
L-Tetromino Tilings and Two-Color Integer Compositions

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Abstract. We find recurrent as well as explicit formulas for the number of tilings of a $2n \times 4$ rectangle using L-tetrominoes when only rotation of tiles is allowed. We show that the problem is equivalent to calculating the number of certain two-color integer compositions.

1. INTRODUCTION. Tiling problems fascinate amateur as well as professional mathematicians. Many of them are accessible to a wide audience and require only elementary mathematics; an excellent collection of such problems is available in the first chapter of Alexander Soifer's book [1]. On the other hand, there are deep results involving nontrivial mathematics such as the celebrated Fisher–Kasteleyn–Temperley formula, which counts the number of tilings of a rectangle using dominoes [2, 3]. Two major recent breakthroughs in the area of tiling are David Smith's discovery of an einstein (a shape that admits tilings of the plane, but only aperiodic ones, [4]), as well as Rachel Greenfeld's and Terence Tao's recent disproof of the periodic tiling conjecture (the discovery of a tile whose translations cover the Euclidean space, but only aperiodically [5]). A nice survey of tiling problems was provided by Federico Ardila and Richard P. Stanley [6]. It is also worth mentioning an excellent computer program for experimenting with tiling problems, the *PolySolver*, written by Jaap Scherphuis [7].

A common type of a tile is a polyomino, which is composed of unit squares. Tetrominoes consist of four unit squares (see Figure 1), and they are familiar to all Tetris players. Because of their shapes, they are known as the I-tetromino, L-tetromino, S-tetromino, T-tetromino, and O-tetromino.

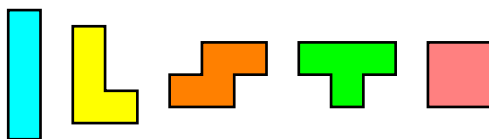


Figure 1. Five possible tetrominoes.

Which of these tetrominoes can be used to tile an $a \times b$ rectangle with integer sides? The following necessary and sufficient conditions are well known:

- An I-tetromino tiling exists if and only if at least one of a , b is divisible by 4 (see [1, Section 1.4]).
- An L-tetromino tiling exists if only if $a, b \geq 2$ and ab is divisible by 8 (see [8]).
- No rectangle can be tiled with S-tetrominoes (this is obvious; there are two ways of covering the upper left corner, both of which lead to a failure).
- A T-tetromino tiling exists if and only if both a and b are divisible by 4 (see [9]).
- An O-tetromino tiling exists if and only if both a and b are even (this is obvious).

Once we know a tiling exists, the next problem is to calculate the number of all possible tilings. In the present paper, we focus on tiling rectangles with L-tetrominoes. These are the only tetrominoes for which it makes sense to distinguish two versions of the tiling problem: either we permit the tiles to be rotated as well as reflected, or we allow only rotation (as in Tetris). In both cases, it is straightforward to calculate the number of all tilings if the length of the shorter side of the rectangle is 2 and the length of the longer one is divisible by 4. Hence, the smallest nontrivial counting problem for L-tetrominoes is that of a $2n \times 4$ rectangle. The case when both rotation and reflection are allowed was analyzed by Christopher Moore in [10], and additional results are available in the OEIS [11], sequence A084480.

As far as we are aware, the second version when only rotation is allowed is new. In this case, there are only four positions of the L-tetromino that we need to consider; see Figure 2.

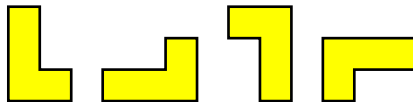


Figure 2. L-tetromino in four available positions.

For each $n \in \mathbb{N}_0$, denote by a_n the number of all tilings of a $2n \times 4$ rectangle with L-tetrominoes when only rotation is allowed. The first values are $a_0 = 1$, $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, etc. It was conjectured by Nicolas Břelohoubek that the even-indexed terms of this sequence (i.e., the numbers of tilings for the $4n \times 4$ rectangles) correspond to the sequence A166482 in the OEIS. We will confirm this conjecture and calculate a_n for all $n \in \mathbb{N}_0$. We demonstrate several possible approaches for solving the problem, and reveal connections to other topics in discrete mathematics, namely two-color integer compositions and nondecreasing Dyck paths.

2. A SYSTEM OF RECURRENCES. Our first solution is similar to [12, Section 7.3, Example 3], which deals with domino tilings of an $n \times 3$ rectangle and involves a system of recurrence relations.

When tiling a $2n \times 4$ rectangle with the L-tetrominoes from Figure 2, the upper left corner can be covered using the first, third, or fourth tile. Each of the three choices inevitably leads to the placement of one or more additional tiles; see Figure 3.

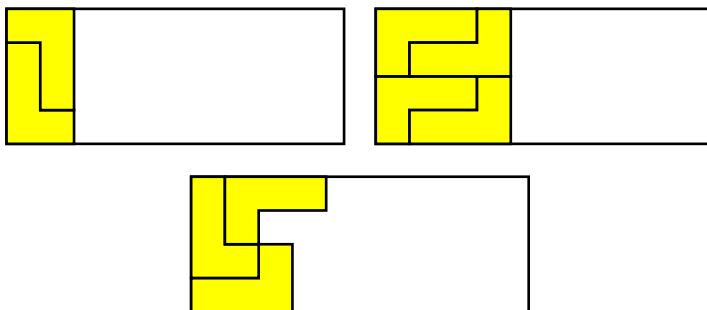


Figure 3. Covering the left end of a $2n \times 4$ rectangle.

In the first case, the remaining empty part is a $(2n - 2) \times 4$ rectangle, which can be tiled in a_{n-1} ways. What remains in the second case is a $(2n - 4) \times 4$ rectangle,

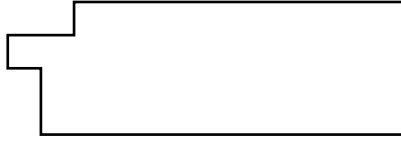


Figure 4. A non-rectangular shape to be tiled by L-tetrominoes.

which can be tiled in a_{n-2} ways. To determine the number of tilings in the third case, suppose that for each $n \in \mathbb{N}_0$, we are able to calculate the number of tilings of a figure obtained from the $2n \times 4$ rectangle by adjoining a vertical tetromino as in Figure 4.

Then the last case in Figure 3 corresponds to b_{n-2} tilings, and the previous analysis leads to the recurrence relation

$$a_n = a_{n-1} + a_{n-2} + b_{n-2}, \quad n \geq 2. \quad (1)$$

It remains to calculate b_n . Figure 5 shows that there are only two possibilities of tiling the left end of the shape from Figure 4.

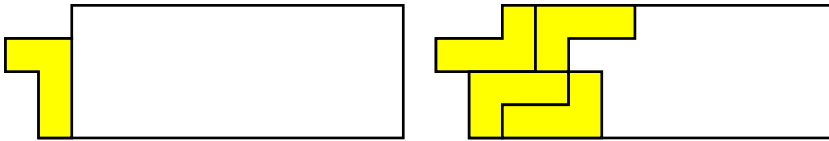


Figure 5. Tiling the non-rectangular shape.

Thus, we see that

$$b_n = a_n + b_{n-2}, \quad n \geq 2. \quad (2)$$

The system of recurrences (1) and (2) together with the initial values $a_0 = a_1 = 1$ and $b_0 = b_1 = 1$ uniquely determines a_n and b_n for all $n \in \mathbb{N}_0$.

In fact, it is possible to eliminate b_n , and obtain a recurrence relation involving only a_n . This is the content of the next theorem, which also provides an explicit formula for a_n .

Theorem 1. *For each $n \in \mathbb{N}_0$, let a_n be the number of tilings of a $2n \times 4$ rectangle using L-tetrominoes when only rotation of tiles is allowed. Then*

$$a_n = a_{n-1} + 3a_{n-2} - a_{n-3} - a_{n-4}, \quad n \geq 4. \quad (3)$$

Moreover, let $\varphi = (1 + \sqrt{5})/2$ be the golden ratio, and $\psi = 1 - \varphi = (1 - \sqrt{5})/2$. Then

$$a_n = \alpha_1 x_1^n + \alpha_2 x_2^n + \alpha_3 x_3^n + \alpha_4 x_4^n, \quad n \in \mathbb{N}_0, \quad (4)$$

where

$$x_{1,2} = \frac{1}{4} \left(1 + \sqrt{5} \pm \sqrt{2 \left(11 + \sqrt{5} \right)} \right) = \frac{1}{2} \left(\varphi \pm \sqrt{\varphi + 5} \right), \quad (5)$$

$$x_{3,4} = \frac{1}{4} \left(1 - \sqrt{5} \pm \sqrt{2 \left(11 - \sqrt{5} \right)} \right) = \frac{1}{2} \left(\psi \pm \sqrt{\psi + 5} \right) \quad (6)$$

(with plus signs corresponding to x_1, x_3 and minus signs to x_2, x_4) and

$$\alpha_{1,2} = \frac{\varphi}{2\sqrt{5}} \left(1 \pm \frac{\varphi}{\sqrt{\varphi + 5}} \right), \quad \alpha_{3,4} = \frac{-\psi}{2\sqrt{5}} \left(1 \pm \frac{\psi}{\sqrt{\psi + 5}} \right) \quad (7)$$

(with plus signs corresponding to α_1, α_3 and minus signs to α_2, α_4).

Proof. Observe that (1) implies $b_{n-2} = a_n - a_{n-1} - a_{n-2}$, and substituting into the right-hand side of (2) yields $b_n = 2a_n - a_{n-1} - a_{n-2}$. Replacing n by $n - 2$ and substituting back to (1), we arrive at (3).

The characteristic polynomial of the recurrence (3) is

$$P(x) = x^4 - x^3 - 3x^2 + x + 1 = (x^2 - \varphi x - 1)(x^2 - \psi x - 1). \quad (8)$$

To verify the second equality, it is helpful to note that $\varphi\psi = -1$ and $\varphi + \psi = 1$, since φ and ψ are the roots of the polynomial $x^2 - x - 1$.

The subsequent calculations are somewhat tedious, and it is best to use a symbolic computation program such as *Wolfram Mathematica*: One finds that the roots of P are given by (5) and (6). Thus, an explicit formula for the sequence $(a_n)_{n=0}^{\infty}$ has the form (4), where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ can be determined from the initial values a_0, \dots, a_3 ; the results are given in (7). ■

Among the four values in (5) and (6), the number with the largest absolute value is x_1 . Thus, for large n , the number of tilings a_n behaves as

$$\alpha_1 x_1^n \approx 0.589363 \cdot 2.09529^n.$$

Here is an interesting curiosity related to the golden ratio: Using the fact that the dominating term on the right-hand side of (4) is $\alpha_1 x_1^n$, one can prove that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_{n-1}}{a_n} = x_1 - \frac{1}{x_1} = \varphi.$$

Table 1 shows the exact values of a_n up to $n = 10$.

Table 1. Number of tilings of a $2n \times 4$ rectangle (A131322 in the OEIS).

n	0	1	2	3	4	5	6	7	8	9	10
a_n	1	1	3	5	12	23	51	103	221	456	965

An OEIS search reveals that these values agree with the sequence A131322. To check that the two sequences indeed coincide for all n , one can use the recurrence (3) and the initial values to obtain the generating function

$$A(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{1 - z^2}{z^4 + z^3 - 3z^2 - z + 1}, \quad (9)$$

which agrees with the generating function from the OEIS. A different proof that $(a_n)_{n=0}^\infty$ coincides with A131322 will be sketched in Section 6.

If we look only at the even-indexed terms a_0, a_2, a_4, \dots , we obtain the generating function

$$\sum_{n=0}^{\infty} a_{2n} z^n = \sum_{n=0}^{\infty} a_{2n} (\sqrt{z})^{2n} = \frac{A(\sqrt{z}) + A(-\sqrt{z})}{2} = \frac{-z^3 + 4z^2 - 4z + 1}{z^4 - 7z^3 + 13z^2 - 7z + 1},$$

which coincides with the generating function of the sequence A166482 from the OEIS. Hence, we have confirmed the conjecture by Nicolas Břelohoubek that the number of tilings in the $4n \times 4$ case indeed corresponds to this sequence. We provide yet another derivation of this fact, which does not involve generating functions. Instead, we show that the numbers e_n , which correspond to the even-indexed terms a_{2n} , satisfy the same recurrence relation as the sequence A166482.

Theorem 2. *For each $n \in \mathbb{N}_0$, let e_n be the number of tilings of a $4n \times 4$ rectangle using L-tetrominoes when only rotation of tiles is allowed. Then*

$$e_n = 7e_{n-1} - 13e_{n-2} + 7e_{n-3} - e_{n-4}, \quad n \geq 4. \quad (10)$$

Proof. Because of (4), we have

$$e_n = a_{2n} = \alpha_1 x_1^{2n} + \alpha_2 x_2^{2n} + \alpha_3 x_3^{2n} + \alpha_4 x_4^{2n}, \quad n \in \mathbb{N}_0. \quad (11)$$

This means that the sequence $(e_n)_{n=0}^\infty$ satisfies a linear recurrence relation, whose characteristic polynomial Q has roots x_1^2, \dots, x_4^2 , i.e., the squares of the roots of the polynomial P given by (8). As explained in [13], we can obtain Q by taking $P(x)P(-x)$ and replacing x^2 by y ; this gives

$$Q(y) = y^4 - 7y^3 + 13y^2 - 7y + 1.$$

Hence, the sequence $(e_n)_{n=0}^\infty$ indeed satisfies (10). ■

3. A SINGLE RECURRENCE. It is possible to gain an additional insight into the structure of all L-tetromino tilings by returning to the analysis in Figure 3. If we begin as in the third case, what are the possible continuations? Two of them are shown in Figure 6. In general, the shaded group of four L-tetrominoes arranged into two mutually shifted 4×2 rectangles can be repeated as many times as we wish. If we take $k \in \mathbb{N}_0$ copies of this group instead of one, we fill a $4(k+1) \times 4$ rectangle.

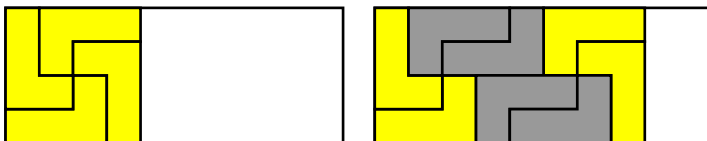


Figure 6. Filling $4(k+1) \times 4$ rectangles with $k \in \mathbb{N}_0$.

We also observe that, as soon as we insert at least one such group, we can continue only by inserting another group, or by placing two L-tetrominoes as in the right part of Figure 6, thereby leaving an unfilled rectangular part.

Taking into account these results as well as the first two cases from Figure 3, we obtain the recurrence

$$a_n = a_{n-1} + a_{n-2} + \sum_{k \geq 0} a_{n-2(k+1)}, \quad n \geq 2,$$

where the sum is over all $k \geq 0$ such that $n - 2(k + 1) \geq 0$. We perform the change of variables $l = k + 1$, where $l \geq 1$ is such that $l \leq \lfloor n/2 \rfloor$, and record the result in the following theorem.

Theorem 3. *For each $n \in \mathbb{N}_0$, let a_n be the number of tilings of a $2n \times 4$ rectangle using L-tetrominoes when only rotation of tiles is allowed. Then*

$$a_n = a_{n-1} + a_{n-2} + \sum_{l=1}^{\lfloor n/2 \rfloor} a_{n-2l}, \quad n \geq 2. \quad (12)$$

This is another recurrence relation which, together with the initial values $a_0 = a_1 = 1$, uniquely determines a_n for all $n \in \mathbb{N}_0$. Although it seems more complicated than the earlier recurrence (3), it has a clear combinatorial meaning. It corresponds to the fact that all L-tetromino tilings are constructed from rectangular building blocks, namely the 2×4 and 4×4 blocks from Figure 3, and $4(k + 1) \times 4$ blocks (with arbitrary $k \in \mathbb{N}_0$) from Figure 6. Hence, we have a block of width 2, two types of blocks of width 4, and additional blocks of widths 8, 12, 16, etc.

Note that (12) can be obtained in a purely algebraic way from (1) by repeated application of (2). Conversely, we can recover the earlier recurrence (3) from (12) as follows: Replacing n by $n - 2$ yields

$$a_{n-2} = a_{n-3} + a_{n-4} + \sum_{l=1}^{\lfloor n/2 \rfloor - 1} a_{n-2(l+1)}, \quad n \geq 4, \quad (13)$$

and subtracting this from (12) leads to

$$a_n - a_{n-2} = a_{n-1} + 2a_{n-2} - a_{n-3} - a_{n-4}, \quad n \geq 4,$$

which is equivalent to (3).

4. TILINGS AS COMPOSITIONS. A consequence of the analysis carried out in the previous section is that the tiling problem is equivalent to counting all ways of writing the number $2n$ (the width of the whole rectangle) as the sum of the numbers 2, 4, 8, 12, 16, ... (the widths of the building blocks), when we have two kinds of 4's. In other words, the number of tilings equals the number of compositions of $2n$ involving the terms 2, 4, 8, 12, 16, ..., where 4 has two possible colors. Dividing all numbers by 2, we obtain the following result.

Theorem 4. *For each $n \in \mathbb{N}_0$, the number of tilings of a $2n \times 4$ rectangle using L-tetrominoes when only rotation of tiles is allowed coincides with the number of compositions of n involving 1, 2, 4, 6, 8, ..., where 2 has two possible colors.*

Clearly, the number of such compositions satisfies the recurrence (12), and therefore also (3). But compositions are an old mathematical topic, so there must be other ways of counting them. A basic result says that the number of monochromatic compositions

of a number n involving k summands is $\binom{n-1}{k-1}$. What about color compositions? Perhaps the most elegant method of counting them is based on generating functions [14, Section 3.5]:

Suppose that each summand of value $i \in \mathbb{N}$ can have $c_i \in \mathbb{N}_0$ possible colors, and consider the generating function $C(z) = \sum_{i=1}^{\infty} c_i z^i$. If t_n denotes the number of color compositions of a number $n \in \mathbb{N}_0$ (where $t_0 = 1$), then $t_n = \sum_{k=1}^n c_k t_{n-k}$ for all $n \in \mathbb{N}$. Therefore, the generating function $T(z) = \sum_{n=0}^{\infty} t_n z^n$ satisfies $T(z) = 1 + C(z)T(z)$, i.e.,

$$T(z) = \frac{1}{1 - C(z)}.$$

The compositions we are interested in correspond to $c_2 = 2$, $c_n = 1$ for all $n \in \{1, 4, 6, \dots\}$, and $c_n = 0$ for all $n \in \{3, 5, \dots\}$, which leads to

$$C(z) = z + 2z^2 + z^4 + z^6 + \dots = z + z^2 + \frac{z^2}{1 - z^2}.$$

Consequently,

$$T(z) = \frac{1}{1 - z - z^2 - \frac{z^2}{1 - z^2}} = \frac{1 - z^2}{z^4 + z^3 - 3z^2 - z + 1},$$

which agrees with the previously derived generating function (9). So, this method gives a quick check of our earlier results, but we did not learn anything new.

Let us try a different way of calculating a_n , i.e., the number of compositions of n involving the summands 1, 2, 4, 6, ..., where 2 has two possible colors. The sum of all even numbers in such a composition is $2j$, where $j \in \{0, \dots, \lfloor n/2 \rfloor\}$, and the remaining summands are $n - 2j$ ones. Denote by $c_{j,k}$ the number of compositions of $2j$ involving exactly k even summands, where 2 has two colors. For each such composition, we can insert the $n - 2j$ ones into $k + 1$ available positions (some of them might be chosen repeatedly), which can be done in $\binom{n - 2j + k}{k}$ ways. This leads to the formula

$$a_n = \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{k=0}^j \binom{n - 2j + k}{k} c_{j,k}, \quad (14)$$

where we assume that $c_{0,0} = 1$ and $c_{j,0} = 0$ for all positive j . How do we calculate the remaining values of $c_{j,k}$? Dividing all summands in a composition of $2j$ by two, we see that $c_{j,k}$ also equals the number of compositions of j involving exactly k numbers having values 1, 2, 3, ..., where 1 has two colors, say red and blue.

Assume there are i red ones; eliminating all of them, we get a monochromatic composition of $j - i$ involving $k - i$ positive integers. One can easily reverse this process: Begin with a monochromatic composition of $j - i$ consisting of $k - i$ summands; there are $\binom{j - i - 1}{k - i - 1}$ possibilities. Next, insert i red ones into $k - i + 1$ possible places (some of them might be chosen repeatedly); this can be done in $\binom{k}{i}$ ways. Summing over all possible i , we get

$$c_{j,k} = \sum_{i=0}^k \binom{j - i - 1}{k - i - 1} \binom{k}{i} = \sum_{l=0}^k \binom{k}{l} \binom{j - k + l - 1}{l - 1}, \quad (15)$$

Table 2. The values $c_{j,k}$, $0 \leq k \leq j$

$j \setminus k$	0	1	2	3	4	5	6
0	1						
1	0	2					
2	0	1	4				
3	0	1	4	8			
4	0	1	5	12	16		
5	0	1	6	18	32	32	
6	0	1	7	25	56	80	64

where the second equality follows from the change of variables $l = k - i$.

There is yet another way of calculating $c_{j,k}$: Each k -term composition of j containing l red or blue ones can be obtained from a $(k - l)$ -term monochromatic composition of $j - l$ involving only the numbers $2, 3, \dots$. It suffices to insert l red or blue ones into $k - l + 1$ available positions (possibly with repetition), which can be done in $\binom{k}{l} 2^l$ ways. The number of $(k - l)$ -term monochromatic compositions of $j - l$ involving $2, 3, \dots$ is the same as the number of $(k - l)$ -term monochromatic compositions of $(j - l) - (k - l) = j - k$ involving $1, 2, \dots$, which is given by $\binom{j-k-1}{k-l-1}$. Summing over all possible l , we get

$$c_{j,k} = \sum_{l=2k-j}^k 2^l \binom{k}{l} \binom{j-k-1}{k-l-1} = \sum_{i=k}^j 2^{2k-i} \binom{k}{i-k} \binom{j-k-1}{j-i}, \quad (16)$$

where the second equality follows from the change of variables $i = 2k - l$.

Substituting (15) or (16) into (14), we obtain the following result.

Theorem 5. For each $n \in \mathbb{N}_0$, let a_n be the number of tilings of a $2n \times 4$ rectangle using L -tetrominoes when only rotation of tiles is allowed. Then

$$\begin{aligned} a_n &= \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{k=0}^j \binom{n-2j+k}{k} \sum_{l=0}^k \binom{k}{l} \binom{j-k+l-1}{l-1} \\ &= \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{k=0}^j \binom{n-2j+k}{k} \sum_{i=k}^j 2^{2k-i} \binom{k}{i-k} \binom{j-k-1}{j-i}. \end{aligned}$$

Thus, we have discovered two additional formulas for calculating the number of compositions or tilings.

The numbers $c_{j,k}$ we have introduced along the way are of independent interest. Table 2 shows their values for small i and j . They coincide with the numbers $T(j+1, k+1)$ listed in the OEIS item A121462 (which contains the formula (15) with j, k replaced by $j+1, k+1$). According to the OEIS, the numbers $T(j, k)$ count nondecreasing Dyck paths of semilength j , having pyramid weight k . The meaning of these terms is as follows: A Dyck path of semilength n is a lattice path from $(0, 0)$ to $(2n, 0)$ consisting of n northeast steps $(1, 1)$ and n southeast steps $(1, -1)$ that never goes below the horizontal axis. Such a path is called nondecreasing if the altitudes of its valleys never decrease. A pyramid of weight k within a Dyck path is a sequence of k northeast steps followed by k southeast steps; it is maximal if it is not contained in

a larger pyramid. Finally, the pyramid weight of the whole path is the sum of weights of its maximal pyramids. For more information on non-decreasing Dyck paths, see [15, 16, 17], and the references therein.

Since $c_{j,k} = T(j+1, k+1)$, it follows that the number of nondecreasing Dyck paths of semilength $j+1$, having pyramid weight $k+1$, equals the number of compositions of j involving exactly k numbers having values $1, 2, 3, \dots$, where 1 has two colors. We have also proved that the number of such Dyck paths can be calculated using the formula (16).

5. MORE ON TWO-COLOR COMPOSITIONS. Let us mention two simple problems involving two-color compositions, which are closely related to the previous exposition. Both of them are probably known, although we were unable to find a reference for the first one.

The numbers $c_{j,k}$ from the previous section count the compositions of a number j involving exactly k summands having values $1, 2, 3, \dots$, where 1 has two colors. What happens if we relax the condition on the number of terms?

Problem 6. What is the number of all compositions of a number $j \in \mathbb{N}_0$ involving summands of values $1, 2, 3, \dots$, where 1 has two colors?

The answer is provided by the row sums of Table 2:

$$1, 2, 5, 13, 34, 89, 233, \dots$$

These are the odd-indexed Fibonacci numbers (see A001519 in the OEIS). Here is a quick proof of this fact: Denote by k_j the number of compositions of j where 1 has two colors. We have the recurrence

$$k_j = 2k_{j-1} + k_{j-2} + \dots + k_0, \quad j \geq 1$$

(where we let $k_0 = 1$). Replacing j by $j-1$, we get

$$k_{j-1} = 2k_{j-2} + k_{j-3} + \dots + k_0, \quad j \geq 2.$$

Subtracting this identity from the previous one gives

$$k_j = 3k_{j-1} - k_{j-2}, \quad j \geq 2.$$

One can check that the Fibonacci numbers F_{2j+1} satisfy the same recurrence (this is also mentioned in the OEIS), and $k_0 = 1 = F_1$, $k_1 = 2 = F_3$. Thus, $k_j = F_{2j+1}$ for all $j \in \mathbb{N}_0$.

Here is a natural modification of the previous problem.

Problem 7. What is the number of compositions of a number $j \in \mathbb{N}_0$ involving only odd summands, where 1 has two colors?

Denoting the solution by l_j , we have the recurrence

$$l_j = 2l_{j-1} + l_{j-3} + l_{j-5} + \dots, \quad j \geq 1.$$

Replacing j by $j-2$ and subtracting the result from the previous identity, we obtain

$$l_j = 2l_{j-1} + l_{j-2} - l_{j-3}, \quad j \geq 3.$$

The first few terms of the sequence $(l_j)_{j=0}^\infty$ are

$$1, 2, 4, 9, 20, 45, 101, \dots$$

The sequence coincides with A052534 in the OEIS, which also mentions its meaning in terms of color compositions.

6. OPEN PROBLEMS. Here are two open problems for the readers.

Problem 8. We know that the sequence $(a_n)_{n=0}^\infty$ coincides with the OEIS entry A131322. According to the OEIS, the terms of this sequence can be also calculated using the formula

$$a_n = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} F_{n-2j+1}. \quad (17)$$

Is there a combinatorial proof of the formula (17) based on tilings or two-color compositions? Note that one can verify (17) algebraically: To evaluate the right-hand side, we rewrite F_{n-2j+1} using Binet's formula, which yields

$$F_{n-2j+1} = \frac{1}{\sqrt{5}}(\varphi^{n-2j+1} - \psi^{n-2j+1}),$$

and then apply the identity

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} \beta^{n-2j} = \frac{1}{\sqrt{\beta^2 + 4}} \left(\left(\frac{\beta + \sqrt{\beta^2 + 4}}{2} \right)^{n+1} - \left(\frac{\beta - \sqrt{\beta^2 + 4}}{2} \right)^{n+1} \right),$$

which holds for every $\beta \in \mathbb{R}$ (see formulas (2.7) and (1.3) in [18]). After some manipulations, one finds that the result coincides with the explicit formula for a_n given in (4).

Problem 9. We have shown that the number of nondecreasing Dyck paths of semi-length $j + 1$, having pyramid weight $k + 1$, equals the number of compositions of j involving exactly k numbers having values $1, 2, 3, \dots$, where 1 has two colors. Is there a bijective proof of this fact?

Throughout the paper, we have provided links to several sequences in the OEIS. Readers interested in tiling problems involving L-tetrominoes might enjoy exploring additional sequences listed in Table 3.

Table 3. Additional sequences in the OEIS involving L-tetrominoes.

A084481	fault-free tilings of a $4 \times 2n$ rectangle with L-tetrominoes
A174248	tilings of a $4 \times n$ rectangle with tetrominoes of any shape
A226322	tilings of a $4 \times n$ rectangle using L- and O-tetrominoes
A232497	tilings of a $4 \times n$ rectangle using L- and S-tetrominoes
A232757	tilings of a $3 \times 4n$ rectangle with $3n$ tetrominoes of any shape
A233191	tilings of a $4 \times n$ rectangle using L- and T-tetrominoes
A233266	tilings of a $4 \times n$ rectangle using L-, T-, and S-tetrominoes
A242636	tilings of a $4 \times n$ rectangle using L-, S-, and O-tetrominoes

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