Spectra

A brief introduction into Gelfand theory

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Outline

Spectrum

Definition (1)

Let V be a vector space and $T : V \to V$ a linear map (operator=matrix). *Spectrum* $\sigma(T)$ of an operator T is the set of $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is not invertible in the ring of operators on V.

- If $\dim(V) < \infty$, spectrum is the set of all eigenvalues.
- If $\dim(V) \not< \infty$, spectrum is more complicated.
- In quantum mechanics, (Hermitian) operators play the role of observations and spectrum consists of all possible observed values.

Another spectrum

Definition (2)

Let R be a ring. The set $Spec(R)$ of its prime ideals is called *Spectrum*. Maximal ideals $Spec_m(R)$ form *maximal spectrum*.

- Spectrum is a natural geometric space attached to R.
- Prime ideals in ring of polynomials $\mathbb{C}[x_1, x_2, \ldots, x_k]$ are in one-to-one correspondence with irreducible algebraic sets in $\mathbb{C}^k$, maximal ideals correspond with points of $\mathbb{C}^k$.
- The set $Spec(R)$ can also be viewed as 'how R looks like from the view of all fields'.

Question

Is there any connection between spectrum of a ring and spectra of operators?

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Spectra

In linear algebra, both spectra are the same.

Proposition

Let $\mathbb{C}[T]$ be an algebra of operators on V ($\dim(V) < \infty$) generated by T. Then Spec($\mathbb{C}[T]$) is in one-to-one correspondence with eigenvalues of V.

Proof.

- Algebra $\mathbb{C}[T]$ is isomorphic to $\mathbb{C}[x]/(m(x))$, where $m(x)$ is the minimal polynomial of T.
- In $\mathbb{C}[x]$, prime ideals are of the form $(x-\lambda)$ where $\lambda \in \mathbb{C}$.
- In $\mathbb{C}[x]/(m(x))$, prime ideals are those $(x-\lambda)$ which contain $(m(x))$.
- $(m(x)) \subset (x-\lambda) \Leftrightarrow (x-\lambda)|(m(x)) \Leftrightarrow m(\lambda) = 0 \Leftrightarrow \lambda \in \sigma(T)$

$\square$

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Spectral theorem (more linear algebra)

If T is normal operator, we have spectral theorem.

Theorem

Let $T : V \to V$ be normal operator ($TT^* = T^*T$) on the complex space V, $\dim(V) < \infty$. Then space V has an orthogonal basis consisting of eigenvectors of T. Equivalently, T has the spectral decomposition:

$$T = \sum_{a \in \sigma(T)} aP_a,$$

where P_a is the orthogonal projection to subspace of eigenvectors of an eigenvalue a. Equivalently,

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Why normal operators?

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Spectral theorem (even more linear algebra)

There is also spectral theorem for several operators.

Theorem

Let $T, S : V \to V$ be normal operators on V such that $TS = ST$. Then space V has an orthogonal basis consisting of common eigenvectors of T and S (!). Equivalently, T and S have spectral decomposition:

$$T = \sum_{a \in \sigma(T), b \in \sigma(S)} a P_a P_b,$$

and

$$S = \sum_{a \in \sigma(T), b \in \sigma(S)} b P_a P_b.$$

Each operator has its own eigenvalues, but they 'share' eigenvectors!

Question

Why T and S should commute?

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Definition

Banach algebra is an associative unital algebra $\mathcal{A}$ over $\mathbb{C}$ which is also a complete normed space such that the algebra multiplication and the norm are related by the following inequality

$$\forall x, y \in \mathcal{A} : \|xy\| \leq \|x\|\|y\|.$$

A *homomorphism* of Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is a bounded linear map $\varphi : \mathcal{A} \to \mathcal{B}$ which satisfies $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathcal{A}$.

The main feature of Banach algebras is that elements have their spectra (like operators).

Definition

Let $\mathcal{A}$ be a Banach algebra with the identity element e. If $x \in \mathcal{A}$ then *spectrum* $\sigma(x)$ is a set of all $\lambda \in \mathbb{C}$ such that $\lambda e - x$ is not invertible.

Characters

- By the *character* of a ring (resp. Banach algebra) $\mathcal{A}$ we mean a nonzero homomorphism from $\mathcal{A}$ to some field F.
- Characters on $\mathcal{A}$ form a category, morphisms are commutative triangles.
- In the case of rings, connected components of this category correspond to prime ideals of $\mathcal{A}$.
- The case of Banach algebra $\mathcal{A}$ is much nicer – the only Banach field is $\mathbb{C}$ and characters are in one-to-one correspondence with maximal ideals of $\mathcal{A}$.

Theorem (Gelfand - Mazur)

If $\mathcal{A}$ is a (possibly noncomutative) Banach algebra in which every nonzero element is invertible, then $\mathcal{A} \cong \mathbb{C}$.

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Gelfand spectrum

Definition (3)

Let $\mathcal{A}$ be a (commutative) Banach algebra. *Gelfand spectrum* $\sigma(\mathcal{A})$ is the set of all characters of $\mathcal{A}$ (homomorphisms from $\mathcal{A}$ to C).

- Gelfand spectrum is in one-to-one correspondence with $Spec_m(\mathcal{A})$ - the maximal spectrum of $\mathcal{A}$.
- Gelfand spectrum is naturally embedded into the unit sphere in A^* and it forms a compact topological space with the topology of pointwise convergence (Banach - Alaoglu theorem).
- Gelfand spectrum forms a functor from *BanAlg* to *Comp*.

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Gelfand transform

Definition

Gelfand transform of an element $x \in \mathcal{A}$ is a function $\hat{x} \in C(\sigma(\mathcal{A}))$ defined as $\hat{x}(h) = h(x)$ for every homomorphism $h \in \sigma(\mathcal{A})$.

Gelfand transformation is a homomorphism of Banach algebras. Image of $\hat{x}$ is $\sigma(x)$. If x and e generate $\mathcal{A}$, then $\sigma(\mathcal{A})$ and $\sigma(x)$ are homeomorphic.

Example

Gelfand spectrum of $C(X)$ where X is a compact space is homeomorphic to X.

Example

Gelfand spectrum of $\ell^1(\mathbb{R})$ is homeomorphic to unit circle and Gelfand transformation becomes Fourier transformation.

Israel Gelfand

C*-algebras

Definition

C^*-algebra is a Banach algebra $\mathcal{A}$ together with a map $^* : \mathcal{A} \to \mathcal{A}$ satisfying for all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ the following conditions:

- $x^{**} = x$
- $(x + y)^* = x^* + y^*$
- $(\lambda x)^* = \bar{\lambda} x^*$
- $(xy)^* = y^* x^*$
- $\|xx^*\| = \|x\| \|x^*\|$.

Question

Why C-algebras?*

Theorem (Gelfand - Naimark)

If $\mathcal{A}$ is a commutative C^*-algebra, then Gelfand transformation $\Gamma_{\mathcal{A}}$ is an isometric isomorphism from $\mathcal{A}$ to $C(\sigma(\mathcal{A}))$.

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Examples of C*-algebras

Example

Algebra of operators on Hilbert space (vector space with scalar product) is C*-algebra.

Example

Algebra $\mathbb{C}(T)$ where T is normal operator is commutative (!) C*-algebra.

Example

Algebra $C(X)$ of continuous functions on compact space is commutative C*-algebra.

Rings vs. C*-algebras

Proposition

The functor $Spec_m$ assigning to every Banach algebra its Gelfand spectrum is a right adjoint to the functor C, which assigns to compact space its algebra of functions. Gelfand transform is the counit a natural "substituting" homomorphism is the unit.

Proposition

The functor $Spec_m$ and the restriction of the functor C to $\mathbf{C^*Alg}^{op}$ form an equivalence of categories $\mathbf{C^*Alg}^{op}$ and $\mathbf{Comp}$.

C*-algebras and compact spaces

$$\begin{tikzcd} C^*alg^{op} \arrow[r, bend left, "Spec_m"] & Comp \arrow[l, bend left, "C"] \end{tikzcd}$$

Rings and algebraic varieties

$$\begin{tikzcd} CRing^{op} \arrow[r, bend left, "Spec"] & Var \arrow[l, bend left, "O"] \end{tikzcd}$$

Noncommutative dream

Question

What if not every algebra is commutative and with 1?

- Gelfand theory works for commutative algebras without 1. Compact spaces are replaced by locally compact spaces.
- Gelfand theory is the theory of '1-dimensional reprezentations.'
- If we study noncommutative C*-algebras, it is better to look at reprezentations of higher dimensions.
- If we ask, what the corresponding spaces are, we 'obtain' noncommutative geometry!
- Noncommutative geometry is a useful tool in physics.

Thanks

Thanks for your attention!

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