Fully Homomorphic Encryption: A Holy Grail of Cryptography

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Definition of FHE

Definition (Fully Homomorphic Encryption)

FHE is a public key encryption scheme which consists of 4 poly-time algorithms $(\mathsf{K}, \mathsf{E}, \mathsf{D}, \mathsf{V})$ where

- given a security parameter λ, K outputs a keypair (pk, sk),
- given pk and $m \in \mathcal{M}$, E outputs randomized* encryption of m,
- given sk and $c \in \mathcal{C}$, an encryption of m, D outputs m,
- given pk, a function $f : \mathcal{M}^t \to \mathcal{M}$ and $(c_1, \ldots, c_t)$ encryptions of $(m_1, \ldots, m_t)$, V outputs c which encrypts $f(m_1, \ldots, m_t)$.

*To achieve CPA security, public key scheme must be randomized.

Goal of FHE

- poly-time computation $\sim$ evaluation of a poly-time evaluable function
- **computation with encrypted data!**
- e.g. in cloud services

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Example

RSA is homomorphic in $\mathbb{Z}/n\mathbb{Z}$ w.r.t. multiplication:

$$\mathrm{E}(m_1 \cdot m_2) = (m_1 \cdot m_2)^e = m_1^e \cdot m_2^e = \mathrm{E}(m_1) \cdot \mathrm{E}(m_2).$$

- initial idea of FHE by Rivest, Adleman and Dertouzos [4] in 1978
- FHE not known to be even possible for decades
- first FHE by Gentry [1] in 2009
  - enormous computational overhead

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Recent Advances

- since Gentry's breakthrough a very active research area
- proposed schemes often proved to be insecure
  - e.g. Liu [2] and Yagisawa [6], both disproved by Wang [5]
- promising framework by Nuida [3]

Introduction to Nuida's FHE Framework

- operations AND and NOT instead
  - sufficient for any computation
- bits encoded into pairs $(x, y) \in G^2$ where $y \neq 1_G$, G group
  - $0 \sim (1_G, y)$
  - $1 \sim (y, y)$
- operations defined using commutator: $[x, y] = x \cdot y \cdot x^{-1} \cdot y^{-1}$
- underlying group G noncommutative!

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Definition of Operations

$$y \neq 1_G \qquad 0 \sim (1_G, y) \qquad 1 \sim (y, y) \qquad [x, y] = x \cdot y \cdot x^{-1} \cdot y^{-1}$$

- (x_1, y_1) AND $(x_2, y_2) := ([x_1, x_2], [y_1, y_2])^{\dagger\ddagger}$
  - if w.l.o.g. $(x_1, y_1) \sim 0$ i.e. $x_1 = 1_G$
    then $[x_1, x_2] = 1_G$ i.e. $([x_1, x_2], [y_1, y_2]) \sim 0$
  - if $(x_1, y_1) \sim (x_2, y_2) \sim 1$ i.e. $x_1 = y_1$ and $x_2 = y_2$
    then $[x_1, x_2] = [y_1, y_2]$
    - note that $[y_1, y_2] \overset{!}{\neq} 1_G$ i.e. $y_1 \overset{!}{\neq} y_2$ and must not commute

[†] Originally $([g x_1 g^{-1}, x_2], [g y_1 g^{-1}, y_2])$ where g is random. Not necessary.
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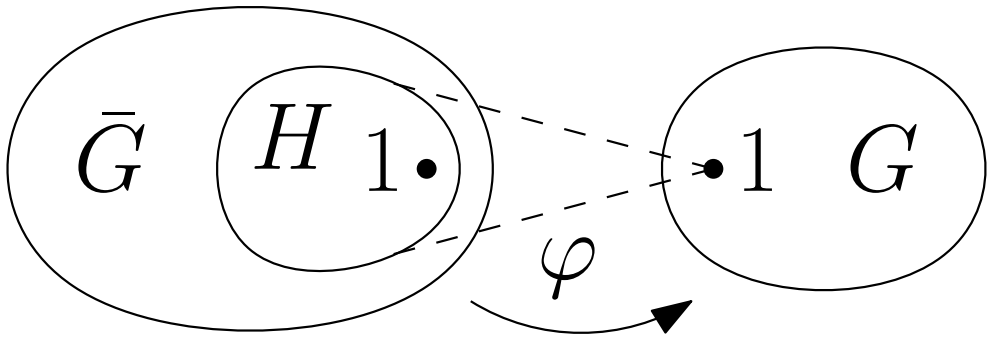
Encryption, Decryption – Most General Setup



$$H = \ker(\varphi)$$

- sofar only encoding, needs encryption s.t. decryption is
  - homomorphic – to preserve operations, and
  - surjective with secret nontriv. kernel – randomization
- let $\varphi : \bar{G} \to G$ be such homomorphism

Encryption, Decryption – Most General Setup

Key generation

- $pk = (h_1, \ldots, h_t, g_1, \ldots, g_u) \in \ker(\varphi)^t \times \bar{G}^u$
- sk – a decisional algorithm $g \overset{?}{\in} \ker(\varphi)$

Encryption

- $h \in \ker(\varphi)$ – a random product of $(h_1, \ldots, h_t)$, and $g \in \bar{G}$ – a random product of $(g_1, \ldots, g_u)$
- $\mathrm{E}(0) = (h, g)$
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Decryption

- $\mathrm{D}(x, y) = x \overset{?}{\notin} \ker(\varphi)$ using sk

Kernel distinguishability is hard $\Rightarrow$ this scheme is secure.

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Nuida's Suggestion – General Setup

Hiding kernel with trapdoor

- let $\bar{\varphi} : \bar{G} \to G$ be a known surjective homomorphism
- take $\tilde{G} > \bar{G}$ and an (inner) automorphism $\tau : \tilde{G} \to \tilde{G}$
  - i.e. conjugation by a secret $t \in \tilde{G} \setminus \bar{G}$
  - $\tau(\tilde{g}) = t^{-1}\tilde{g}t$
- $\varphi : (\tilde{G}) \to G$, $\varphi := \bar{\varphi} \circ \tau$
  - $\ker(\varphi) = \tau^{-1}(\ker(\bar{\varphi})) = t \ker(\bar{\varphi})t^{-1}$

Summary

- $g \in G$ – encoding
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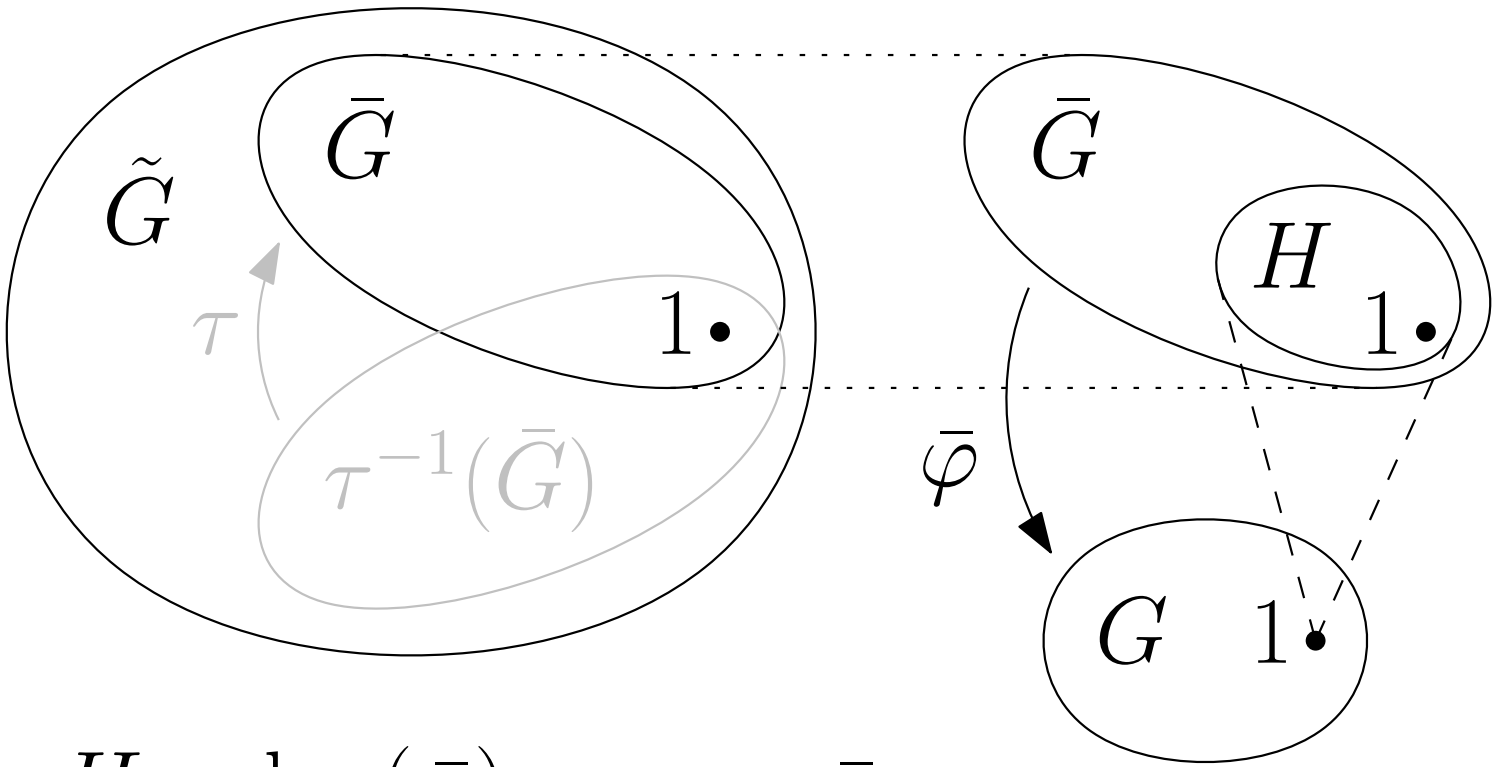
General Setup Depicted



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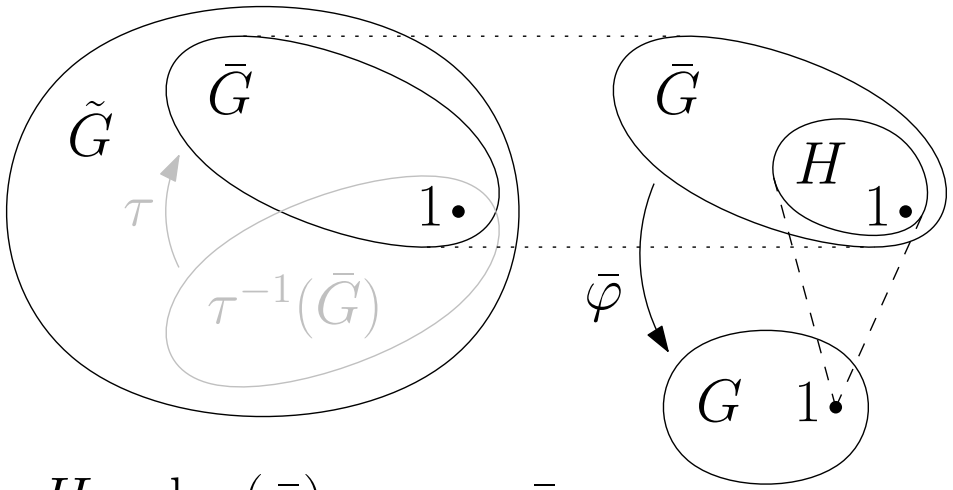
What properties do we need?

Required Properties of General Setup



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- $\ker(\bar{\varphi}) = H \triangleleft \bar{G} < \tilde{G}$
  $\Rightarrow$ $\bar{G}$ shall have a nontrivial normal subgroup
  - way to achieve: $G = K \times H$
  - (construct $\bar{\varphi}$ using H and 1st homomorphism theorem)
- AND gives $([x_1, x_2], [y_1, y_2])$ i.e. moves to commutator subgroup
  $\Rightarrow$ $\bar{G}$ shall be perfect ($\bar{G}$ equals to its commutator subgroup)

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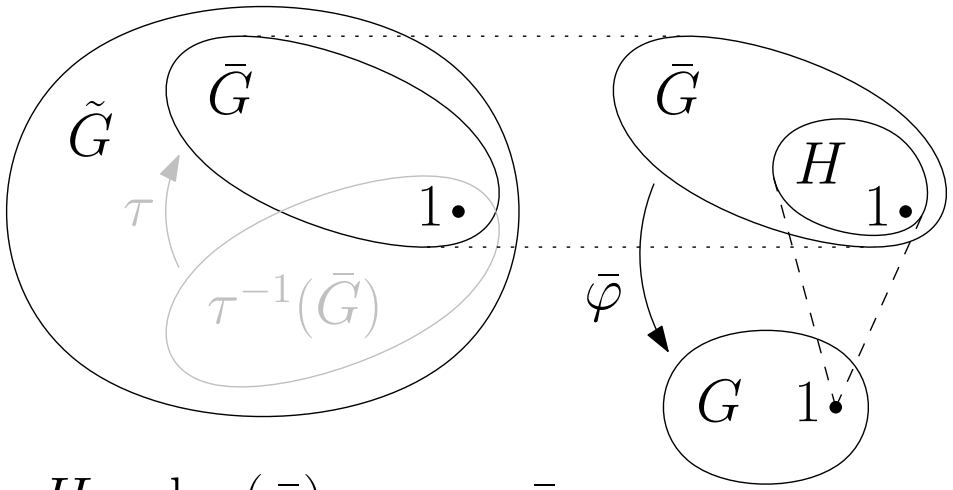

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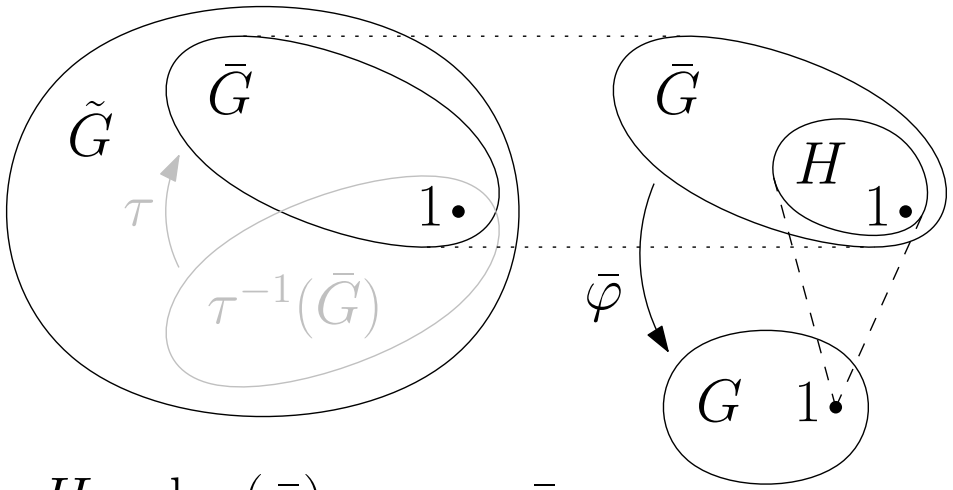

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Proposal – Specific Setup

Nuida mentioned special linear group $SL(2, \mathbb{F})$ – perfect group

$$SL(2, \mathbb{F}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{F}, \det(A) = 1 \right\}.$$

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- $H \triangleleft \bar{G} \ldots \bar{G} = K \times H$
- $\bar{G}, H$ perfect $\ldots K, H = SL(2, \mathbb{F})$, $\bar{G}$ perfect as well
- φ with unknown kernel $\ldots \varphi = \bar{\varphi} \circ \tau$
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  - τ automorphism $\ldots$ inner automorphism i.e. $\tau(\tilde{g}) = t^{-1}\tilde{g}t$, t secret

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- $\bar{G} = K \times H \simeq \left\{ \begin{pmatrix} A_1 & \Theta \\ \Theta & A_2 \end{pmatrix} \middle| A_{1,2} \in SL(2, \mathbb{F}) \right\}$, $\Theta =$ zero matrix
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Cryptanalysis of Specific Setup

Does $\tau(M) := T^{-1}MT$

meet security requirements?

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- problem to be hard: given $M \in \tau^{-1}(\bar{G})$, decide $M \stackrel{?}{\in} \ker(\varphi) = \tau^{-1}(H) = THT^{-1}$
- such $M = T \begin{pmatrix} R & \Theta \\ \Theta & S \end{pmatrix} T^{-1}$ for some $R, S \in SL(2, \mathbb{F})$ and secret $T \in \tilde{G}$
- $\Rightarrow$ decide $R \stackrel{?}{=} I$

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There is an effective way to decide the previous decision problem without the knowledge of T with overwhelming probability.

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Proof.

Note that

$$M - I = T \begin{pmatrix} R & \Theta \\ \Theta & S \end{pmatrix} T^{-1} - TT^{-1} = T \begin{pmatrix} R - I & \Theta \\ \Theta & S - I \end{pmatrix} T^{-1}.$$

Here if $R = I$, then the resulting matrix $M - I$ has rank ≤ 2.
So if $\mathrm{rank}(M - I) \leq 2$, then $R = I$ with overwhelming probability since R, S are pseudorandom with determinant $= 1$. (The other options are $S = I$ or $\det(R - I) = \det(S - I) = 0$, both negl.) □

Cryptanalysis of Specific Setup

$$\Rightarrow \text{testing } \operatorname{rank}(M - I) \overset{?}{\leq} 2$$

leads to plaintext recovery w.h.p.

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