Partitions and Modular Forms

Kristýna Zemková

November 19, 2015

Partitions

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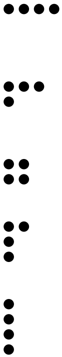
\begin{aligned}
4 &= 4 \\
&= 3 + 1 \\
&= 2 + 2 \\
&= 2 + 1 + 1 \\
&= 1 + 1 + 1 + 1
\end{aligned}
$$



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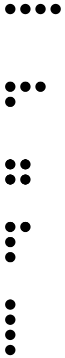

Definition

A partition of the positive integer n is any nonincreasing sequence of positive integers $\lambda_1, \ldots, \lambda_r$ such that $n = \lambda_1 + \cdots + \lambda_r$.

Partition Function

Definition

The partition function $p(n)$ is the number of partitions of n.
By convention is $p(0) = 1$ and $p(-n) = 0$ for each $n > 0$.

$$
\begin{aligned}
p(1) &= 1: \quad 1 \\
p(2) &= 2: \quad 2 = 1 + 1 \\
p(3) &= 3: \quad 3 = 2 + 1 = 1 + 1 + 1 \\
p(4) &= 5: \quad 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1 \\
p(5) &= 7: \quad 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = \dots \\
p(6) &= 11: \quad 6 = 5 + 1 = 4 + 2 = 4 + 1 + 1 = 3 + 2 + 1 = \dots \\
&\vdots \\
p(10) &= 42 \\
p(20) &= 627 \\
p(50) &= 204\,226 \\
p(100) &= 190\,569\,292 \\
p(200) &= 3\,972\,999\,029\,388
\end{aligned}
$$

Generating Function

- We know that if $|x| < 1$, then

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots.$$

- We expand the infinite product:

$$\begin{aligned}\prod_{n=1}^{\infty} \frac{1}{1-x^n} &= (1 + x + x^2 + x^3 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots)\dots \\ &= 1 + x + 2x^2 + 3x^3 + 5x^4 + \dots.\end{aligned}$$

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- Then the coefficient of x^n is equal to $p(n)$ and we get the following lemma:

Lemma (Generating function)

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n}$$

Number of Partitions with Distinct and Odd Parts

- Denote by $p(\mathcal{D}, n)$ the number of all partitions of n with distinct parts.
- E.g. the partition $3 + 2 + 1$ of 6 has distinct parts but $4 + 1 + 1$ does not.

Lemma (Distinct parts)

$$\sum_{n=0}^{\infty} p(\mathcal{D}, n)x^n = \prod_{n=1}^{\infty} (1 + x^n)$$

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Lemma (Distinct parts)

$$\sum_{n=0}^{\infty} p(\mathcal{D}, n)x^n = \prod_{n=1}^{\infty} (1 + x^n)$$

- Similarly, denote by $p(\mathcal{O}, n)$ the number of all partitions of n with only odd parts.
- E.g. the partition $3+1+1+1$ of 6 has only odd parts but $4+1+1$ does not.

Lemma (Odd parts)

$$\sum_{n=0}^{\infty} p(\mathcal{O}, n)x^n = \prod_{n=1}^{\infty} \frac{1}{1 - x^{2n-1}}$$

Number of Partitions with Distinct and Odd Parts

- Compute:

$$\prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty}\frac{1-x^{2n}}{1-x^n} = \prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}}$$

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- But we know:

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- We have just proved the following lemma:

Lemma

$$p(\mathcal{D},n)=p(\mathcal{O},n)$$

Reccurent Formula

Theorem (Euler's Pentagonal Number Theorem)

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k-\infty}^{\infty}(-1)^k x^{(3k^2-k)/2} = 1 - x - x^2 + x^5 + x^7 - x^{12} - \dots$$

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- Combining with the generating function $\sum_{n=0}^{\infty} p(n)x^n=\prod_{n=1}^{\infty}\frac{1}{1-x^n}$ we get

$$\begin{aligned}1&=\left(\prod_{n=1}^{\infty}\frac{1}{1-x^n}\right)\left(\prod_{n=1}^{\infty}(1-x^n)\right)\\&=\left(\sum_{n=0}^{\infty}p(n)x^n\right)\left(1-x-x^2+x^5+x^7-x^{12}-\dots\right).\end{aligned}$$

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$$\begin{aligned} 1 &= \left(\prod_{n=1}^{\infty}\frac{1}{1-x^n}\right)\left(\prod_{n=1}^{\infty}(1-x^n)\right) \\ &= \left(\sum_{n=0}^{\infty}p(n)x^n\right)\left(1 - x - x^2 + x^5 + x^7 - x^{12} - \dots\right). \end{aligned}$$

- This implies:

Corollary

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + \cdots$$

Ramanujan's Congruences

- Arithmetic properties of the partition function?

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- Computation of the first 10,000 values:
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  - 3,325 are congruent 1,
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1	1	2	3	5
7	11	15	22	30
42	56	77	101	135
176	231	297	385	490
627	792	1002	1255	1575
1958	2436	3010	3718	4565

Ramanujan's Congruences

- Ramanujan computed the first 200 values of $p(n)$.
- He discovered and proved the following congruences:

Theorem (Ramanujan's congruences)

$$p(5n+4) \equiv 0 \pmod{5}$$
$$p(7n+5) \equiv 0 \pmod{7}$$
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- To prove the first two congruences he used the following identities:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=0}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6},$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{n=0}^{\infty} \frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q\prod_{n=0}^{\infty} \frac{(1-q^{7n})^7}{(1-q^n)^8}.$$

Partition function (Revision)

- Partition of n - a finite nonincreasing sequence of positive integers, i.e. the sum of them is equal to n.
- Partition function $p(n)$ - the number of partitions of n.
- Generating function

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n}.$$

Modular Forms (Revision)

Definition

We denote by $\mathcal{H}$ the upper-half plane, i.e. the set of complex numbers with positive imaginary part.

Function f on $\mathcal{H}$ is called modular form of weight $2k$ if it satisfies following conditions:

- $f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,
- f is holomorphic on $\mathcal{H}$,
- f is holomorphic at ∞.

Modular form is called cusp form if it satisfies also

- $f(\infty) = 0$.

Eisenstein Series (Revision)

Definition

Let $z \in \mathcal{H}$. For an integer $k \geq 2$ we define the *Eisenstein series* of index k by the following series:

$$E_k(z) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(mz+n)^{2k}}.$$

- E_k is a modular form of weight $2k$.
- $E_k(\infty) = 2\zeta(2k)$, where ζ denotes the Riemann zeta function.

Definition

$$\Delta = (60E_2)^3 - 27(140E_3)^2$$

- Δ is a cusp form of weight 12.

Jacobi Theorem

Theorem (Jacobi)

Let $z \in \mathcal{H}$, then

$$\Delta(z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

where $q = e^{2\pi i z}$.

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Let $z \in \mathcal{H}$, then

$$\Delta(z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

where $q = e^{2\pi i z}$.

Proof (main idea)

- We know that Δ is a cusp form of weight 12.
- $\dim S_k = 1$ (the space of cusp forms of weight 12).
- We put $F(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$.
- It suffices to show that F is a modular form of weight 12.
- We only have to prove that $F(-1/z) = z^{12} F(z)$.
- It follows from a computation with double series.

$\square$

Ramanujan Function

Definition

The Ramanujan function is the function $\tau : \mathbb{N} \to \mathbb{Z}$ defined as the nth coefficient of the cusp form $F(z) = (2\pi)^{-12}\Delta(z)$. Thus

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

where $q = e^{2\pi i z}$.

n	1	2	3	4	5	6	7	8	9
$\tau(n)$	1	-24	252	-1,472	4,830	-6,048	-16,744	84,480	-113,643

Ramanujan Function

Theorem (Properties of $\tau(n)$)

1. $\tau(n) = O(n^{11/2+\epsilon})$ *for every* $\epsilon > 0$,
2. $\tau(mn) = \tau(m)\tau(n)$ *if* $GCD(m,n) = 1$,
3. $\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1})$ *for* p *prime,* $n > 1$.

- The first property follows from:
  - (Hecke theorem) If a_n is the nth coefficient of a cusp form of the weight $2k$, then
  $$a_n = O(n^{k-1/2+\epsilon}).$$
  - $\tau(n)$ is defined as the nth coefficient of a cusp form of the weight 12.
- The second property says that $\tau(n)$ is multiplicative.

L-functions (Revision)

Definition

Let $X = (a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers and put

$$L(s,X) = \sum_{n\in\mathbb{N}} \frac{a_n}{n^s}.$$

Furthermore suppose that following holds:

- $L(s,X)$ is absolutely convergent for $\mathbf{Re}(s) > k, k \in \mathbb{N}$
- $L(s,X)$ has analytic continuation to $\mathbb{C}$
- $L(s,X) = \gamma(s,X)L(k-s,X')$ for some "elementary" function γ, $k \in \mathbb{N}$
- $L(s,X) = \prod_{p\in\mathbb{P}} h_p(p^{-s})^{-1}$ where h_p is a polynomial for each $p \in \mathbb{P}$.

Then $L(s,X)$ is called a (general) L-function.

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- We have $F(z) = \sum_{n=1}^{\infty} \tau(n)q^n = (2\pi)^{-12}\Delta(z)$.
- Put $L(s, F) = \sum_{n=1}^{\infty} \tau(n)n^{-s}$.
- Could it be a L-function?
- We will check the last condition.

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- Put $h_p(x) = 1 - \tau(p)x + p^{11}x^2$.
- At first we would like to show that $\sum_{m=0}^{\infty} \tau(p^m)p^{-ms} = \frac{1}{h_p(p^{-s})}$.

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$$\left(\sum_{n=0}^{\infty} \tau(p^n)x^n\right)\left(1 - \tau(p)x + p^{11}x^2\right)$$

$$= \sum_{n=0}^{\infty} \tau(p^n)x^n - \sum_{n=0}^{\infty} \tau(p^n)\tau(p)x^{n+1} + \sum_{n=0}^{\infty} p^{11}\tau(p^n)x^{n+2}$$

$$= \tau(p^0)x^0 - \tau(p)x + \tau(p)x + \sum_{n=1}^{\infty}\left(\tau(p^{n+1}) - \tau(p)\tau(p^n) + p^{11}\tau(p^{n-1})\right)x^n \overset{(3)}{=} 1$$

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- We get $L(s,F) = \sum_{n=1}^{\infty} \tau(n)n^{-s} = \prod_{p\in\mathbb{P}} \frac{1}{h_p(p^{-s})}$.

Connection between Partition Function and Modular Forms

- Lets look at these two identities:
  - $\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$,
  - $\sum_{n=1}^{\infty} \tau(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{24}$.

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- We can put them together:

$$\left(\sum_{n=0}^{\infty} p(n)q^n\right)^{24} \left(\sum_{n=1}^{\infty} \tau(n)q^n\right) = q\prod_{n=1}^{\infty} \frac{(1-q^n)^{24}}{(1-q^n)^{24}}$$

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- So we get:

Theorem

$$\sum_{n=1}^{\infty} \tau(n)q^{n-1} = \frac{1}{\left(\sum_{n=1}^{\infty} p(n)q^n\right)^{24}}$$

Back to Ramanujan's Congruences

- Recall that:

Theorem (Ramanujan's congruences)

$$p(5n+4) \equiv 0 \pmod 5$$
$$p(7n+5) \equiv 0 \pmod 7$$
$$p(11n+6) \equiv 0 \pmod{11}$$

- Ramanujan expected that these three congruences are the only congruences of this form.
- In his own words:

"It appears that there are no equally simple properties for any moduli involving primes other than these three."

Back to Ramanujan's Congruences

- Formally, there are not any congruences of the form

$$p(\ell n + \beta) \equiv 0 \pmod{\ell}$$

for all $n \in \mathbb{Z}$, $\ell \neq 5, 7, 11$ prime, and some fixed $\beta \in \mathbb{Z}$.

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Back to Ramanujan's Congruences

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  - Let $\ell = 3$.
  - Then $\beta \in \{0, 1, 2\}$.
  - $p(3n + \beta) \equiv 0 \pmod{\ell}$ has to be true for all $n \in \mathbb{Z}$, specially for $n = 0$.
  - But $p(0) = p(1) = 1$, $p(2) = 2$ and $1, 2 \not\equiv 0 \pmod{3}$.

Back to Ramanujan's Congruences

- Other cases are much more difficult.
- It can be shown that if there exists a $\beta \in \mathbb{Z}$ such that

$$p(\ell n + \beta) \equiv 0 \pmod{\ell}$$

  holds for all n, then $24\beta \equiv 1 \pmod{\ell}$.
- The proof is based on the function $\Delta(z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}$.

Back to Ramanujan's Congruences

- Ramanujan also made the following conjecture:

Ramanujan's conjecture

If $\delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1 \pmod{\delta}$, then $p(\delta n + \lambda) \equiv 0 \pmod{\delta}$.

Back to Ramanujan's Congruences

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- It follows easily from the cases when the moduli are powers of 5, 7, or 11.
- But this conjecture is not quite correct.
- Counterexample: $p(243) \not\equiv 0 \pmod{7^3}$.
- We know the following:

Theorem

If $\delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1 \pmod{\delta}$, then $p(\delta n + \lambda) \equiv 0 \pmod{5^a 7^{\lfloor \frac{b}{2} \rfloor + 1} 11^c}$.

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If ℓ is prime, then there is at least one nonnegative integer n_ℓ, for which

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Theorem

For any prime $\ell \geq 5$, there exist infinitely many congruences of the form

$$p(An + B) \equiv 0 \pmod{\ell}.$$

Srinivasa Ramanujan

- 22 December 1887 - 26 April 1920
- Indian mathematician.
- Almost no formal training in pure mathematics.
- Repeatedly failing college exams in other subjects and losing scholarship.
- Work as a clerk.
- First paper published in 1911 in the *Journal of the Indian Mathematical Society*.
- A real genius or a crank?

- Approval from a famous mathematician G. H. Hardy.
- Five-year collaboration with Hardy - explicit formula for $p(n)$.
- Problems with health in England, return to India (1919).

- Most of his papers unpublished until 1957, resp. 1988.
- Some of his work discovered by other mathematicians.