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ON A CHARACTERISTIC PROPERTY OF FINITE-DIMENSIONAL BANACH SPACES*

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Abstract. This paper is inspired by a counter example of J. Kurzweil published in [**5**], whose intention was to demonstrate that a certain property of linear operators on finite-dimensional spaces need not be preserved in infinite dimension. We obtain a stronger result, which says that no infinite-dimensional Banach space can have the given property. Along the way, we will also derive an interesting proposition related to Dvoretzky's theorem.

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1. Introduction. Let X be a real Banach space and $\mathcal{L}(X)$ the space of all bounded linear operators on X. Let I denote the identity operator. We say that X has the property (JK), if the following statement is true:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $n \in \mathbb{N}$ and $Z_1, \ldots, Z_n \in \mathcal{L}(X)$ are operators satisfying

$$\|(I + Z_{j_p})(I + Z_{j_{p-1}}) \cdots (I + Z_{j_1}) - I\| \leq \delta$$

for every $p \in \{1, \ldots, n\}$ and every p-tuple $1 \leq j_1 < j_2 < \cdots < j_p \leq n$, then

$$\sum_{j=1}^{n} \|Z_j\| \leq \varepsilon.$$

In short, the property (JK) guarantees that the sum $\sum_{j=1}^{n} \|Z_j\|$ is small whenever all the 'products' $(I + Z_{j_p})(I + Z_{j_{p-1}}) \cdots (I + Z_{j_1})$ are close to the identity operator.

The property (JK) plays an important role in product-integration theory (see [**3**, **5**, **6**]). Its first appearance seems to be in a paper by J. Jarník and J. Kurzweil (see [**3**]), who have investigated the case $X = \mathbb{R}^n$ and $\mathcal{L}(X) = \mathbb{R}^{n\times n}$. They showed that this space possesses the property (JK); since all norms on a finite-dimensional space are equivalent, their result implies that every finite-dimensional space has the property (JK).

On the other hand, the paper of Š. Schwabik (see [**5**]) contains an example of J. Kurzweil, which shows that the space c_0 does not have the property (JK). Our main

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goal is to investigate other infinite-dimensional Banach spaces and see whether they have the property (JK).

2. Main results. The argument that lies at the core of J. Kurzweil's example can be stated as follows:

LEMMA 1. *Let X be a Banach space and $\{c_n\}_{n=1}^{\infty}$ a sequence of positive numbers such that* $\lim_{n\to\infty}(c_n/n)=0$. *Assume that for every $n\in\mathbb{N}$, there exists operators $E_1,\dots,E_n\in\mathcal{L}(X)$ satisfying the following conditions:*

(i) $\|E_i\|\geq 1$ *for every $i\in\{1,\dots,n\}$,*

(ii) $\left\|\sum_{k=1}^{p}E_{j_k}\right\|\leq c_n$ *for every $p\in\{1,\dots,n\}$ and every p-tuple $1\leq j_1<j_2<\cdots<j_p\leq n$,*

(iii) $E_iE_j=0$ *whenever $i>j$.*

Then, the space X does not have the property (JK).

Proof. Assume for contradiction that X has the property (JK). Choose an arbitrary $\varepsilon>0$ and let $\delta>0$ be the corresponding constant from the definition of the property (JK). Put $Z_i=\delta/c_n\cdot E_i$ for $i\in\{1,\dots,n\}$. It follows from the assumptions that for every $p\in\{1,\dots,n\}$ and every p-tuple $1\leq j_1<j_2<\cdots<j_p\leq n$, we have

$$\|(I+Z_{j_p})(I+Z_{j_{p-1}})\cdots(I+Z_{j_1})-I\|=\left\|\sum_{k=1}^{p}Z_{j_k}\right\|=\delta/c_n\cdot\left\|\sum_{k=1}^{p}E_{j_k}\right\|\leq\delta.$$

Thus, by taking n such that $c_n/n<\delta/\varepsilon$ (remember that $\lim_{n\to\infty}(c_n/n)=0$), we have found n operators $Z_1,\dots,Z_n$ such that

$$\|(I+Z_{j_p})(I+Z_{j_{p-1}})\cdots(I+Z_{j_1})-I\|\leq\delta$$

for every p-tuple $1\leq j_1<j_2<\cdots<j_p\leq n$, but

$$\sum_{k=1}^{n}\|Z_j\|\geq n\delta/c_n>\varepsilon,$$

a contradiction. Therefore, X does not have the property (JK). □

In the following example, we use the previous Lemma to prove that the space c_0 does not have the property (JK); this is the example of J. Kurzweil (see [**5**]).

EXAMPLE 2. Let $X=c_0$, i.e. the space of all real sequences $\{a_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty}a_n=0$. The space is equipped with the norm

$$\|\{a_i\}_{i=1}^{\infty}\|=\sup_{i\in\mathbb{N}}|a_i|.$$

Given $n\in\mathbb{N}$, we define operators $E_1,\dots,E_n\in\mathcal{L}(X)$ in the following way:

$$E_k\big(\{a_i\}_{i=1}^{\infty}\big)=\{b_i\}_{i=1}^{\infty},$$

where $b_i=0$ for $i\neq 2k-1$ and $b_{2k-1}=a_{2k}$, i.e. the operator E_k sets all components of the given sequence except the $2k$-th one to zero, and then shifts the result to the left. It is easy to see that $E_iE_j = 0$ when $i \neq j$, $\|E_i\| = 1$ for every $i \in \{1, \dots, n\}$, and $\| \sum_{k=1}^{p} E_{j_k} \| = 1$ for every $p \in \{1, \dots, n\}$ and every p-tuple $1 \leq j_1 < j_2 < \cdots < j_p \leq n$. Thus, by Lemma 1, the space c_0 does not have the property (JK).

A close inspection of the previous example reveals that a similar argument works in a more general setting. As a prerequisite, we need the following projection theorem of Kadets and Snobar. Recall that a projection of a space X onto a subspace V is a linear mapping $P : X \to V$ such that $P^2 = P$ and the range of P is V.

THEOREM 3 (Kadets–Snobar theorem). *Let X be a Banach space and V a finite-dimensional subspace of X. Then, there exists a projection P of X onto V such that $\|P\| \leq \sqrt{\dim V}$.*

Proof. See the original paper [**4**] or the monograph [**1**]. □

Note the following obvious fact: Since the range of P is V, every $v \in V$ can be written as $v = P(w)$ for some $w \in X$. It follows that $P(v) = P^2(w) = P(w) = v$, i.e. the restriction of P to V is the identity operator.

LEMMA 4. *Let X be a Banach space and $c > 0$, $d > 0$ two constants such that for every $m \in \mathbb{N}$, there exist vectors $x_1, \dots, x_m \in X$ such that*

(i) *$\{x_1, \dots, x_m\}$ is a linearly independent set,*
(ii) *$\|x_i\| = 1$ for every $i \in \{1, \dots, m\}$,*
(iii) *$\| \sum_{i \in I} \alpha_i x_i \| \leq c \| \sum_{i=1}^{m} \alpha_i x_i \|$ for every $I \subset \{1, \dots, m\}$ and $\alpha_1, \dots, \alpha_m \in \mathbb{R}$,*
(iv) *$\| \sum_{i=1}^{m-1} \alpha_{i+1} x_i \| \leq d \| \sum_{i=1}^{m} \alpha_i x_i \|$ for every m-tuple $\alpha_1, \dots, \alpha_m \in \mathbb{R}$.*

Then, the space X does not have the property (JK).

Proof. Let $n \in \mathbb{N}$ be a given number. In order to prove the statement, we are going to construct operators $E_1, \dots, E_n$ satisfying the assumptions of Lemma 1.

Taking $m = 2n$, let $x_1, \dots, x_{2n} \in X$ be some vectors having the properties (*i*)–(*iv*). Let V be the $2n$-dimensional subspace of X spanned by $x_1, \dots, x_{2n}$. For $k \in \{1, \dots, n\}$, we define the operator $E'_k : V \to V$ by

$$E'_k \left(\sum_{i=1}^{2n} \alpha_i x_i \right) = \alpha_{2k} x_{2k-1}.$$

It is clear that $\|E'_k\| \geq \|E'_k(x_{2k})\| = \|x_{2k-1}\| = 1$. On the other hand, the assumption (*iii*) implies

$$\|\alpha_{2k} x_{2k-1}\| = |\alpha_{2k}| = \|\alpha_{2k} x_{2k}\| \leq c \left\| \sum_{i=1}^{2n} \alpha_i x_i \right\|,$$

i.e. $\|E'_k\| \leq c$ for every $k \in \{1, \dots, n\}$. Now, consider a $p \in \{1, \dots, n\}$ and a p-tuple $1 \leq j_1 < j_2 < \cdots < j_p \leq n$. Take an arbitrary $x \in V$ with $\|x\| = 1$, and write it as $x = \sum_{i=1}^{2n} \alpha_i x_i$. Then,

$$\left\| \left(\sum_{k=1}^{p} E'_{j_k} \right) \left(\sum_{i=1}^{2n} \alpha_i x_i \right) \right\| = \left\| \sum_{k=1}^{p} \alpha_{2j_k} x_{2j_k - 1} \right\| \leq cd.$$

(We have used assumptions (*iii*) and (*iv*).) Therefore,

$$\left\|\sum_{k=1}^{p} E'_{j_k}\right\| \le cd.$$

Finally, it is clear that $E'_i E'_j = 0$, whenever $i \ne j$.

Now, let P be a projection of X onto V such that $\|P\| \le \sqrt{2n}$. We define operators $E_1, \dots, E_n : X \to X$ by

$$E_k(x) = E'_k(P(x)), \quad x \in X, \quad k \in \{1, \dots, n\}.$$

These operators are linear and bounded, because

$$\|E_k\| \le \|E'_k\| \cdot \|P\| \le c\sqrt{2n}, \quad k \in \{1, \dots, n\}.$$

Since $E_k(x) = E'_k(x)$ for $x \in V$, we have a lower bound

$$\|E_k\| \ge 1, \quad k \in \{1, \dots, n\}.$$

For $i \ne j$ and $x \in X$, we have

$$E_i E_j(x) = E'_i(P(E'_j(P(x))) = E'_i(E'_j(P(x))) = 0.$$

Finally, if $x \in X$ and $\|x\| = 1$, then $\|P(x)\| \le \sqrt{2n}$, and thus,

$$\left\|\sum_{k=1}^{p} E_{j_k}(x)\right\| = \left\|\left(\sum_{k=1}^{p} E'_{j_k}\right)(P(x))\right\| \le \sqrt{2n} \cdot \left\|\sum_{k=1}^{p} E'_{j_k}\right\| \le cd\sqrt{2n}$$

for every $p \in \{1, \dots, n\}$ and every p-tuple $1 \le j_1 < j_2 < \cdots < j_p \le n$, which means that

$$\left\|\sum_{k=1}^{p} E_{j_k}\right\| \le cd\sqrt{2n}.$$

□

The following examples show that certain familiar infinite-dimensional Banach spaces do not have the property (JK). In each case, we suggest a choice of vectors $x_1, \dots, x_m$ (where $m \in \mathbb{N}$ is arbitrary) and leave it up to the reader to check that these vectors satisfy the assumptions of Lemma 4.

Example 5. For $X = \ell^p$, $p \in [1, \infty)$, there is a natural choice: Let

$$x_k = \{\delta_{kn}\}_{n=1}^{\infty}, \quad k \in \{1, \dots, m\},$$

where δ_{kn} denotes the Kronecker symbol. This choice also works when $X = \ell^\infty$, $X = c$ or $X = c_0$.

Example 6. Let $X = \mathcal{L}^p([a, b])$, where $p \in [1, \infty)$. Then, we can choose

$$x_k = \frac{m}{b-a} \cdot f_k, \quad k \in \{1, \dots, m\},$$

where $f_k : [a, b] \to \mathbb{R}$ is the characteristic function of interval $(a + (k-1)(b-a)/m, a + k(b-a)/m)$.

EXAMPLE 7. When $X = \mathcal{C}([a, b])$, we can take

$$x_k = f_k, \quad k \in \{1, \dots, m\},$$

where $f_k : [a, b] \to \mathbb{R}$ is a function, which is zero outside $I = (a + (k-1)(b-a)/m, a + k(b-a)/m)$, it equals 1 at the midpoint of I and is linear on both halves of I. This choice also works when $X = \mathcal{L}^\infty([a, b])$.

It should be clear that whenever an infinite-dimensional Banach space X contains an isometric copy of one of the spaces mentioned in the previous examples, then X does not have the propery (JK). Unfortunately, not every Banach space contains an isometric copy of ℓ^p or c_0. To overcome this difficulty, we use the following Dvoretzky's theorem, which says that an infinite-dimensional Banach space contains an 'almost-isometric' copy of ℓ^2_m for every $m \in \mathbb{N}$ (where ℓ^2_m denotes the space $\mathbb{R}^m$ equipped with the Euclidean norm).

THEOREM 8 (Dvoretzky's theorem). *Let X be an infinite-dimensional Banach space. Then, for every $\varepsilon > 0$ and every $m \in \mathbb{N}$, there is an m-dimensional subspace $Y \subset X$ and an isomorphism $T : Y \to \ell^2_m$ such that $\|T\| \cdot \|T^{-1}\| \le 1 + \varepsilon$.*

Proof. See the original paper [**2**] or the monograph [**1**]. □

The following proposition will be used to obtain our main result, but it is also interesting in its own right. It implies that, given one of the finite-dimensional subspaces whose existence is guaranteed by Dvoretzky's theorem (which says that $c = \|T\| \cdot \|T^{-1}\| \le 1 + \varepsilon$), we can find a basis whose properties are very similar to the properties of the canonical basis of ℓ^2_m (where the statements (*ii*)–(*iii*) below are true with $c = 1$).

THEOREM 9. *Let Y be an m-dimensional Banach space, $T : Y \to \ell^2_m$ an isomorphism and $c = \|T\| \cdot \|T^{-1}\|$. Then Y has a basis $\{x_1, \dots, x_m\}$ with the following properties:*

(i) $\|x_i\| = 1$ *for every* $i \in \{1, \dots, m\}$,
(ii) $\|\sum_{i\in I} \alpha_i x_i\| \le c \|\sum_{i=1}^m \alpha_i x_i\|$ *for every* $I \subset \{1, \dots, m\}$ *and* $\alpha_1, \dots, \alpha_m \in \mathbb{R}$,
(iii) $\|\sum_{i=1}^{m-1} \alpha_{i+1} x_i\| \le c^2 \|\sum_{i=1}^m \alpha_i x_i\|$ *for every m-tuple* $\alpha_1, \dots, \alpha_m \in \mathbb{R}$.

Proof. Note that by replacing T by a suitable multiple, we may assume that $\|T\| = 1$ and $\|T^{-1}\| = c$. Let $e_1, \dots, e_m$ be the canonical basis of ℓ^2_m and put

$$x_i = \frac{T^{-1}(e_i)}{\|T^{-1}(e_i)\|}, \quad i \in \{1, \dots, m\}.$$

It is clear that $\|x_i\| = 1$ for every $i \in \{1, \dots, m\}$ and that $\{x_1, \dots, x_m\}$ is a basis. Note that

$$e_i = \|T^{-1}(e_i)\| T(x_i), \quad i \in \{1, \dots, m\}.$$

Given an arbitrary $I \subset \{1, \dots, m\}$ and $\alpha_1, \dots, \alpha_m \in \mathbb{R}$, we have

$$\begin{aligned}
\left\|\sum_{i\in I}\alpha_i x_i\right\| &= \left\|\sum_{i\in I}\frac{\alpha_i T^{-1}(e_i)}{\|T^{-1}(e_i)\|}\right\| = \left\|T^{-1}\left(\sum_{i\in I}\frac{\alpha_i e_i}{\|T^{-1}(e_i)\|}\right)\right\| \\
&\le c\left\|\sum_{i\in I}\frac{\alpha_i e_i}{\|T^{-1}(e_i)\|}\right\| = c\sqrt{\sum_{i\in I}\frac{\alpha_i^2}{\|T^{-1}(e_i)\|^2}} \le c\sqrt{\sum_{i=1}^{m}\frac{\alpha_i^2}{\|T^{-1}(e_i)\|^2}} \\
&= c\left\|\sum_{i=1}^{m}\frac{\alpha_i e_i}{\|T^{-1}(e_i)\|}\right\| = c\left\|\sum_{i=1}^{m}\alpha_i T(x_i)\right\| = c\left\|T\left(\sum_{i=1}^{m}\alpha_i x_i\right)\right\| \\
&\le c\left\|\sum_{i=1}^{m}\alpha_i x_i\right\|.
\end{aligned}$$

To verify the third condition, note that for every $i \in \{1, \dots, m\}$ we have

$$1 = \|e_i\| = \|T(T^{-1}(e_i))\| \le \|T\| \cdot \|T^{-1}(e_i)\| = \|T^{-1}(e_i)\|,$$

i.e. $1/\|T^{-1}(e_i)\| \le 1$. Now, for any choice of $\alpha_1, \dots, \alpha_m \in \mathbb{R}$, we obtain

$$\begin{aligned}
\left\|\sum_{i=1}^{m-1}\alpha_{i+1}x_i\right\| &= \left\|\sum_{i=1}^{m-1}\frac{\alpha_{i+1}T^{-1}(e_i)}{\|T^{-1}(e_i)\|}\right\| = \left\|T^{-1}\left(\sum_{i=1}^{m-1}\frac{\alpha_{i+1}e_i}{\|T^{-1}(e_i)\|}\right)\right\| \\
&\le c\left\|\sum_{i=1}^{m-1}\frac{\alpha_{i+1}e_i}{\|T^{-1}(e_i)\|}\right\| = c\sqrt{\sum_{i=1}^{m-1}\frac{\alpha_{i+1}^2}{\|T^{-1}(e_i)\|^2}} \le c\sqrt{\sum_{i=1}^{m-1}\alpha_{i+1}^2} \\
&\le c\sqrt{\sum_{i=1}^{m}\alpha_i^2} \le c\max_{i\in\{1,\dots,m\}}\|T^{-1}(e_i)\|\sqrt{\sum_{i=1}^{m}\frac{\alpha_i^2}{\|T^{-1}(e_i)\|^2}} \\
&\le c^2\sqrt{\sum_{i=1}^{m}\frac{\alpha_i^2}{\|T^{-1}(e_i)\|^2}} = c^2\left\|\sum_{i=1}^{m}\frac{\alpha_i e_i}{\|T^{-1}(e_i)\|}\right\| = c^2\left\|\sum_{i=1}^{m}\alpha_i T(x_i)\right\| \\
&= c^2\left\|T\left(\sum_{i=1}^{m}\alpha_i x_i\right)\right\| \le c^2\left\|\sum_{i=1}^{m}\alpha_i x_i\right\|.
\end{aligned}$$

□

Choose an arbitrary $\varepsilon > 0$. Given an infinite-dimensional space X, we can combine the previous theorem with Dvoretzky's theorem to see that the assumptions of Lemma 4 are satisfied (note that ε might be arbitrarily large; we are using Dvoretzky's theorem only to ensure that the values $c = 1 + \varepsilon$ and $d = (1 + \varepsilon)^2$ in Lemma 4 do not depend on m). Thus, we have proved the following corollary.

COROLLARY 10. *Let X be an arbitrary infinite-dimensional Banach space. Then X does not have the property* (JK).

Since we know that every finite-dimensional space has the property (JK), we arrive at the following conclusion.

COROLLARY 11. *A Banach space has the property* (JK) *if and only if it is finite-dimensional.*

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