scales containing right-dense points, and also for solutions which go backward in time.

Let $\ell^\infty(\mathbb{Z})$ denote the space of all bounded real sequences $\{u_n\}_{n\in\mathbb{Z}}$ equipped with the supremum norm

$$\|u\|_\infty = \sup_{n\in\mathbb{Z}} |u_n|, \quad u \in \ell^\infty(\mathbb{Z}).$$

Also, let $\ell^1(\mathbb{Z})$ denote the space of all real sequences $\{u_n\}_{n\in\mathbb{Z}}$ such that $\sum_{n\in\mathbb{Z}} |u_n|$ is finite. This space is equipped with the ℓ^1 norm

$$\|u\|_1 = \sum_{n\in\mathbb{Z}} |u_n|, \quad u \in \ell^1(\mathbb{Z}).$$

Note that $\ell^1(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$ and $\|u\|_\infty \le \|u\|_1$ for every $u \in \ell^1(\mathbb{Z})$.

For $p = 1$ or $p = \infty$, consider the linear operator $A : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ given by

$$A(\{u_n\}_{n\in\mathbb{Z}}) = \{au_{n+1} + bu_n + cu_{n-1}\}_{n\in\mathbb{Z}}. \tag{3.2}$$

This operator will play an important role in our calculations. For both choices of p, the operator A is bounded and $\|A\| = |a| + |b| + |c|$.

In the next lemma, we use the following notation: Given a function $U : \mathbb{T} \to \ell^p(\mathbb{Z})$, the symbol $U(t)_x$ denotes the x-th component of the sequence $U(t)$, and should not be confused with the derivative of U with respect to x (which never appears in this paper).

Lemma 3.1. *Let $p = 1$ or $p = \infty$. If $U : \mathbb{T} \to \ell^p(\mathbb{Z})$ is a solution of the dynamic equation*

$$U^\Delta(t) = AU(t),$$

then the function given by $u(x,t) = U(t)_x$, $x \in \mathbb{Z}$, $t \in \mathbb{T}$, is a solution of Eq. (3.1).

Proof. Recalling the definition of the norm in $\ell^p(\mathbb{Z})$, we see that differentiability of U implies that its components are differentiable. We have

$$(U(t)_x)^\Delta = (U^\Delta(t))_x = (AU(t))_x = aU(t)_{x+1} + bU(t)_x + cU(t)_{x-1}, \quad x \in \mathbb{Z},\ t \in \mathbb{T},$$

which means that u is a solution of Eq. (3.1). □

Note that the converse statement need not be true: Given a solution $u : \mathbb{Z} \times \mathbb{T} \to \mathbb{R}$ of Eq. (3.1), the function $U : \mathbb{T} \to \ell^p(\mathbb{Z})$ given by $U(t) = \{u(x,t)\}_{x\in\mathbb{Z}}$ need not satisfy $U^\Delta(t) = AU(t)$. The reason is that the differentiability of $t \mapsto u(x,t)$ for every $x \in \mathbb{Z}$ does not necessarily imply that U is differentiable.

Lemma 3.2. *For every $\delta \in [0, \frac{1}{|a|+|b|+|c|})$, the operator $I + \delta A$ is invertible.*

Proof. Recall that $\|A\| = |a| + |b| + |c|$ for both $p = 1$ and $p = \infty$. Hence,

$$\|I - (I + \delta A)\| = \delta\|A\| < 1,$$

and it follows that $I + \delta A$ is invertible. □

The next theorem provides a sufficient condition for the existence of solutions of Eq. (3.1). Moreover, it guarantees that an initial condition from $\ell^p(\mathbb{Z})$ generates a solution which stays in $\ell^p(\mathbb{Z})$ for all $t \in \mathbb{T}$.

Theorem 3.3. *Consider an interval $[T_1, T_2]_\mathbb{T} \subset \mathbb{T}$ and a point $t_0 \in [T_1, T_2]_\mathbb{T}$. Let $u^0 \in \ell^p(\mathbb{Z})$, where $p = 1$ or $p = \infty$. Assume that $\mu(t) < \frac{1}{|a|+|b|+|c|}$ for every $t \in [T_1, t_0)_\mathbb{T}$. Let $U : [T_1, T_2]_\mathbb{T} \to \ell^p(\mathbb{Z})$ be given by $U(t) = e_A(t, t_0)u^0$, $t \in [T_1, T_2]_\mathbb{T}$. Then*

$$u(x,t) = U(t)_x, \quad x \in \mathbb{Z},\ t \in [T_1, T_2]_\mathbb{T},$$

is a bounded solution of Eq. (3.1) *and satisfies $u(x, t_0) = u^0_x$ for every $x \in \mathbb{Z}$.*

Proof. According to the previous lemma, the condition $\mu(t) < \frac{1}{|a|+|b|+|c|}$ implies that $I+\mu(t)A$ is invertible for $t \in [T_1, t_0)_\mathbb{T}$. Therefore, the exponential function $t \mapsto e_A(t, t_0)$ is well defined on $[T_1, T_2]_\mathbb{T}$, and U is a solution of the initial-value problem

$$\begin{aligned} U^\Delta(t) &= AU(t), \quad t \in [T_1, T_2]_\mathbb{T}, \\ U(t_0) &= u^0. \end{aligned}$$

The fact that u is a solution of Eq. (3.1) follows from Lemma 3.1. Since U is continuous on $[T_1, T_2]_\mathbb{T}$, it is bounded in the $\ell^p(\mathbb{Z})$ norm. For both $p = 1$ and $p = \infty$, it follows that the solution u is bounded. □

We now proceed to uniqueness of solutions. It is known that for the classical diffusion equation with continuous time and space, initial-value problems on the whole real line do not have unique solutions (this was shown by Tychonoff [29]; see also [13]). The following construction demonstrates that solutions of Eq. (3.1) with a given initial condition are not necessarily unique. Note that Eq. (3.1) represents a countable system of linear dynamic equations; the fact that initial-value problems for countable linear systems of differential equations need not have unique solutions was also observed by Tychonoff [28].

Consider the time scale $\mathbb{T} = \mathbb{R}$. Choose a pair of infinitely differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f^{(i)}(0) = 0$ and $g^{(i)}(0) = 0$ for every $i \in \mathbb{N}_0$. Let $u(0,t) = f(t)$ and $u(1,t) = g(t)$ for every $t \in \mathbb{R}$. It remains to define $u(x,t)$ for the remaining values of x so that

$$u_t(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t) \tag{3.3}$$

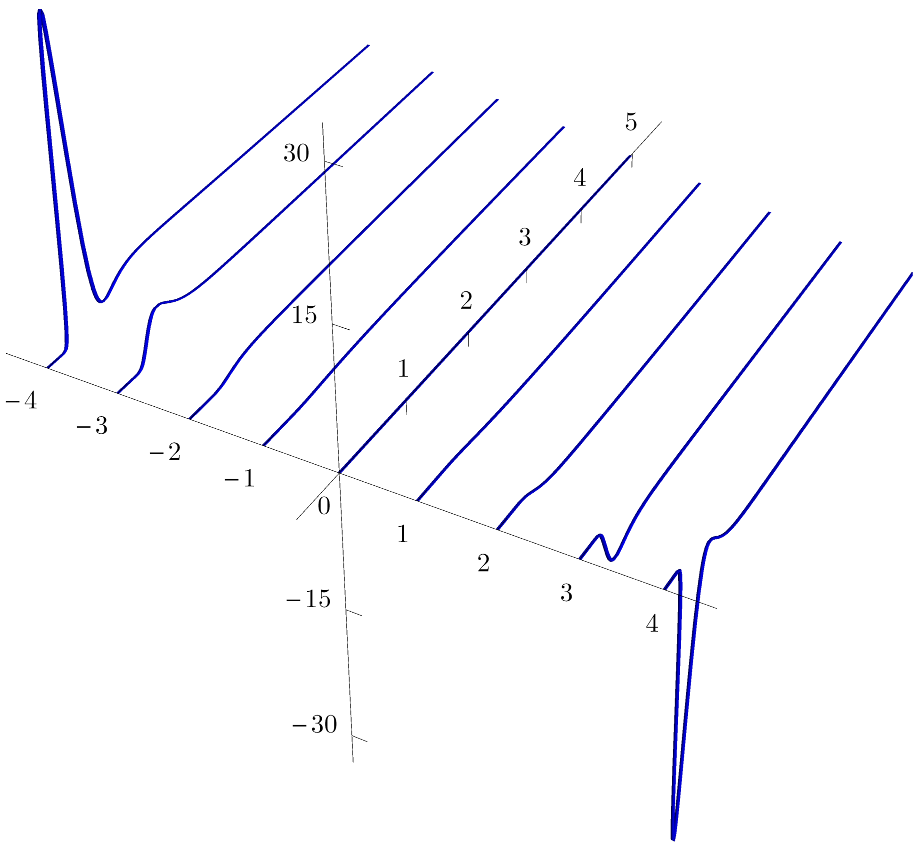

Figure 1: Unbounded solution of the semidiscrete diffusion equation with zero initial condition

holds for all $x \in \mathbb{Z}$ and $t \in \mathbb{R}$. We consider only the case when $a, c \neq 0$. Then, the formulas

$$u(x+1,t) = \frac{1}{a}\left(u_t(x,t) - bu(x,t) - cu(x-1,t)\right), \quad x \geq 1,\ t \in \mathbb{R}, \tag{3.4}$$

$$u(x-1,t) = \frac{1}{c}\left(u_t(x,t) - au(x+1,t) - bu(x,t)\right), \quad x \leq 0,\ t \in \mathbb{R}, \tag{3.5}$$

determine the remaining values $u(x,t)$ uniquely. By the properties of f and g, we have $u(x,0) = 0$ for every $x \in \mathbb{Z}$. Since there are infinitely many possibilities of choosing f and g, it follows that Eq. (3.3) has infinitely many solutions corresponding to the zero initial condition at $t = 0$. The solution obtained from Theorem 3.3 is the zero solution, which corresponds to $f = g = 0$. It can be shown (for example, it follows from Theorem 3.5 below) that all nonzero solutions display a curious behavior: they are unbounded on all sets of the form $\mathbb{Z} \times [0, \varepsilon]$, where $\varepsilon > 0$ can be arbitrarily small. Conversely, it turns out that if we restrict our attention to solutions which are bounded on all sets of the form $\mathbb{Z} \times [a, b]$, where $[a, b] \subset \mathbb{R}$, then all initial-value problems with bounded initial conditions have a unique solution; this is the content of Theorem 3.5 below.

At this point, it is worth mentioning that there are other reasons why unbounded solutions are pathological in a certain sense. For example, consider the previous construction with f identically zero and $g(t) = -e^{-1/t^2}$. The corresponding solution $u : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is given by the identities (3.4) and (3.5); see Figure 1. One curious fact to note is that the initial condition at $t = 0$ is symmetric with respect to the origin, but the solution does not maintain this property (cf. Theorem 5.1) and is odd in x. Also, the solution violates the maximum and minimum principles (cf. Theorem 4.7).

Before we state our main result, we prove an auxiliary lemma.

Lemma 3.4. *Consider an interval* $[\tau_1, \tau_2]_{\mathbb{T}} \subset \mathbb{T}$ *and a point* $t \in [\tau_1, \tau_2]_{\mathbb{T}}$ *such that* $|t - \tau_i| < \frac{1}{2(|a|+|b|+|c|)}$ *for* $i \in \{1, 2\}$. *Assume that* $u_1, u_2 : \mathbb{Z} \times [\tau_1, \tau_2]_{\mathbb{T}} \to \mathbb{R}$ *are bounded solutions of Eq.* (3.1). *If* $u_1(x,t) = u_2(x,t)$

for every $x \in \mathbb{Z}$, then u_1 and u_2 coincide on $\mathbb{Z} \times [\tau_1, \tau_2]_{\mathbb{T}}$.

Proof. For every $x \in \mathbb{Z}$ and $r \in [\tau_1, \tau_2]_{\mathbb{T}}$, we have

$$u_1(x,r) - u_2(x,r) = u_1(x,t) - u_2(x,t) + \int_t^r (u_1^\Delta(x,s) - u_2^\Delta(x,s))\Delta s = \int_t^r (u_1^\Delta(x,s) - u_2^\Delta(x,s))\Delta s$$

$$= \int_t^r (au_1(x+1,s) + bu_1(x,s) + cu_1(x-1,s) - au_2(x+1,s) - bu_2(x,s) - cu_2(x-1,s))\Delta s.$$

Combining this equality with the estimate

$$|au_1(x+1,s) + bu_1(x,s) + cu_1(x-1,s) - au_2(x+1,s) - bu_2(x,s) - cu_2(x-1,s)|$$

$$\leq |a| \cdot |u_1(x+1,s) - u_2(x+1,s)| + |b| \cdot |u_1(x,s) - u_2(x,s)| + |c| \cdot |u_1(x-1,s) - u_2(x-1,s)|$$

$$\leq (|a| + |b| + |c|) \sup_{x \in \mathbb{Z},\, s \in [\tau_1, \tau_2]_{\mathbb{T}}} |u_1(x,s) - u_2(x,s)|,$$

we obtain

$$|u_1(x,r) - u_2(x,r)| \leq |r - t| \cdot (|a| + |b| + |c|) \sup_{x \in \mathbb{Z},\, s \in [\tau_1, \tau_2]_{\mathbb{T}}} |u_1(x,s) - u_2(x,s)|$$

$$\leq \frac{1}{2} \sup_{x \in \mathbb{Z},\, s \in [\tau_1, \tau_2]_{\mathbb{T}}} |u_1(x,s) - u_2(x,s)|.$$

Passing to the supremum, we conclude that

$$\sup_{x \in \mathbb{Z},\, r \in [\tau_1, \tau_2]_{\mathbb{T}}} |u_1(x,r) - u_2(x,r)| \leq \frac{1}{2} \sup_{x \in \mathbb{Z},\, s \in [\tau_1, \tau_2]_{\mathbb{T}}} |u_1(x,s) - u_2(x,s)|.$$

Clearly, this inequality holds only if both suprema vanish, i.e., if u_1 and u_2 coincide. □

We are ready to prove the uniqueness of forward and backward bounded solutions.

Theorem 3.5. *Consider an interval $[T_1, T_2]_{\mathbb{T}} \subset \mathbb{T}$ and a point $t_0 \in [T_1, T_2]_{\mathbb{T}}$. Let $u^0 \in \ell^\infty(\mathbb{Z})$. Assume that $\mu(t) < \frac{1}{|a|+|b|+|c|}$ for every $t \in [T_1, t_0)_{\mathbb{T}}$.*

Then, there exists a unique bounded solution $u : \mathbb{Z} \times [T_1, T_2]_{\mathbb{T}} \to \mathbb{R}$ of Eq. (3.1) such that $u(x, t_0) = u_x^0$ for every $x \in \mathbb{Z}$.

Proof. The existence of a bounded solution $u : \mathbb{Z} \times [T_1, T_2]_{\mathbb{T}} \to \mathbb{R}$ was already proven in Theorem 3.3. Now, consider a pair of bounded solutions $u_1, u_2 : \mathbb{Z} \times [T_1, T_2]_{\mathbb{T}} \to \mathbb{R}$.

First, assume that u_1, u_2 do not coincide on $\mathbb{Z} \times (t_0, T_2]_{\mathbb{T}}$; let

$$t = \inf\{s \in (t_0, T_2];\ u_1(x,s) \neq u_2(x,s) \text{ for some } x \in \mathbb{Z}\}.$$

We claim that $u_1(x,t) = u_2(x,t)$ for every $x \in \mathbb{Z}$. If $t = t_0$, the statement is true. If $t > t_0$ and t is left-dense, then the statement follows from continuity. Finally, if $t > t_0$ and t is left-scattered, then $u_1(x, \rho(t)) = u_2(x, \rho(t))$, and the statement follows from the fact that $u_1^\Delta(x, \rho(t)) = u_2^\Delta(x, \rho(t))$. Now, if t is right-scattered, then $u_1(x,t) = u_2(x,t)$ and $u_1^\Delta(x,t) = u_2^\Delta(x,t)$ imply $u_1(x, \sigma(t)) = u_2(x, \sigma(t))$, a contradiction with the definition of t. On the other hand, if t is right-dense, there is a point $\tau \in (t, t + \frac{1}{2(|a|+|b|+|c|)})_{\mathbb{T}}$, and Lemma 3.4 (with $\tau_1 = t$, $\tau_2 = \tau$) leads to a contradiction again.

Before we proceed to the uniqueness in the backward direction, we make the following observation: Let u be a bounded solution of Eq. (3.1) on $\mathbb{Z} \times [T_1, t_0]_{\mathbb{T}}$. Denote $U(t) = \{u(x,t)\}_{x \in \mathbb{Z}}$. If $t \in [T_1, t_0)_{\mathbb{T}}$ is a right-scattered point, we have

$$U(\sigma(t)) = \{u(x, \sigma(t))\}_{x \in \mathbb{Z}} = \{u(x,t) + u^\Delta(x,t)\mu(t)\}_{x \in \mathbb{Z}}$$

$$= \{a\mu(t)u(x+1,t) + (1 + b\mu(t))u(x,t) + c\mu(t)u(x-1,t)\}_{x \in \mathbb{Z}} = (I + \mu(t)A)U(t).$$

By Lemma 3.2, the operator $I+\mu(t)A$ is invertible. Hence,

$$U(t)=(I+\mu(t)A)^{-1}U(\sigma(t)).$$

In other words, the values of the solution at time $\sigma(t)$ uniquely determine the values at time t.

It remains to discuss the possibility that u_1, u_2 do not coincide on $\mathbb{Z}\times[T_1,t_0)_{\mathbb{T}}$; let

$$t=\sup\{s\in[T_1,t_0);\ u_1(x,s)\neq u_2(x,s) \text{ for some } x\in\mathbb{Z}\}.$$

We claim that $u_1(x,t)=u_2(x,t)$ for every $x\in\mathbb{Z}$. If $t=t_0$, the statement is true. If $t<t_0$ and t is right-dense, then the statement follows from continuity. Finally, if $t<t_0$ and t is right-scattered, we use our observation. Now, if t is left-scattered, then $u_1(x,t)=u_2(x,t)$ and our observation implies $u_1(x,\rho(t))=u_2(x,\rho(t))$, a contradiction. On the other hand, if t is left-dense, there is a point $\tau\in(t-\frac{1}{2(|a|+|b|+|c|)},t)_{\mathbb{T}}$, and Lemma 3.4 (with $\tau_1=\tau$, $\tau_2=t$) leads to a contradiction. □

The graininess condition $\mu(t)<\frac{1}{|a|+|b|+|c|}$ in Theorem 3.5 applies to backward solutions only, and can neither be omitted nor improved. For example, let $a=c=1$, $b=-2$, and consider the time scale $\mathbb{T}=\frac{1}{4}\mathbb{Z}=\{\frac{n}{4},\ n\in\mathbb{Z}\}$, which violates the graininess condition:

- For the zero initial condition at $t=0$, it is easy to check that $u(x,-1/4)=(-1)^x\alpha$, $x\in\mathbb{Z}$, satisfies Eq. (3.1) for every $\alpha\in\mathbb{R}$. This shows that in general, bounded backward solutions need not be unique.
- Consider the initial condition

  $$u(x,0)=\begin{cases}1 & \text{if } x=0,\\ 0 & \text{if } x\neq 0.\end{cases}$$

  Let us try to find a bounded sequence $\{u(x,-1/4)\}_{x\in\mathbb{Z}}$ so that Eq. (3.1) is satisfied. Denote

  $$u(0,-1/4)=\alpha,\quad u(1,-1/4)=\beta. \tag{3.6}$$

  From Eq. (3.1), it follows that

  $$\frac{u(x,0)-u(x,-1/4)}{1/4}=u(x+1,-1/4)-2u(x,-1/4)+u(x-1,-1/4),\quad x\in\mathbb{Z}. \tag{3.7}$$

  Taking into account the initial condition, we see that

  $$u(x+1,-1/4)=-2u(x,-1/4)-u(x-1,-1/4),\quad x\in\{1,2,3,\dots\}.$$

  This is a linear recurrence equation, whose characteristic polynomial $\lambda^2+2\lambda+1$ has the double root $\lambda=-1$. Hence, the general solution has the form $u(x,-1/4)=(-1)^x(px+q)$. Using the initial conditions (3.6), we find $q=\alpha$ and $p=-\alpha-\beta$. The sequence $\{u(x,-1/4)\}_{x\geq 0}$ will be bounded if and only if p vanishes, which leads to the condition $\alpha+\beta=0$. Now, let us examine what happens for the negative values of x. From Eq. (3.7), we calculate $u(-1,-1/4)=4-2\alpha-\beta$ and

  $$u(x-1,-1/4)=-2u(x,-1/4)-u(x+1,-1/4),\quad x\in\{-1,-2,-3,\dots\}.$$

  Again, we obtained a linear recurrence equation, whose general solution has the form $u(x,-1/4)=(-1)^x(rx+s)$. This time, taking into account the values of $u(-1,-1/4)$ and $u(0,-1/4)$, we get $s=\alpha$ and $r=4-\alpha-\beta$. The sequence $\{u(x,-1/4)\}_{x\leq 0}$ will be bounded if and only if $r=0$, which is equivalent to $\alpha+\beta=4$. This condition is incompatible with our first condition $\alpha+\beta=0$. Therefore, we see that in general, bounded backward solutions need not exist once the graininess condition is violated.

Remark 3.6. From Theorem 3.3, we know that the unique bounded solution of Eq. (3.1) on $\mathbb{Z}\times[T_1,T_2]_{\mathbb{T}}$ is given by the formula $U(t)=e_A(t,t_0)u^0$, where u^0 is the initial condition. Now, consider a pair of initial conditions $u_1^0,u_2^0\in\ell^\infty(\mathbb{Z})$. Then, the corresponding solutions U_1,U_2 satisfy

$$\|U_1(t)-U_2(t)\|_\infty\leq\left(\sup_{s\in[T_1,T_2]_{\mathbb{T}}}\|e_A(s,t_0)\|\right)\|u_1^0-u_2^0\|_\infty,\quad t\in[T_1,T_2]_{\mathbb{T}}.$$

In other words, the solutions depend continuously on the initial condition.

Our final result in this section is the superposition principle. Eq. (3.1) is linear, and it is clear that every finite linear combination of solutions is a solution again. The next theorem shows that under certain assumptions, it makes sense to consider infinite linear combinations as well. The idea of the proof is taken over from [20, Theorem 3.1].

Theorem 3.7. *Let $u_k : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$, $k \in \mathbb{N}$, be a sequence of bounded solutions of Eq.* (3.1)*. Assume there exists a $\beta > 0$ such that $\sum_{k=1}^{\infty} |u_k(x, t_0)| \le \beta$ for every $x \in \mathbb{Z}$. Then, for every bounded sequence $\{c_k\}_{k=1}^{\infty}$, the function $u(x, t) = \sum_{k=1}^{\infty} c_k u_k(x, t)$ is a solution of Eq.* (3.1) *on $\mathbb{Z} \times [t_0, T]_{\mathbb{T}}$.*

Proof. Let $M > 0$ be such that $|c_k| \le M$, $k \in \mathbb{N}$. Let $\{d_k\}_{k=1}^{\infty}$ be an arbitrary sequence of numbers such that $|d_k| \le M$. We consider the sums

$$u^{(m)}(x, t) = \sum_{k=1}^{m} d_k u_k(x, t), \quad x \in \mathbb{Z},\ t \in [t_0, T]_{\mathbb{T}},\ m \in \mathbb{N}.$$

Linearity implies that $u^{(m)}$ is a solution of Eq. (3.1), i.e.,

$$u^{(m)}(x, t) = u^{(m)}(x, t_0) + \int_{t_0}^{t} (a u^{(m)}(x + 1, s) + b u^{(m)}(x, s) + c u^{(m)}(x, s - 1))\Delta s,$$

which leads to the estimate

$$\sup_{x\in\mathbb{Z}} |u^{(m)}(x, t)| \le \sup_{x\in\mathbb{Z}} |u^{(m)}(x, t_0)| + (|a| + |b| + |c|) \int_{t_0}^{t} \sup_{x\in\mathbb{Z}} |u^{(m)}(x, s)|\Delta s.$$

By Gronwall's inequality (see [7, Corollary 6.7]), we have

$$\sup_{x\in\mathbb{Z}} |u^{(m)}(x, t)| \le \sup_{x\in\mathbb{Z}} |u^{(m)}(x, t_0)| \cdot e_{|a|+|b|+|c|}(t, t_0) \le \beta M e_{|a|+|b|+|c|}(t, t_0),$$

and therefore

$$|u^{(m)}(x, t)| \le \beta M e_{|a|+|b|+|c|}(t, t_0), \quad x \in \mathbb{Z},\ t \in [t_0, T]_{\mathbb{T}},\ m \in \mathbb{N}.$$

For an arbitrary fixed pair (x, t), we can choose $d_k = |c_k| \operatorname{sgn} u_k(x, t)$, $k \in \mathbb{N}$. Then, the previous inequality reduces to

$$\sum_{k=1}^{m} |c_k| \cdot |u_k(x, t)| \le \beta M e_{|a|+|b|+|c|}(t, t_0), \quad x \in \mathbb{Z},\ t \in [t_0, T]_{\mathbb{T}},\ m \in \mathbb{N}.$$

This means that the series $\sum_{k=1}^{\infty} c_k u_k(x, t)$ in the definition of u is absolutely convergent.

As a next step, consider the sum

$$\sum_{k=1}^{m} d_k u_k^{\Delta}(x, t) = \sum_{k=1}^{m} d_k (a u_k(x + 1, t) + b u_k(x, t) + c u_k(x - 1, t)).$$

We have the estimate

$$\begin{aligned}\left|\sum_{k=1}^{m} d_k u_k^{\Delta}(x, t)\right| &\le |a| \sum_{k=1}^{m} |d_k| \cdot |u_k(x + 1, t)| + |b| \sum_{k=1}^{m} |d_k| \cdot |u_k(x, t)| + |c| \sum_{k=1}^{m} |d_k| \cdot |u_k(x - 1, t)| \\ &\le (|a| + |b| + |c|)\beta M e_{|a|+|b|+|c|}(t, t_0).\end{aligned}$$

In particular, for $d_k = |c_k| \operatorname{sgn} u_k^{\Delta}(x, t)$, $k \in \mathbb{N}$, the inequality reduces to

$$\sum_{k=1}^{m} |c_k| \cdot |u_k^{\Delta}(x, t)| \le (|a| + |b| + |c|)\beta M e_{|a|+|b|+|c|}(t, t_0).$$

Again, we see that the series $\sum_{k=1}^{\infty} c_k u_k^{\Delta}(x, t)$ is absolutely convergent. Moreover, by Lebesgue's dominated convergence theorem, the series can be integrated term by term. Therefore,

$$\sum_{k=1}^{\infty} c_k u_k(x, t_0) + \int_{t_0}^{t} \left(\sum_{k=1}^{\infty} c_k u_k^{\Delta}(x, s)\right) \Delta s = \sum_{k=1}^{\infty} c_k u_k(x, t_0) + \sum_{k=1}^{\infty} c_k (u_k(x, t) - u_k(x, t_0)) = u(x, t).$$

One consequence of this relation is that u is continuous with respect to t (since the integral on the left-hand side is a continuous function of its upper bound). Since

$$\sum_{k=1}^{\infty} c_k u_k^{\Delta}(x,t) = \sum_{k=1}^{\infty} c_k(au_k(x+1,t) + bu_k(x,t) + cu_k(x-1,t)) = au(x+1,t) + bu(x,t) + cu(x-1,t),$$

it follows that $\sum_{k=1}^{\infty} c_k u_k^{\Delta}(x,t)$ is continuous with respect to t. Hence, we can differentiate the equality

$$u(x,t) = \sum_{k=1}^{\infty} c_k u_k(x,t_0) + \int_{t_0}^{t} \left(\sum_{k=1}^{\infty} c_k u_k^{\Delta}(x,s) \right) \Delta s$$

with respect to t and obtain

$$u^{\Delta}(x,t) = \sum_{k=1}^{\infty} c_k u_k^{\Delta}(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t). \qquad \square$$

Corollary 3.8. *Let $u : \mathbb{Z} \times [t_0,T]_{\mathbb{T}} \to \mathbb{R}$ be the unique bounded solution of Eq.* (3.1) *corresponding to the initial condition*

$$u(x,t_0) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

If $\{c_k\}_{k\in\mathbb{Z}}$ is an arbitrary bounded real sequence, then

$$v(x,t) = \sum_{k\in\mathbb{Z}} c_k u(x-k,t)$$

is the unique bounded solution of Eq. (3.1) *corresponding to the initial condition $v(x,t_0) = c_x$, $x \in \mathbb{Z}$.*

4 Sum-preserving right-hand sides

In this section, we focus our attention on equations of the form (3.1) where $a,b,c \in \mathbb{R}$ satisfy $a+b+c = 0$. Motivated by the next theorem, we call them equations with sum-preserving right-hand sides because, for every solution u, the sum $\sum_{x\in\mathbb{Z}} u(x,t)$ is the same for all t.

Theorem 4.1. *Let $u : \mathbb{Z} \times [T_1,T_2]_{\mathbb{T}} \to \mathbb{R}$ be a bounded solution of Eq.* (3.1) *with $a+b+c = 0$. Assume that the following conditions are satisfied:*

- *For a certain $t_0 \in [T_1,T_2]_{\mathbb{T}}$, the sum $\sum_{x\in\mathbb{Z}} |u(x,t_0)|$ is finite.*
- *$\mu(t) < \frac{1}{|a|+|b|+|c|}$ for every $t \in [T_1,t_0)_{\mathbb{T}}$.*

Then $\sum_{x\in\mathbb{Z}} u(x,t) = \sum_{x\in\mathbb{Z}} u(x,t_0)$ for every $t \in [T_1,T_2]_{\mathbb{T}}$.

Proof. The sequence $u^0 = \{u(x,t_0)\}_{x\in\mathbb{Z}}$ is an element of $\ell^1(\mathbb{Z})$. Using Theorem 3.3 and uniqueness of bounded solutions, we infer that $u(x,t) = U(t)_x$, where $U : [T_1,T_2]_{\mathbb{T}} \to \ell^1(\mathbb{Z})$ is given by $U(t) = e_A(t,t_0)u^0$.

For every $t \in [T_1,T_2]_{\mathbb{T}}$, we have $U(t), U^{\Delta}(t) \in \ell^1(\mathbb{Z})$, and thus $\sum_{x\in\mathbb{Z}} u(x,t)$ and $\sum_{x\in\mathbb{Z}} u^{\Delta}(x,t)$ are absolutely convergent.

Since U is continuous on $[T_1,T_2]_{\mathbb{T}}$, there exists an $M > 0$ such that $\|U(t)\|_1 \le M$ for every $t \in [T_1,T_2]_{\mathbb{T}}$. Consequently, for every $I \subset \mathbb{Z}$, we have

$$\sum_{x\in I} |u^{\Delta}(x,t)| \le \|U^{\Delta}(t)\|_1 = \|AU(t)\|_1 \le \|A\| \cdot \|U(t)\|_1 \le (|a|+|b|+|c|)M, \quad t \in [T_1,T_2]_{\mathbb{T}}.$$

By Lebesgue's dominated convergence theorem, the series $\sum_{x\in\mathbb{Z}} u^{\Delta}(x,t)$ can be integrated term by term, and we obtain

$$\sum_{x\in\mathbb{Z}} u(x,t) = \sum_{x\in\mathbb{Z}} u(x,t_0) + \int_{t_0}^{t} \left(\sum_{x\in\mathbb{Z}} u^{\Delta}(x,s) \right) \Delta s$$

$$= \sum_{x\in\mathbb{Z}} u(x,t_0) + \int_{t_0}^{t} \left(\sum_{x\in\mathbb{Z}} (au(x+1,s) + bu(x,s) + cu(x-1,s) \right) \Delta s$$

$$= \sum_{x\in\mathbb{Z}} u(x,t_0) + \int_{t_0}^{t} (a+b+c) \left(\sum_{x\in\mathbb{Z}} u(x,s) \right) \Delta s = \sum_{x\in\mathbb{Z}} u(x,t_0),$$

since $a+b+c=0$. □

The graininess condition in the previous theorem applies to backward solutions only; for forward solutions, the sum is always preserved. Moreover, the condition can neither be omitted nor improved: For $a=c=1$ and $b=-2$, we have $|a|+|b|+|c|=4$. Consider again the time scale $\mathbb{T}=\frac{1}{4}\mathbb{Z}$, and a zero initial condition at $t=0$. Recall that we no longer have uniqueness of bounded solutions at $t=-1/4$ (cf. Theorem 3.5). For example, it is easy to check that $u(x,-1/4)=(-1)^x$, $x\in\mathbb{Z}$, satisfies Eq. (3.1). For this choice, the sum $\sum_{x\in\mathbb{Z}} u(x,-1/4)$ does not converge.

Our next goal is to prove the maximum and minimum principles for equations with sum-preserving right-hand sides. To this end, we introduce a partial ordering on $\ell^\infty(\mathbb{Z})$ as follows: For $u,v\in\ell^\infty(\mathbb{Z})$, we write $u\le v$ if and only if $u_n\le v_n$ for every $n\in\mathbb{Z}$.

Consider an operator $A:\ell^\infty(\mathbb{Z})\to\ell^\infty(\mathbb{Z})$. We say that A is nonnegative, if $u\ge 0$ implies $Au\ge 0$ (where 0 stands for the zero sequence). Further, we say that A is monotone if $u\le v$ implies $Au\le Av$. When A is linear, it is easy to observe that A is nonnegative if and only if it is monotone. Also, note the following simple facts:

- The composition of nonnegative operators is nonnegative.
- The limit of nonnegative operators is nonnegative.

Lemma 4.2. *Let a, b, c be such that $a,c\ge 0$, $b\le 0$. Consider the operator $A:\ell^\infty(\mathbb{Z})\to\ell^\infty(\mathbb{Z})$ given by* (3.2)*. Then, for every $\delta\in[0,-1/b]$, the operator $I+\delta A$ is nonnegative.*

Proof. Consider a nonnegative sequence $\{u_n\}_{n\in\mathbb{Z}}\in\ell^\infty(\mathbb{Z})$. By the definition of A, we have

$$(I+\delta A)(\{u_n\}_{n\in\mathbb{Z}}) = \{\delta a u_{n+1} + (1+\delta b)u_n + \delta c u_{n-1}\}_{n\in\mathbb{Z}}.$$

Since $\delta b\ge -1$, the sequence is nonnegative. □

Lemma 4.3. *Let a, b, c be such that $a,c\ge 0$, $b\le 0$. Consider the operator $A:\ell^\infty(\mathbb{Z})\to\ell^\infty(\mathbb{Z})$ given by* (3.2)*, and a time scale interval $[t_0,t]_\mathbb{T}$ such that $\mu(\tau)\le -1/b$ for every $\tau\in[t_0,t)_\mathbb{T}$. Then, the operator $e_A(t,t_0)$ is nonnegative.*

Proof. From Theorem 2.2, we know that $e_A(t,t_0)=\lim_{\|D\|\to 0} P(D)$, where $P(D)$ is given by Eq. (2.5). For every $n\in\mathbb{N}$, let $\delta_n=-\frac{1}{bn}$ and consider an arbitrary partition $D_n\in\mathcal{P}_{\delta_n}(t_0,t)$. From the definition of $P(D_n)$, we see that $P(D_n)$ is nonnegative (by Lemma 4.2, $P(D_n)$ is a composition of nonnegative operators). Consequently, $e_A(t,t_0)=\lim_{n\to\infty} P(D_n)$ is nonnegative. □

We get immediately that bounded solutions preserve the sign of the initial condition.

Corollary 4.4. *Let a, b, c be such that $a,c\ge 0$, $b\le 0$. Consider a bounded solution $u:\mathbb{Z}\times[T_1,T_2]_\mathbb{T}\to\mathbb{R}$ of Eq.* (3.1)*. Moreover, assume that $\mu(t)\le -1/b$ for every $t\in[T_1,T_2)_\mathbb{T}$. If $u(x,T_1)\ge 0$ for every $x\in\mathbb{Z}$, then $u(x,t)\ge 0$ for all $t\in[T_1,T_2]_\mathbb{T}$, $x\in\mathbb{Z}$.*

Before we proceed to the maximum and minimum principles, we need two simple lemmas.

Lemma 4.5. *For arbitrary $K\in\mathbb{R}$, the function $u(x,t)=Ke_{a+b+c}(t,t_0)$ is the unique locally bounded solution of Eq.* (3.1) *satisfying $u(x,t_0)=K$ for every $x\in\mathbb{Z}$.*

Proof. Let u be the solution of the given initial-value problem. First, we note that for every fixed t, the function $x\mapsto u(x,t)$ must be constant. Indeed, consider an arbitrary $\Delta x\in\mathbb{Z}$ and let $v(x,t)=u(x+\Delta x,t)$. Then v is a solution of the same initial-value problem as v, and it follows from uniqueness that $u(x,t)=u(x+\Delta x,t)$.

Using the fact that $u(x+1,t)=u(x,t)=u(x-1,t)$, Eq. (3.1) reduces to $u^\Delta(x,t)=(a+b+c)u(x,t)$, whose unique solution satisfying $u(x,t_0)=K$ is given by $u(x,t)=Ke_{a+b+c}(t,t_0)$. □

Lemma 4.6. *Let $\alpha \in \mathbb{R}$. The exponential function e_α has the following properties:*

- *If $\alpha > 0$, then $e_\alpha(t, t_0) \geq 1$ for all $t \geq t_0$.*
- *If $\alpha = 0$, then $e_\alpha(t, t_0) = 1$ for all $t \geq t_0$.*
- *If $\alpha < 0$ and $\mu(t) \leq \frac{1}{|\alpha|}$ for every $t \in [t_0, T)_{\mathbb{T}}$, then $0 \leq e_\alpha(t, t_0) \leq 1$ for all $t \in [t_0, T]_{\mathbb{T}}$.*

Proof. All statements follow from the fact that $e_\alpha(t, t_0)$ is a limit of products of the form

$$(1 + \alpha(t_1 - t_0))(1 + \alpha(t_2 - t_1)) \cdots (1 + \alpha(t_m - t_{m-1})),$$

where $t_0 < t_1 < \cdots < t_m$ is a partition of $[t_0, t]_{\mathbb{T}}$. For $\alpha > 0$ or $\alpha = 0$, these products are always greater or equal to 1, respectively. For $\alpha < 0$, the products take values in $[0, 1]$, provided that the lengths of all subintervals do not exceed $1/|\alpha|$. $\square$

We can now state the desired minimum and maximum principles.

Theorem 4.7. *Let a, b, c be such that $a, c \geq 0$, $b \leq 0$. Consider a bounded solution $u : \mathbb{Z} \times [T_1, T_2]_{\mathbb{T}} \to \mathbb{R}$ of Eq.* (3.1)*. Moreover, assume that $\mu(t) \leq -1/b$ for every $t \in [T_1, T_2)_{\mathbb{T}}$. Then the following statements are true for all $K \geq 0$:*

- *If $a + b + c \geq 0$ and $u(x, T_1) \geq K$ for every $x \in \mathbb{Z}$, then $u(x, t) \geq K$ for all $t \in [T_1, T_2]_{\mathbb{T}}$, $x \in \mathbb{Z}$.*
- *If $a + b + c \leq 0$ and $u(x, T_1) \leq K$ for every $x \in \mathbb{Z}$, then $u(x, t) \leq K$ for all $t \in [T_1, T_2]_{\mathbb{T}}$, $x \in \mathbb{Z}$.*

Proof. Let $u(t) = \{u(x, t)\}_{x \in \mathbb{Z}}$, $t \in [T_1, T_2]_{\mathbb{T}}$. From Theorem 3.3, we know that

$$u(t) = e_A(t, T_1)u(T_1), \quad t \in [T_1, T_2]_{\mathbb{T}},$$

where A is given by (3.2). By Lemma 4.3, the operator $e_A(t, T_1)$ is nonnegative, and hence monotone. By Lemma 4.5, we have $e_A(t, T_1)\{K\}_{x \in \mathbb{Z}} = \{K e_{a+b+c}(t, T_1)\}_{x \in \mathbb{Z}}$.

Consequently, if $u(T_1) \geq \{K\}_{x \in \mathbb{Z}}$ and $a + b + c \geq 0$, then

$$u(t) = e_A(t, T_1)u(T_1) \geq e_A(t, T_1)\{K\}_{x \in \mathbb{Z}} = \{K e_{a+b+c}(t, T_1)\}_{x \in \mathbb{Z}} \geq \{K\}_{x \in \mathbb{Z}},$$

which proves the first statement; the second can be obtained by reversing the inequalities (note that $\mu(t) \leq -1/b \leq 1/(-a - b - c) = 1/|a + b + c|$, and therefore the assumption from Lemma 4.6 is satisfied). $\square$

If $a+b+c = 0$, both minimum and maximum principles hold, and we get two important consequences:

- *Stability of solutions.* If u, v is a pair of solutions of Eq. (3.1) such that $\|u(x, T_1) - v(x, T_1)\| \leq \varepsilon$ for every $x \in \mathbb{Z}$, we can apply Theorem 4.7 to the function $u - v$ and conclude that $\|u(x, t) - v(x, t)\| \leq \varepsilon$ for all $x \in \mathbb{Z}$, $t \geq T_1$.
- *Global boundedness.* We know from Theorem 3.5 that for every $t_1 \in [t_0, \infty)_{\mathbb{T}}$, Eq. (3.1) has a unique bounded solution on $\mathbb{Z} \times [t_0, t_1]_{\mathbb{T}}$. It follows from Theorem 4.7 that these solutions are always bounded by the same constant independent of t_1. Hence, Eq. (3.1) has a unique bounded solution on $\mathbb{Z} \times [t_0, \infty)_{\mathbb{T}}$. On the other hand, when $a + b + c \neq 0$, we still have a solution on $\mathbb{Z} \times [t_0, \infty)_{\mathbb{T}}$, but it need not be globally bounded (consider the case $a = c = 0$, $b = 1$).

The graininess condition in Theorem 4.7 can neither be omitted nor improved: Assume that t_0 is an arbitrary right-scattered point, and consider the nonnegative initial condition

$$u(x, t_0) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Then it follows from Eq. (3.1) that $u(0, \sigma(t_0)) = \mu a u(1, t_0) + (1 + \mu b)u(0, t_0) + \mu c u(-1, t_0) = 1 + \mu b$. Clearly, this value is nonnegative if and only if $\mu(t_0) \leq -1/b$.

Note that the maximum and minimum principles cannot hold simultaneously for equations with $a + b + c \neq 0$ (if both principles hold, then constant initial condition should give constant solutions, but this is true only if $a + b + c = 0$).

For backward solutions, the maximum and minimum principles are no longer valid. For example, when $\mathbb{T} = \mathbb{R}$, $a = c = 1$ and $b = -2$, it is easy to check that the function $u(x,t) = e^{-2t} I_x(2t)$, where I_x is the modified Bessel function of the first kind, is a solution of Eq. (3.1); see [21, Example 3.1]. At this moment, we need the fact that for a fixed t, the function $x \mapsto I_x(2t)$ attains its maximum at $x = 0$ (see [19, paragraph 10.37]). Also, the function $u(0,t) = e^{-2t} I_0(2t)$ is decreasing in t. Hence, if we go backward in time, the maximum value of u increases, which violates the maximum principle.

Before we finish this section, we recall that for $a = 0$, $b = -k$ and $c = k$, our Eq. (3.1) reduces to the transport equation, which has been studied in [26]. We point out that our Theorem 4.1 generalizes [26, Theorem 6.6]. Also, the condition $\mu(t) \leq -1/b$ from Theorem 4.7 reduces to $\mu(t) \leq 1/k$, and hence Theorem 4.7 generalizes [26, Theorem 6.3].

5 Symmetric right-hand sides

In this part, we focus on a special case of Eq. (3.1) with $a = c$, i.e., we study the equation

$$u^{\Delta}(x,t) = au(x+1,t) + bu(x,t) + au(x-1,t), \tag{5.1}$$

where a, b are real numbers.

It is reasonable to expect that symmetric initial conditions give rise to symmetric solutions. We discuss symmetry with respect to the origin, but any other point can serve the same purpose.

Theorem 5.1. *Let $u : \mathbb{Z} \times [T_1, T_2]_{\mathbb{T}} \to \mathbb{R}$ be a bounded solution of Eq.* (5.1). *Assume that the following conditions are satisfied:*

- *For a certain $t_0 \in [T_1, T_2]_{\mathbb{T}}$, we have $u(x, t_0) = u(-x, t_0)$ for every $x \in \mathbb{N}$.*
- *$\mu(t) < \frac{1}{2|a|+|b|}$ for every $t \in [T_1, t_0)_{\mathbb{T}}$.*

Then $u(x,t) = u(-x,t)$ for every $t \in [T_1, T_2]_{\mathbb{T}}$ and $x \in \mathbb{N}$.

Proof. We claim that the function $v : \mathbb{Z} \times [T_1, T_2]_{\mathbb{T}} \to \mathbb{R}$ given by $v(x,t) = u(-x,t)$ is a solution of Eq. (5.1). Indeed,

$$v^{\Delta}(x,t) = u^{\Delta}(-x,t) = au(-x+1,t) + bu(-x,t) + au(-x-1,t) = av(x-1,t) + bv(x,t) + av(x+1,t).$$

Also, u and v have the same values for $t = t_0$. By the uniqueness of solution (Theorem 3.5), we have $u(x,t) = v(x,t) = u(-x,t)$ for every $t \in [T_1, T_2]_{\mathbb{T}}$ and $x \in \mathbb{N}$. $\square$

The graininess condition in the previous theorem cannot be omitted. For example, let $\mathbb{T} = \frac{1}{2}\mathbb{Z}$, and consider Eq. (5.1) with $a = c = 1$, $b = -2$, and the zero initial condition at $t = 0$. We no longer have uniqueness at $t = -1/2$ (cf. Theorem 3.5); one possible choice is

$$u(x, -1/2) = \begin{cases} 0 & \text{if } x \text{ is even}, \\ (-1)^k & \text{if } x = 2k+1, \end{cases}$$

which is not symmetric with respect to the origin.

Also, the boundedness of u in the previous theorem cannot be left out; see the example of unbounded solution described after Theorem 3.3 and depicted in Figure 1.

Our second result for equations with symmetric right-hand sides characterizes the maxima of solutions corresponding to the initial condition

$$u(x, t_0) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases} \tag{5.2}$$

First, we need some facts about isometric time scale intervals. We say that two adjacent time scale intervals $[t_0, t_0 + t]_{\mathbb{T}}$ and $[t_0 + t, t_0 + 2t]_{\mathbb{T}}$ are isometric, if the following conditions are satisfied:

- If $\tau \in [t_0, t_0 + t]_{\mathbb{T}}$, then $\tau + t \in [t_0 + t, t_0 + 2t]_{\mathbb{T}}$.
- If $\tau \in [t_0, t_0 + t)_{\mathbb{T}}$, then $\mu(\tau) = \mu(\tau + t)$.

Under these assumptions, it can be shown that the function $s(\tau) = \tau + t$ maps $[t_0, t_0 + t]_{\mathbb{T}}$ onto $[t_0 + t, t_0 + 2t]_{\mathbb{T}}$. This fact implies that

$$\int_{t_0}^{t_0+t} f(\tau)\Delta\tau = \int_{t_0+t}^{t_0+2t} f(\tau - t)\Delta\tau, \tag{5.3}$$

for every integrable function $f : [t_0, t_0 + t]_{\mathbb{T}} \to \mathbb{R}$ (see Lemma 2.2 and Lemma 2.3 in [22]).

The next theorem is well known in the context of discrete random walks (see the calculation in Section 1.2.3 of [27]). We show that almost the same proof still works in a more general setting.

Theorem 5.2. *Assume that the intervals $[t_0, t_0 + t]_{\mathbb{T}}$, $[t_0 + t, t_0 + 2t]_{\mathbb{T}}$ are isometric, and $u : \mathbb{Z} \times [t_0, t_0 + 2t]_{\mathbb{T}} \to \mathbb{R}$ is the unique bounded solution of Eq.* (5.1) *corresponding to the initial condition* (5.2)*. Then $|u(x, t_0 + 2t)| \le u(0, t_0 + 2t)$ for every $x \in \mathbb{Z}$.*

Proof. Using the fact that $[t_0, t_0 + t]_{\mathbb{T}}$ and $[t_0 + t, t_0 + 2t]_{\mathbb{T}}$ are isometric and its consequence in Eq. (5.3), we conclude that the function $v : \mathbb{Z} \times [t_0 + t, t_0 + 2t]_{\mathbb{T}} \to \mathbb{R}$ given by $v(x, s) = u(x, s - t)$ is a solution of Eq. (5.1) corresponding to the initial condition

$$v(x, t_0 + t) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

According to the superposition principle, we have

$$u(x, t_0 + 2t) = \sum_{k\in\mathbb{Z}} u(k, t_0 + t)v(x - k, t_0 + 2t) = \sum_{k\in\mathbb{Z}} u(k, t_0 + t)u(x - k, t_0 + t).$$

The proof is now finished by a simple manipulation involving the Cauchy-Schwarz inequality and symmetry of u with respect to the origin:

$$|u(x, t_0 + 2t)| \le \sum_{k\in\mathbb{Z}} |u(k, t_0 + t)u(x - k, t_0 + t)| \le \left(\sum_{k\in\mathbb{Z}} u(k, t_0 + t)^2\right)^{1/2} \left(\sum_{k\in\mathbb{Z}} u(x - k, t_0 + t)^2\right)^{1/2}$$

$$= \sum_{k\in\mathbb{Z}} u(k, t_0 + t)^2 = \sum_{k\in\mathbb{Z}} u(k, t_0 + t)u(-k, t_0 + t) = u(0, t_0 + 2t). \qquad \square$$

We illustrate Theorem 5.2 by two examples.

Example 5.3. Let $\mathbb{T} = \mathbb{R}$ and consider Eq. (5.1) with $a = 1$, $b = -2$. For an arbitrary interval $[t_0, t_0 + t] \subset \mathbb{R}$, let $u : \mathbb{Z} \times [t_0, t_0 + t] \to \mathbb{R}$ be the unique bounded solution of Eq. (5.1) with the initial condition (5.2). The intervals $[t_0, t_0 + (t - t_0)/2]$ and $[t_0 + (t - t_0)/2, t]$ are isometric; hence, by Theorem 5.2, the function $x \mapsto u(x, t)$ attains its maximum value at $x = 0$. This also follows from the fact that the explicit solution is given by $u(x, t) = e^{-2(t-t_0)} I_x(2(t - t_0))$; see [21, Example 3.1].

Example 5.4. Let $\mathbb{T} = \mathbb{Z}$ and consider Eq. (5.1) with $a = 1$, $b = -2$. Take an arbitrary $t_0 \in \mathbb{Z}$ and $t \in \mathbb{Z}$ of the form $t = t_0 + 2k$, $k \in \mathbb{N}$. Let $u : \mathbb{Z} \times [t_0, t_0 + t]_{\mathbb{Z}} \to \mathbb{R}$ be the unique solution of Eq. (5.1) with the initial condition (5.2). The intervals $[t_0, t_0 + k]_{\mathbb{Z}}$ and $[t_0 + k, t_0 + 2k]_{\mathbb{Z}}$ are isometric, and it follows from Theorem 5.2 that $x \mapsto u(x, t)$ attains its maximum value at $x = 0$. This fact is confirmed by Table 1, which shows the values of u (horizontal direction corresponds to spatial location, and the upward direction corresponds to increasing values of time, starting with $t = t_0$).

When $t - t_0$ is odd, the maximum of $x \mapsto u(x, t)$ is no longer at $x = 0$. Although it is true that $x \mapsto |u(x, t)|$ attains its maximum at $x = 0$ for all t, the symmetric random walk (1.2) shows that this is not true in general.

0	1	−7	28	−77	161	−266	357	−393	357	−266	161	−77	28	−7	1	0
0	0	1	−6	21	−50	90	−126	141	−126	90	−50	21	−6	1	0	0
0	0	0	1	−5	15	−30	45	−51	45	−30	15	−5	1	0	0	0
0	0	0	0	1	−4	10	−16	19	−16	10	−4	1	0	0	0	0
0	0	0	0	0	1	−3	6	−7	6	−3	1	0	0	0	0	0
0	0	0	0	0	0	1	−2	3	−2	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	−1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0

Table 1: Solution of the discrete diffusion equation $u^{\Delta}(x,t) = u(x+1,t) - 2u(x,t) + u(x-1,t)$

6 Conclusion and open problems

In the course of our investigations, we have discovered that the graininess of $\mathbb{T}$ influences the behavior of solutions to dynamic diffusion equations in a substantial way. Table 2 summarizes some of the results obtained in the previous sections.

Property	**Condition**
Unique bounded forward solution (Theorem 3.5)	
Forward sum preservation for $a+b+c=0$ (Theorem 4.1)	Always
Symmetric forward solutions for $a=c$ (Theorem 5.1)	
Forward maximum and minimum principles (including sign preservation) for $a,c \geq 0$, $a+b+c=0$ (Theorem 4.7)	$\mu(t) \leq -1/b$
Unique bounded backward solution (Theorem 3.5)	
Backward sum preservation for $a+b+c=0$ (Theorem 4.1)	$\mu(t) < \frac{1}{\lvert a\rvert+\lvert b\rvert+\lvert c\rvert}$
Symmetric backward solutions for $a=c$ (Theorem 5.1)	
Backward maximum and minimum principles (including sign preservation) for $a,c \geq 0$, $a+b+c=0$	Never

Table 2: Summary of the main results

Here are some open problems related to the topic of the present paper:

1. One question which wasn't dealt within this manuscript is the problem of convergence. Let $\{\mathbb{T}_n\}_{n=1}^{\infty}$ be a sequence of time scales such that $\mathbb{T}_n \to \mathbb{R}$ in some sense. If $\{u_n\}_{n=1}^{\infty}$ and u are the corresponding solutions of Eq. (3.1) with the same initial condition, is it true that $u_n \to u$? For ordinary dynamic equations, it is known that solutions depend continuously on the choice of the time scale (see [1]). In view of Theorem 3.1, it would be enough to check whether the continuous dependence results are still valid for equations whose solutions take values in infinite-dimensional spaces.

2. Throughout the paper, we have restricted our attention to bounded initial conditions and bounded solutions; the main reason was that boundedness guarantees uniqueness of solutions. For the classical diffusion equation with continuous space and time, uniqueness can be obtained under the weaker hypothesis $|u(x,t)| \leq Me^{\alpha x^2}$. Also, it is known that for a nonnegative initial condition, there exists a unique nonnegative solution (both results can be found in [13, Section 7.1]). Is it possible to relax the boundedness condition for Eq. (3.1) in a similar way?

3. In Section 3, we have shown that for $\mathbb{T} = \mathbb{R}$, Eq. (3.1) has infinitely many solutions corresponding to the zero initial condition. We conjecture that for any time scale with a right-dense point t_0, there are infinitely many solutions corresponding to the zero initial condition at t_0. An inspection of the construction in Section 3 shows that it is enough to prove the existence of a nonzero infinitely

Δ-differentiable function $f : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ such that $f^{\Delta^n}(t_0) = 0$ for all $n \in \mathbb{N}_0$. For example, if $q > 1$, $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, and f is an infinitely differentiable function defined on $\mathbb{R}$, then the Δ-derivative $f^{\Delta^n}(0)$ can be expressed in terms of the ordinary derivative $f^{(n)}(0)$; see [14]. In general, the values of the two derivatives differ, but nonetheless, $f^{(n)}(0) = 0$ implies $f^{\Delta^n}(0) = 0$. Hence, our conjecture is true for $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$.

4. Theorem 5.2 provides a partial information concerning the location of $\max_{x \in \mathbb{Z}} u(x, t)$, where u is the solution which is initially concentrated at $x = 0$. However, a more complete information about the location of maxima is missing. The answer depends not only on the time scale, but on the coefficients a, b, c as well. For example, when $\mathbb{T} = \mathbb{Z}$, $a = c = 1/2$ and $b = -1$, then $x \mapsto u(x, t)$ attains its maximum at $x = 0$ if and only if t is even; for $a = c = 1/4$ and $b = -1/2$, the maximum is always at $x = 0$. For an arbitrary time scale, it is easy to check that when $a+b+c = 0$, the condition $b \geq -1/(2\mu(t))$ is necessary to guarantee that the maximum remains at $x = 0$ for all $t \geq t_0$. Is this condition sufficient? A related open problem is the following: When u is the unique bounded solution of $u^{\Delta}(x, t) = au(x + 1, t) + bu(x, t) + cu(x - 1, t)$ corresponding to the initial condition (5.2), what is the necessary and sufficient condition to ensure that $\int_{t_0}^{\infty} u(0, t)\Delta t \geq \int_{t_0}^{\infty} u(x, t)\Delta t$ for every $x \in \mathbb{Z}$?

5. We have focused our attention on partial dynamic equations where the spatial domain is $\mathbb{Z}$, while the time domain is a general time scale. We leave it as an open problem to investigate equations with different spatial domains, such as $\mathbb{R}$ or $q^{\mathbb{Z}} \cup \{0\}$, or to study the general case where space is an arbitrary time scale.

Readers interested in the topic of partial dynamic equations on discrete-space domains are invited to check the recent articles [9, 21, 30], which were published while the present paper was under review.

In [21], we present two methods for finding explicit solutions of Eq. (3.1) once a particular time scale is given. These methods are then used to examine the asymptotic behavior of solutions as well as finiteness of their time integrals $\int_0^{\infty} u(x, t)\Delta t$. We also consider multidimensional diffusion equations and prove a slight generalization of G. Pólya's famous result on the recurrence of symmetric random walks in $\mathbb{Z}^N$.

In [9], we point out that the results obtained in the present paper (such as existence and uniqueness, sign preservation, and maximum principle) can be extended to a larger class of linear partial dynamic equations. We prove a general theorem concerning the continuous dependence of solutions on initial values, coefficients on the right-hand side, as well as the choice of time scale. Finally, we show that under certain conditions, solutions of linear diffusion-type equations describe probability distributions of nonhomogeneous Markov processes, and their time integrals remain the same for all underlying regular time scales.

In [30], J. Volek studies the nonlinear transport equation with discrete space and continuous time. The paper is primarily concerned with maximum and minimum principles, existence and uniqueness of solutions, and related questions such as stability or sign preservation.

Acknowledgements

We thank the anonymous referees for their careful reading of the manuscript and valuable comments, which helped to improve the clarity of exposition. The latter author gratefully acknowledges the support by the Czech Science Foundation, Grant No. 201121757.

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