Coextensions of totally ordered monoids

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Basic definitions

Every structure in this talk will be a commutative one.

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Definition

A quadruple $(S; \cdot, \leq, 1)$ is called a *partially ordered monoid*, or shortly a *pomonoid*, if the reduct $(S; \cdot, 1)$ is a commutative monoid and $\leq$ is a partial order on S, such that $x \leq y$ implies $xz \leq yz$ for all $x, y, z \in S$.

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- A pomonoid S is *negative* if $x \leq 1$ for all $x \in S$,
- *togroup* - a tomonoid, where $(S; \cdot, 1)$ is a group.

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- standard MV-algebra, Chang's MV-algebra,
- t-norms.

Definition

A *t-norm* is a function $\odot : [0,1] \times [0,1] \to [0,1]$ which satisfies for all $x, y, z, w \in [0,1]$

- $x \odot y = y \odot x$,
- $x \odot y \leq z \odot w$ if $x \leq z$ and $y \leq w$,
- $x \odot (y \odot z) = (x \odot y) \odot z$,
- $x \odot 1 = x$.

Examples - Gödel, product, Łukasiewitz.

Definition

By a *homomorphism* φ *of pomonoids* $(E; \cdot_E, \leq_E, 1_E), (S; \cdot_S, \leq_S, 1_S)$, resp. *of tomonoids*, we mean an order preserving homomorphism of monoids $\varphi : E \to S$, i.e. $x \leq_E y$ implies $\varphi(x) \leq_S \varphi(y)$ for all $x, y \in E$.

Definition

Let $(E; \cdot_E, 1_E)$ and $(S; \cdot_S, 1_S)$ be monoids such that there exists a surjective homomorphism $\pi : E \to S$. Then we call E a *monoid coextension* of S.

Definition

Let $(E; \cdot_E, \leq_E, 1_E)$ and $(S; \cdot_S, \leq_S, 1_S)$ be pomonoids, resp. tomonoids, such that there exists a surjective pomonoid homomorphism $\pi : E \to S$. Then we call E a *pomonoid coexstension* , resp. *tomonoid coextension of* S. In that case π is called a *natural projection* of coextension E onto S.

A direct system

Definition

Let $(S, \leq)$ be a directed set. Let $M = (M_a)_{a \in S}$ be a set of objects and let $f = \{f^a_b : M_a \to M_b \mid a \leq b, a, b \in S\}$ be a set of homomorphisms such that

(i) $f^a_a = \mathrm{id}_{M_a}$,

(ii) $f^a_c = f^b_c \circ f^a_b$

for every $a, b, c \in S, a \leq b \leq c$. Then the pair (M, f) is called a *direct system* over $(S, \leq)$.

A natural order

Definition

On every monoid $(S; \cdot, 1)$, we can define a *natural preoder* $\leq_{\mathcal{H}}$ by

$$a \leq_{\mathcal{H}} b \text{ if there exists } c \in S, \text{ such that } a = bc. \qquad (1)$$

If a pomonoid $(S; \cdot, \leq, 1)$ is negative, then $x \leq_{\mathcal{H}} y$ implies $x \leq y$ and $\leq_{\mathcal{H}}$ is a partial order.

Let S be a semigroup.

- $\mathcal{H}$*-class* H – a congruence class H of congruence $\sim_{\mathcal{H}}$ induced by a natural preoder $\leq_{\mathcal{H}}$,

Marcel-Paul Schützenberger

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- *$\mathcal{H}$-class H* – a congruence class H of congruence $\sim_{\mathcal{H}}$ induced by a natural preoder $\leq_{\mathcal{H}}$,
- *a Schützenberger group $\overline{H}$* of $\mathcal{H}$-class H – group of all transformations on H induced by elements $t \in S^1$ such that $tH \subseteq H$.

Definition

Let $(S; \cdot, 1)$ be a monoid such that $\leq_{\mathcal{H}}$ is a partial order and (G, γ) a direct system of groups over $(S, \leq_{\mathcal{H}})$. A *monoid coextension* E of a S by the direct system of groups (G, γ) consists of a monoid coextension E of S and for each $a \in S$ an action $\cdot$ of G_a on congruence class E_a such that

(i) for any $x, y \in E_a$, there exists unique $g \in G_a$ such that $g \cdot x = y$ and

(ii) for any $x \in E_a, y \in E_b$ and $g \in G_a$ it holds $(g \cdot x) + y = \gamma_b^a(g) \cdot (x + y)$.

An unordered case

Theorem

Let $(S;\cdot,1)$ be a commutative monoid, (G,φ) be a direct system of groups over $(S,\leq_{\mathcal{H}})$ and $\sigma=(\sigma_{a,b})_{a,b\in S}$ where $\sigma_{a,b}\in G_{ab}$ and

$$\sigma_{a,b}=\sigma_{b,a} \tag{2}$$

$$\varphi^{ab}_{abc}\sigma_{a,b}+\sigma_{ab,c}=\sigma_{a,bc}+\varphi^{bc}_{abc}\sigma_{b,c} \tag{3}$$

for all $a,b,c\in S$. Let $E[G,\sigma]$ be a set of all ordered pairs (a,x) with $a\in S$, $x\in G_a$, with multiplication

$$(a,x)(b,y)=(ab,\varphi^{a}_{ab}x+\varphi^{b}_{ab}y+\sigma_{a,b}). \tag{4}$$

Then $E[G,\sigma]$ is a coextension of S by the direct system of groups (G,φ) with a factor set σ.
If conversely E is a coextension of S by some direct system of groups (G,φ) with a factor set σ, then E is equivalent to $E[G,\sigma]$.

Definition

Let $(G; +, \leq, 0)$ be a togroup. We say that G is *Archimedean* if for any $x, y \in G$ such that $0 < x, y$, there exists $n \in \mathbb{N}$ such that $x \leq ny$.

Let $S = (S; \cdot, \leq, 1)$ be a negative pomonoid. Let (G, φ) be a direct system of togroups on $(S, \leq_{\mathcal{H}})$, such that for every $a \in S$, G_a is Archimedean. For every $a, b \in S$, let us have $\sigma_{a,b} \in G_{ab}$ and let $E[S, G, \sigma]$ be the set of all ordered pairs (a, x), $x \in G_a, a \in S$. Let us define an operation

$$(a, x)(b, y) := (ab, \varphi^a_{ab}(x) + \varphi^b_{ab}(y) + \sigma_{a,b})$$

and a relation $\leq_E$ lexicographically i.e.

$$(a, x) \leq_E (b, y) \text{ if } a \leq b \text{ or } a = b \text{ and } x \leq_a y.$$

An ordered case - groups

Theorem

A set $E[S, G, \sigma]$ with the defined operation is a pomonoid coextension of S if and only if for all $a, b, c \in S$ the following conditions are satisfied

(1) $\sigma_{a,b} = \sigma_{b,a}$,

(2) $\varphi^{ab}_{abc}\sigma_{a,b} + \sigma_{ab,c} = \sigma_{a,bc} + \varphi^{bc}_{abc}\sigma_{b,c}$,

(3) $G_1 = \{0\}$,

(4) *if* $a < b$ *and* $ca = cb$ *then* $\varphi^{a}_{ca} = \varphi^{b}_{cb} = 0$ *and* $\sigma_{a,c} \leq_{ab} \sigma_{b,c}$.

Definition

Let $(S; \cdot, \leq, 1)$ be a pomonoid and (G, γ) a direct system of togroups over $(S, \leq_{\mathcal{H}})$. A *pomonoid coextension E of a S by a direct system of togroups* (G, γ) is a monoid coextension E of S by direct system of groups (G, γ) such that E is a pomonoid coextension of S.

Theorem

Let S be a commutative pomonoid and $E[S, G, \sigma]$ be a pomonoid coextension of S. Then $E[S, G, \sigma]$ is a pomonoid coextension of S by a direct system of togroups (G, φ). On the other side, every pomonoid coextension E' of S' by a direct system of togroups (G', φ') is isomorphic to one given by construction from the previous theorem, i.e $E[S', G', \sigma']$.

An example

Example

- Let $S = L_4$, i.e. $L_4 = \{0, -1, -2, -3\}$ with the operation $a +_{L_4} b := (a + b) \vee -3$.
- Let G be given by $G_0 = G_{-3} = \{0\}$ and $G_a = \mathbb{R}$ otherwise considered with the usual multiplication and order of reals.
- The only homomorphism we can choose is φ^{-1}_{-2}.
- The only admissible nonzero $\sigma_{a,b}$ is $\sigma_{-1,-1}$ and it can be chosen in an arbitrary way.

Let $S = (S; \cdot, \leq, 1)$ be a negative pomonoid. Let (M, φ) be a direct system of tomonoids over $(S, \leq_{\mathcal{H}})$, $\sigma = (\sigma_{a,b})_{a,b \in S}$ where $\sigma_{a,b} \in M_{ab}$.
We define $E[S, M, \sigma]$ as a set of all ordered pairs (a, x), $x \in M_a, a \in S$ with operation given for every $(a, x), (b, y) \in E[S, M, \sigma]$ by

$$(a, x)(b, y) := (ab, \varphi^{a}_{ab}(x) + \varphi^{b}_{ab}(y) + \sigma_{a,b}) \tag{5}$$

and a relation $\leq_E$ lexicographically i.e.

$$(a, x) \leq_E (b, y) \text{ if } a \leq b \text{ or } a = b \text{ and } x \leq_a y. \tag{6}$$

Theorem

Then $E[S, M, \sigma]$ is a negative pomonoid coextension of S if and only if the following conditions are satisfied for all $a, b, c \in S$

(M1) $\sigma_{a,b} = \sigma_{b,a}$,

(M2) $\varphi^{ab}_{abc}\sigma_{a,b} + \sigma_{ab,c} = \sigma_{a,bc} + \varphi^{bc}_{abc}\sigma_{b,c}$,

(M3) $\sigma_{1,a} = 0$,

(M4) *M_1 is negative tomonoid,*

(M5) *if $a < b$ and $ca = cb$, then $\varphi^{a}_{ca}(x) + \sigma_{a,c} \leq_{ca} \varphi^{b}_{cb}(y) + \sigma_{a,b}$.*

The example of the pomonoid coextension

Example

- Let $S = L_5$, i.e. $L_5 = \{0, -1, -2, -3, -4\}$ with $a +_{L_5} b := (a + b) \vee -4$.
- Let M be given by $M_{-4} = \{0\}$ and $M_a = (0, 1] \subseteq \mathbb{R}$ for all $a \in S - \{-4\}$ considered with the usual multiplication of reals.
- We chose the homomorphisms as identity embeddings, i.e. $\varphi^a_b : (0, 1] \to (0, 1], \varphi^a_b(x) = x$ for all $x \in M_a, a \in S, b \in S - \{-4\}, a \leq b$ and $\varphi^a_{-4} = 0$.
- For every σ such that $E[S, M, \sigma]$ is pomonoid coextension it holds $\sigma_{0,a} = 0$ and $\sigma_{-2,-2} = \sigma_{-2,-3} = \sigma_{-3,-3} = \sigma_{-3,-4}$ $= \sigma_{-4,-4} \in M_{-4} = \{0\}$. The only admissible nonzero $\sigma_{a,b}$ are $\sigma_{-1,-1}$ and $\sigma_{-1,-2}$, which can be chosen in an arbitrary way from M_{-2}, resp. M_{-3}.
- Let us choose $\sigma_{-1,-1} := \frac{1}{2}$ and $\sigma_{-1,-2} := \frac{1}{4}$.

An example - Schreier coextension

Let $(S, \cdot, 1)$ be a monoid and $\rho \subseteq S \times S$ a congruence relation on S. Let N_1 be a congruence class of ρ containing a neutral element 1. We call ρ a *normal* if each of its congruence class is in the form aN_1 for some $a \in S$ where a acts simply on N_1. Let us define a *Schreier coextension* $Sch[S, M]$ of monoid S by monoid M as a monoid coextension $Sch[S, M]$ of S such that the congruence $\rho, Sch[S, M]/\rho = S$ is a normal with congruence class N_1 being submonoid isomorphic to M.

An example - Schreier coextension

Theorem

Let $(S; \cdot, \leq_S, 1), (M; +, \leq_M, 0)$ *be a disjoint commutative tomonoids and let* $Sch[S, M]$ *their Schreier coextension with partial order* $\leq$ *on* $Sch[S, M]$ *such that* $(Sch[S, M]; \cdot, \leq, (1, 0))$ *is a negative tomonoid,* $\leq_{|M} = \leq_M$*, and natural projection* $\pi : Sch[S, M] \to S$ *is isotone mapping. Then* $Sch[S, M]$ *can be described as a coextension* $E[S, \mathcal{M}, \sigma] \cong Sch[S, M]$ *of* S *by system* $\mathcal{M} = \{M_a\}_{a \in S}$ *where* $M_a \cong M$ *for all* $a \in S$*.*

Corollary

Let $(S; \cdot_S, \leq_S, 1), (M; +, \leq_M, 0)$ *be a commutative negative tomonoids. A Schreier coextension* $Sch[S, M]$ *can be organized into a tomonoid with respect to partial order* $\leq$ *such that* $\leq_{|M} = \leq_M$ *and natural projection* π *is isotone if and only if for any* $a, b, c \in S$*,* $a <_S b$ *and* $ca = cb$ *holds* $x + \sigma_{a,c} \leq_M y + \sigma_{a,b}$ *for all* $x, y \in M$*.*

Thank you for your attention!