WALNUT PROVER

- Written by Hamoon Mousavi
- Available on webside of Jeffrey O. Shallit *https://cs.uwaterloo.ca/~shallit/papers.html*
- Most of examples in this representation come from this webside and the Walnut documentation.

How to run Walnut?

- Walnut is used by typing command in command line.
- Java is needed to run the Walnut.
- We run Walnut by write:

```
cd C:/.../Walnut/bin
>java Main.prover
.
.
.
>exit
```

Commands

List of all commands and their parameters:

```
eval <name> <predicate>
def <name> <predicate>
reg <name> <number system> <regular expression>
reg <name> <alphabet> <regular expression>
load <file name>
exit
```

Command eval

- Eval is basic command.
- It has two parameters: name and predicade.
- Example:

```
eval example "b=a+1";
```

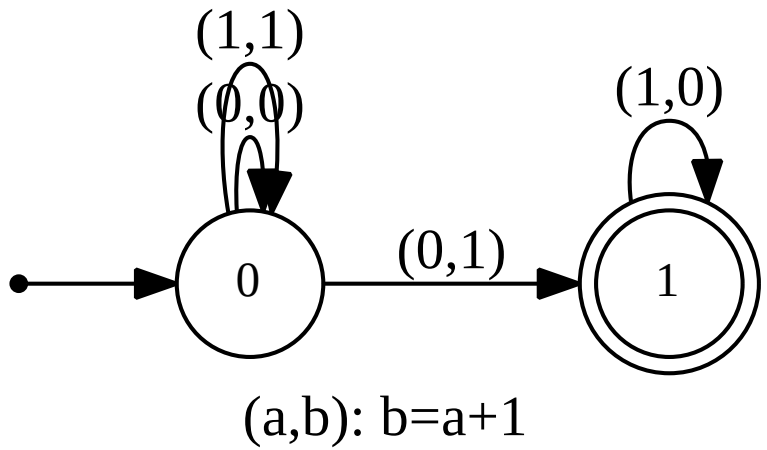
- As an output we get three files in the directory .../Walnut/Result:
  - *example.txt*
  - *example.gv*
  - *example_ log.txt*

Output

table-form description
(example.txt)

msd_2	msd_2		
0	0		
0	0	->	0
0	1	->	1
1	1	->	0
1	1		
1	0	->	1

graph-form description
(example.gv)



(a,b): b=a+1

inputed words:

a = 101101111
b = 101110000

Overlaps in the Thue-Morse infinite word

Theorem

The Thue-Morse word is overlap-free.

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- The following predicade sais "In $\mathbf{t}$ is overelap of order n, starting at position i":
  $(n > 0) \wedge \forall k : k \leq n \Rightarrow T[i + k] = T[i + n + k]$

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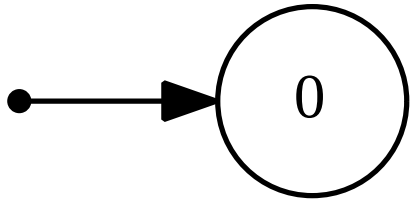
- The resultant automaton:



(i,n): n>0 & Ak k<= n => T[i+k]=T[i+k+n]

- The automaton defines an empty language, so there are no overlaps in the Thue-Morse infinite word.

Overlaps in the Thue-Morse infinite word

- In the file *log* we can find some information about the proces of computing the automaton.

```
n>0 has 2 states: 453ms
 k<=n has 2 states: 0ms
  T[(i+k)]=T[((i+k)+n)] has 12 states: 56ms
   (k<=n=>T[(i+k)]=T[((i+k)+n)]) has 25 states: 2ms
    (A k (k<=n=>T[(i+k)]=T[((i+k)+n)])) has 1 states: 9ms
     (n>0\&(A k (k<=n=>T[(i+k)]=T[((i+k)+n)]))) has 1 states: 0ms
total computation time: 583ms
```

- Computation is devided into several steps following the structure of the predicade.

Squares in the Thue-Morse infinite word

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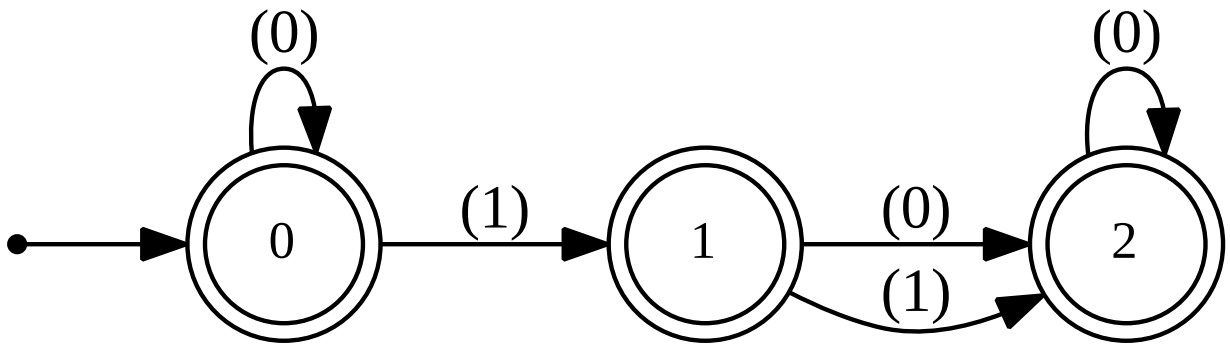

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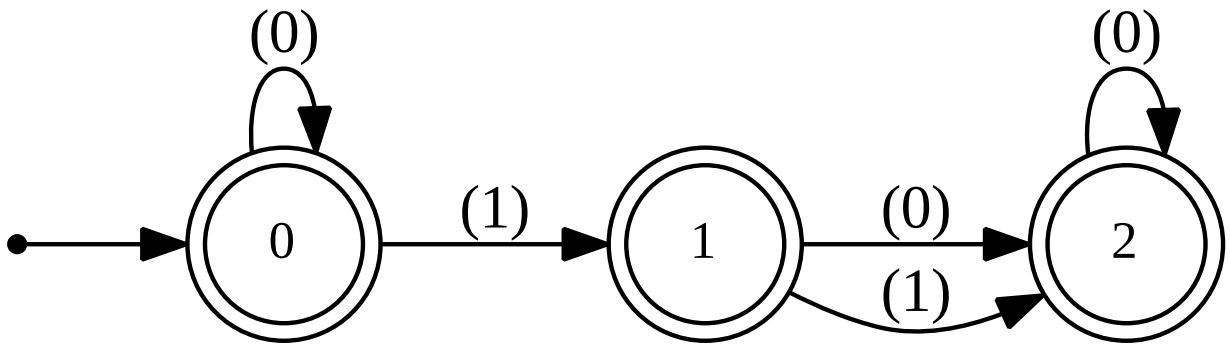

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- In t, there are squares of orders 2^n and 3^n for all $n \in \mathbb{N}^0$.

Comamnd def

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- The difference between *eval* and *def* is that outcome automaton is saved in an automaton library and can be used later.

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- *Example*
  instead of:

```
eval complicatedEq "a = 3*((b+3)/2)";
```

  we can write:

```
def simpleEq "c =(b+3)/2";
eval simpleEq2 "Ec a=3*c & $simpleEq(c,b)";
```

command reg

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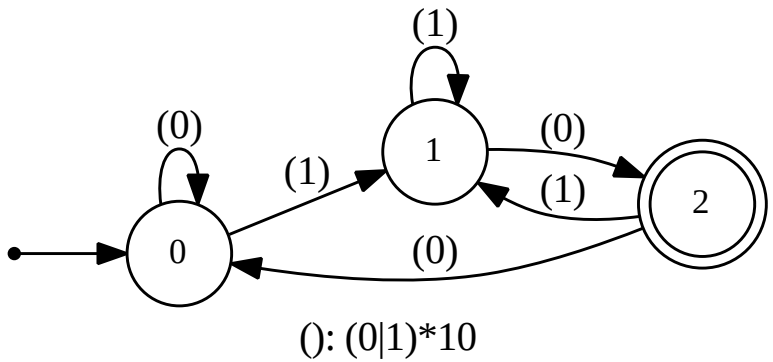
- Command and automaton:

```
reg res2mod4 msd_2"(0|1)*10";
```



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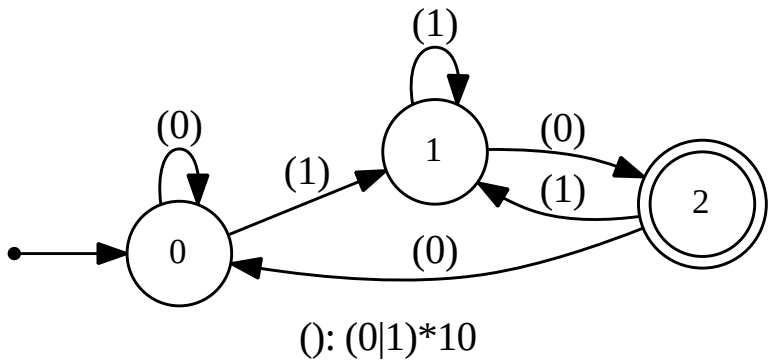

- Evaluation of reg is also saved in an automaton library.

Fibonacci base

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- So we can uniquely express every natural number in Fibonacci base.
- Example:

$$85321$$

$$10_{dec} = 10010_{fib} = 8_{dec} + 2_{dec}$$

Example in Fibonacci base

- What orders are squares in the Fibonacci word?
- We definine Fibonacci base by typing *?msd_fib* at the begining of predicade parametr.
- Command:

```
orderOfSquareInF "?msd_fib Ei
(Ak k<n => F[i+k]=F[i+n+k])";
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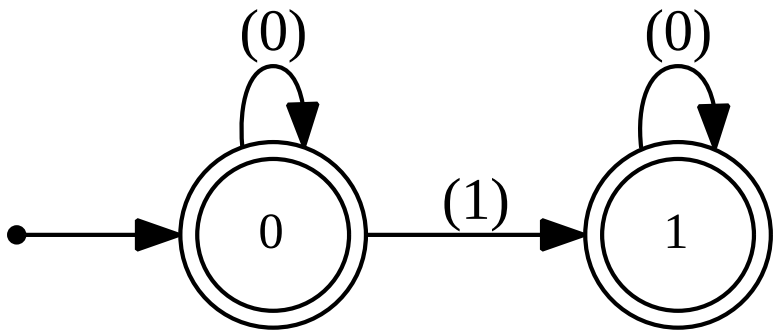
- automaton:



(n): ?msd_fib Ei Ak k<n => F[i+k]=F[i+n+k]

- The automaton defines language 0^*10^*, whichs are exactly Fibonacci numbers.

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- Intuitively, periodicity would imply, that frequency of 0 is rational. However, we know it is $\frac{1}{\varphi}$, where φ is golden ratio, which is irrational number.
- The predicade, coding the theorem:
  $(p > 0) \wedge \exists n \forall i \geq n : F[i] = F[i + p]$
- Log:

```
p>0 has 3 states: 0ms
p>0 has 2 states: 15ms
 i>=n has 2 states: 0ms
  F[i]=F[(i+p)] has 10 states: 63ms
   (i>=n=>F[i]=F[(i+p)]) has 22 states: 15ms
    (A i (i>=n=>F[i]=F[(i+p)])) has 1 states: 250ms
     (E n (A i (i>=n=>F[i]=F[(i+p)]))) has 1 states: 0ms
      (p>0&(E n (A i (i>=n=>F[i]=F[(i+p)])))) has 1 states: 0ms
total computation time: 32ms
```

Palindromes in the infinity Fibonacci word

Theorem

The infinite Fibonacci word $\mathbf{f}$ *has exactly one palindromic factor of length* n *for* n *even and exactly two palindromic factor of length* n *for* n *odd.*

- Theorem is proved by Walnut in two steps. Firstly we verify the even case, than the odd case.

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- Theorem is proved by Walnut in two steps. Firstly we verify the even case, than the odd case.
- The following predicade defines n such that f has exactly one palindromic factor of lenght n

$$\exists i(\forall t < n : f[i+t] = f[i+n-1-t]) \wedge$$

$$\forall j(\forall s < n : f[j+s] = f[j+n-1-s]) \Rightarrow$$

$$\Rightarrow (\forall u < n : f[i+u] = f[j+u])$$

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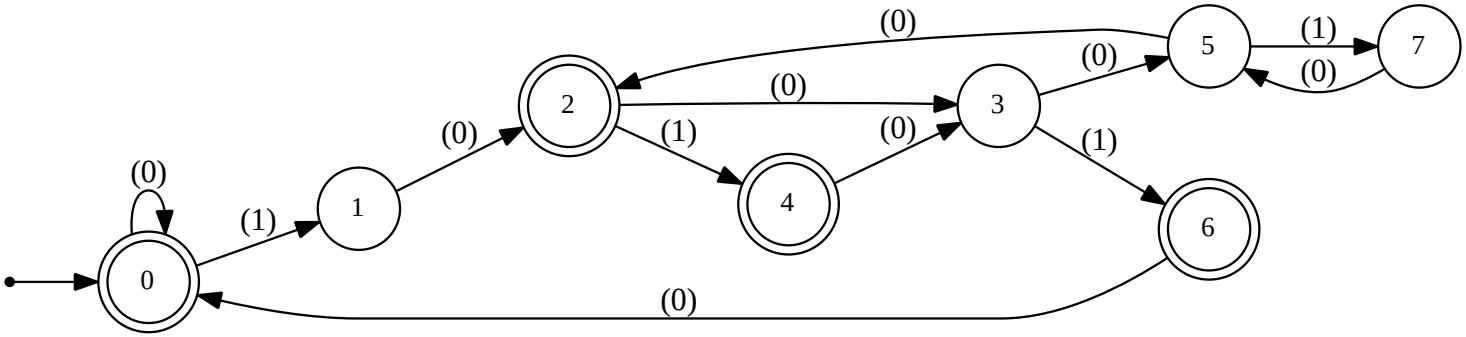

(n): ?msd_fib Ei((At t<n => F[i+t]=F[i+n-1-t])& (Aj (As s<=n => F[j+s]=F[j+n-1-s])=> (Au u<n => F[i+u]=F[j+u])))

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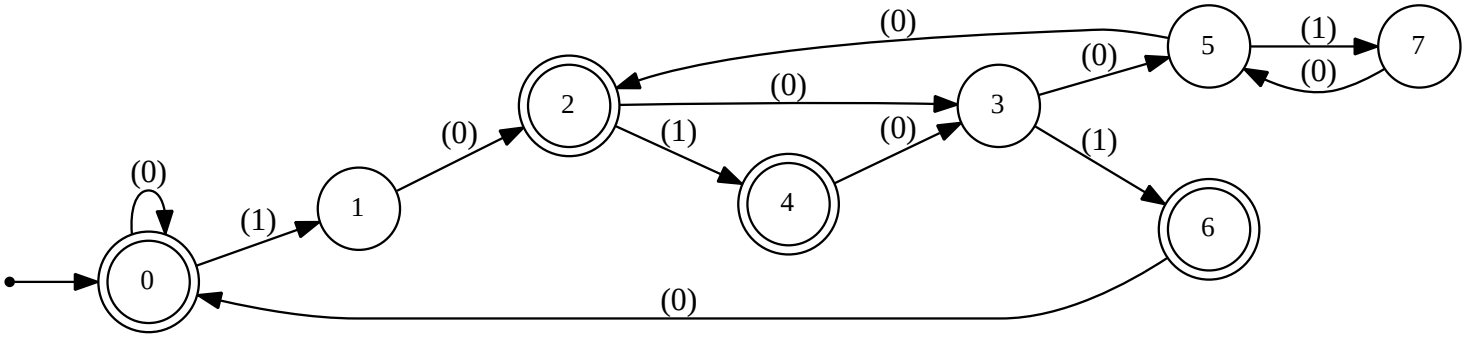

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- We can easily check, that this automaton defines even numbers, by compute automaton of predicade $\exists m(n = 2 * m)$ in Fibonacci base and compare them.
- The case of odd palindromes could be verify in similar way.

Antipalindromes in the infinity Fibonacci word

Definition

Word $x \in \{0,1\}^*$ is antipalindrome if $x = \bar{x}^R$. That is x is invariant under composition of reverse and negation.

Theorem

The only nonempty antipalindromes in $\mathbf{f}$ *are* 01, 10, 0101 *and* 1010.

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Theorem

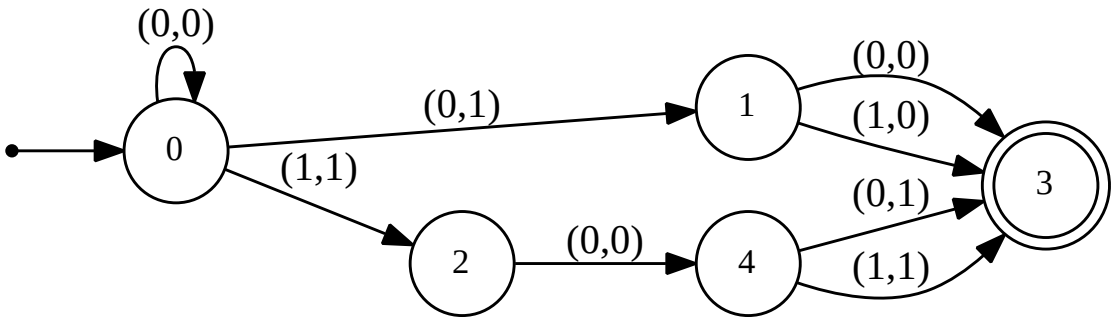
The only nonempty antipalindromes in $\mathbf{f}$ *are* 01, 10, 0101 *and* 1010.

- Following predicade sais: There is antipalindrom $f[i, \ldots, i-1+n]$ and it is the first occurence of such antipalindrom in f.

$$(n > 0) \wedge (\forall j < n : f[i+j] \neq f[i+n-1-j]) \wedge$$

$$\wedge (\forall k < i, \exists j < n : f[k+j] \neq f[i+j])$$

One more example



(i,n): ?msd_fib (n>0) & (Aj j<n => F[i+j]!=F[i+n-1-j]) & (Ak (k<i => (Ej j<n & F[k+j]!= F[i+j])))

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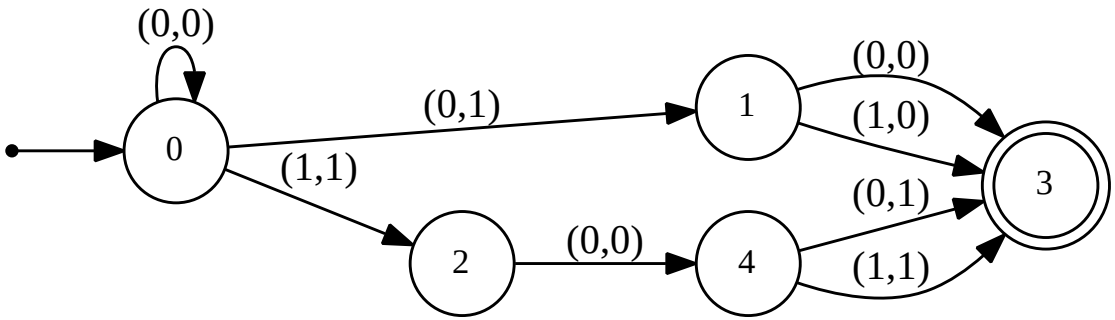

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- It can be easily seen that the automaton accepts pairs: $(0^*101, 0^*100)$, $(0^*101, 0^*101)$, $(0^*10, 0^*00)$ and $(0^*10, 0^*01)$ in Fibonacci base.

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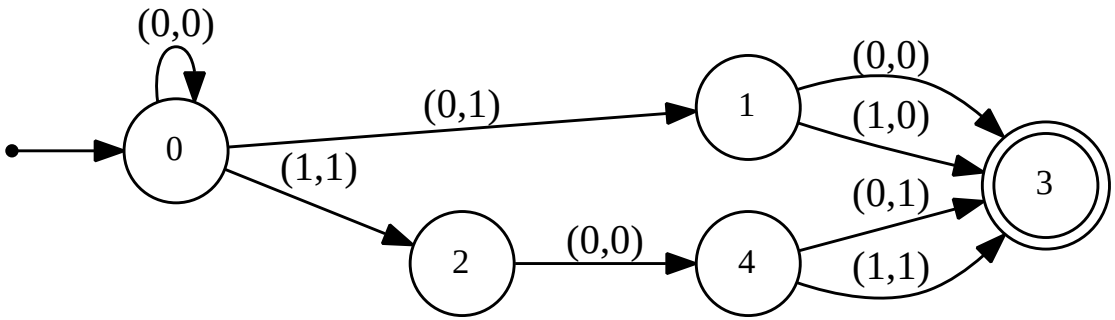

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- In decadic base: $(4, 3)$, $(4, 4)$, $(2, 0)$ and $(2, 1)$.

$$0100101001001010010 1\ldots$$
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