Undirected Loop Conditions

Miroslav Olšák

Algebra = set + operations

Examples:

groups $(G, \cdot, {}^{-1}, 1)$

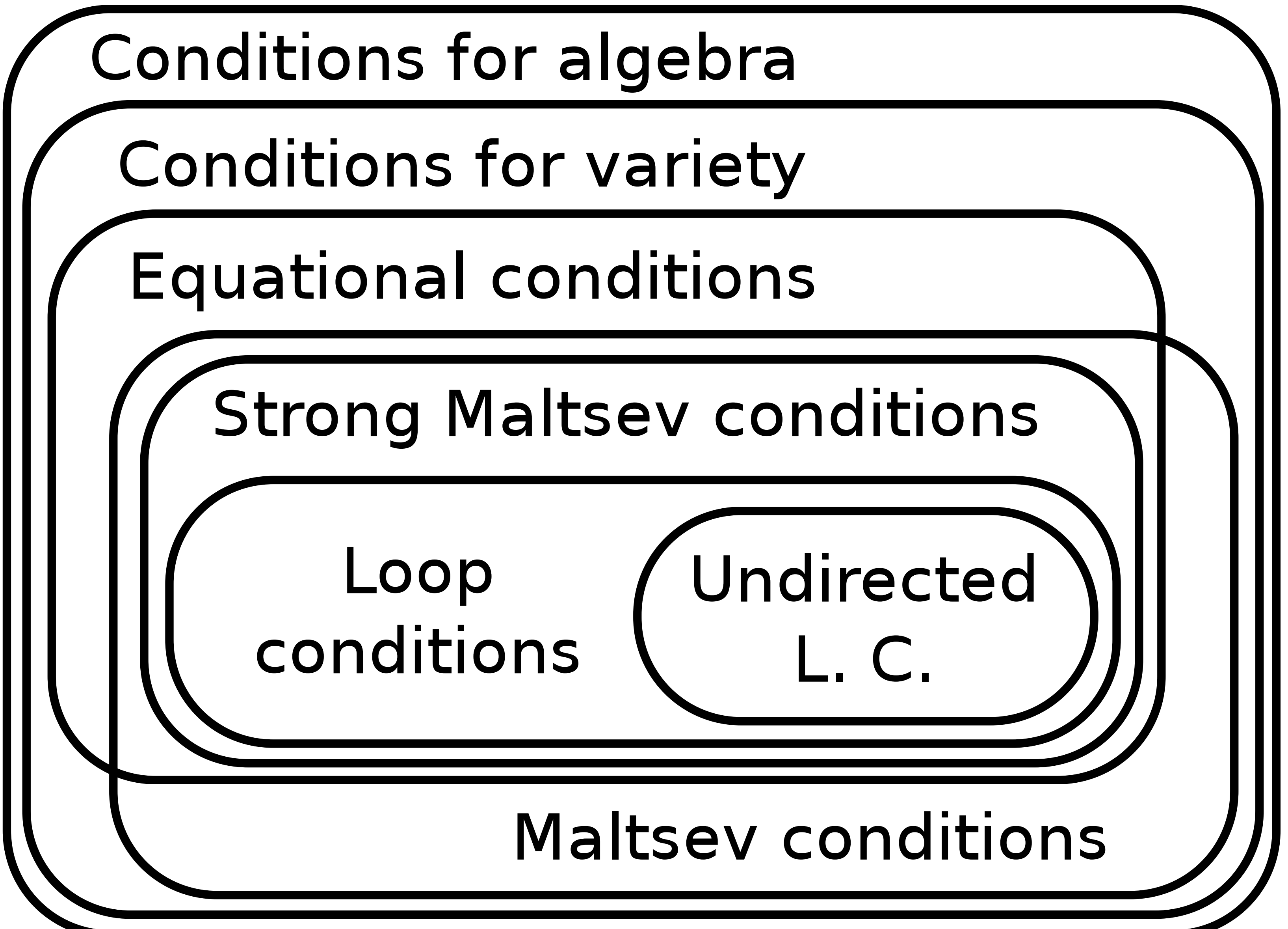
rings $(R, +, \cdot, -, 0, 1)$

lattices $(L, \vee, \wedge)$

sets (S)

Algebra = set + operations

Examples:		Terms
groups	$(G, \cdot, {}^{-1}, 1)$	$t(a,b,c) = (ab^{-1})c$
rings	$(R, +, \cdot, -, 0, 1)$	$t(a,b) = a \cdot (b + a)$
lattices	$(L, \vee, \wedge)$	$t(a,b,c) = (a \wedge c) \vee a$
sets	(S)	$t(a,b,c) = b$


Conditions for algebra
Conditions for variety
Equational conditions
Strong Maltsev conditions
Loop conditions
Undirected L. C.
Maltsev conditions

Loop Condition

There is a term t satisfying

$$
\begin{aligned}
& t(\text{variables}) \\
= \; & t(\text{variables})
\end{aligned}
$$

Loop Condition

There is a term t satisfying

$$t(\text{variables})$$
$$= t(\text{variables})$$

Examples:

Commutativity:
$$c(x, y)$$
$$= c(y, x)$$

Maltsev term:
$$m(x, y, y)$$
$$= m(z, z, x)$$

Loop Condition

There is a term t satisfying

$$t(\text{variables})$$
$$= t(\text{variables})$$

Examples:

Commutativity:
$$c(x, y)$$
$$= c(y, x)$$

$$c(a, b) = a \wedge b$$

Maltsev term:
$$m(x, y, y)$$
$$= m(z, z, x)$$

$$m(a, b, c) = (ab^{-1})c$$

Observations

Ordering of columns is irrelevant

$$\begin{aligned} & m(x, y, y) \\ = \; & m(z, z, x) \end{aligned}$$

$$m(a, b, c) = (ab^{-1})c$$

Observations

Ordering of columns is irrelevant

$$\begin{aligned} & m(x, y, y) \\ = \; & m(z, x, z) \end{aligned}$$

$$m(a, b, c) = (ac^{-1})b$$

Observations

Ordering of columns is irrelevant

$$\begin{aligned} & m(x, y, y) \\ = \; & m(z, x, z) \end{aligned}$$

$$m(a, b, c) = (ac^{-1})b$$

Duplicite columns are irrelevant

$$\begin{aligned} & t(x, y, y) \\ = \; & t(y, x, x) \end{aligned}$$

$$t(a, b, c) = (a \wedge c) \wedge b$$

Observations

Ordering of columns is irrelevant

$$\begin{aligned} & m(x, y, y) \\ = \; & m(z, x, z) \end{aligned}$$

$$m(a, b, c) = (ac^{-1})b$$

Duplicite columns are irrelevant

$$\begin{aligned} & t(x, y, y) \\ = \; & t(y, x, x) \end{aligned}$$

$$t(a, b, c) = (a \wedge b) \wedge b$$

Observations

Ordering of columns is irrelevant

$$m(x, y, y) = m(z, x, z)$$

$$m(a, b, c) = (ac^{-1})b$$

Duplicite columns are irrelevant

$$t(x, y) = t(y, x)$$

$$t(a, b) = (a \wedge b) \wedge b$$

Observations

Ordering of columns is irrelevant

$$\begin{aligned} & m(x,y,y) \\ = \ & m(z,x,z) \end{aligned} \qquad m(a,b,c) = (ac^{-1})b$$

Duplicite columns are irrelevant

$$\begin{aligned} & t(x,y) \\ = \ & t(y,x) \end{aligned} \qquad t(a,b) = (a \wedge b) \wedge b$$

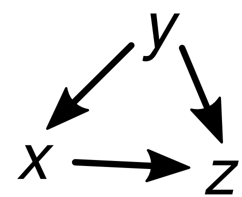
Constant column causes triviality

$$\begin{aligned} & t(x,y,y) \\ = \ & t(y,y,x) \end{aligned} \qquad t(a,b,c) = b$$

Observations $\Rightarrow$ Graph

Ordering of columns is irrelevant

$$m(x,y,y) = m(z,x,z) \qquad m(a,b,c) = (ac^{-1})b$$

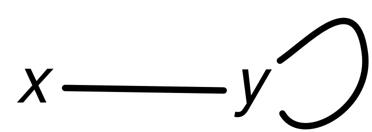


Duplicite columns are irrelevant

$$t(x,y) = t(y,x) \qquad t(a,b) = (a \wedge b) \wedge b$$

Constant column causes triviality

$$t(x,y,y) = t(y,y,x) \qquad t(a,b,c) = b$$



Graph Compatible with Algebra

Vertices: Universe of $\mathbf{A}$

Edges: Subalgebra of $\mathbf{A}^2$

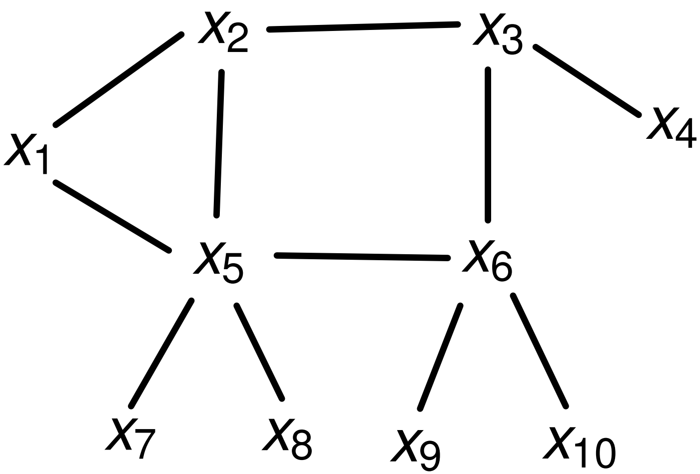
$\iff$ Operation applied to edges is an edge

Graph Compatible with Algebra

Vertices: Universe of $\mathbf{A}$

Edges: Subalgebra of $\mathbf{A}^2$

$\Leftrightarrow$ Operation applied to edges is an edge

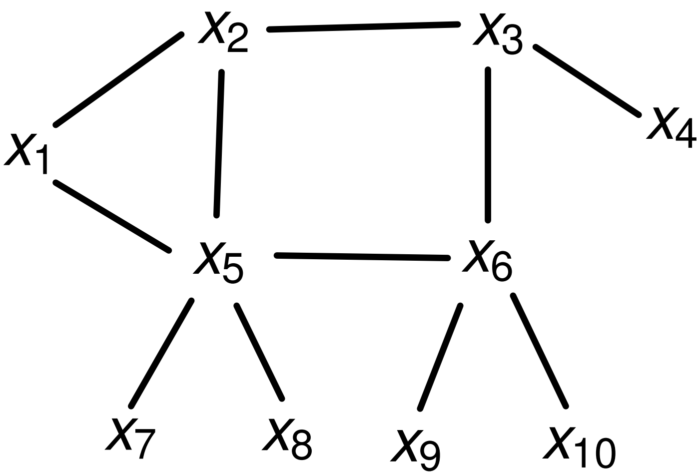


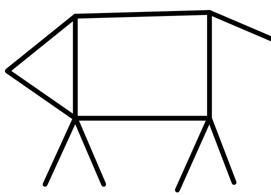
Graph Compatible with Algebra

Vertices: Universe of $\mathbf{A}$

Edges: Subalgebra of $\mathbf{A}^2$

$\Leftrightarrow$ Operation applied to edges is an edge





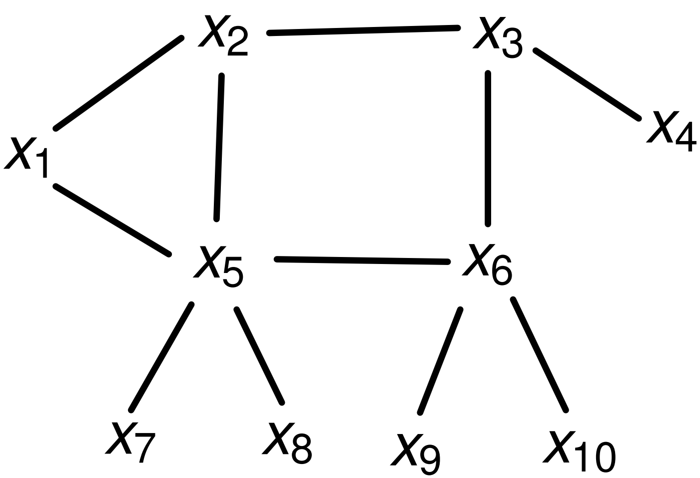
- loop condition

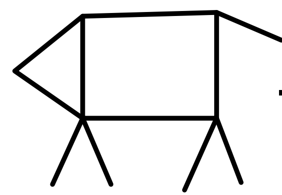
Graph Compatible with Algebra

Vertices: Universe of $\mathbf{A}$

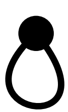
Edges: Subalgebra of $\mathbf{A}^2$

$\Leftrightarrow$ Operation applied to edges is an edge





- loop condition



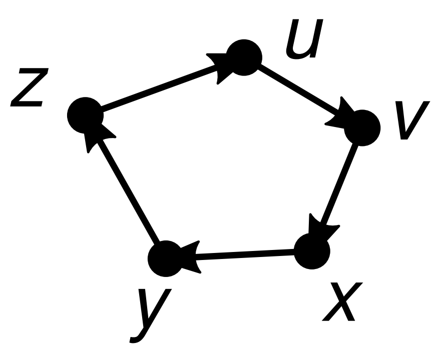
$$t(x_1, x_2, x_2, x_3, x_3, x_4, x_2, x_5, x_3, x_6, x_1, x_5, x_5, x_6, x_5, x_7, x_5, x_8, x_6, x_9, x_6, x_{10})$$
$$= t(x_2, x_1, x_3, x_2, x_4, x_3, x_5, x_2, x_6, x_3, x_5, x_1, x_6, x_5, x_7, x_5, x_8, x_5, x_9, x_6, x_{10}, x_6)$$

Strength Ordering

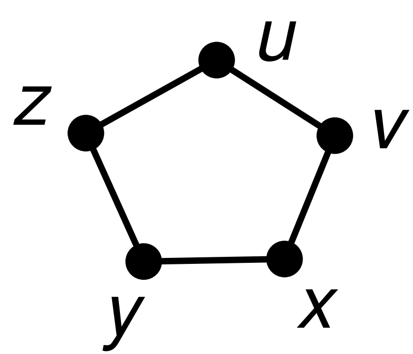
$$G \leq H \quad \Longrightarrow \quad G \Rightarrow H$$

$$t(u,v,x,y,z) = t(v,x,y,z,u)$$

$$t(u,v,v,x,x,y,y,z,z,u) = t(v,u,x,v,y,x,z,y,u,z)$$



$$\Longrightarrow$$



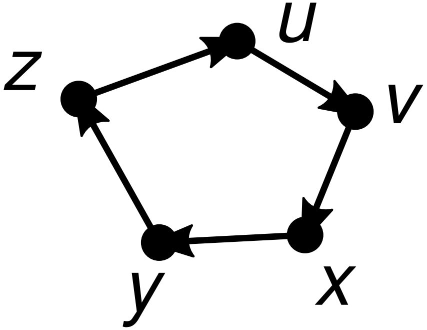
Strength Ordering

$$G \leq H \quad \Longrightarrow \quad G \Rightarrow H$$

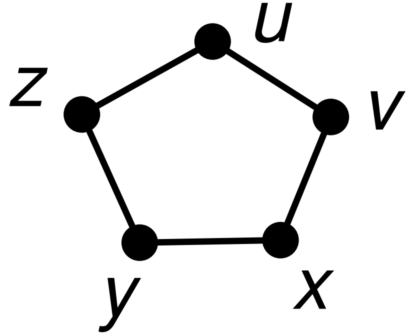
$$t(u, v, x, y, z) = t(v, x, y, z, u)$$

$$t(u, v, v, x, x, y, y, z, z, u) = t(v, u, x, v, y, x, z, y, u, z)$$

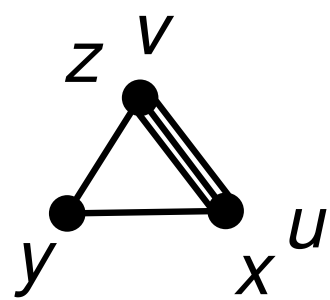
$$t(x, y, y, x, x, y, y, z, z, x) = t(y, x, x, y, y, x, z, y, x, z)$$



$\Longrightarrow$



$\Longrightarrow$



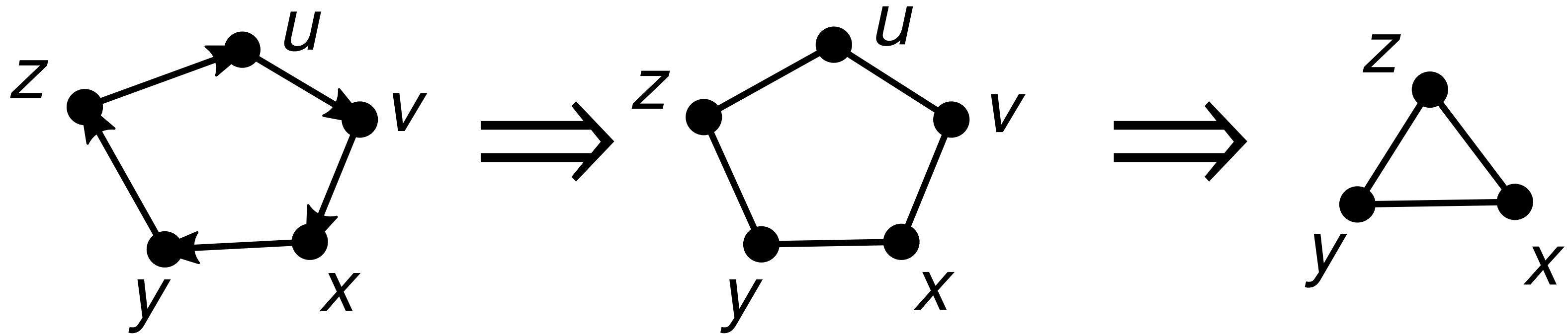
Strength Ordering

$$G \leq H \quad \Longrightarrow \quad G \Rightarrow H$$

$$t(u, v, x, y, z) = t(v, x, y, z, u)$$

$$t(u, v, v, x, x, y, y, z, z, u) = t(v, u, x, v, y, x, z, y, u, z)$$

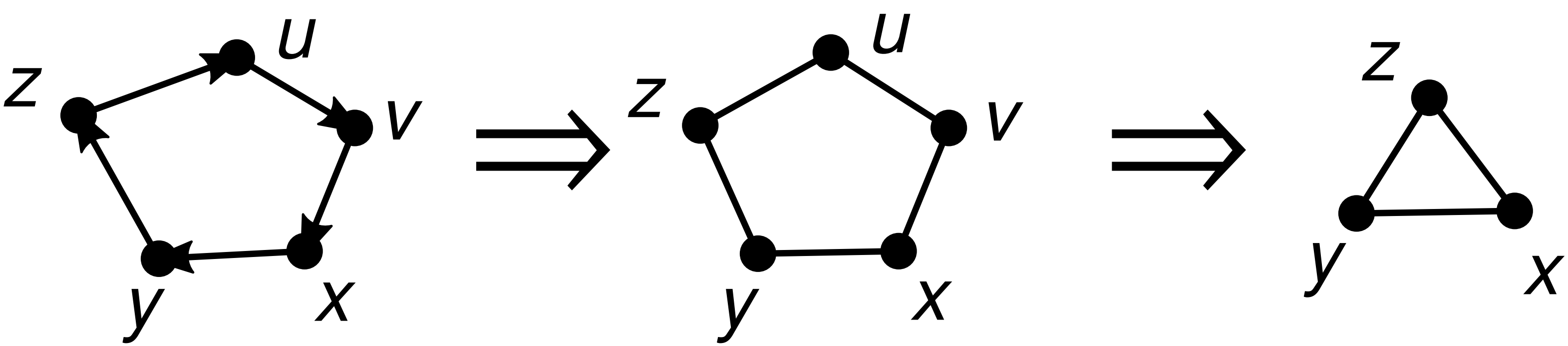
$$t(x, y, y, z, z, x) = t(y, x, z, y, x, z)$$



Strength Ordering

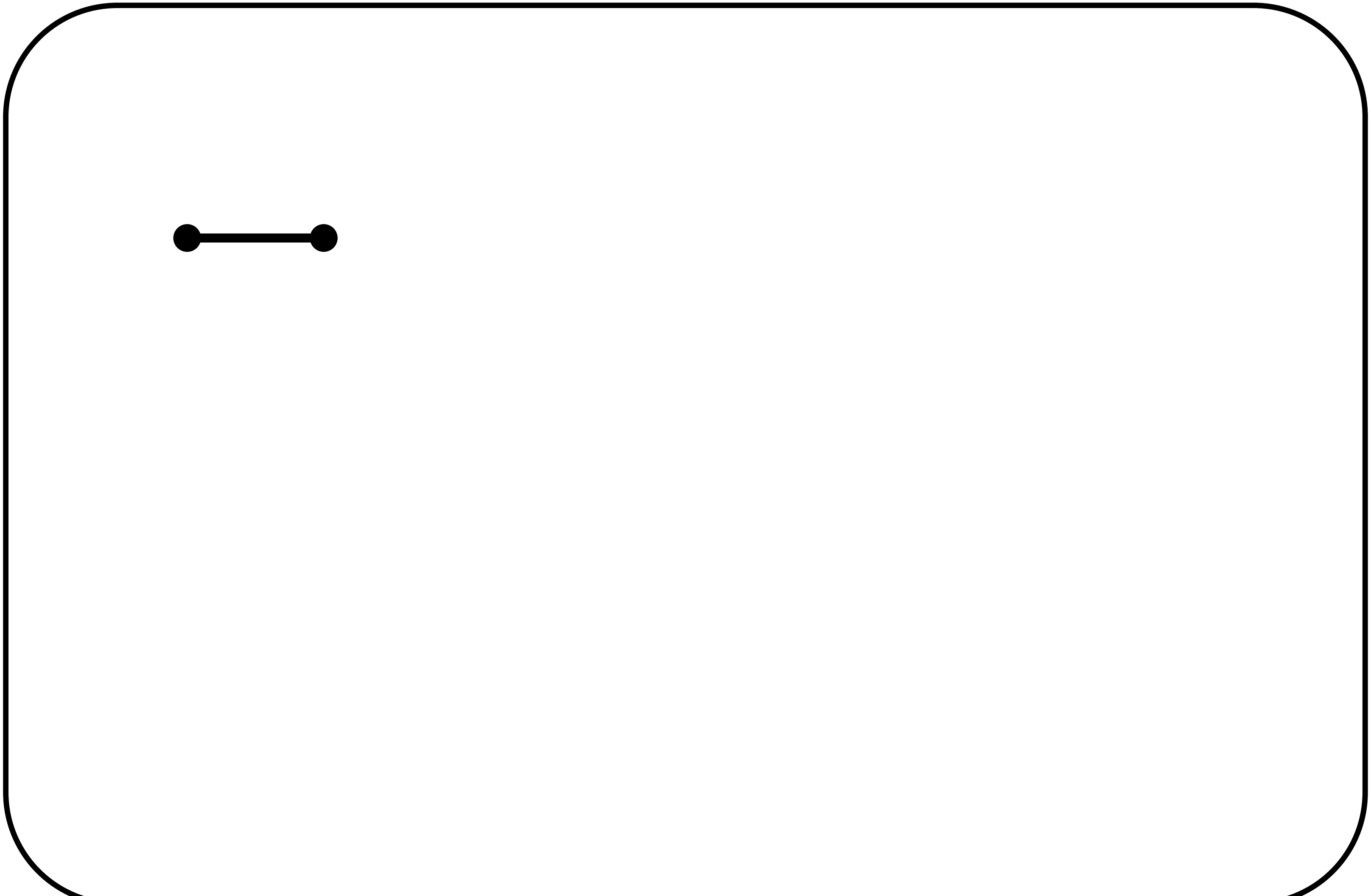
$$\left.\begin{array}{l} G \le H \\ G \to H \end{array}\right\} \Longrightarrow G \Rightarrow H$$

$$t(u,v,x,y,z) = t(v,x,y,z,u) \qquad t(u,v,v,x,x,y,y,z,z,u) = t(v,u,x,v,y,x,z,y,u,z) \qquad t(x,y,y,z,z,x) = t(y,x,z,y,x,z)$$

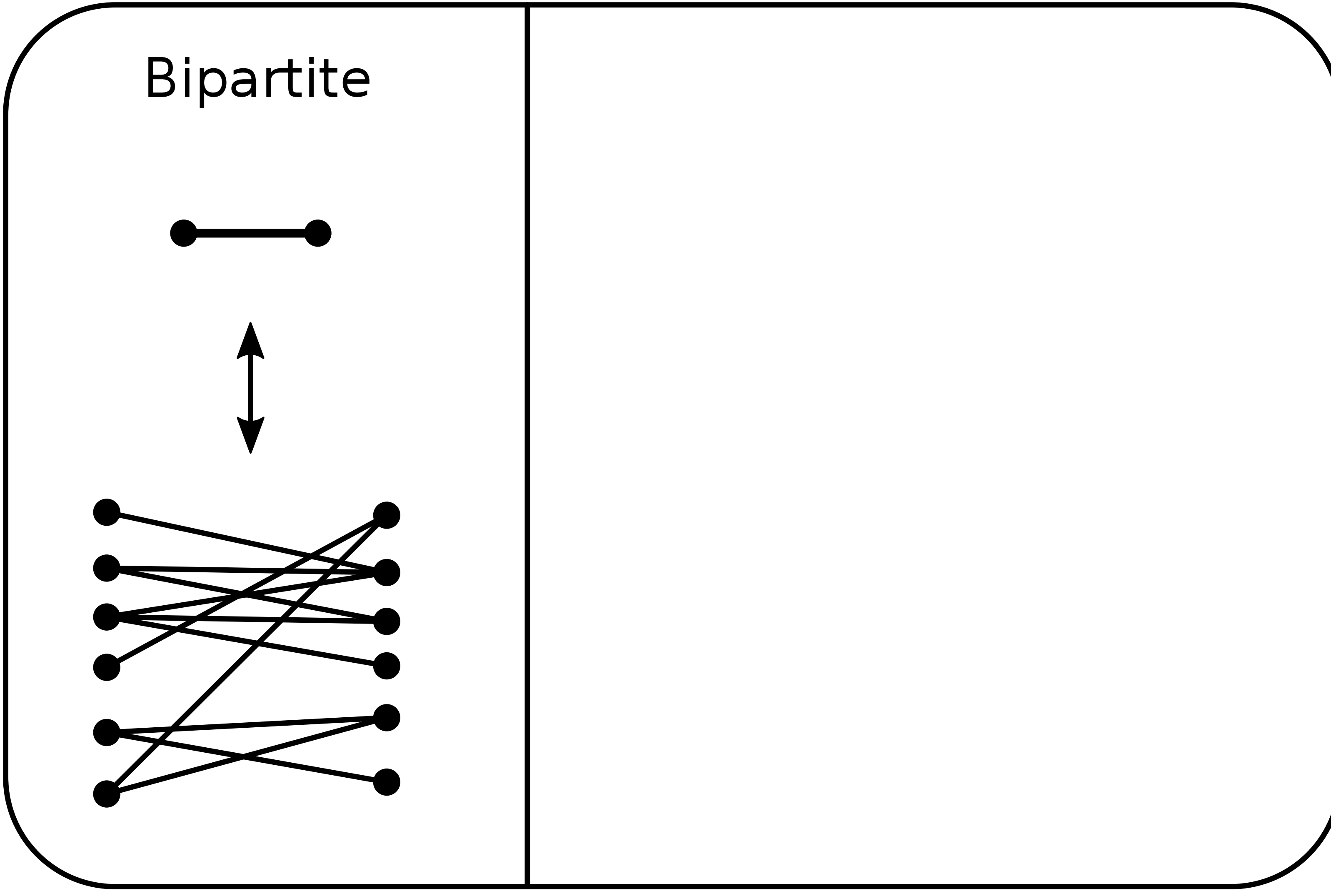


Non-trivial Undirect Loop Conditions

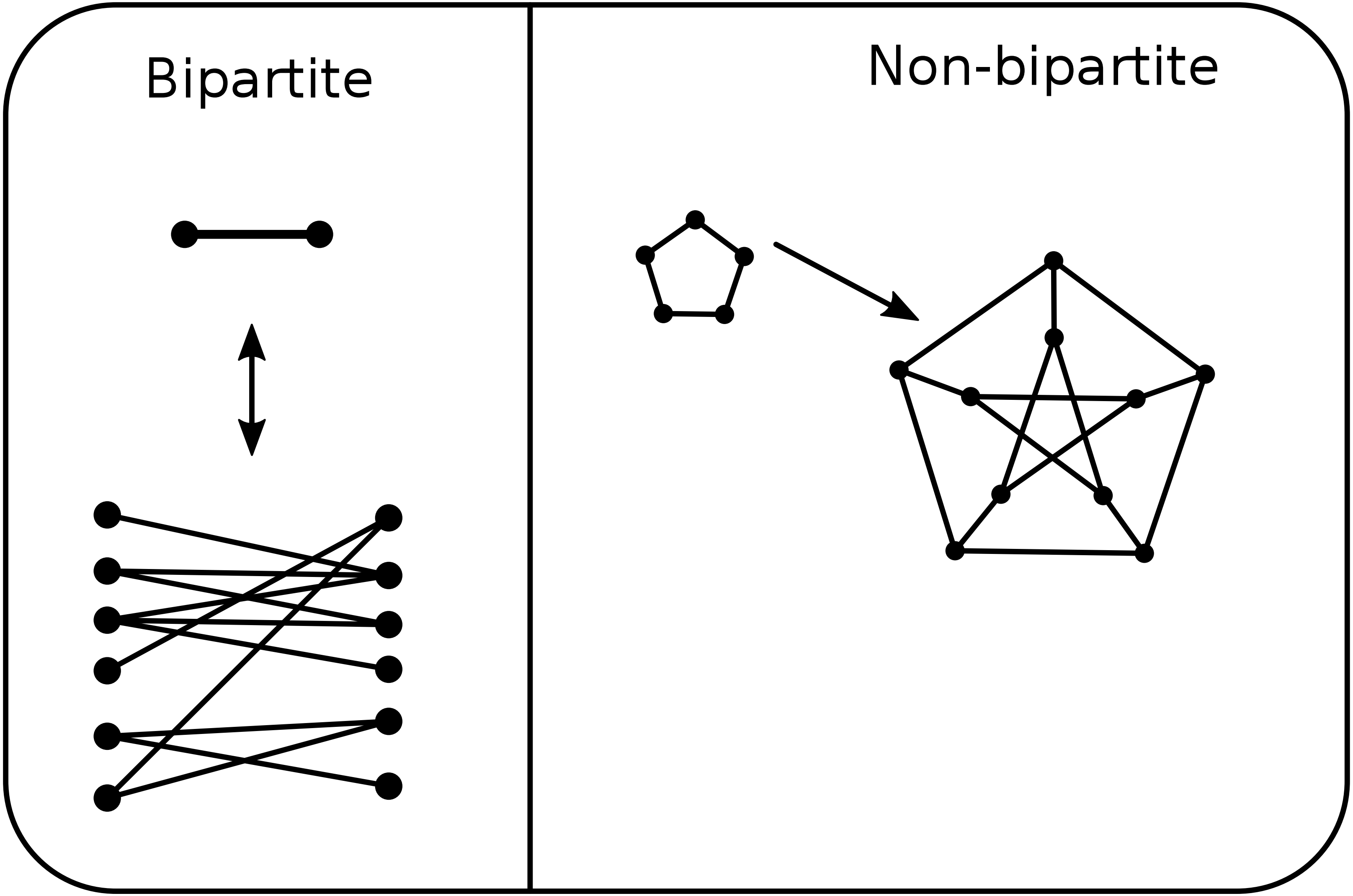
Non-trivial Undirect Loop Conditions



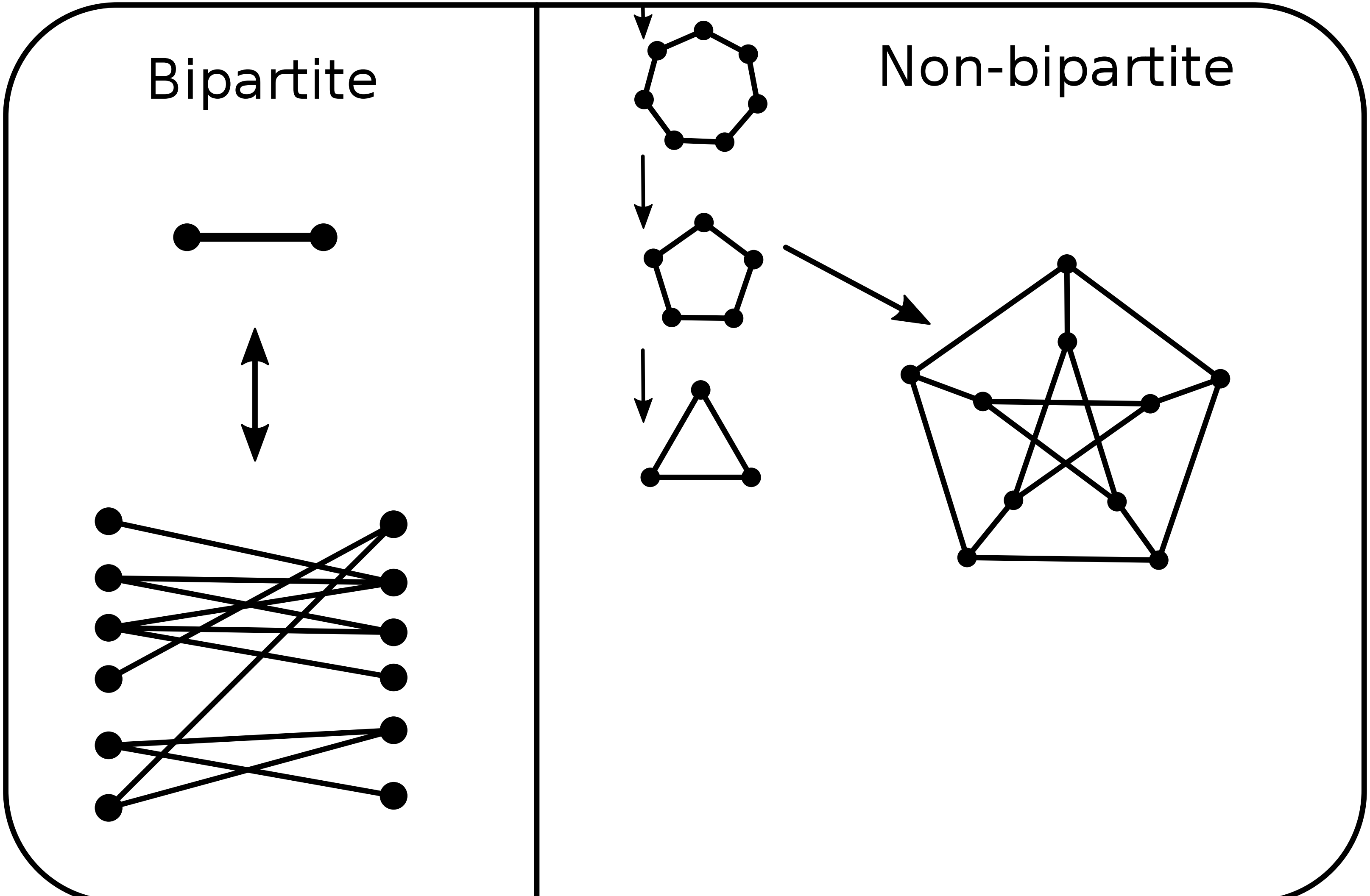
Non-trivial Undirect Loop Conditions

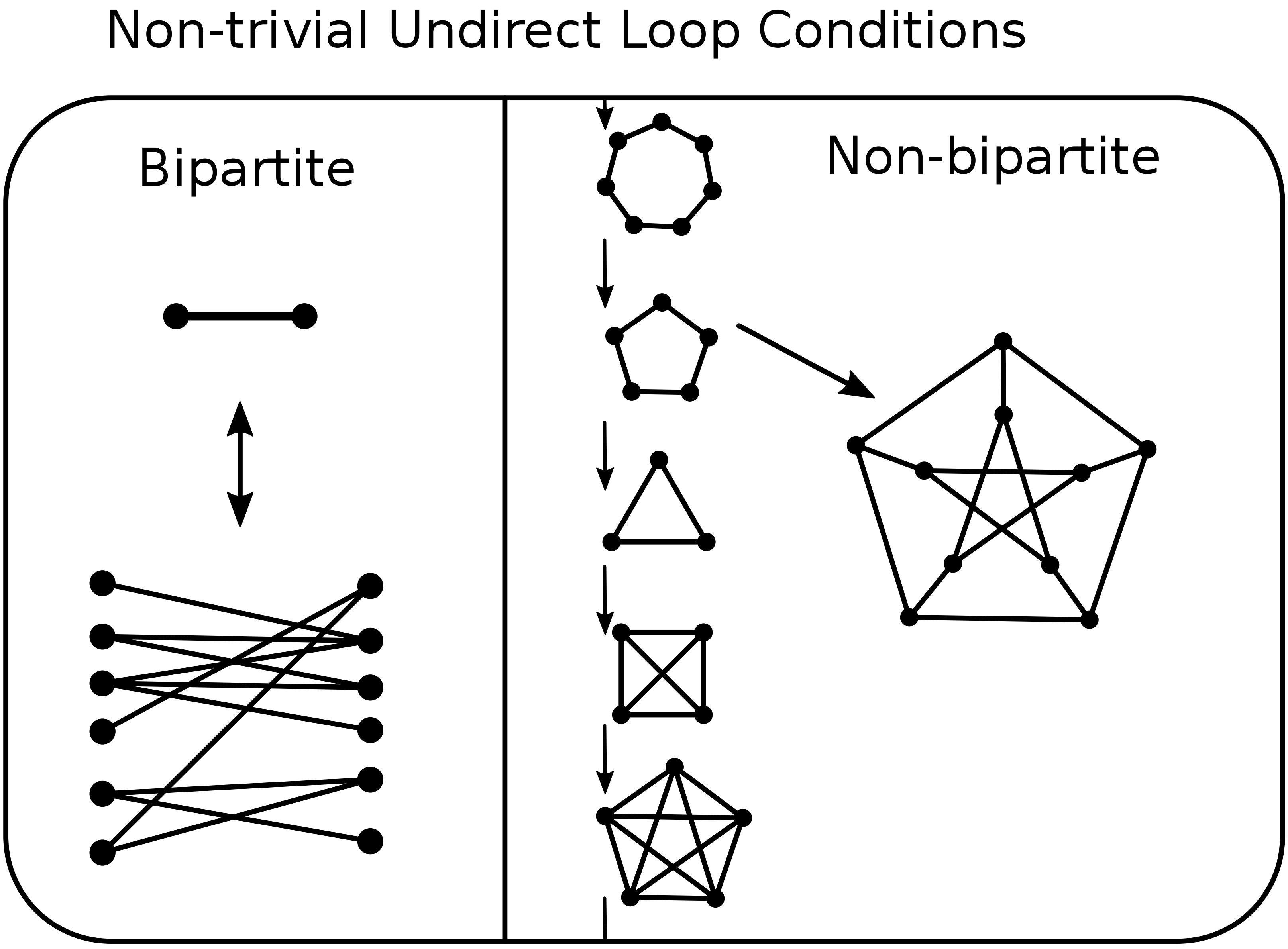


Non-trivial Undirect Loop Conditions



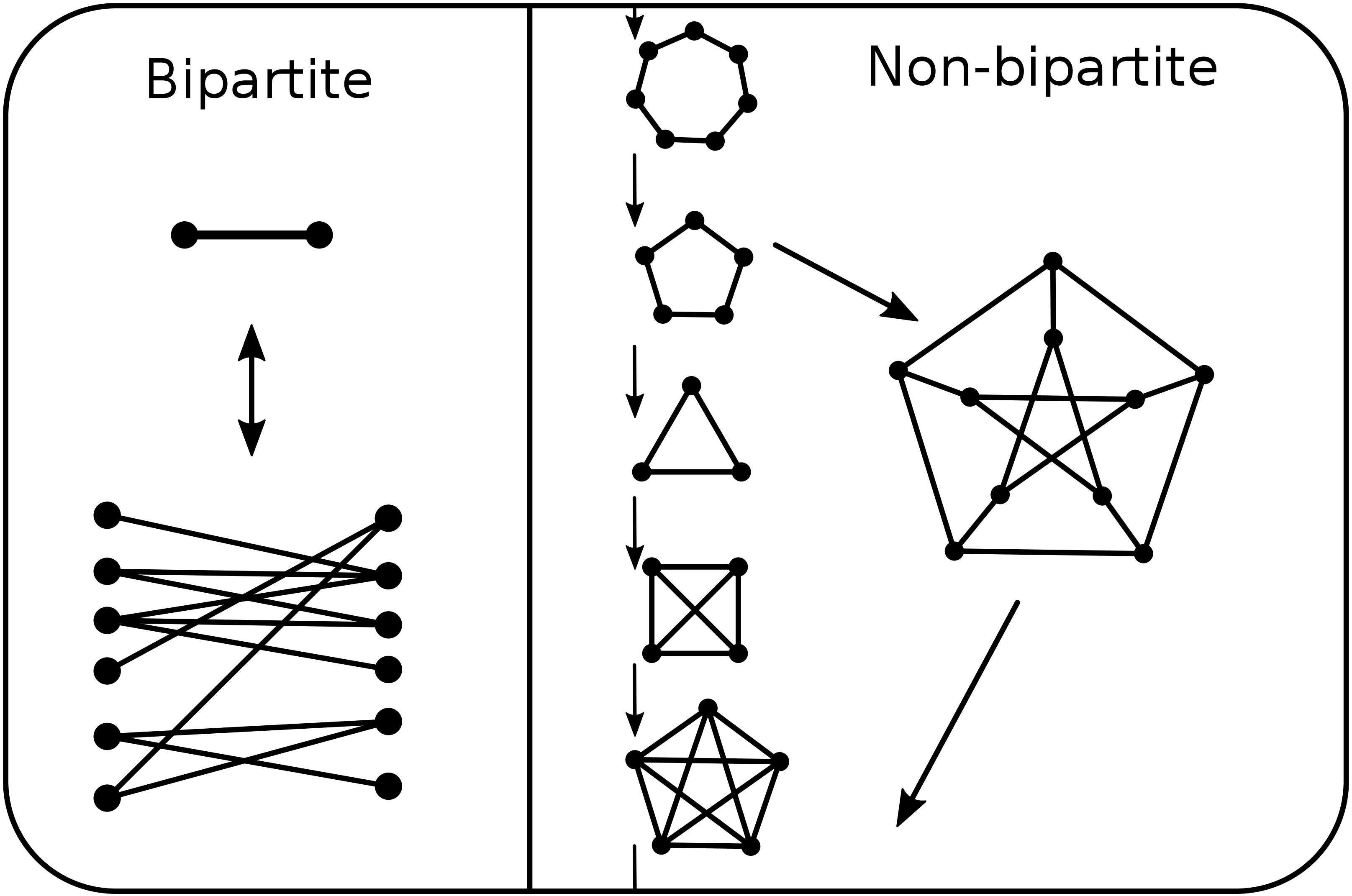
Non-trivial Undirect Loop Conditions

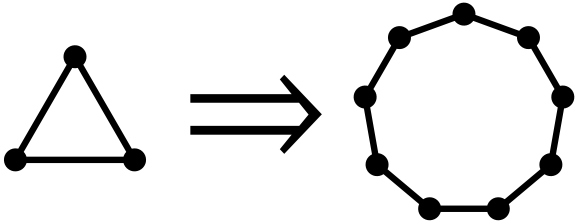


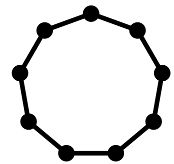
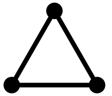


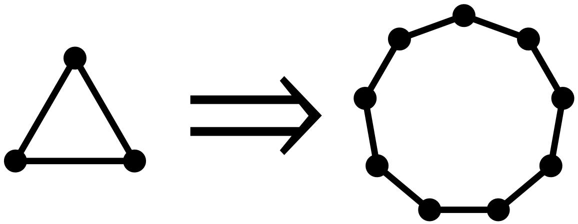
Non-trivial Undirect Loop Conditions
Bipartite
Non-bipartite

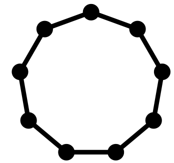
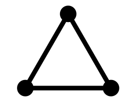
Non-trivial Undirect Loop Conditions

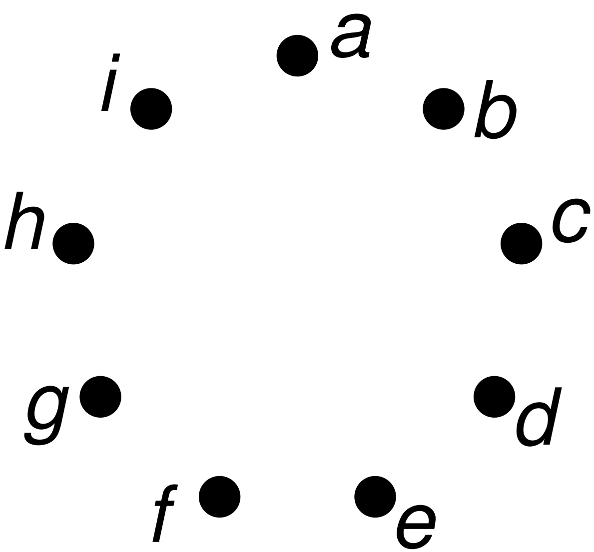


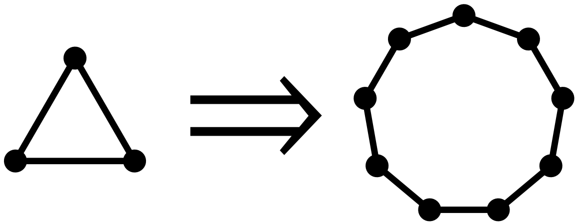


Free algebra over  modulo 

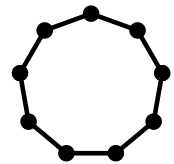
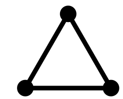


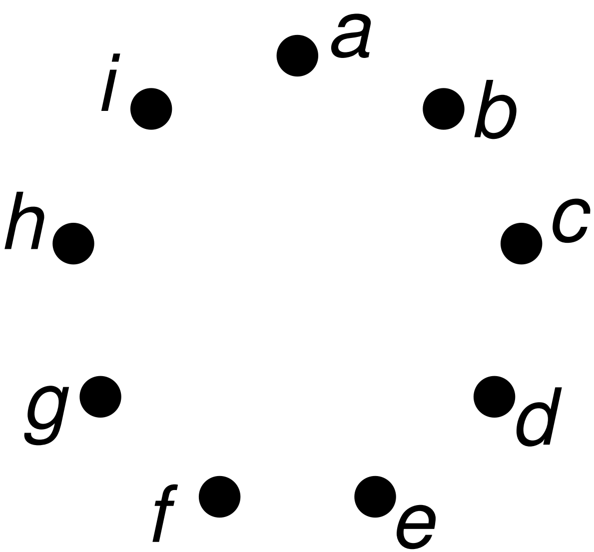
Free algebra over  modulo 



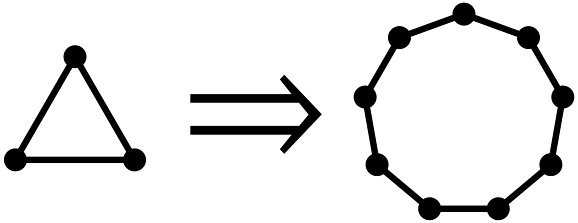


$$s(x, y, y, z, z, x) = s(y, x, z, y, x, z)$$

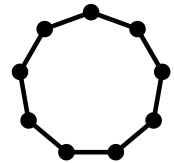
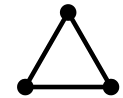
Free algebra over  modulo 

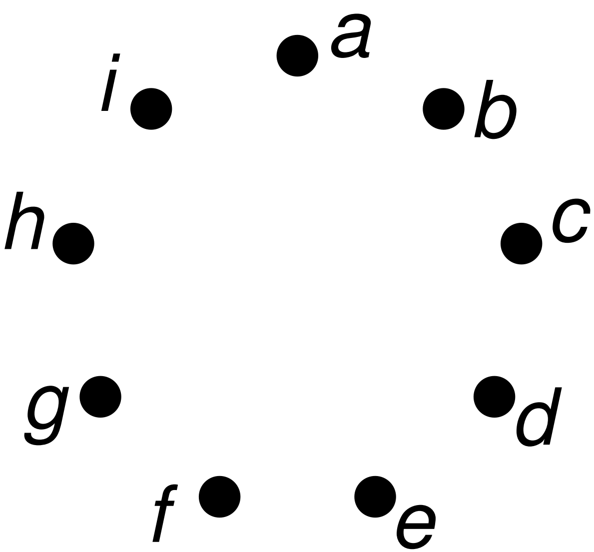


- $s(a, b, e, f, c, h)$
- $s(b, a, d, g, b, g)$



$$s(x,y,y,z,z,x) = s(y,x,z,y,x,z)$$

Free algebra over  modulo 

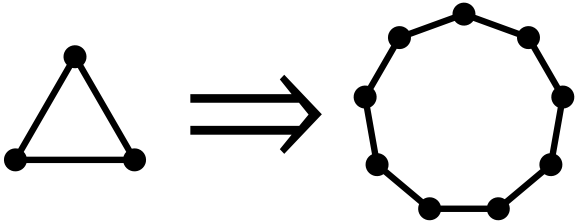


Edges of graph:

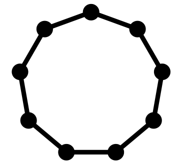
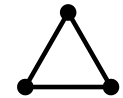
$$t(a,b,b,c,c,d,d,e,e,f,f,g,g,h,h,i,i,a)$$
$$t(b,a,c,b,d,c,e,d,f,e,g,f,h,g,i,h,a,i)$$

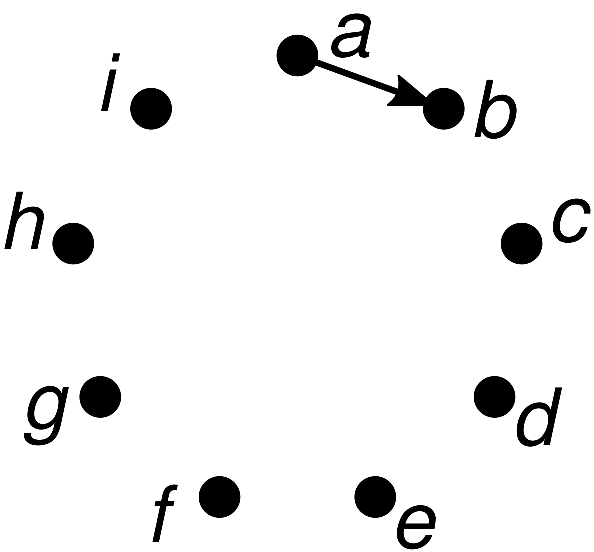
for all terms

- $s(a,b,e,f,c,h)$
- $s(b,a,d,g,b,g)$



$$s(x,y,y,z,z,x) = s(y,x,z,y,x,z)$$

Free algebra over  modulo 



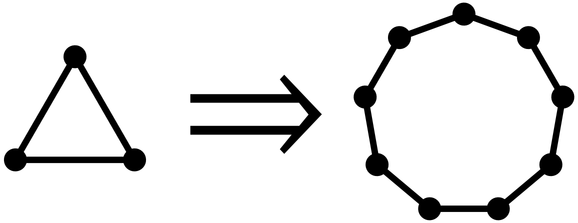
Edges of graph:

$t(a,b,b,c,c,d,d,e,e,f,f,g,g,h,h,i,i,a)$

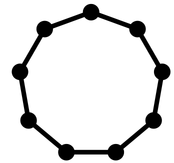
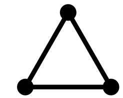
$t(b,a,c,b,d,c,e,d,f,e,g,f,h,g,i,h,a,i)$

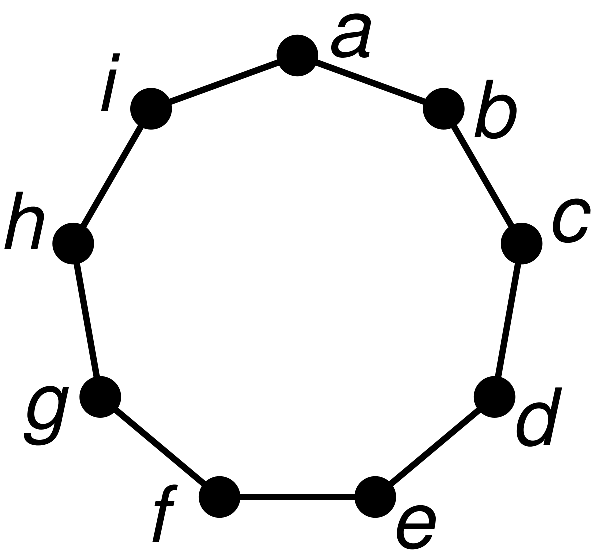
for all terms

- $s(a,b,e,f,c,h)$
- $s(b,a,d,g,b,g)$



$$s(x, y, y, z, z, x)$$
$$= s(y, x, z, y, x, z)$$

Free algebra over  modulo 

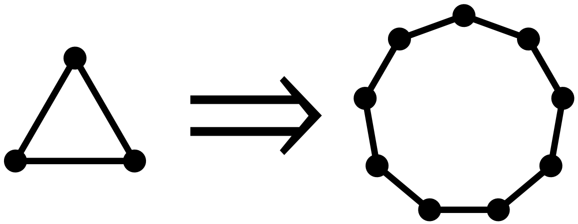


Edges of graph:

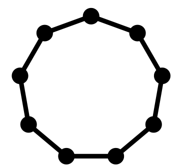
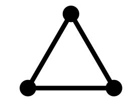
$$t(a, b, b, c, c, d, d, e, e, f, f, g, g, h, h, i, i, a)$$
$$t(b, a, c, b, d, c, e, d, f, e, g, f, h, g, i, h, a, i)$$

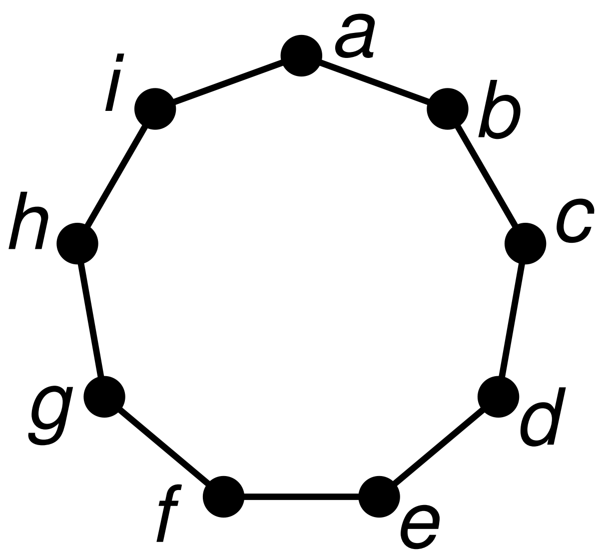
for all terms

- $s(a, b, e, f, c, h)$
- $s(b, a, d, g, b, g)$



$$s(x,y,y,z,z,x)$$
$$= s(y,x,z,y,x,z)$$

Free algebra over  modulo 

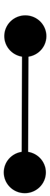


Edges of graph:

$t(a,b,b,c,c,d,d,e,e,f,f,g,g,h,h,i,i,a)$

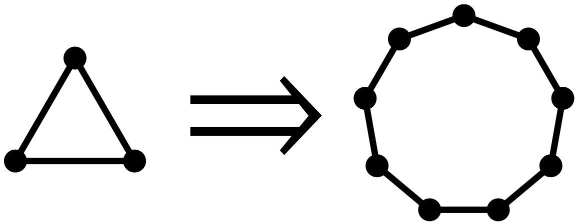
$t(b,a,c,b,d,c,e,d,f,e,g,f,h,g,i,h,a,i)$

for all terms

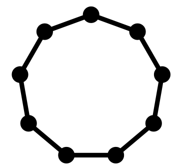
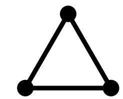


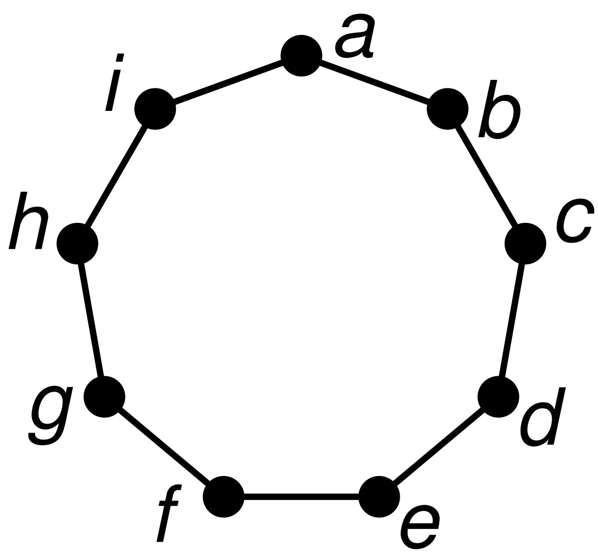
$s(a,b,e,f,c,h)$

$s(b,a,d,g,b,g)$



$$s(x, y, y, z, z, x)$$
$$= s(y, x, z, y, x, z)$$

Free algebra over  modulo 

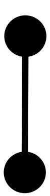


Edges of graph:

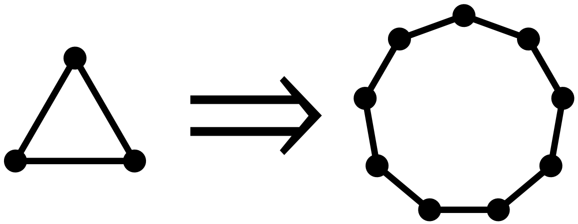
$$t(a, b, b, c, c, d, d, e, e, f, f, g, g, h, h, i, i, a)$$
$$t(b, a, c, b, d, c, e, d, f, e, g, f, h, g, i, h, a, i)$$

for all terms

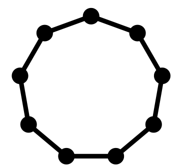
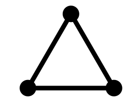
It is compatible

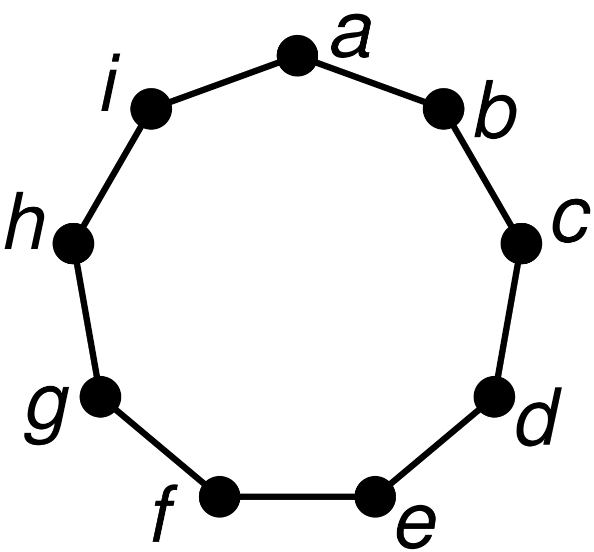


$$s(a, b, e, f, c, h)$$
$$s(b, a, d, g, b, g)$$



$$s(x, y, y, z, z, x)$$
$$= s(y, x, z, y, x, z)$$

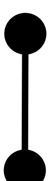
Free algebra over  modulo 



Edges of graph:

$$t(a, b, b, c, c, d, d, e, e, f, f, g, g, h, h, i, i, a)$$
$$t(b, a, c, b, d, c, e, d, f, e, g, f, h, g, i, h, a, i)$$

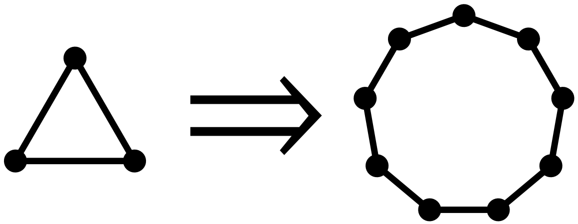
for all terms

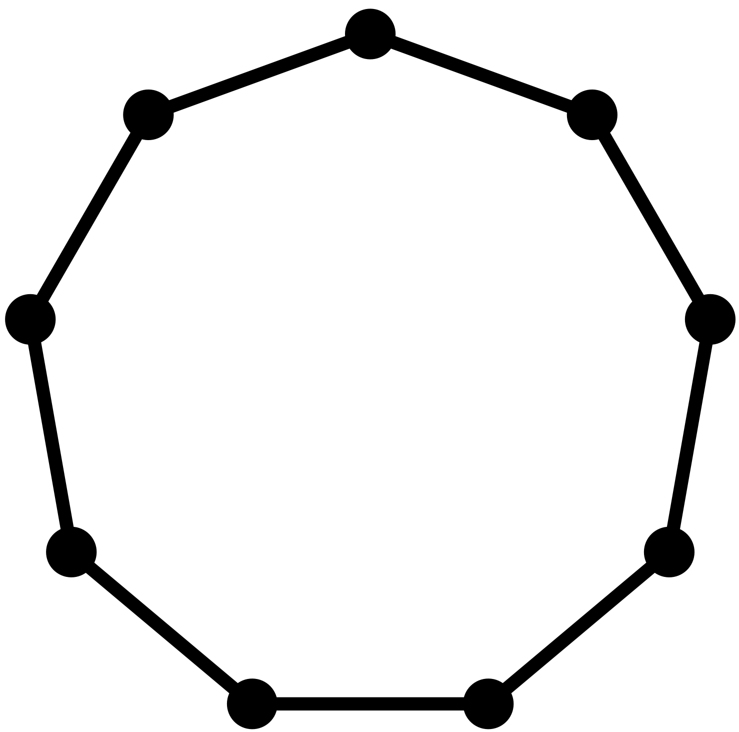


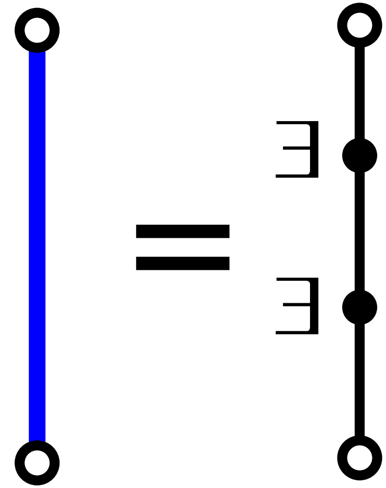
$$s(a, b, e, f, c, h)$$
$$s(b, a, d, g, b, g)$$

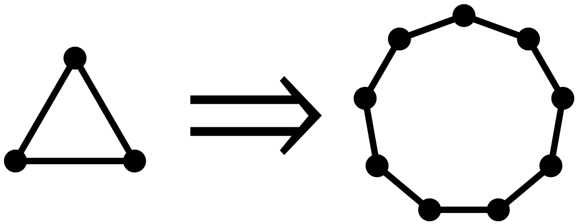
It is compatible

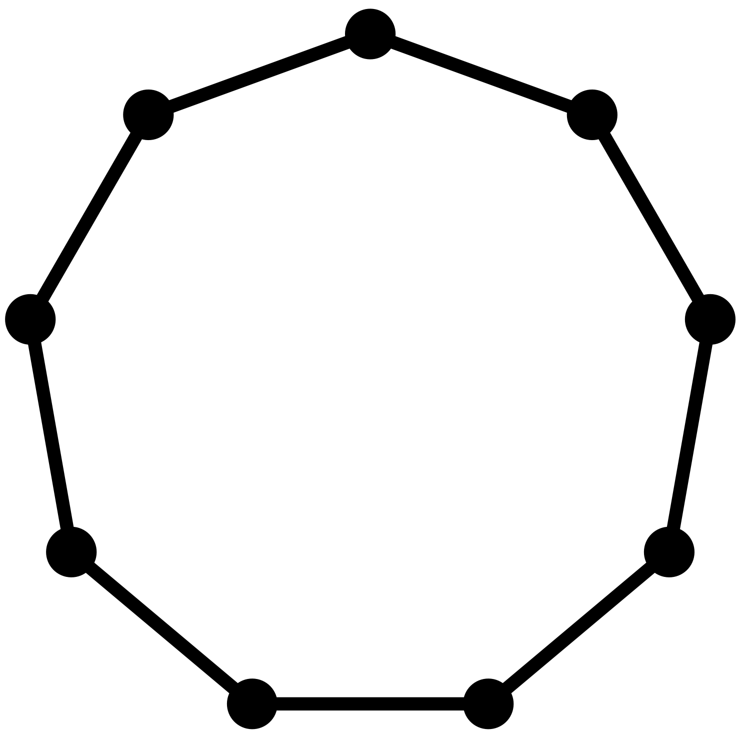
We want a loop!

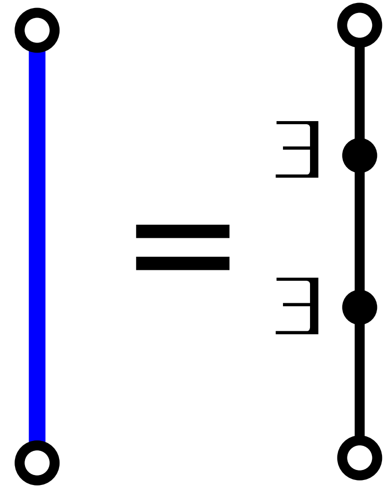


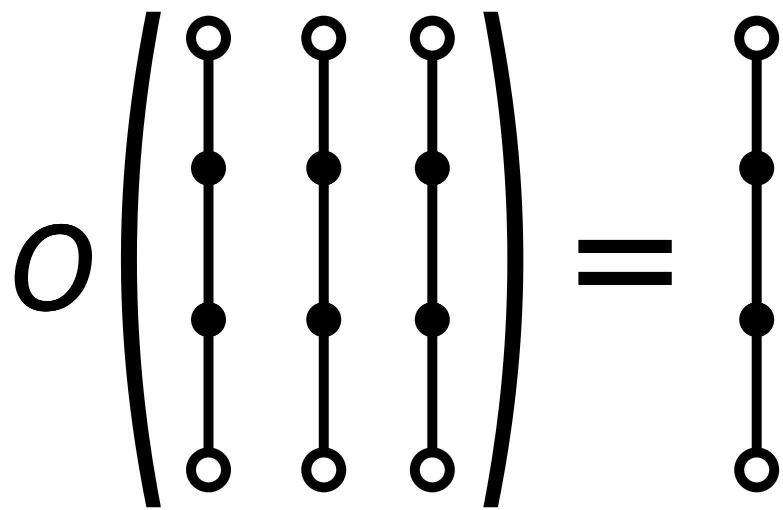


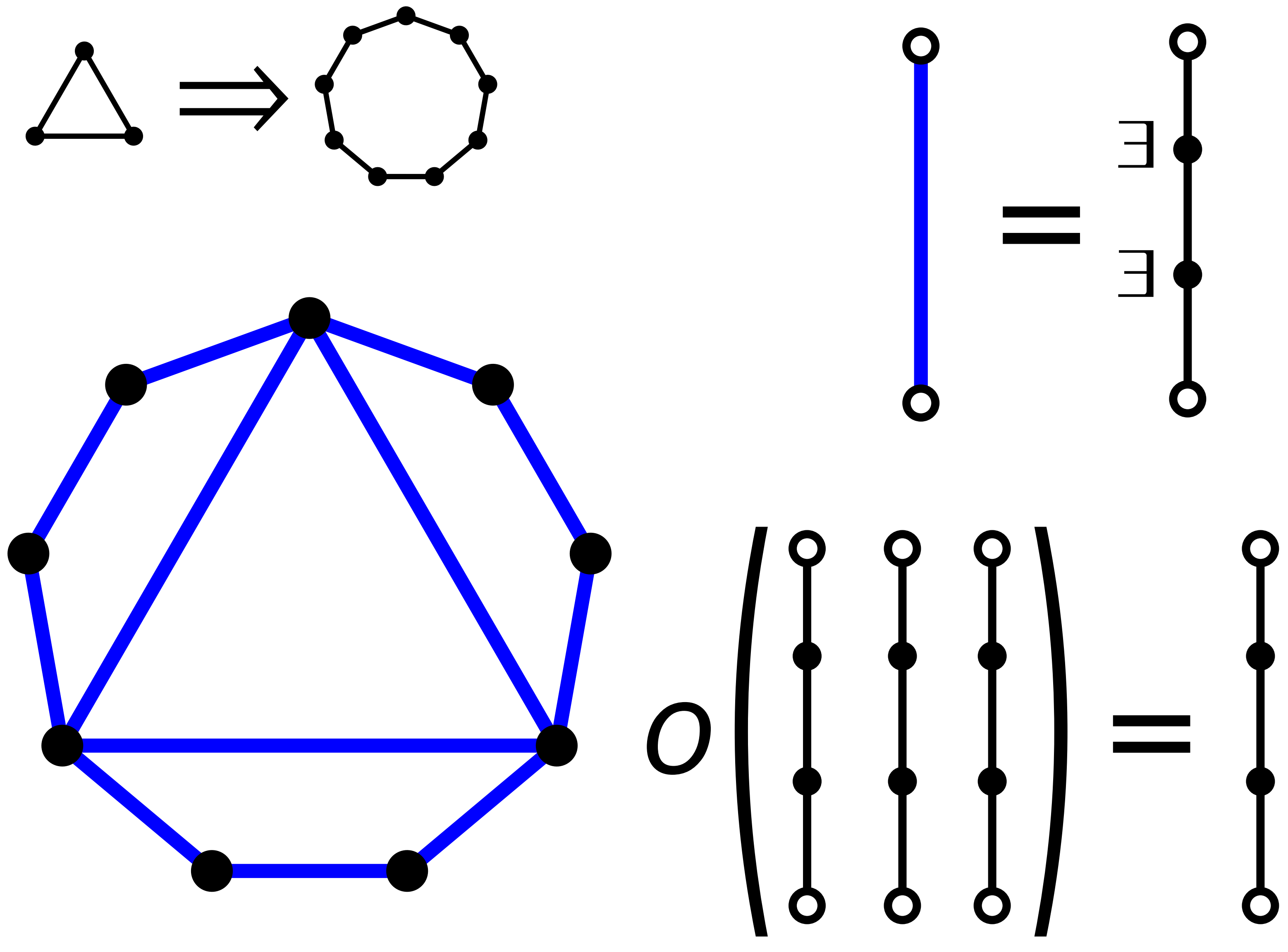


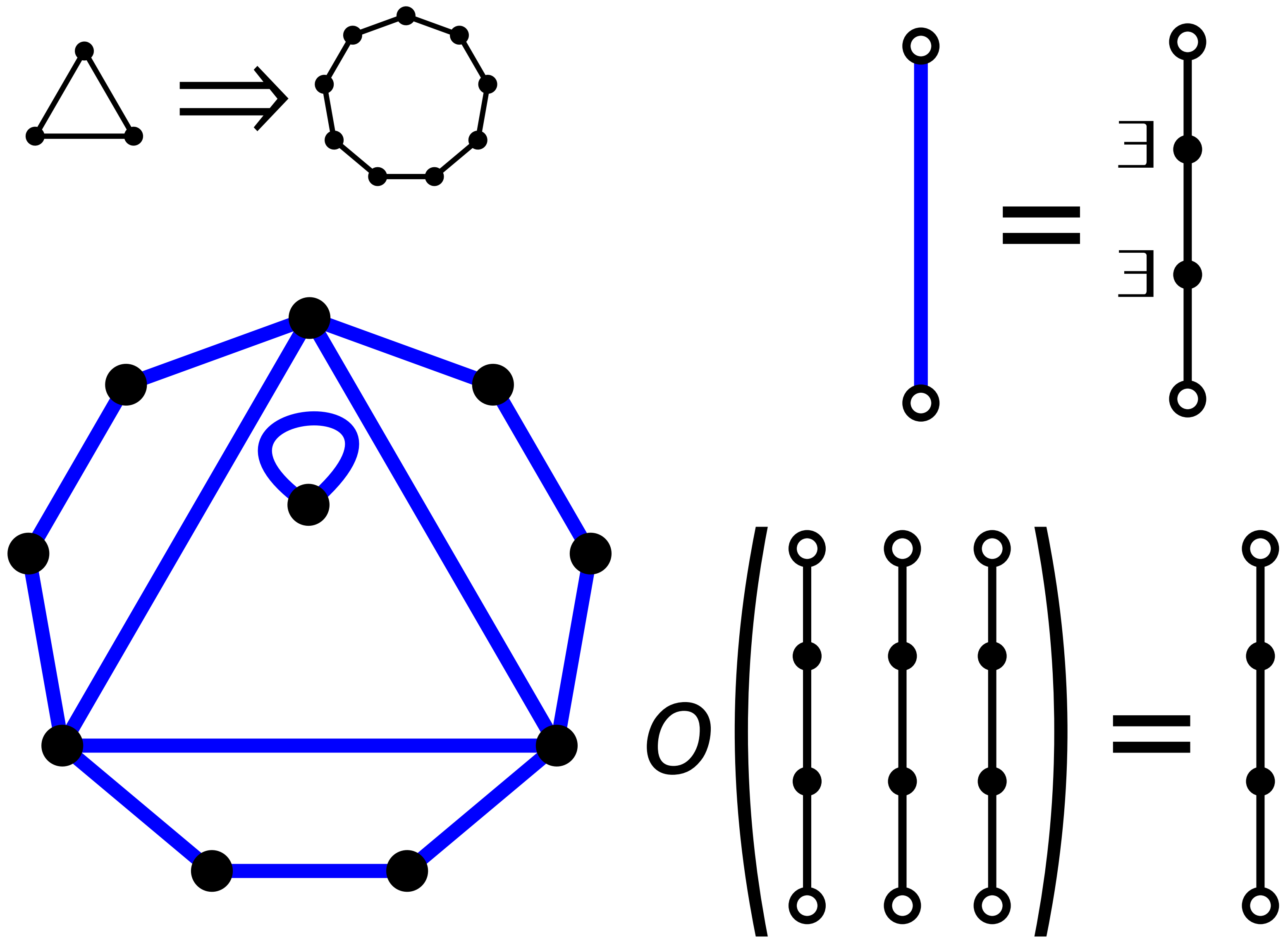


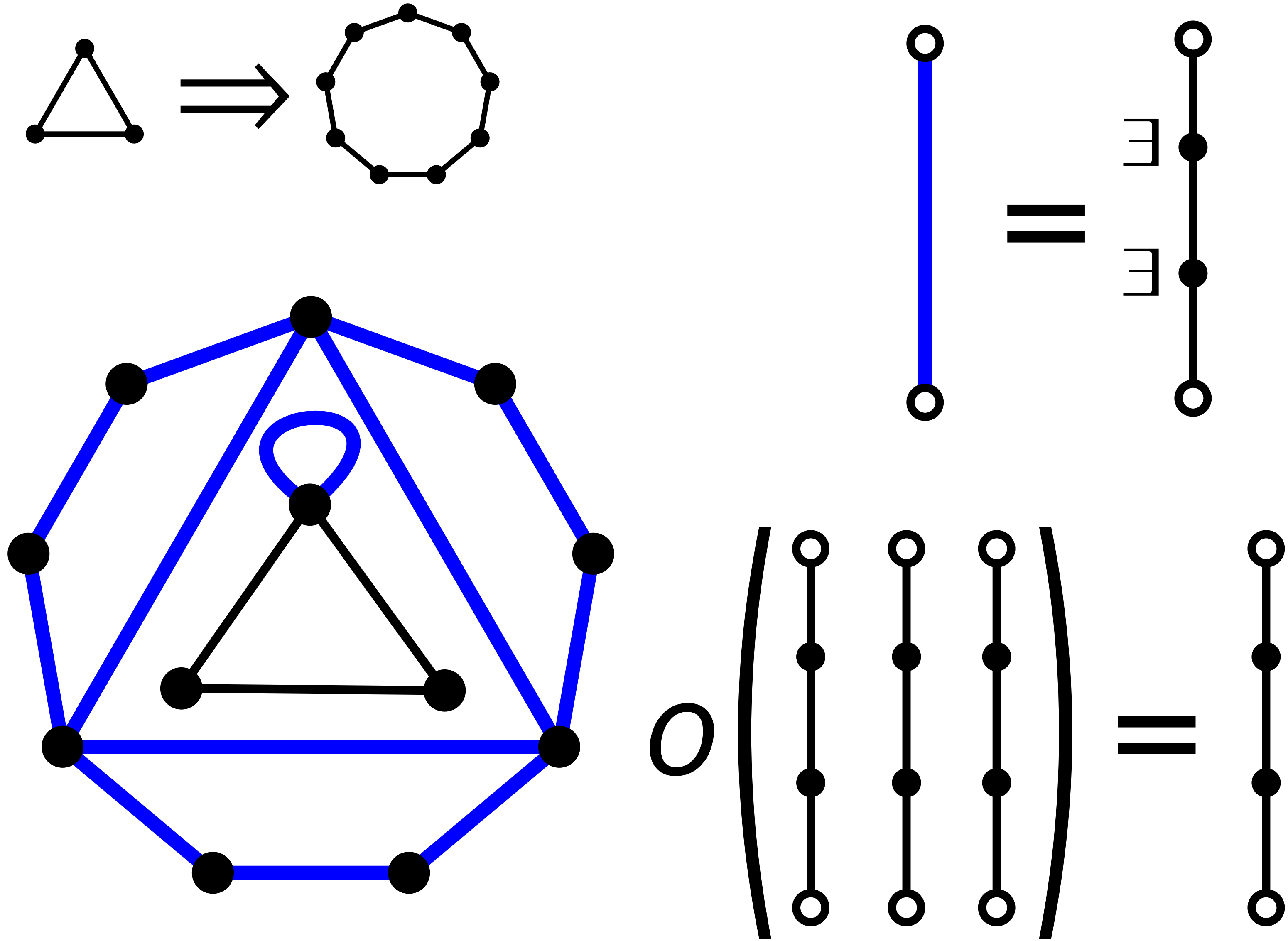


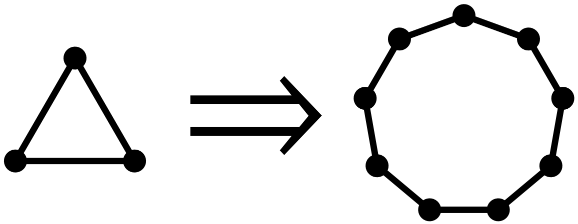

=
∃
∃

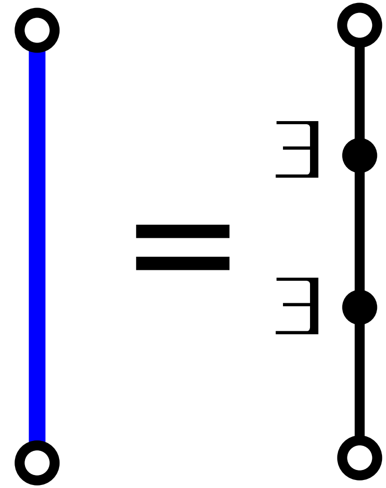

O
=

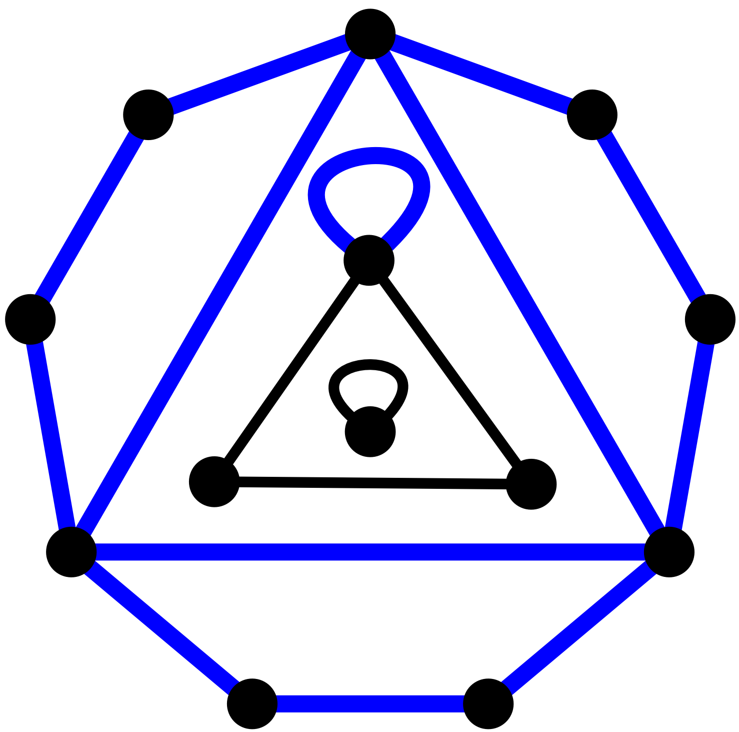

∃
∃
O

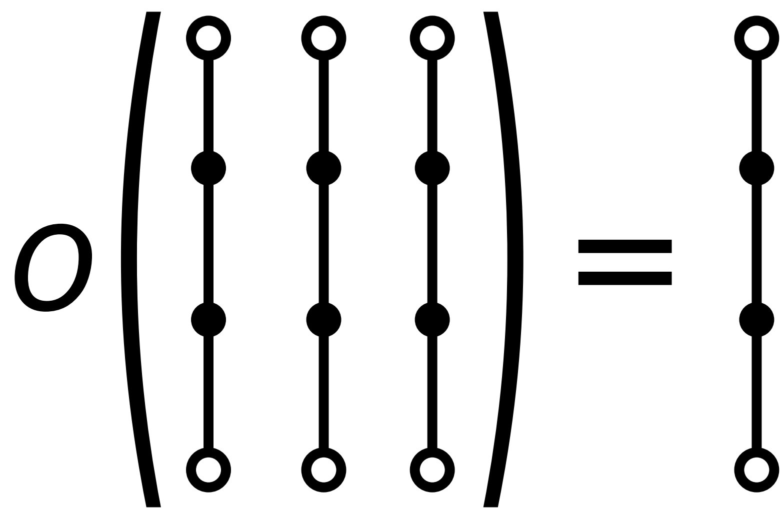

∃
∃
O

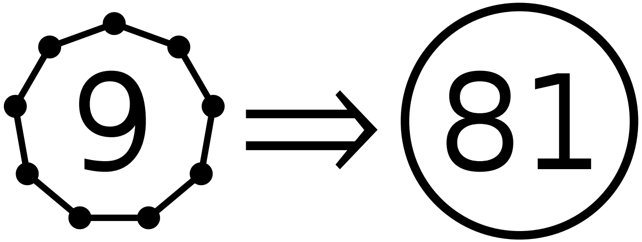

∃
∃
O

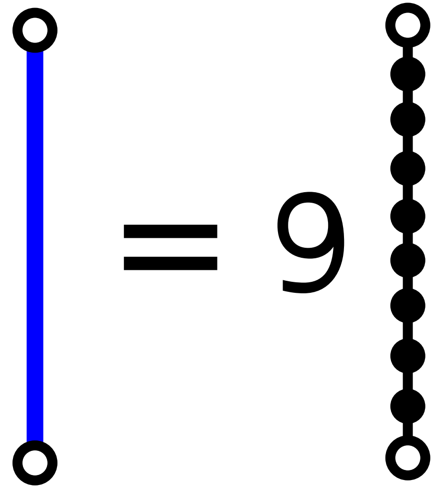



=
∃
∃

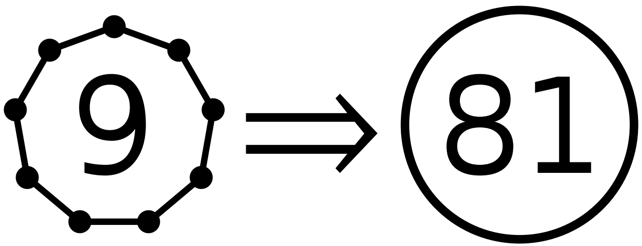


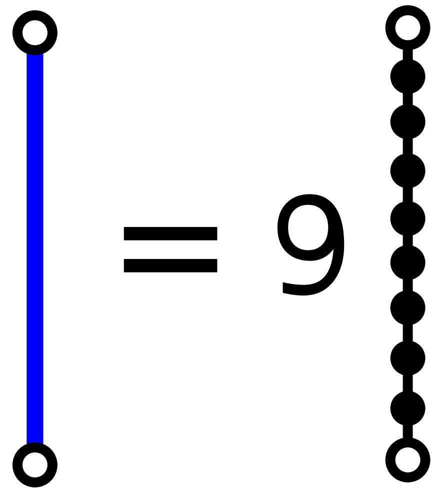

o
=


9
81

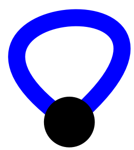

= 9

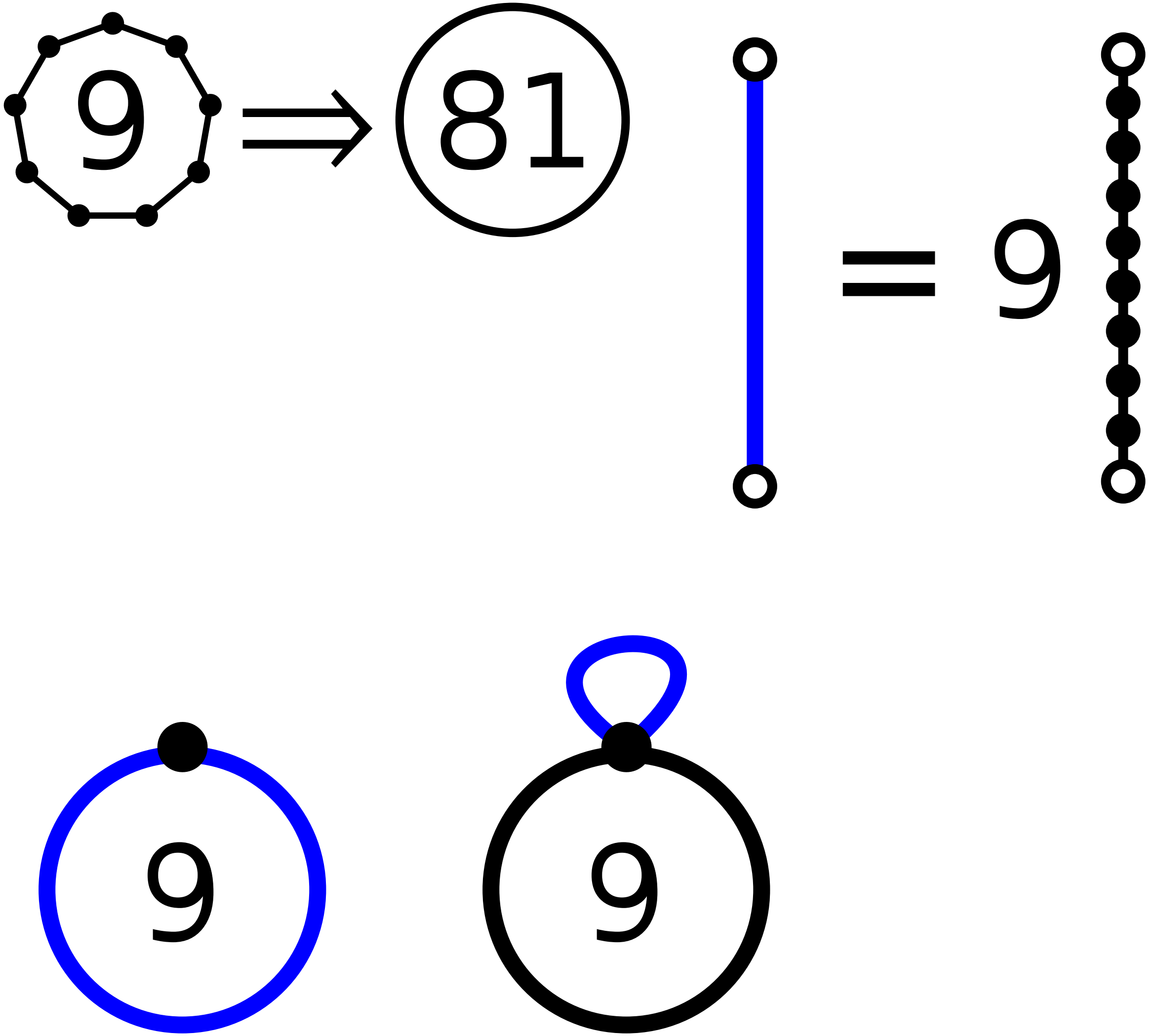

9

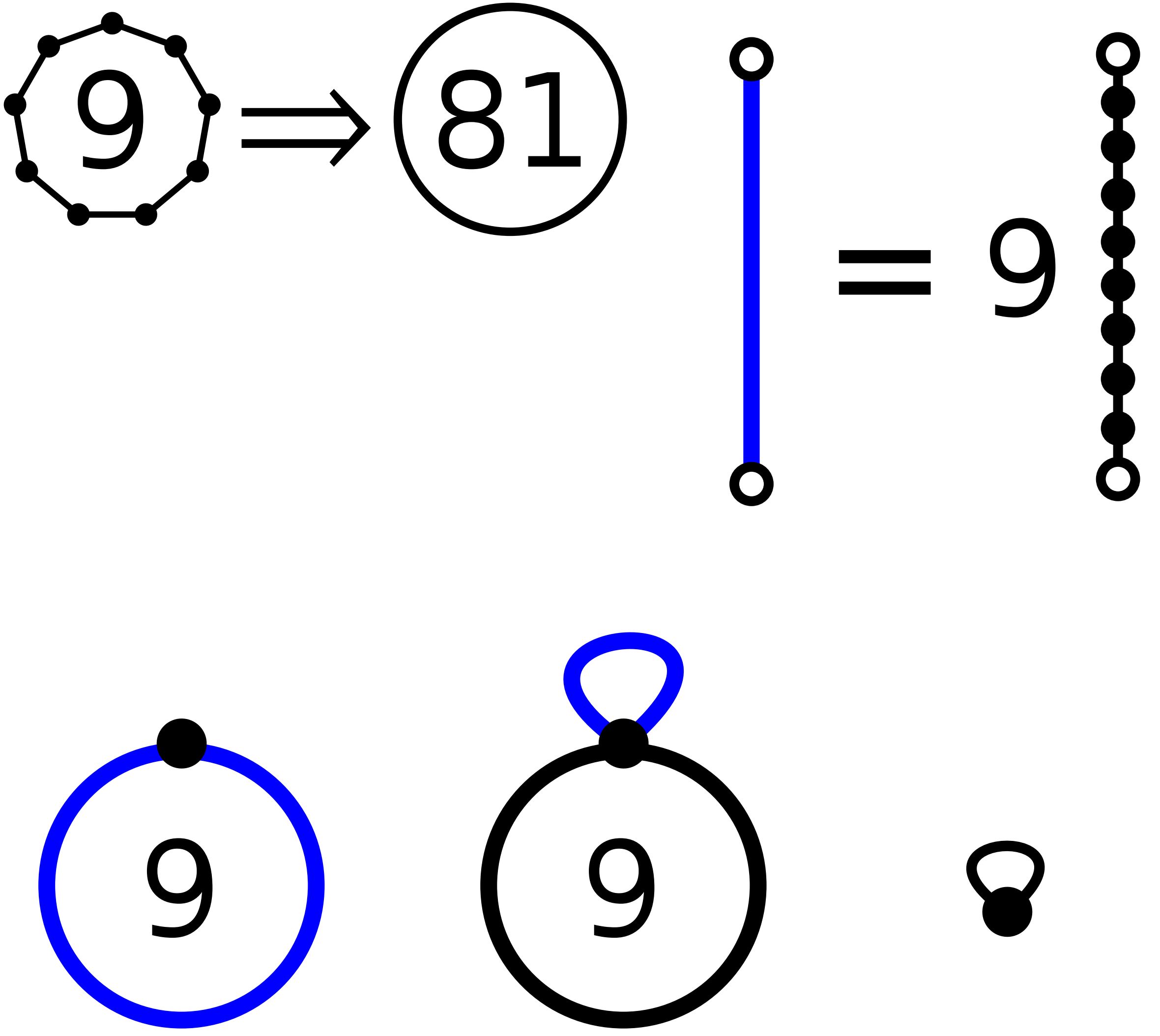

9
⇒
81

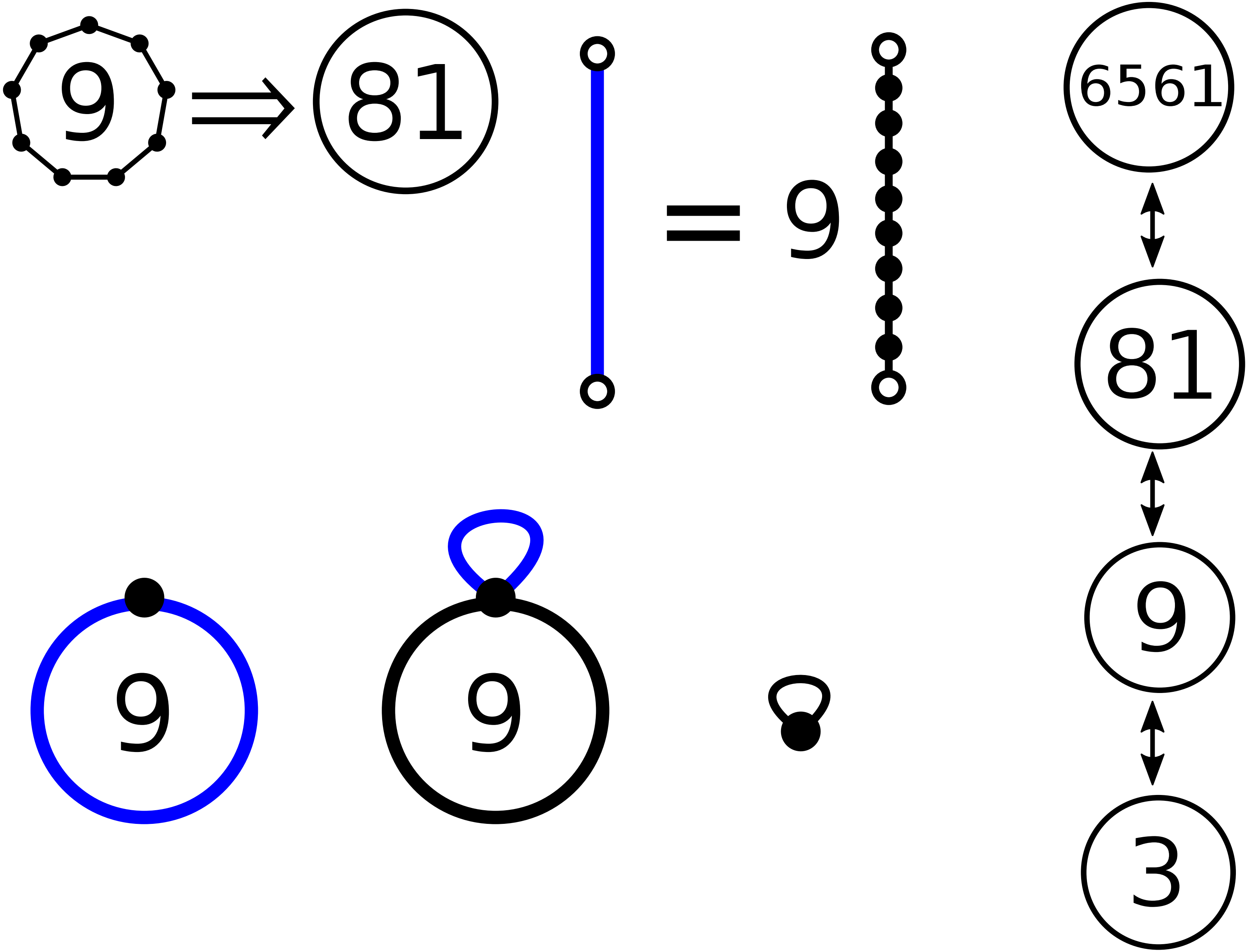

= 9


9

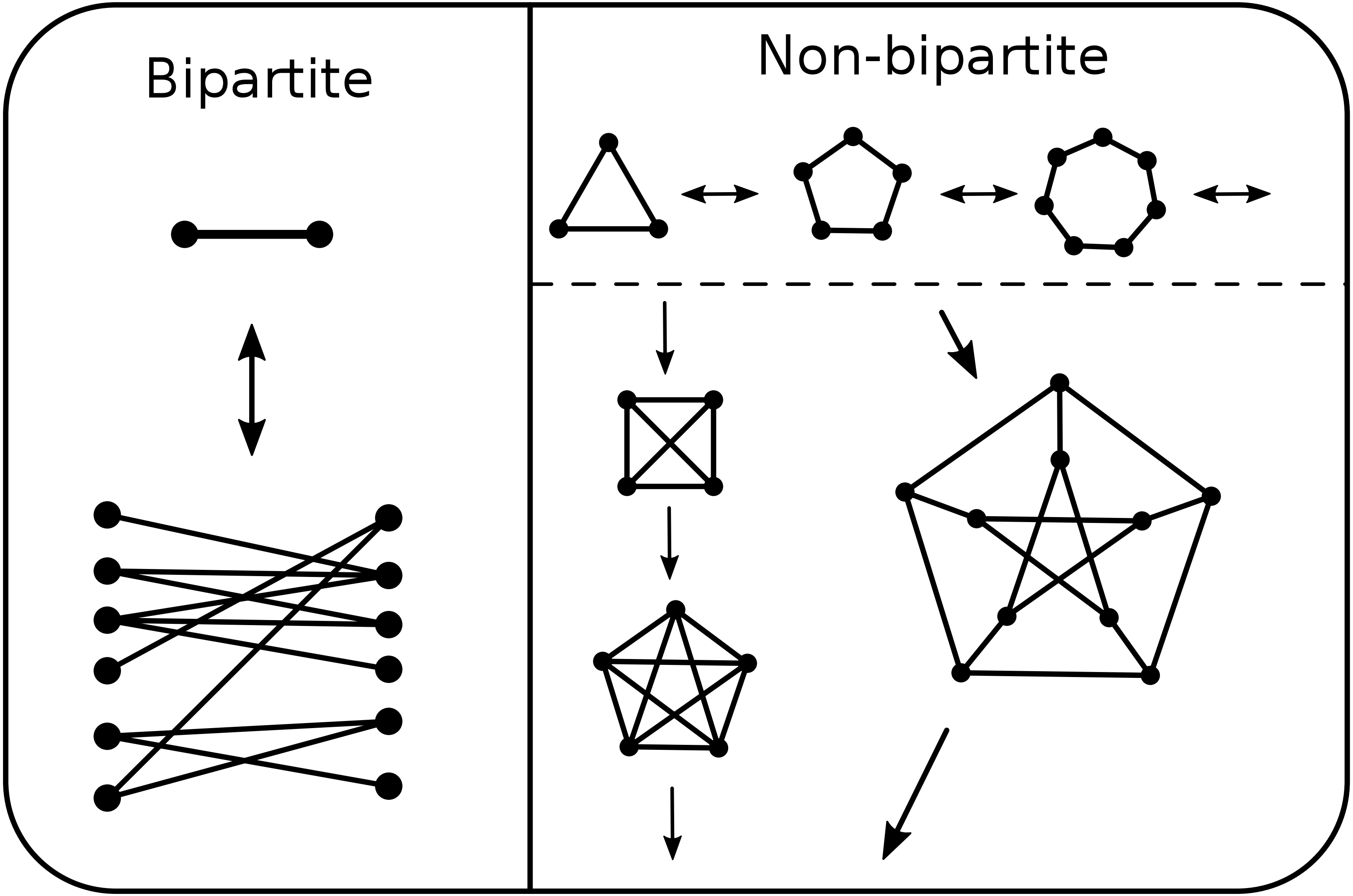


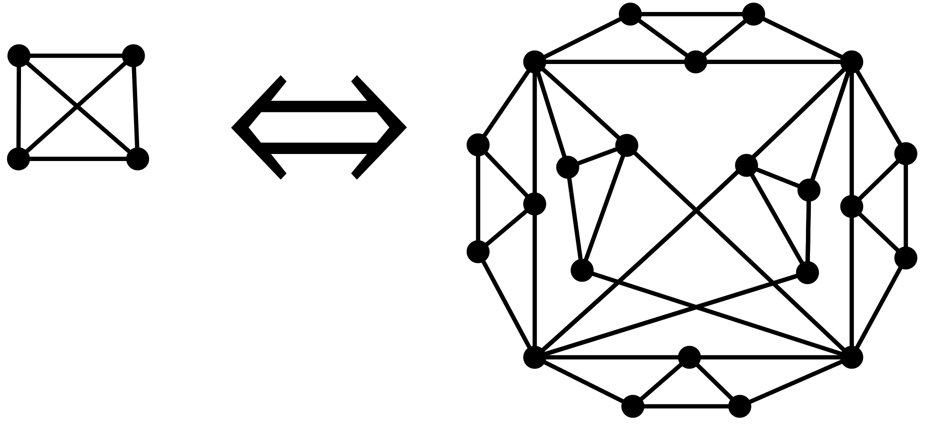

9
⇒
81
= 9
9
9

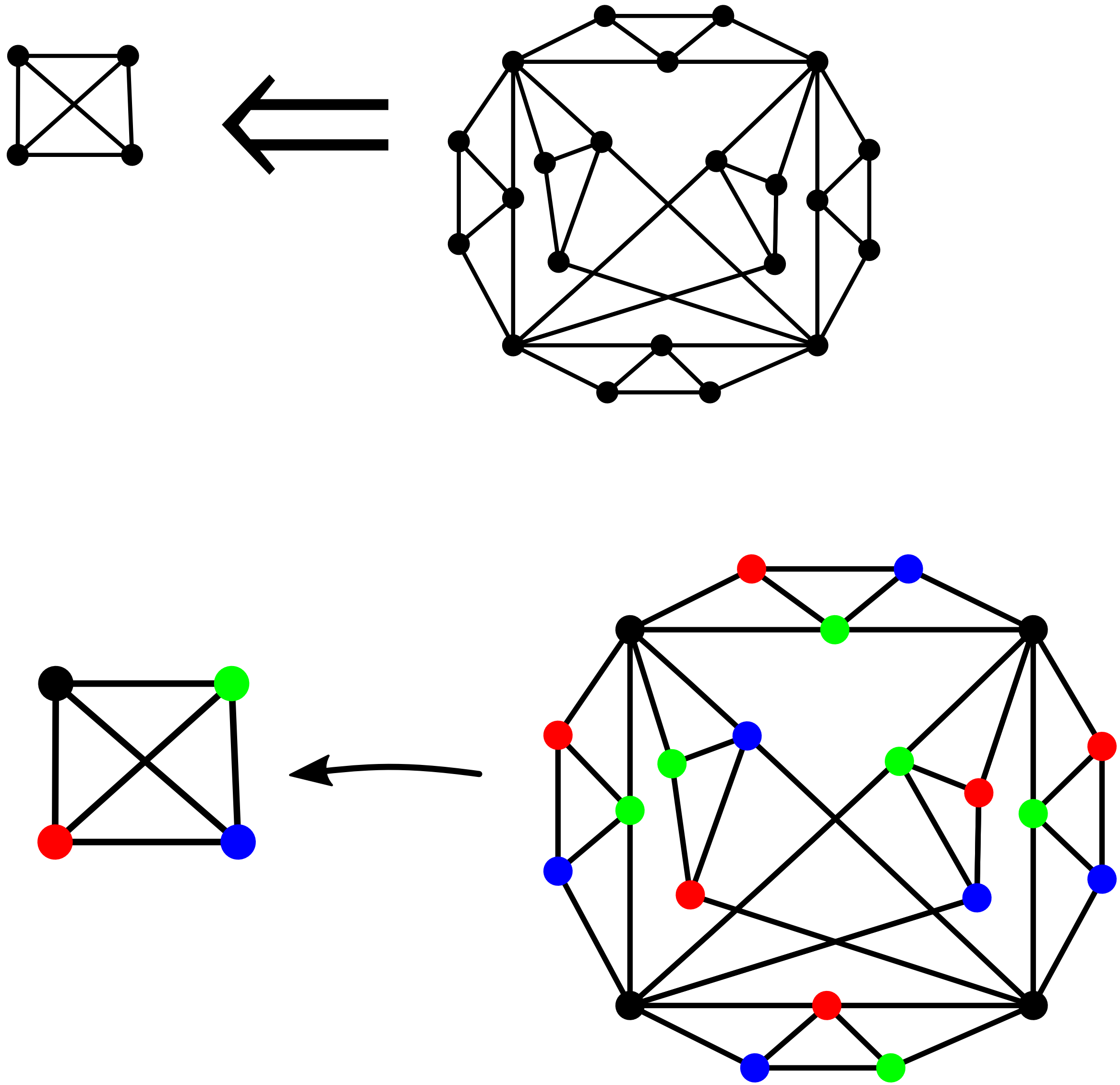

9
⇒
81
= 9
9
9

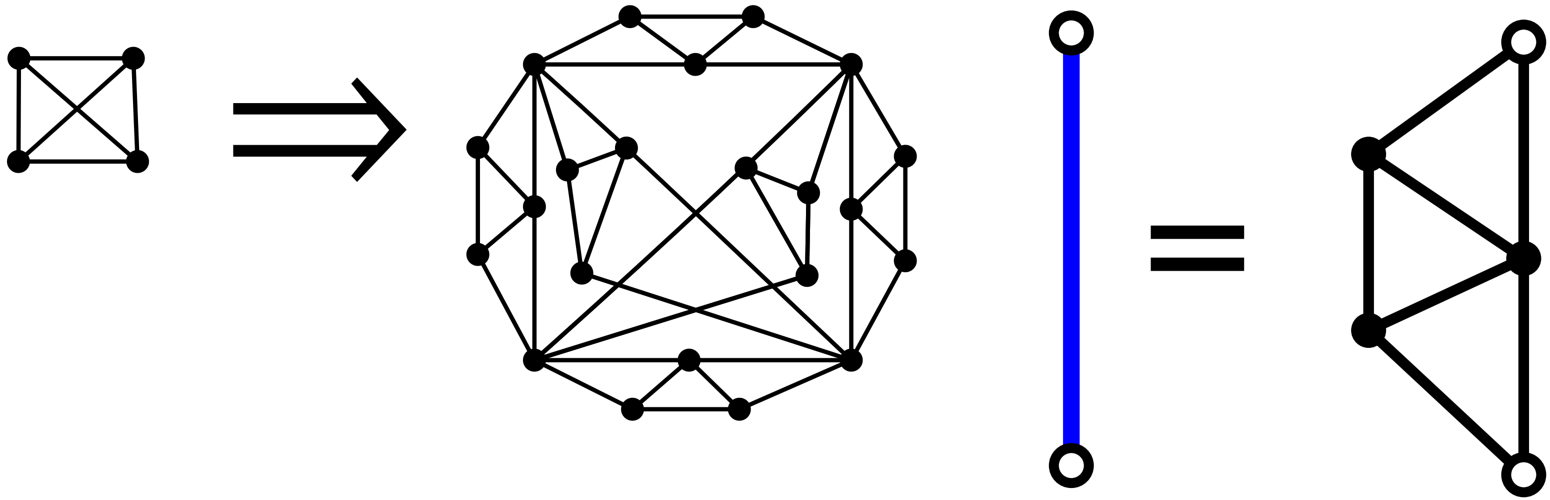

9
⇒
81
= 9
6561
81
9
3
9
9

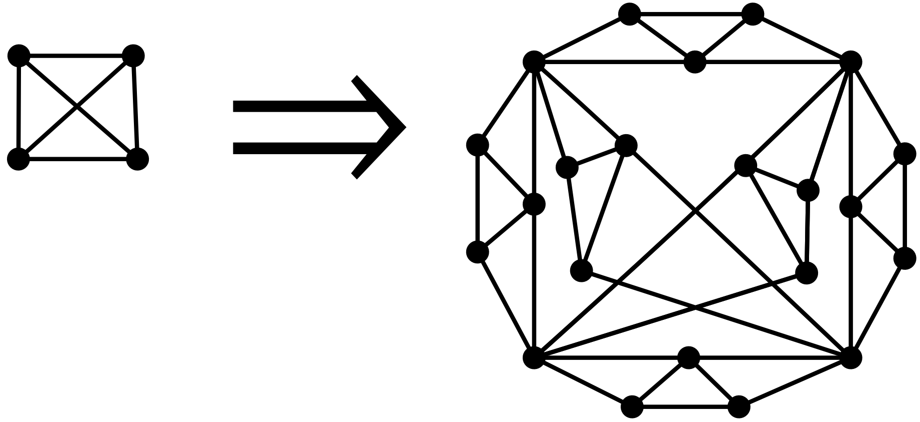
Non-trivial Undirect Loop Conditions

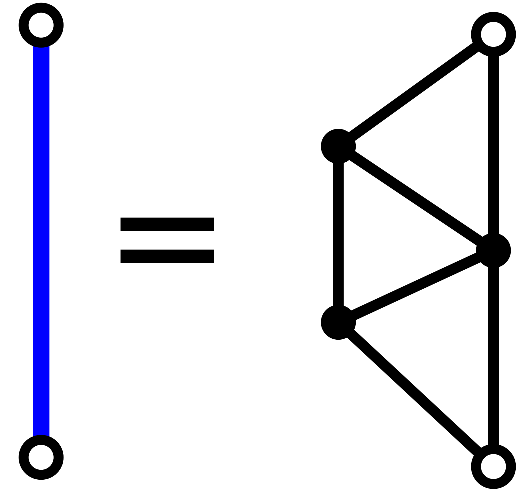


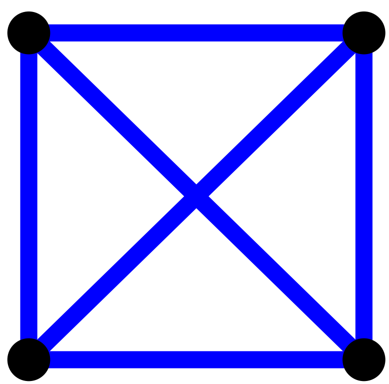


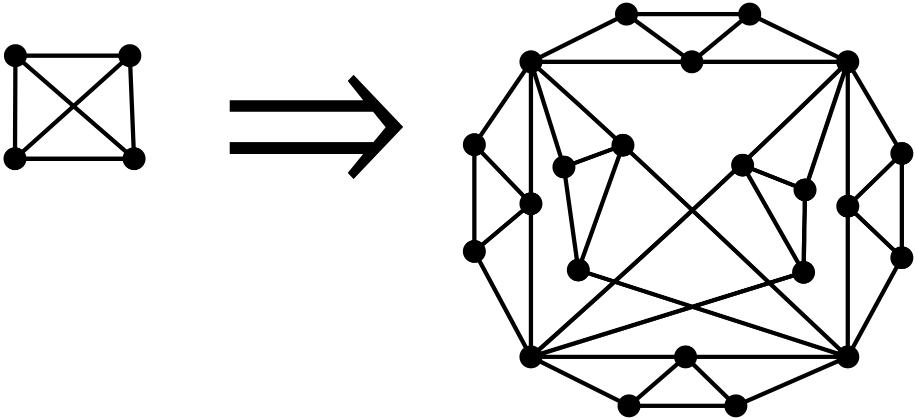


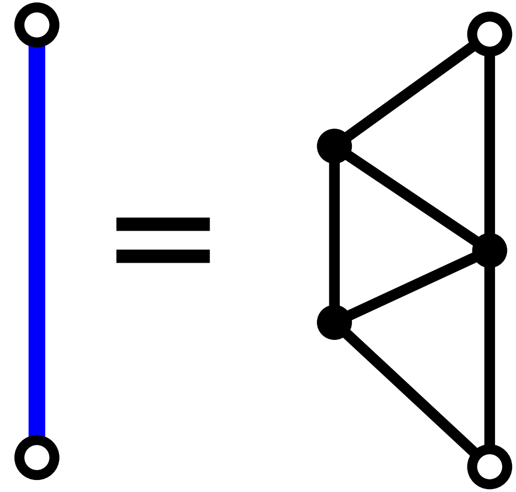


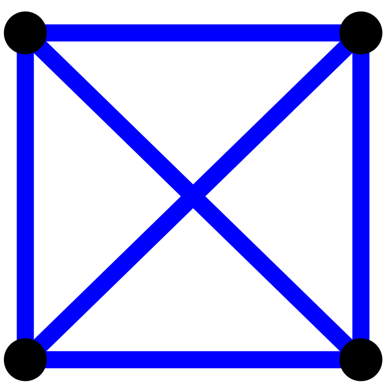


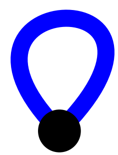


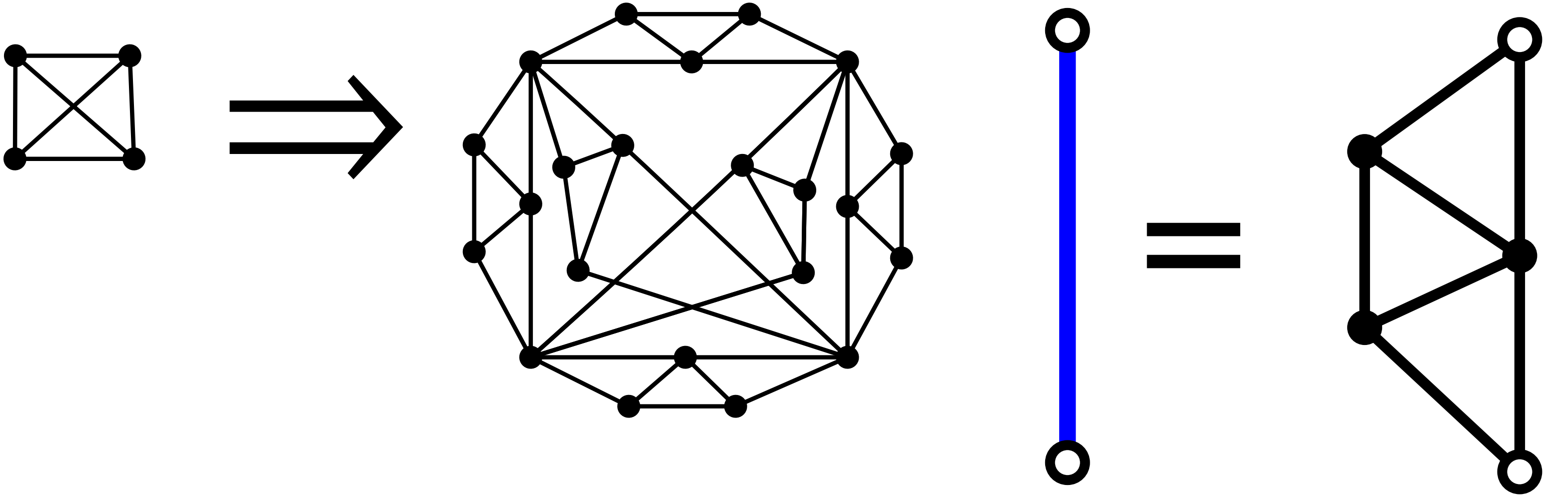


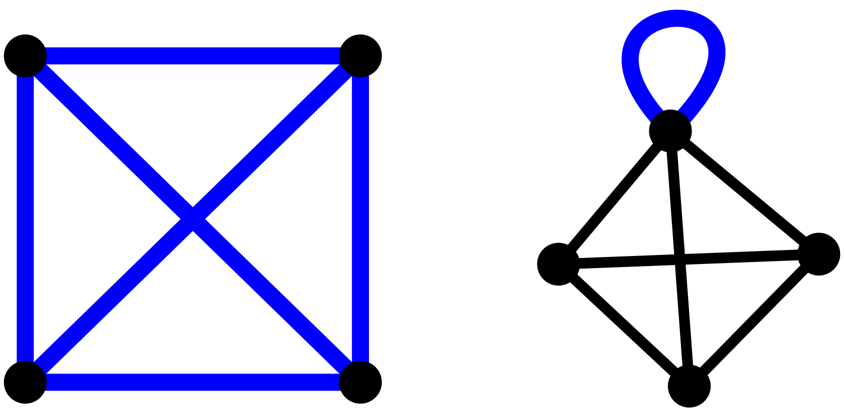


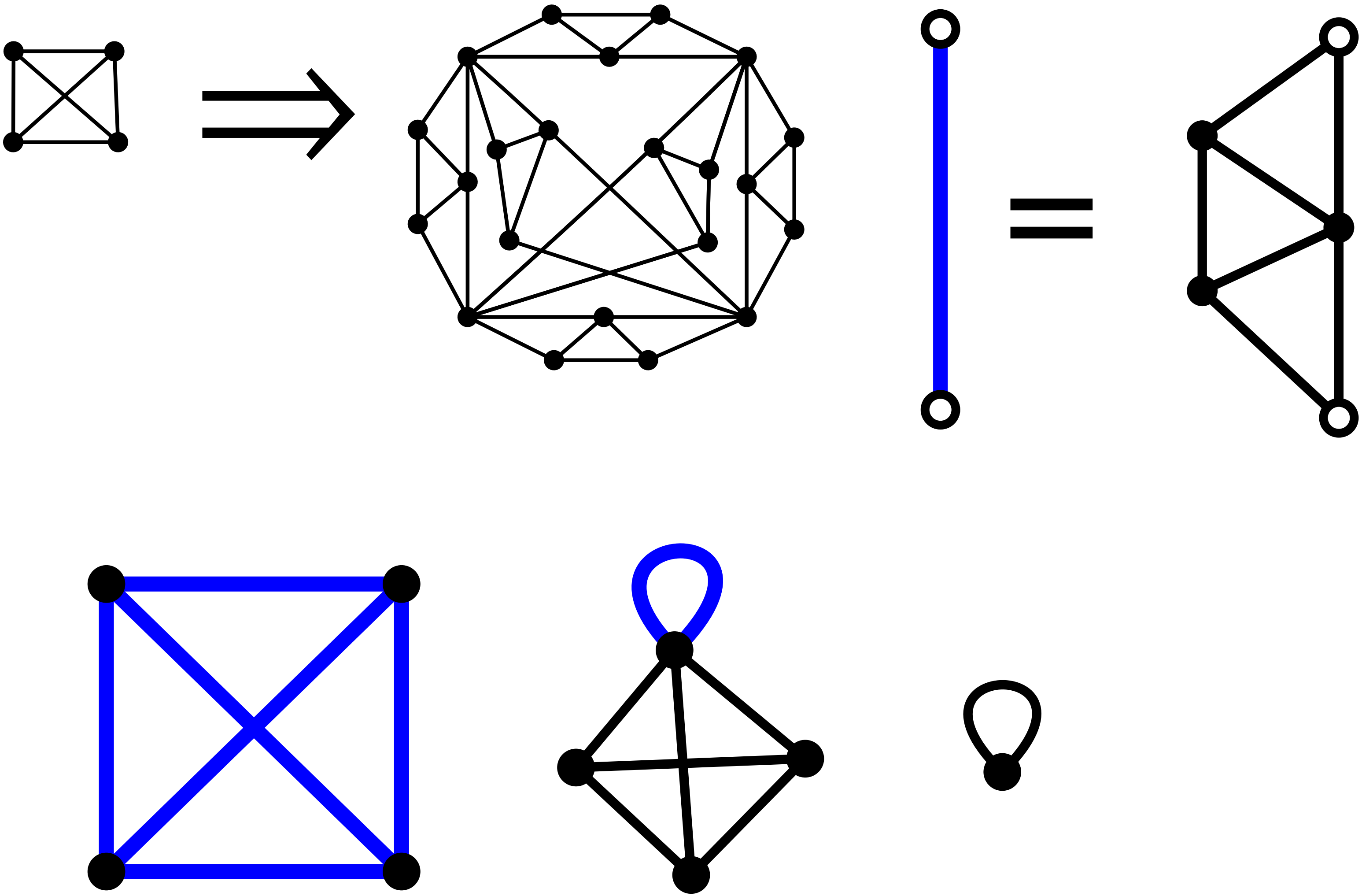


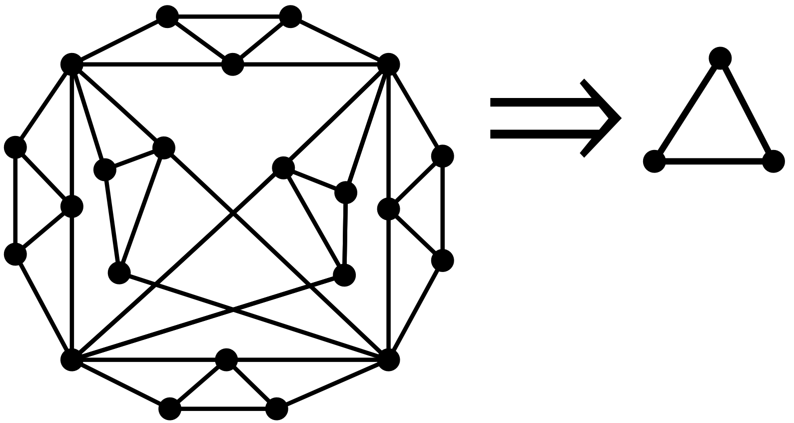




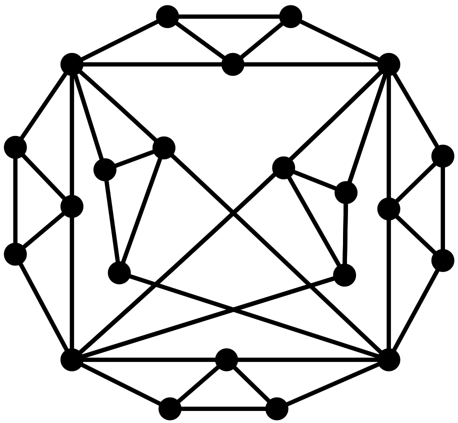
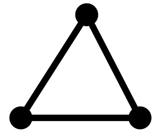






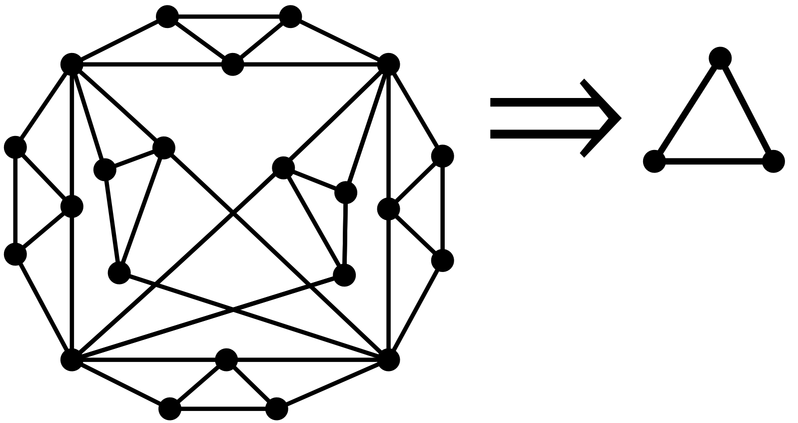


Homomorphism?

 $\Rightarrow$ 

Homomorphism?

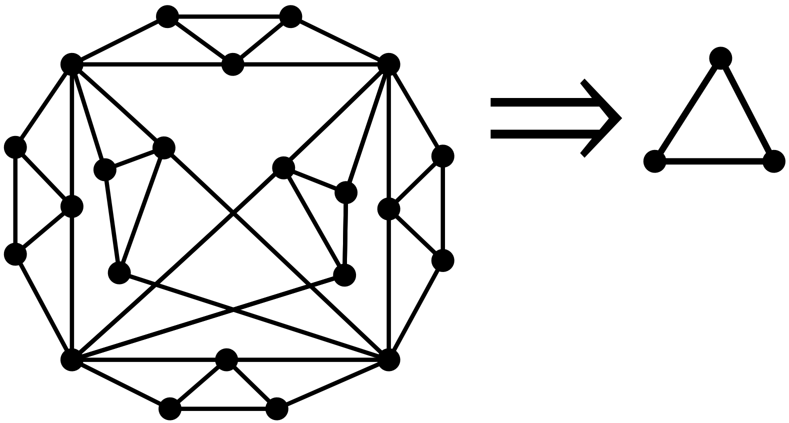
$\Leftrightarrow$ 3-colorable?



Homomorphism?

$\Leftrightarrow$ 3-colorable?

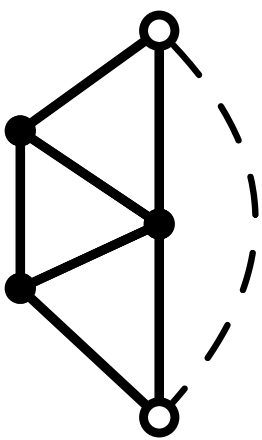
No!

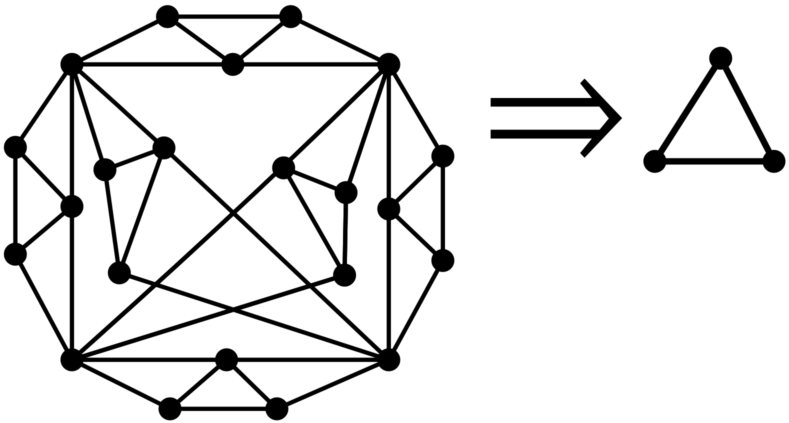


Homomorphism?

$\Leftrightarrow$ 3-colorable?

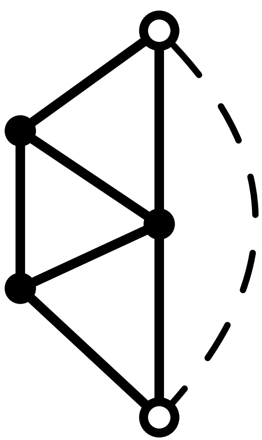
No!

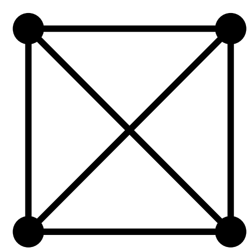


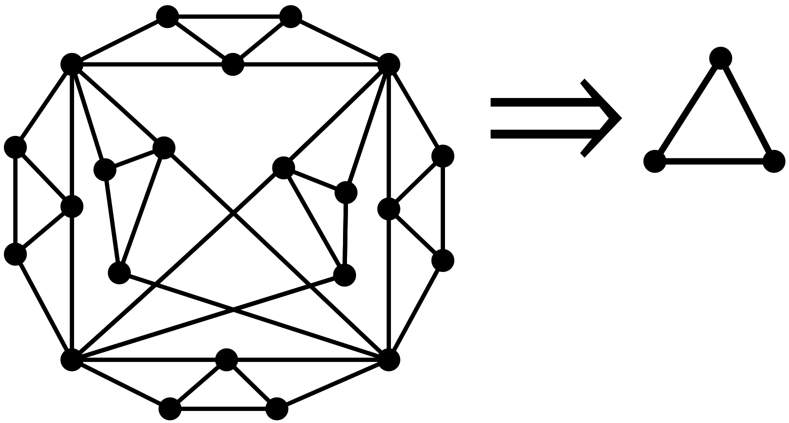


Homomorphism? $\Leftrightarrow$ 3-colorable?

No!



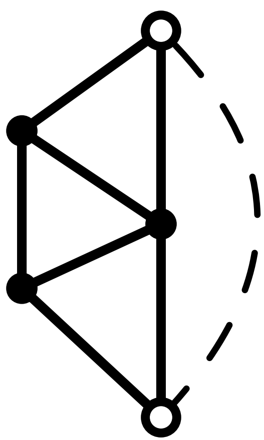


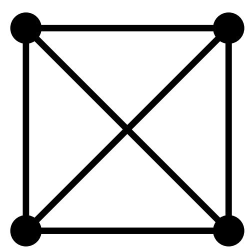


Homomorphism?

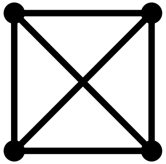
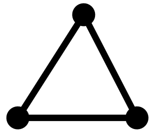
$\Leftrightarrow$ 3-colorable?

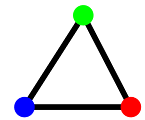
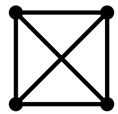
No!

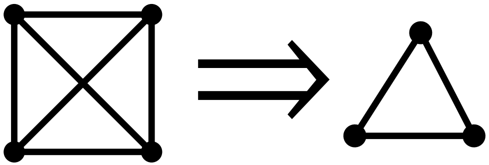


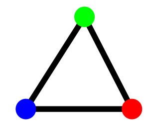
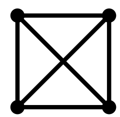


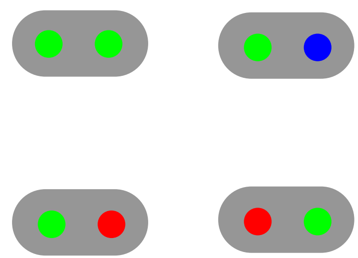
Something more
advanced required...

 $\Rrightarrow$ 

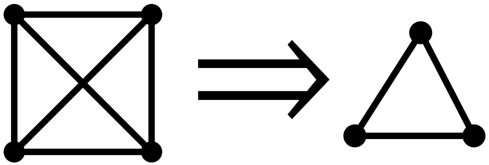
A = Free algebra over 
modulo 

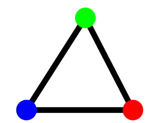
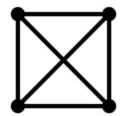


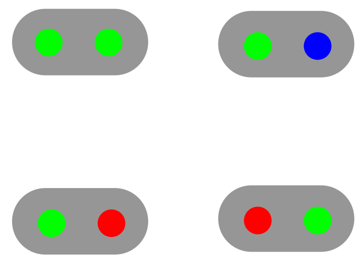
$\mathbf{A}$ = Free algebra over 
modulo 



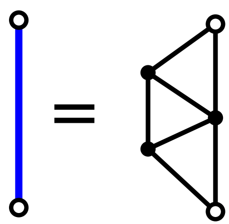
Consider $\mathbf{A}^2$

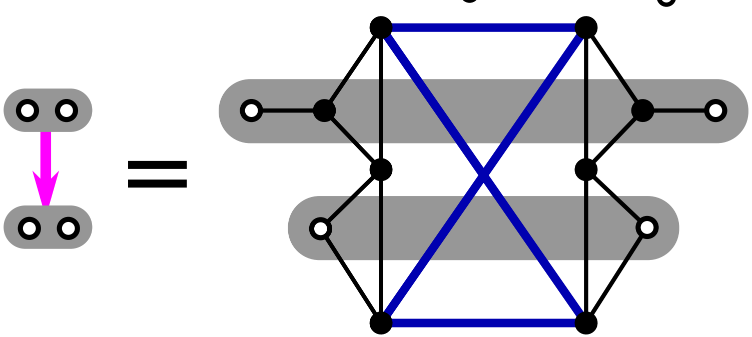


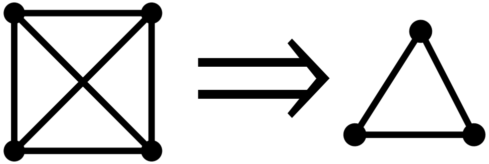
$\mathbf{A}$ = Free algebra over 
modulo 

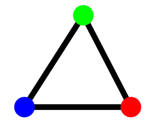


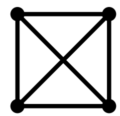
Consider $\mathbf{A}^2$



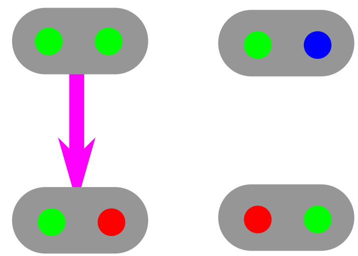


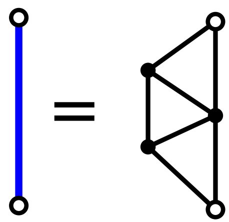


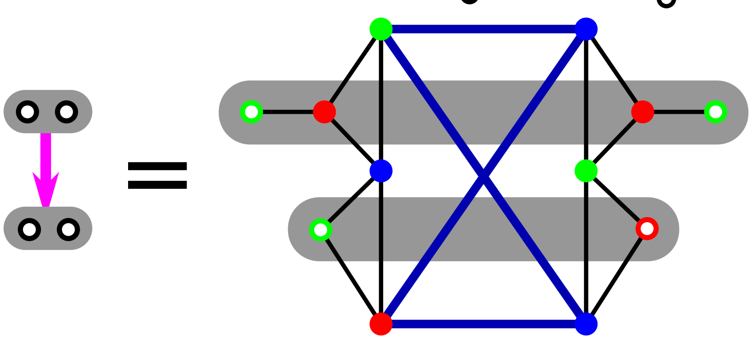
$\mathbf{A}$ = Free algebra over 

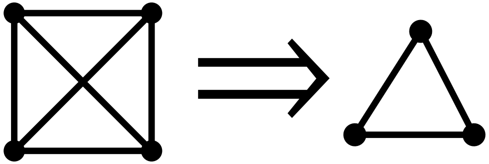
modulo 

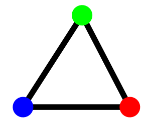
Consider $\mathbf{A}^2$

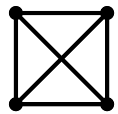


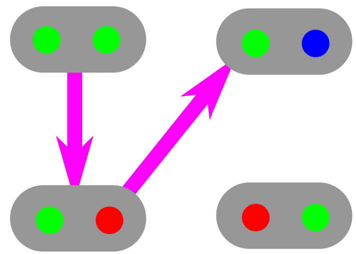




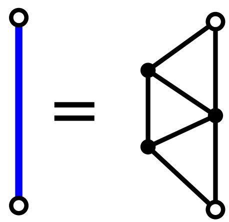


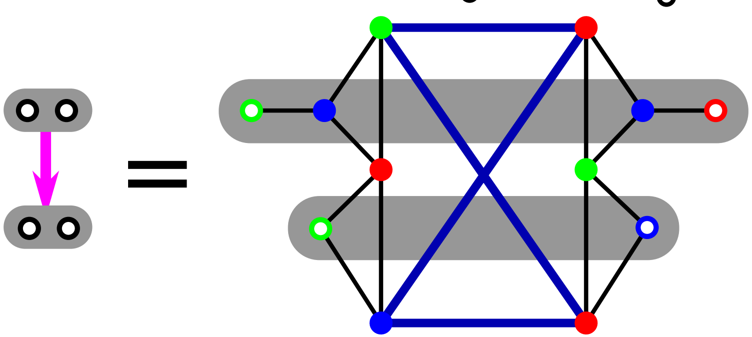
$\mathbf{A}$ = Free algebra over 

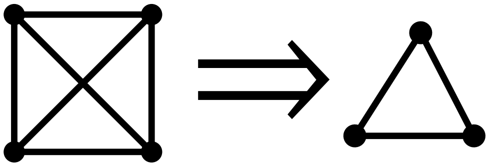
modulo 

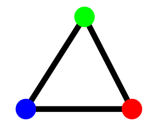


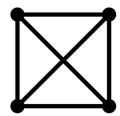
Consider $\mathbf{A}^2$



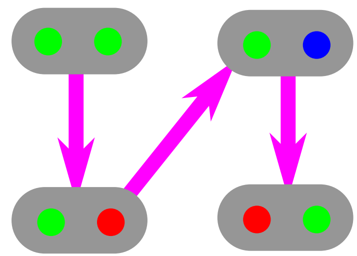


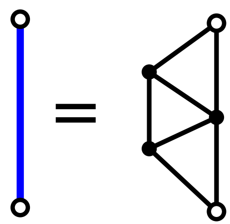


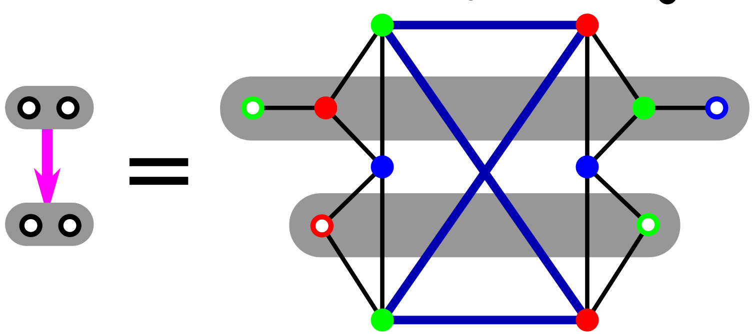
$\mathbf{A}$ = Free algebra over 

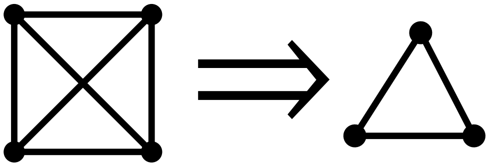
modulo 

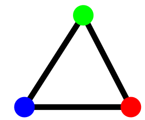
Consider $\mathbf{A}^2$

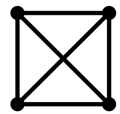




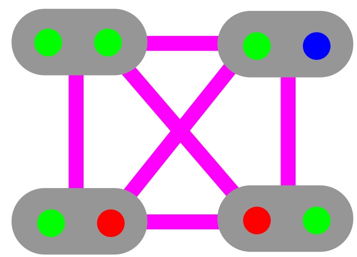


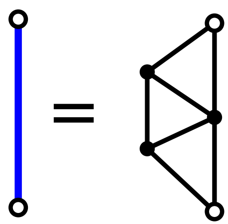


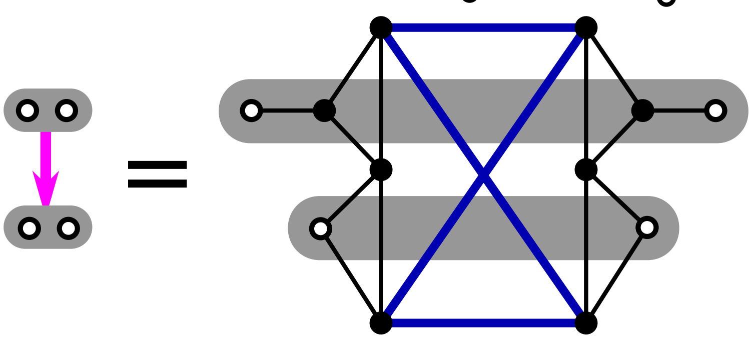
$\mathbf{A}$ = Free algebra over 

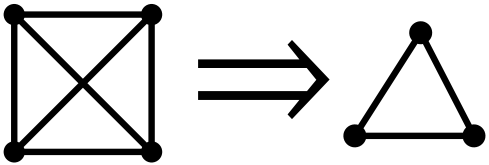
modulo 

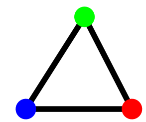
Consider $\mathbf{A}^2$

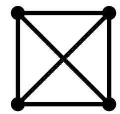


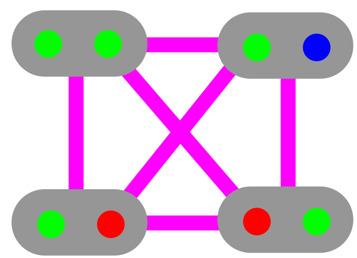




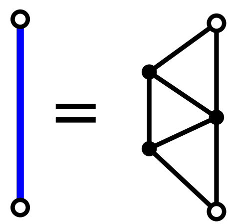


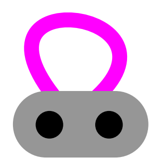
$\mathbf{A}$ = Free algebra over 

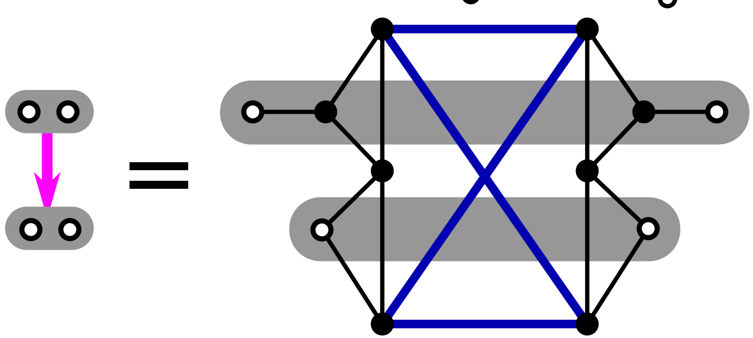
modulo 

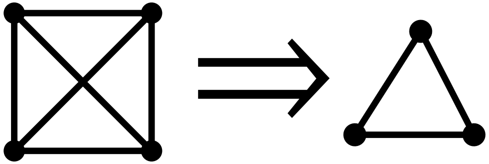


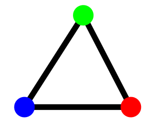
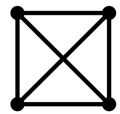
Consider $\mathbf{A}^2$



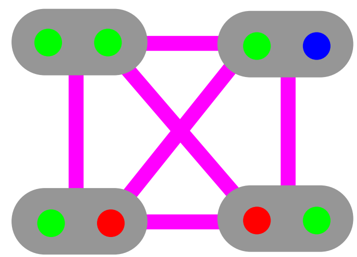


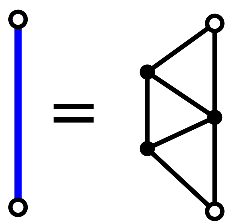


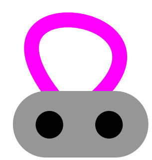


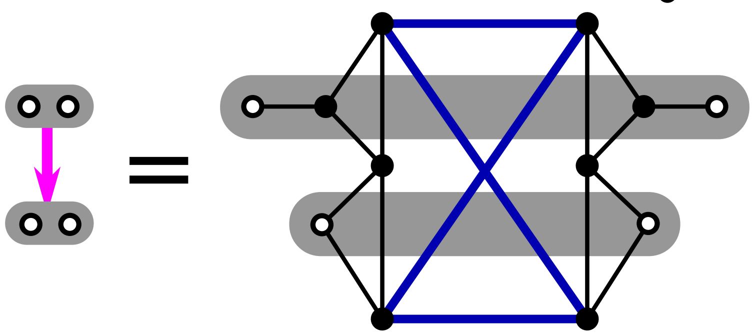
$\mathbf{A}$ = Free algebra over 
modulo 

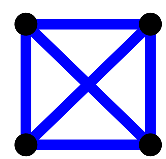
Consider $\mathbf{A}^2$

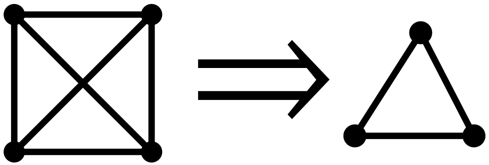


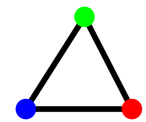
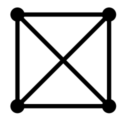


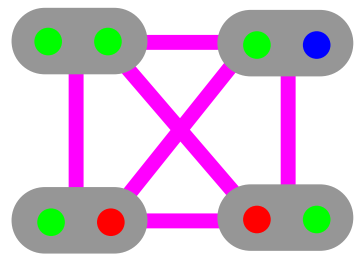




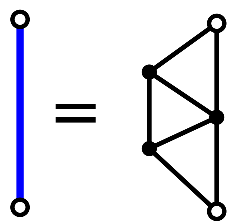


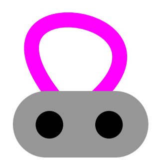


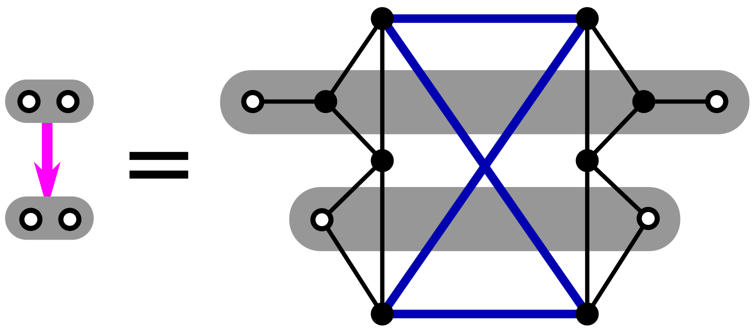
$\mathbf{A}$ = Free algebra over 
modulo 

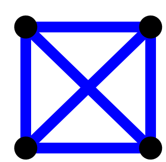


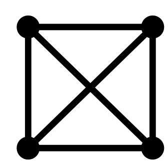
Consider $\mathbf{A}^2$

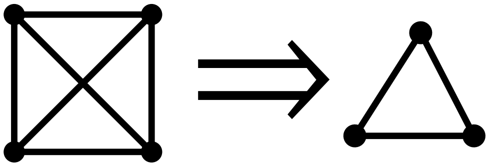


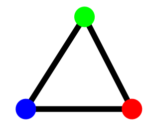


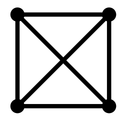




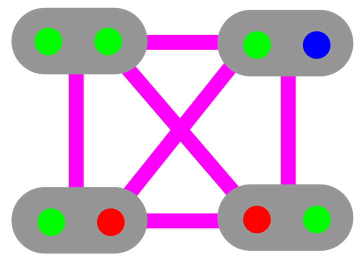


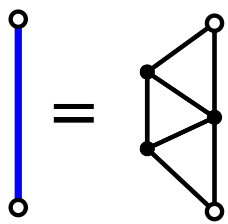


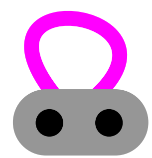
$\mathbf{A}$ = Free algebra over 

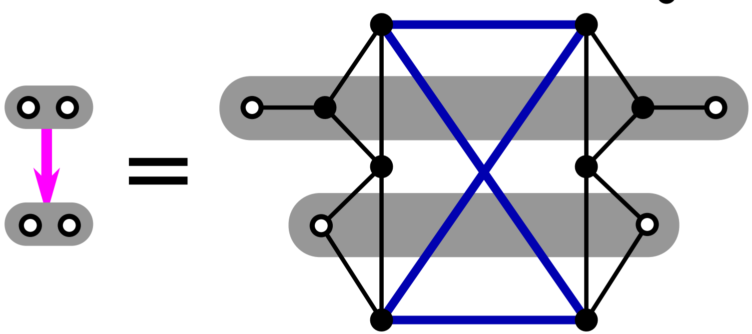
modulo 

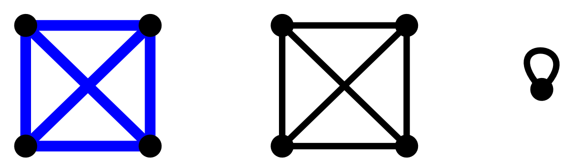
Consider $\mathbf{A}^2$

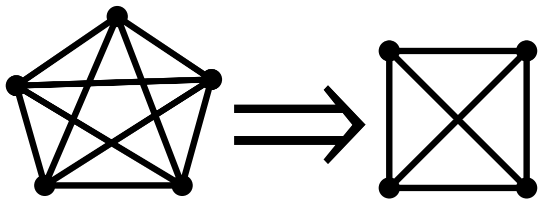




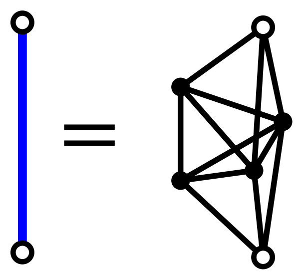


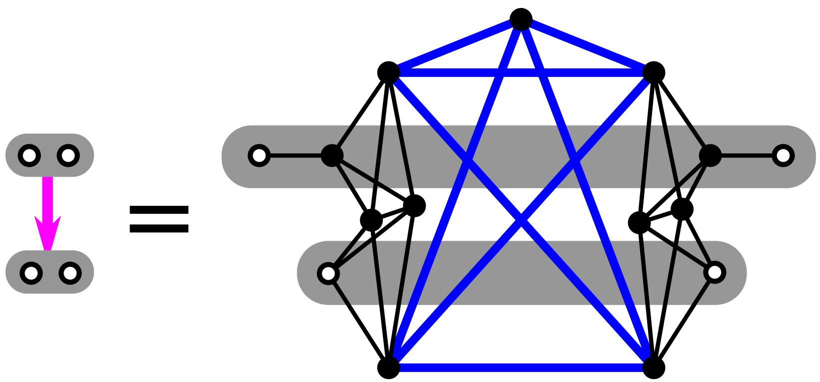


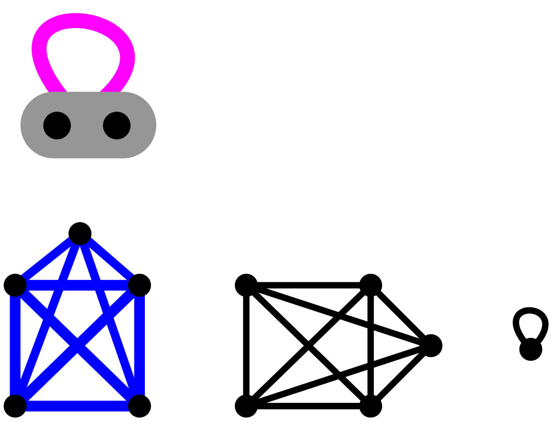


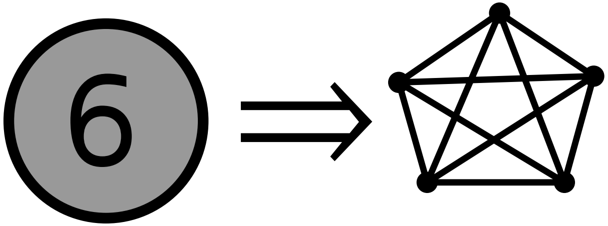


Generalize...

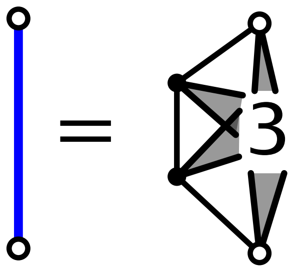


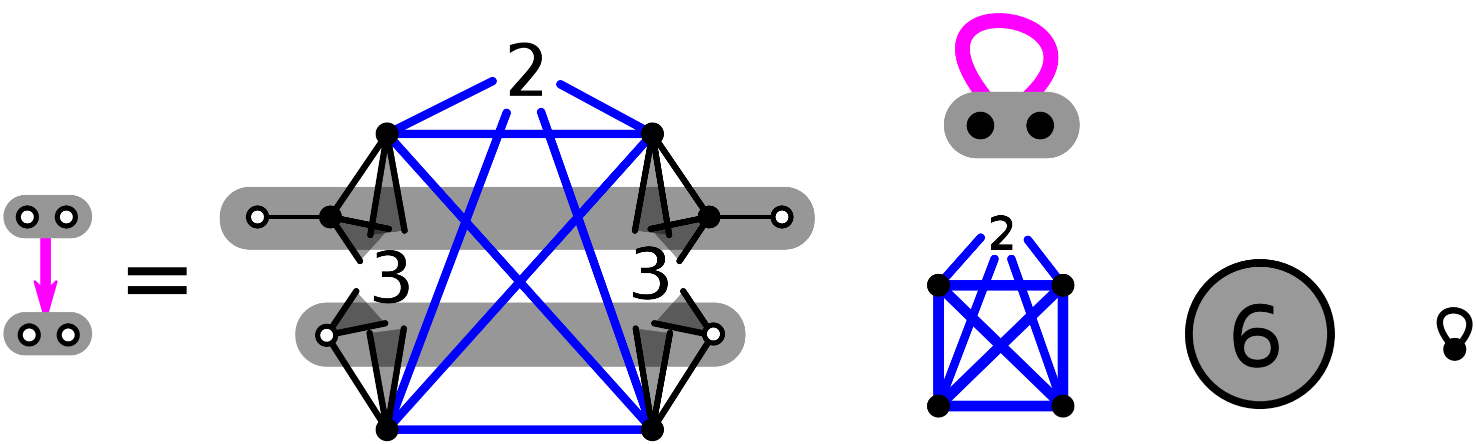




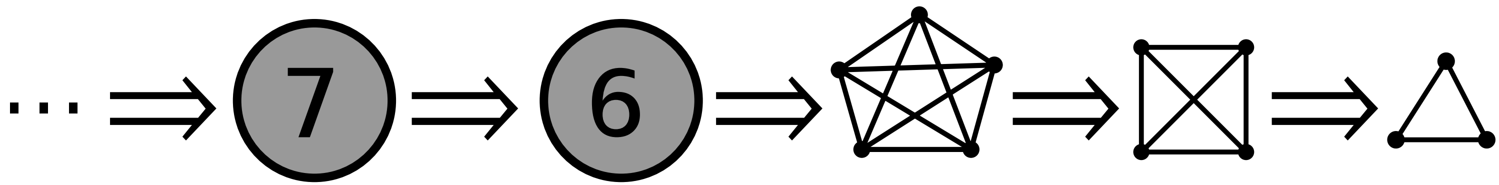


Generalize...

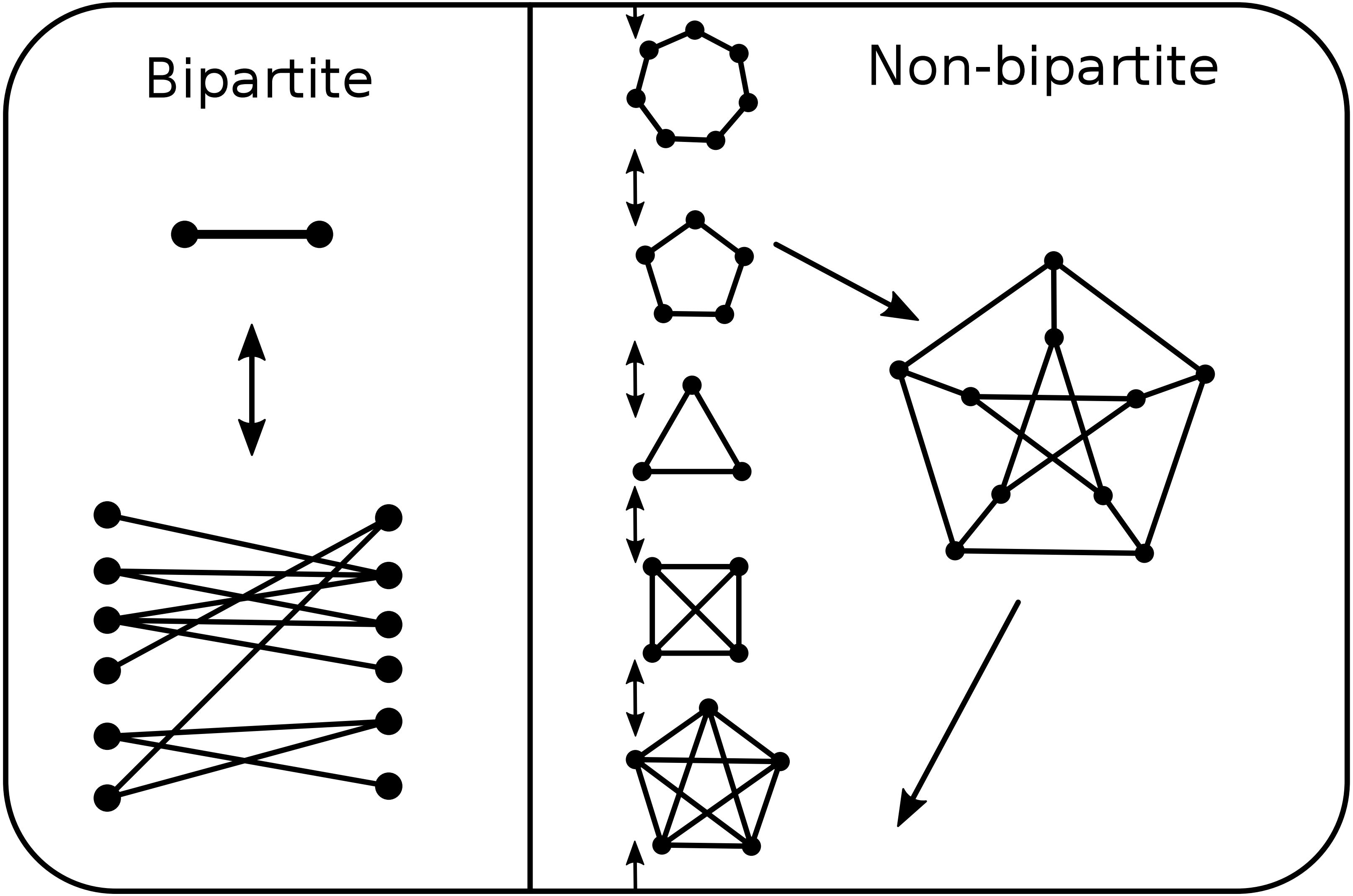




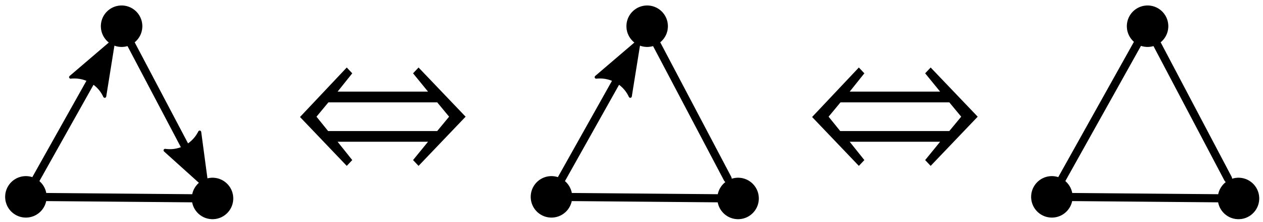
Generalize...



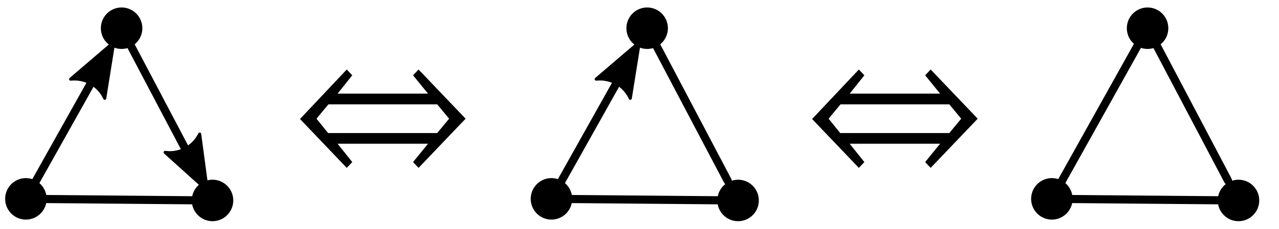
Non-trivial Undirect Loop Conditions



Directed finite case



Directed finite case



open for infinite...

Thank you for your attention

Questions?

Miroslav Olšák