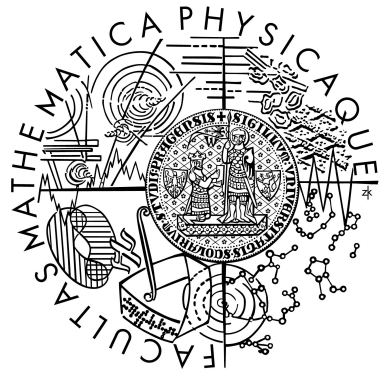
Attacking RSA

Barbora Hudcová
Igor Eržiak



FACULTY
OF MATHEMATICS
AND PHYSICS
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Let $N = pq$ be the product of two primes. Let e, d be two integers satisfying

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where $\phi(N) = (p-1)(q-1)$.

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- $\langle N, d \rangle$ ... private key

Meet the Crew

Bob

Alice

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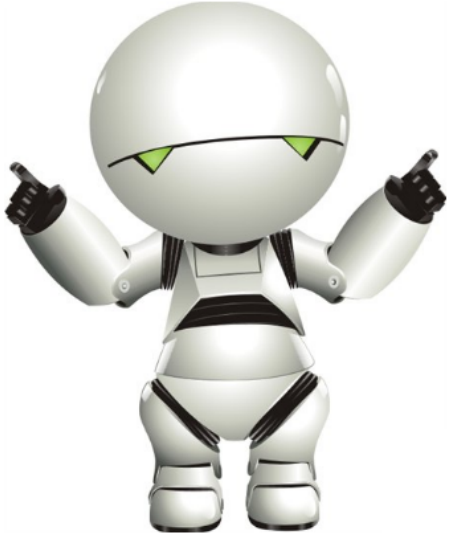
Marvin



Alice

Introduction

$\langle N, e \rangle$

$\langle N, d \rangle$

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$M \in \mathbb{Z}_N^*$

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$C^d \pmod{N}$

Indeed, from Euler's theorem it follows that

$$C^d \equiv M^{ed} \equiv M \pmod{N}.$$

Low Exponent RSA with Related Messages

M. Franklin

M. Reiter

Low Exponent RSA with Related Messages

M. Franklin

M. Reiter

Artist: Milan Boháček

Low Exponent RSA with Related Messages

What does Marvin know?

Low Exponent RSA with Related Messages

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- public key $\langle N, e \rangle$

Low Exponent RSA with Related Messages

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- k ciphertexts of different messages produced using the same public key $\langle N, e \rangle$

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What is Marvin's goal?

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What does Marvin know?

- public key $\langle N, e \rangle$
- k ciphertexts of different messages produced using the same public key $\langle N, e \rangle$
- polynomial relation between the messages

What is Marvin's goal?

- to recover the plaintext messages

Two messages with affine relation

$m_1, m_2 \in \mathbb{Z}_N^*, \quad m_2 = \alpha m_1 + \beta, \quad \alpha, \beta \in \mathbb{Z}_N^*$

$$\begin{aligned} c_1 &\equiv m_1^e \pmod N \\ c_2 \equiv m_2^e &\equiv (\alpha m_1 + \beta)^e \pmod N \end{aligned}$$

$$\begin{aligned} m_1^e - c_1 &\equiv 0 \pmod N \\ (\alpha m_1 + \beta)^e - c_2 &\equiv 0 \pmod N \end{aligned}$$

Let z denote the unknown message m_1:

$$\begin{aligned} z^e - c_1 &\equiv 0 \pmod N \\ (\alpha z + \beta)^e - c_2 &\equiv 0 \pmod N \end{aligned}$$

Applying Euclidean algorithm should yield the linear polynomial $z - m_1$.

$$z - m_1 = \gcd\left(z^e - c_1, (\alpha m_1 + \beta)^e - c_2\right) \quad \in \mathbb{Z}_N[x]$$

Two messages with polynomial relation

We have $p \in \mathbb{Z}_N[x], \quad m_2 = p(m_1), \quad \deg(p) = \delta.$

$$z^e - c_1 \equiv 0 \pmod{N}$$
$$(p(z))^e - c_2 \equiv 0 \pmod{N}$$

Euclidean algorithm should yield $z - m_1$.

$$z - m_1 = \gcd\left(z^e - c_1, (p(z))^e - c_2\right) \quad \in \mathbb{Z}_N[x]$$

Resultant

$$p(x) = p_m x^m + \ldots + p_1 x + p_0, \quad deg(p) = m$$
$$q(x) = q_n x^n + \ldots + q_1 x + q_0, \quad deg(q) = n$$

Resultant

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$$q(x) = q_n x^n + \ldots + q_1 x + q_0, \quad deg(q) = n$$

$$Res_x(p,q) = \begin{vmatrix} p_m & 0 & 0 & \ldots & 0 & q_n & 0 & 0 & \ldots & 0 \\ p_{m-1} & p_m & 0 & \ldots & 0 & q_{n-1} & q_n & 0 & \ldots & 0 \\ p_{m-2} & p_{m-1} & p_m & \ldots & 0 & q_{n-2} & q_{n-1} & q_n & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_1 & p_2 & p_3 & \ldots & \vdots & q_1 & q_2 & q_3 & \ldots & \vdots \\ p_0 & p_1 & p_2 & \ldots & \vdots & q_0 & q_1 & q_2 & \ldots & \vdots \\ 0 & p_0 & p_1 & \ldots & \vdots & 0 & q_0 & q_1 & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ddots & p_1 & \vdots & \vdots & \vdots & \ddots & q_1 \\ 0 & 0 & 0 & \ldots & p_0 & 0 & 0 & 0 & \ldots & q_0 \end{vmatrix}.$$

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Fact: The resultant of two polynomials with coefficients in an integral domain is zero if and only if they have a common divisor of positive degree.

Implicit polynomial relation

What if m_1 and m_2 satisfy an implicit polynomial relation?

$$p(m_1, m_2) \equiv 0 \pmod{N}, \quad deg(p) = \delta$$

Then we have:

$$\begin{aligned} P_1 &= p(m_1, m_2) \equiv 0 \pmod{N} \\ P_2 &= m_1^e - c_1 \equiv 0 \pmod{N} \\ P_3 &= m_2^e - c_2 \equiv 0 \pmod{N} \end{aligned}$$

Implicit polynomial relation

$$P_1 = p(x, y) \equiv 0 \pmod{N}$$
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Resultant of $P_1(x, y)$ and $P_2(x)$ with respect to the variable x will yield a polynomial $P_4(y), \deg(P_4) \leq \delta e$.

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$\gcd(P_3, P_4)$ should yield the linear polynomial $y - m_2$

$\gcd(P_1, P_2)$ should yield the linear polynomial $x - m_1$

Arbitrary polynomial relationship among messages

We have $m_1, m_2, \ldots, m_k$; $\quad p(m_1, \ldots, m_k) = 0$.

$$
\begin{aligned}
P_0(x_1, \ldots, x_k) = p(x_1, \ldots, x_k) &\equiv 0 \pmod{N} \\
P_1(x_1) = x_1^e - c_1 &\equiv 0 \pmod{N} \\
&\vdots \\
P_k(x_k) = x_k^e - c_k &\equiv 0 \pmod{N}
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\begin{aligned}
Q_0(x_1, \ldots, x_k) &= P_0 \\
&\vdots \\
Q_i(x_{i+1}, \ldots, x_k) &= Res_{x_i}(Q_{i-1}, P_i) \\
&\vdots \\
Q_{k-1}(x_k)
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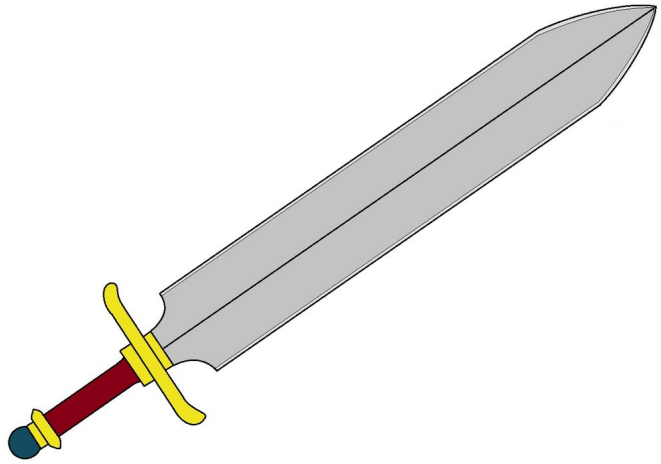
$$\begin{aligned} \gcd\left(Q_{k-1}(x_k), P_k(x_k)\right) &= x_k - m_k \\ &\vdots \\ \gcd\left(P_i(x_i), Q_{i-1}(x_i, m_{i+1}, \ldots, m_k)\right) &= x_i - m_i \end{aligned}$$

Wiener's Attack



Wiener's Attack

Theorem (M. Wiener)

Let $N = pq$ with $q < p < 2q$. Let $d < \frac{1}{3}N^{\frac{1}{4}}$. Given a public key $\langle N, e\rangle$ with $ed \equiv 1 \pmod{\phi(N)}$, an adversary can efficiently recover d.

$$ed - k\phi(N) = 1 \quad k\phi(N) = ed - 1 \quad e < \phi(N) \implies k < d < \frac{1}{3}N^{\frac{1}{4}}$$

$$\left|\frac{e}{\phi(N)} - \frac{k}{d}\right| = \frac{1}{d\phi(N)}$$

$$\left|\frac{e}{N} - \frac{k}{d}\right| = \left|\frac{ed - k\phi(N) + k\phi(N) - kN}{Nd}\right| =$$

$$\left|\frac{k(N - \phi(N)) - 1}{Nd}\right| \leq \left|\frac{3k\sqrt{N}}{Nd}\right| = \frac{3k}{d\sqrt{N}} \leq \frac{1}{dN^{\frac{1}{4}}} < \frac{1}{2d^2}$$

Wiener's Attack

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$\Longrightarrow$ $\frac{e}{N}$ has maximum of $\log_2 N$ convergents

$\Longrightarrow$ we obtain k and d

$\Longrightarrow$ we obtain $\phi(N)$

$\Longrightarrow$ we can factor N

Countermeasures

1. $\langle N, e \rangle \rightarrow \langle N, e^{'} \rangle, \quad e^{'} = e + t\phi(N)$ for some large t.
   If $e^{'} > N^{1.5}$ attack cannot be mounted.

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   If $e^{'} > N^{1.5}$ attack cannot be mounted.
2. CRT: choose d so that

$$d_p = d \bmod p - 1 \qquad \textit{and} \qquad d_q = d \bmod q - 1$$

   are both small.

Decryption:

$$M_p = C^{d_p} \pmod{p}$$
$$M_q = C^{d_q} \pmod{q}$$

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Decryption:

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Using CRT find $M \in \mathbb{Z}_N$ satisfying:

$$M \equiv M_p \pmod{p} \quad \& \quad M \equiv M_q \pmod{q}$$

$$M = C^d \pmod{N}$$

d_p and d_q are small but d mod $\phi(N)$ can be large.

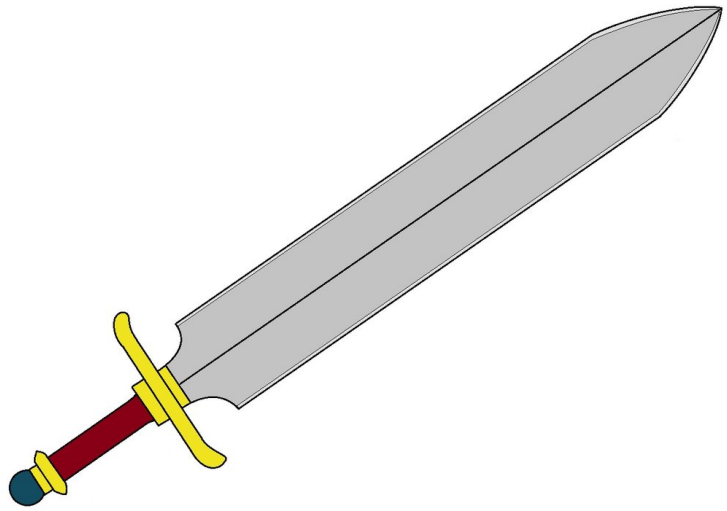
Wiener's Attack

Boneh and Durfee: $d < N^{0.292}$

Open Problem

Let $N = pq$ and $d < N^{0.5}$. If Marvin is given $\langle N, e \rangle$ with $ed \equiv 1 \pmod{\phi(N)}$ and $e < \phi(N)$, can he recover d?

Coppersmith's Attack



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- $f \in \mathbb{Z}[x]$ is monic, $\deg(f) = \delta$
- $X := N^{\frac{1}{\delta} - \epsilon}$ for some $\epsilon \geq 0$

Then, given $\langle N, f \rangle$, Marvin can efficiently find all integers $|x_0| < X$ satisfying

$$f(x_0) \equiv 0 \pmod{N}$$

The running time is dominated by the time it takes to run the LLL algorithm on a lattice of dimension $O(w)$ with
$w = min(\frac{1}{\epsilon}, log_2(N))$.

Lemma 6

Let $h(x) \in \mathbb{Z}[x]$ be a polynomial of degree δ, and let X be a positive integer. Suppose $\|h(xX)\| < \frac{N}{\sqrt{\delta}}$. If $|x_0| < X$ satisfies $h(x_0) \equiv 0 \pmod{N}$ then $h(x_0) = 0$ holds over the integers.

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Theorem (Cauchy-Schwarz)

For each $u_1, \ldots, u_n \in \mathbb{C}$ and $v_1, \ldots, v_n \in \mathbb{C}$:

$$\sum_{i=1}^{n} \left| u_i^T v_i \right| \leq \sum_{j=1}^{n} |u_j|^2 \sum_{k=1}^{n} |v_k|^2$$

$$f(x_0) \equiv 0 \mod N$$
$$f(x_0)^k \equiv 0 \mod N^k$$
$$g_{u,v}(x) := N^{m-v} x^u f(x)^v$$

where $0 \leq v \leq m$ and $0 \leq u$.

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GOAL Find an integer linear combination $h(x)$ of polynomials $g_{u,v}(x)$ such that $\|h(xX)\| < \frac{N^m}{\sqrt{deg(h)}}$.

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How to find $h(x)$?

Lattices

Definition

Let $u_1, \ldots, u_\omega \in \mathbb{Z}^\omega$ be linearly independent vectors. A (full-rank) lattice $\mathcal{L}$ spanned by $\langle u_1, \ldots, u_\omega \rangle$ is defined as:

$$\mathcal{L} := a_1 u_1 + \cdots + a_\omega u_\omega \in \mathbb{Z}.$$

$$\det(\mathcal{L}) := \det \begin{pmatrix} - & u_1 & - \\ & \vdots & \\ - & u_\omega & - \end{pmatrix}$$

LLL Algorithm

INPUT $b_1, \ldots, b_\omega$ - basis of $\mathcal{L}$ ($\mathcal{L}$ is spanned by $\langle b_1, \ldots, b_\omega \rangle$)

OUTPUT $b'_1, \ldots, b'_\omega$ - basis of $\mathcal{L}$ satisfying:

$$\|b'_1\| \leq \|b'_2\| \leq \cdots \leq \|b'_i\| \leq 2^{\frac{\omega(\omega-1)}{4(\omega+1-i)}} \det(\mathcal{L})^{\frac{1}{\omega}}$$

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$$\|b'_1\| \leq 2^{\frac{\omega}{4}} \det(\mathcal{L})^{\frac{1}{\omega}}$$

Our case

We view polynomials $g_{u,v}(x)$ as vectors and study the lattice $\mathcal{L}$ spanned by them.
We let:

$$v = 0, \ldots, m$$

$$u = 0, \ldots, \delta - 1$$

Hence the lattice has dimension $\omega = \delta(m + 1)$.

Some big matrix called G

$g_{u,v}(x) := N^{m-v} x^u f(x)^v \qquad v = 0, \ldots, m \qquad u = 0, \ldots, \delta - 1$
$\omega = \delta(m+1)$

$$
\begin{array}{c}
 \\
g_{0,0}(xX) \\
g_{1,0}(xX) \\
g_{2,0}(xX) \\
\vdots \\
g_{\delta-1,0}(xX) \\
g_{0,1}(xX) \\
\vdots \\
g_{\delta-1,m}
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc}
1 & x^1 & x^2 & \cdots & x^{\delta-1} & x^{\delta} & \cdots & x^{\delta(m+1)-1}
\end{array} \\
\begin{pmatrix}
N^m & & & & & & & \\
\vdots & N^m X & & & & & & \\
\vdots & \ddots & N^m X^2 & & & & & \\
\vdots & & \ddots & \ddots & & & & \\
\vdots & & & \ddots & N^m X^{\delta-1} & & & \\
\vdots & & & & \ddots & N^{m-1} X^{\delta} & & \\
\vdots & & & & & \ddots & \ddots & \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & X^{\delta(m+1)-1}
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\begin{pmatrix}
N^m & & & & & & & \\
\vdots & N^m X & & & & & & \\
\vdots & \ddots & N^m X^2 & & & & & \\
\vdots & & \ddots & \ddots & & & & \\
\vdots & & & \ddots & N^m X^{\delta-1} & & & \\
\vdots & & & & \ddots & N^{m-1} X^{\delta} & & \\
\vdots & & & & & \ddots & \ddots & \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & X^{\delta(m+1)-1}
\end{pmatrix}
\end{array}
$$

$$det(G) = N^{\frac{1}{2}\delta m(m+1)} X^{\frac{1}{2}(\delta(m+1)-1)(\delta(m+1))} = N^{\frac{1}{2}m\omega} X^{\frac{1}{2}(\omega-1)\omega}$$

Applying the LLL algorithm on matrix the G we obtain a polynomial $h(xX) \in \mathcal{L}$:

$$\|h(xX)\| \leq 2^{\frac{\omega}{4}} \det(G)^{\frac{1}{\omega}} =$$

$$= 2^{\frac{\omega}{4}} N^{\frac{1}{2}m} X^{\frac{1}{2}(\omega-1)} \leq \frac{N^m}{\sqrt{\omega}} \ldots \text{ for large enough } m$$

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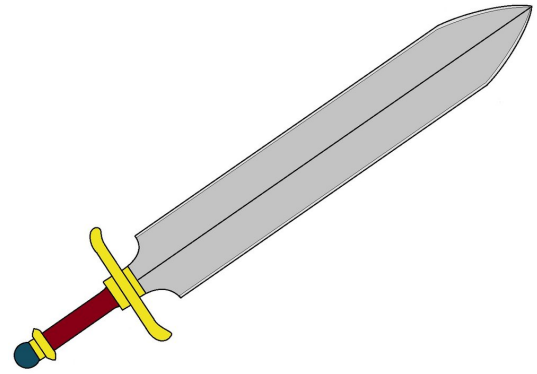
Let $h(x) \in \mathbb{Z}[x]$ be a polynomial of degree δ, and let X be a positive integer. Suppose $\|h(xX)\| < \frac{N}{\sqrt{\delta}}$. If $|x_0| < X$ satisfies $h(x_0) \equiv 0 \pmod{N}$ then $h(x_0) = 0$ holds over the integers.

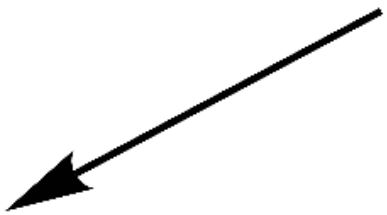
Applications of Coppersmith's Theorem

Håstad's Broadcast Attack

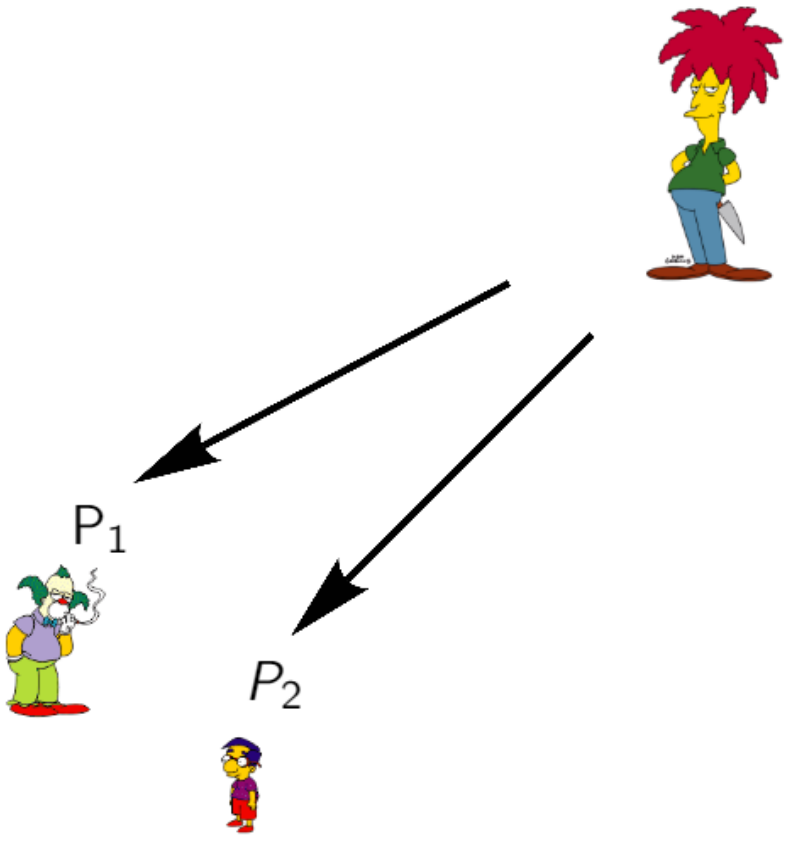


M . . . message

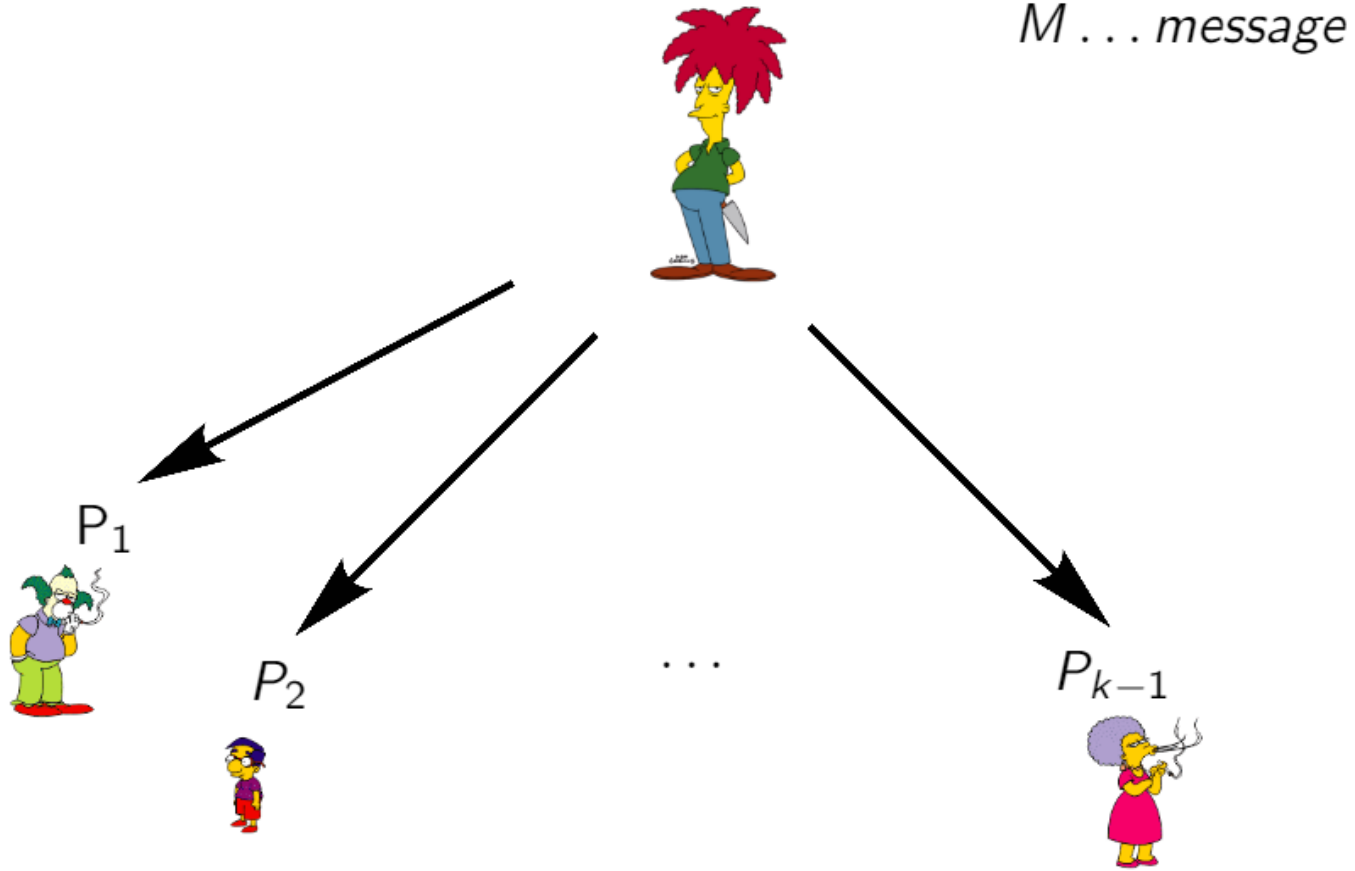
M . . . message

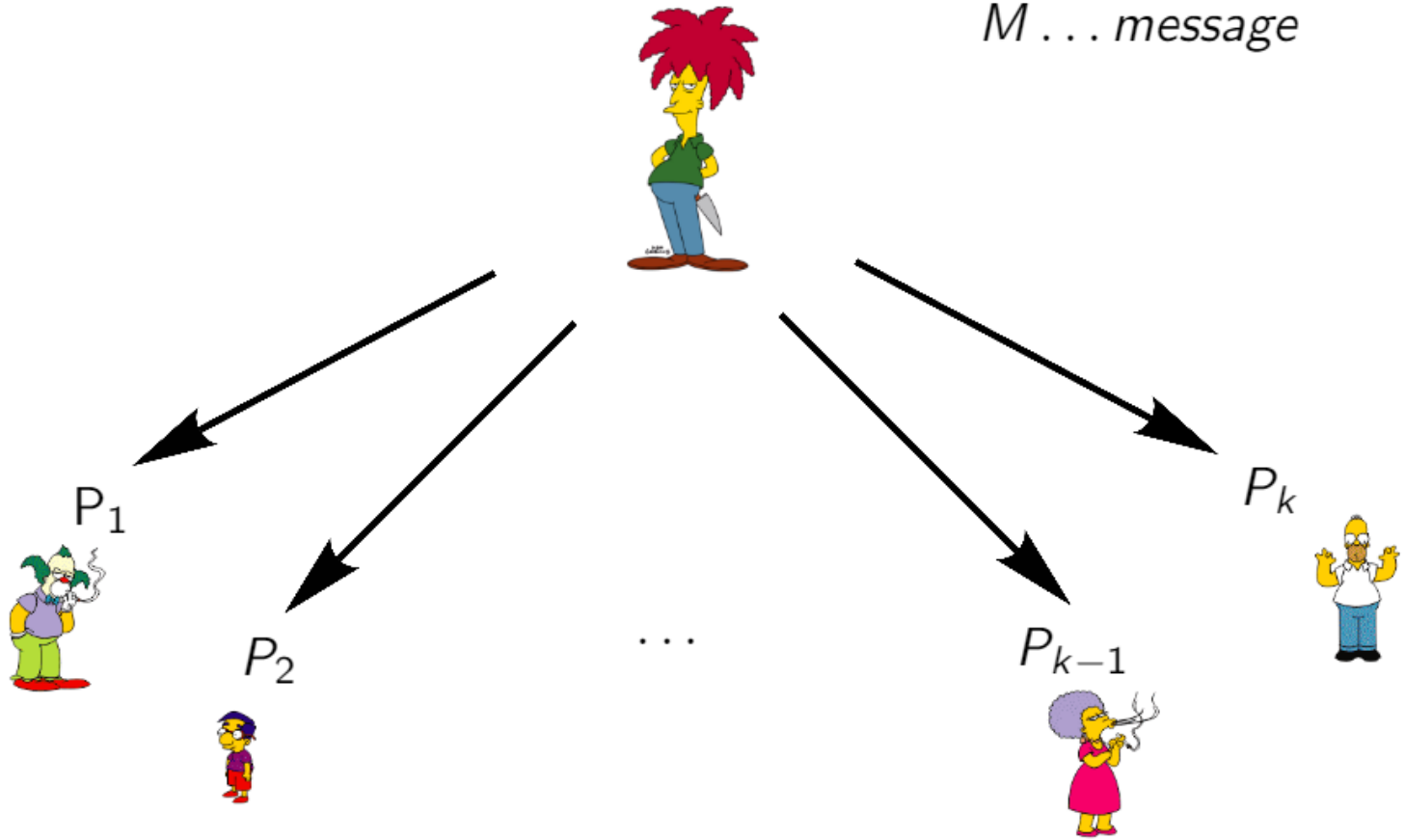


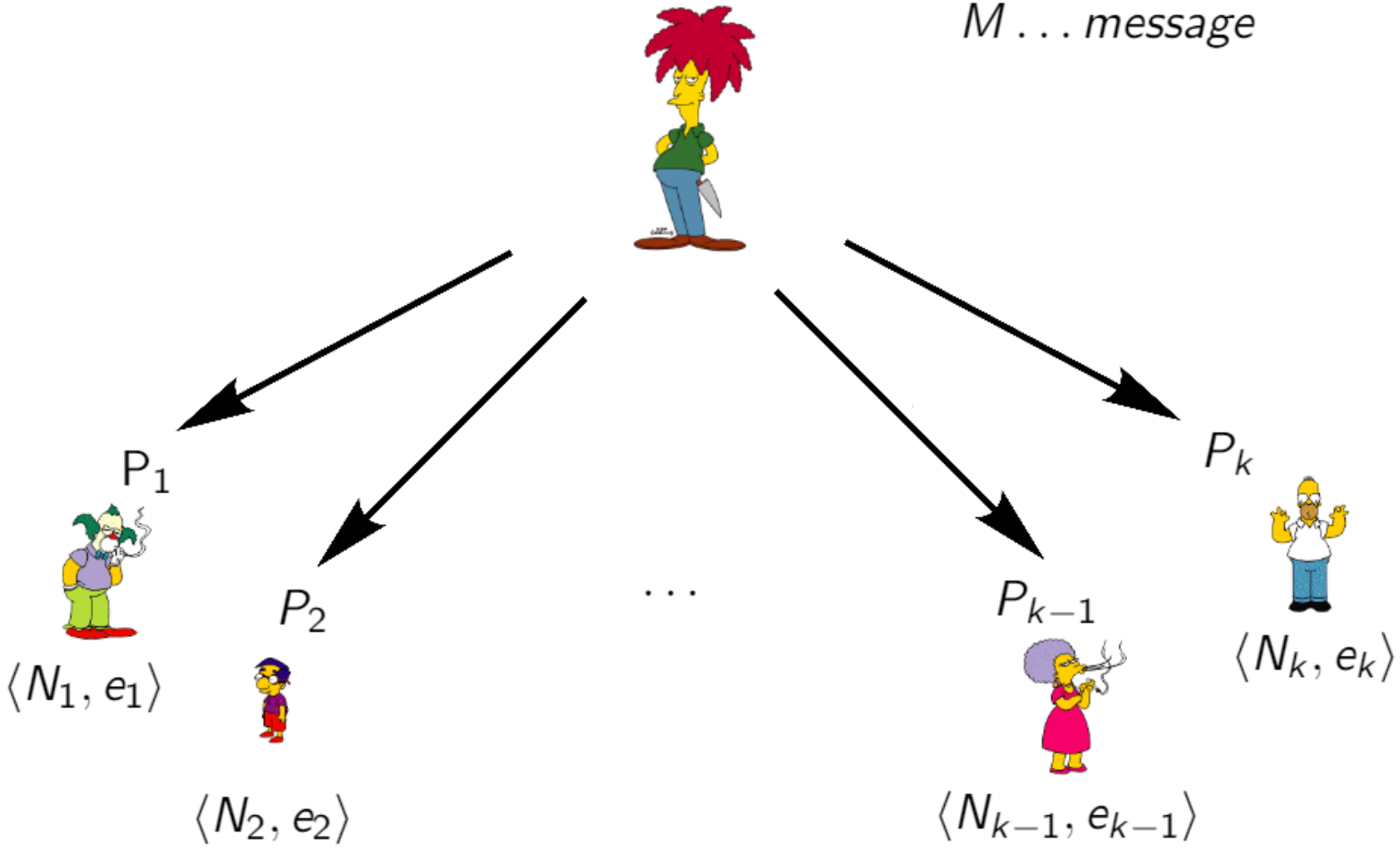
P_1

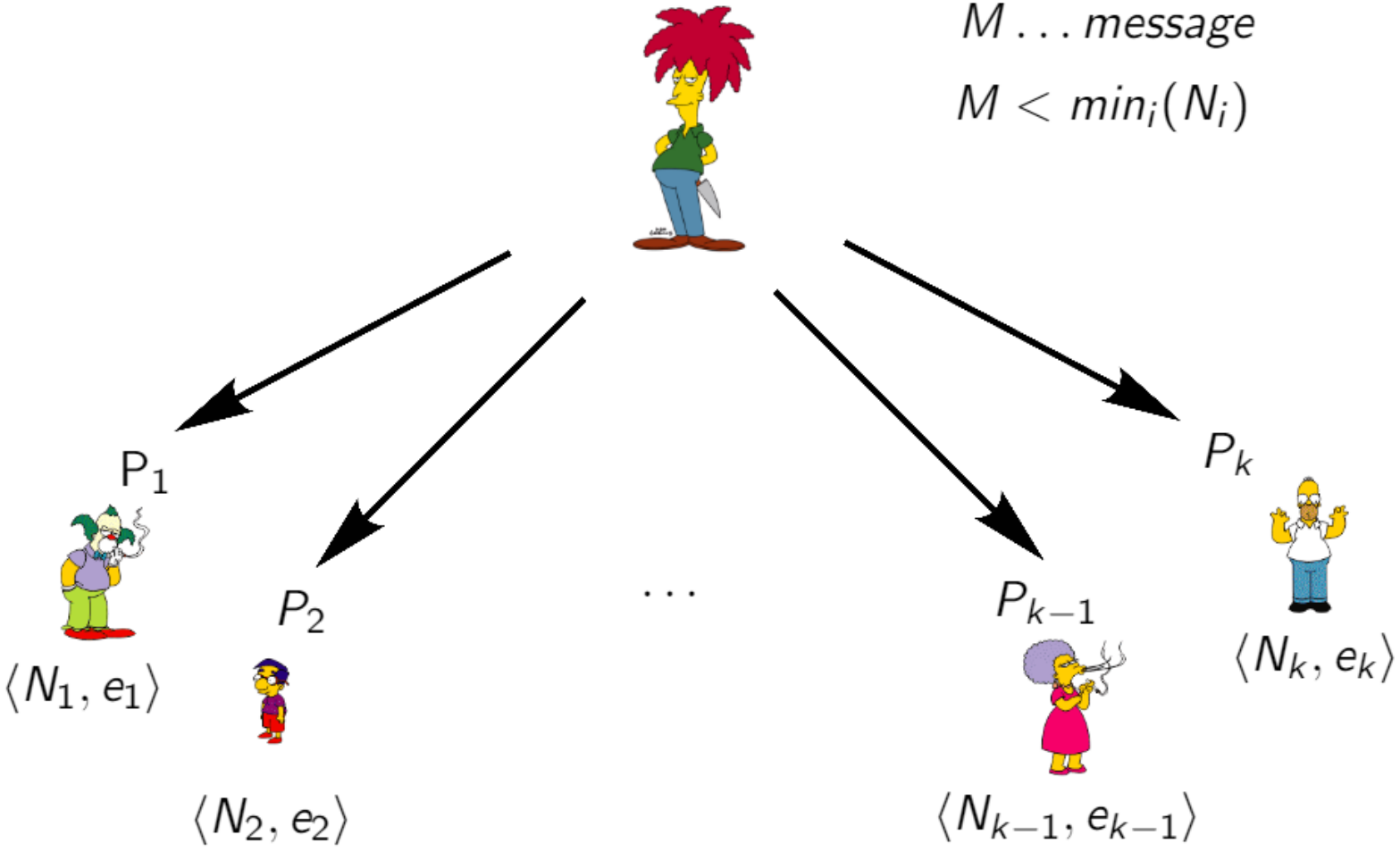


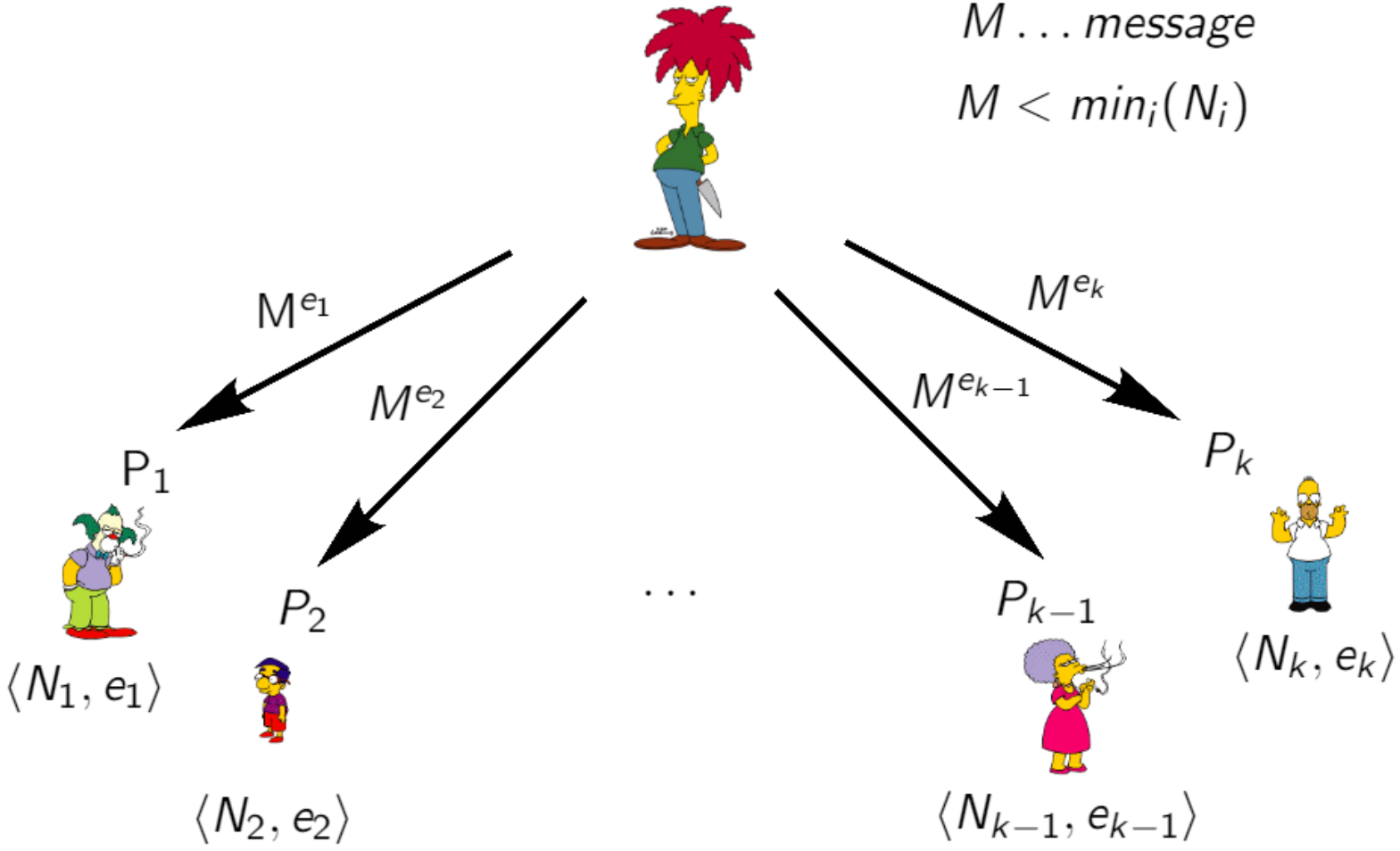
M . . . message


M . . . message
P1
P2
. . .
Pk−1


M . . . message
P1
P2
. . .
Pk−1
Pk


M . . . message
P1
P2
. . .
Pk−1
Pk
⟨N1, e1⟩
⟨N2, e2⟩
⟨Nk−1, ek−1⟩
⟨Nk, ek⟩


$M \ldots message$
$M < min_i(N_i)$
P_1
P_2
$\ldots$
P_{k-1}
P_k
$\langle N_1, e_1 \rangle$
$\langle N_2, e_2 \rangle$
$\langle N_{k-1}, e_{k-1} \rangle$
$\langle N_k, e_k \rangle$


$M \ldots message$
$M < min_i(N_i)$
M^{e_1}
M^{e_2}
$M^{e_{k-1}}$
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P_1
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Håstad's Broadcast Attack

Suppose $e_i = 3 \quad \forall i \in \{1, 2, \ldots, k\}, \quad k \geq 3.$
Marvin obtains: C_1, C_2, C_3 where:

$$C_1 \equiv M^3 \pmod{N_1}$$

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$$\Longrightarrow \; M = \sqrt[3]{C'}$$

Håstad's Broadcast Attack

Suppose that for each participant P_i, Bob has a fixed polynomial $f_i \in \mathbb{Z}_{N_i}[x]$.

$$M \to (f_i(M))^{e_i} = C_i$$

Suppose that Marvin learns C_i.
Håstad showed that if enough parties are involved, Marvin can recover M.

Håstad's Broadcast Attack

Theorem (Håstad)

Let $N_1, \ldots, N_k$ be pairwise relatively prime integers, and set $N_{min} = min_i(N_i)$. Let $g_i \in Z_{N_i}[x]$ be k polynomials of maximum degree δ. Suppose there exists a unique $M < N_{min}$ satisfying

$$g_i(M) \equiv 0 \pmod{N_i} \quad \text{for all } i = 1, \ldots, k.$$

Under the assumption that $k > \delta$, Marvin can efficiently find M given $\langle N_i, g_i \rangle_{i=1}^{k}$.

Håstad's Broadcast Attack

Proof.

$$\bar{N} := N_1 N_2 \cdots N_k$$

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$$g(x) := \sum_{i=1}^{k} T_i g_i(x),$$

$$T_i = \begin{cases} 1 \bmod N_j, & i = j \\ 0 \bmod N_j, & i \neq j \end{cases}$$

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g is monic $\pmod{\bar{N}}$, $\quad g(M) = 0 \pmod{\bar{N}}$

$$M < N_{min} \leq \bar{N}^{\frac{1}{k}} < N^{\frac{1}{\delta}}$$

Conclusion - Use random padding !

Coppersmith's Short Pad Attack

Suppose $e = 3$; R, R' random pads such that $R' = R + r$, where $|r| < N^{\frac{1}{9}}$; $M \in \mathbb{Z}_N^*$

$$c \equiv m^3 \equiv (2^k M + R)^3 \pmod{N}$$
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$Res_m(m^3 - c, (m + r)^3 - c') =$
$r^9 + (3c - 3c')r^6 + (3c^2 + 21cc' + 3(c')^2)r^3 + (c - c')^3 \equiv 0 \pmod{N}$
$|r| < N^{\frac{1}{9}} \rightarrow$ recover r with Coppersmith's method
And then use Franklin-Reiter method to recover m.

Partial Key Exposure

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$$ed - k(N - p - q + 1) \equiv 1 \pmod{2^l}, \ \text{for l} \in 1, \cdots, 300$$
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$$\text{We compute } r^{-1} \pmod{N}$$
$$r^{-1}p = x + r^{-1}t \equiv 0 \pmod{p}$$
$$f(x) := x + r^{-1}t \equiv 0 \pmod{N}$$

Thank you for your attention!