Combinatorics on words and automated proving I

Basic definitions and examples

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Autumn School

Outline

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Alphabet and Words

- alphabet A, letters, words
- length $|w|$, empty word ε, concatenation
- A^* is the set of all (finite) words
- v is a *factor* of a word w when $w = u_1 v u_2$

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Squares and Overlaps

- a *square* is a word of the form ww, $w \in A^*$
- an *overlap* is a word of the form $awawa$, $a \in A$, $w \in A^*$
- *overlap-free* and *square-free* words

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Infinite Words

- considering $a : \mathbb{N} \to A$, we define an *infinite word* as

$$a(0)a(1)a(2)\cdots = a_0 a_1 a_2 \ldots$$

- A^ω is the set of all infinite words over A
- an infinite words has a property P if all its factors do

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Fibonacci Word

Over the two-letters alphabet $A = \{a, b\}$ we define:

Definition

The *Fibonacci word* is the infinite word obtained as the limit

$$S_\infty = \lim_{n \to \infty} S_n$$

where

$$S_0 = a, \quad S_1 = ab, \quad S_n = S_{n-1} S_{n-2} \text{ for } n \geq 2.$$

Fibonacci Word

S_0	a
S_1	ab
S_2	aba
S_3	$abaab$
S_4	$abaababa$
S_5	$abaababaabaab$
	$\vdots$
S_∞	$abaababaabaababaababaabaababaabaab\ldots$

Fibonacci Word

From the construction (similar to Fibonacci numbers) can be easily seen:

- the length of S_n is F_{n+2}
- the number of a's (b's) in S_n is F_{n+1} (F_n)

		$\|S_n\|$	$\#a$	$\#b$
S_0	a	$F_2 = 1$	$F_1 = 1$	$F_0 = 0$
S_1	ab	$F_3 = 2$	$F_2 = 1$	$F_1 = 1$
S_2	aba	$F_4 = 3$	$F_3 = 2$	$F_2 = 1$
S_3	$abaab$	$F_5 = 5$	$F_4 = 3$	$F_3 = 2$
S_4	$abaababa$	$F_6 = 8$	$F_5 = 5$	$F_4 = 3$
S_5	$abaababaabaab$	$F_7 = 13$	$F_6 = 8$	$F_5 = 5$

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S_0	a	$F_2 = 1$	$F_1 = 1$	$F_0 = 0$
S_1	ab	$F_3 = 2$	$F_2 = 1$	$F_1 = 1$
S_2	aba	$F_4 = 3$	$F_3 = 2$	$F_2 = 1$
S_3	$abaab$	$F_5 = 5$	$F_4 = 3$	$F_3 = 2$
S_4	$abaababa$	$F_6 = 8$	$F_5 = 5$	$F_4 = 3$
S_5	$abaababaabaab$	$F_7 = 13$	$F_6 = 8$	$F_5 = 5$

Fibonacci Word

Some other properties:

- $\phi = \frac{|S_\infty|}{\#a} = \frac{\#a}{\#b}$
- hiding last two letters of any S_n, we get a *palindrom*
  - e.g., $S_5 = abaababaaba\cancel{ab}$
- the number $0.010010100\ldots$, whose decimals are built with the digits ($a \leftrightarrow 0$, $b \leftrightarrow 1$) of S_∞, is transcendental

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Thue-Morse Word

Again, let $A = \{a, b\}$ and $\overline{w}$ denote a word made from w by exchanging a's and b's.

Definition

We define the *Thue-Morse word* as the limit

$$\mathbf{t} = \lim_{n \to \infty} U_n$$

where

$$U_0 = a, \quad U_n = U_{n-1}\overline{U}_{n-1} \text{ for } n \geq 1.$$

Thue-Morse Word

$$
\begin{array}{ll}
U_0 & a \\
U_1 & ab \\
U_2 & abba \\
U_3 & abbabaab \\
U_4 & abbabaabbaababba \\
U_5 & abbabaabbaababbabaababbaabbabaab \\
 & \vdots \\
\mathbf{t} = & abbabaabbaababbabaababbaabbabaabbaababba\ldots
\end{array}
$$

Outline

Bitwise Negation

For convenience, let $A = \{0, 1\}$.

Definition (1)

We define the *Thue-Morse word* as the limit

$$\mathbf{t} = \lim_{n \to \infty} U_n$$

where

$$U_0 = 0, \quad U_n = U_{n-1}\overline{U_{n-1}} \text{ for } n \geq 1.$$

U_0	0
U_1	01
U_2	0110
U_3	01101001
U_4	0110100110010110

Binary Expansion

Definition (2)

Let

$$t_n = \begin{cases} 0 & \text{if \#1 in binary expansion of } n \text{ is even,} \\ 1 & \text{if \#1 is odd.} \end{cases}$$

Then $\mathbf{t} = t_0 t_1 t_2 \dots$ is the Thue-Morse word.

n	n_2	#1	t_n
0	0	0	0
1	1	1	1
2	10	1	1
3	11	2	0
4	100	1	1
5	101	2	0
6	110	2	0

Recurrence Relation

Definition (3)

Let

$$t_0 = 0, \quad t_{2n} = t_n, \quad t_{2n+1} = 1 - t_n.$$

Then $\mathbf{t} = t_0 t_1 t_2 \dots$ is the Thue-Morse word.

$$\boxed{\begin{array}{ll} t_0 & = 0 \\ t_1 = 1 - t_0 & = 1 \\ t_2 = t_1 & = 1 \\ t_3 = 1 - t_1 & = 0 \\ t_4 = t_2 & = 1 \\ t_5 = 1 - t_2 & = 0 \\ t_6 = t_3 & = 0 \\ t_7 = 1 - t_3 & = 1 \end{array}}$$

Morphism

Definition (4)

Let $\mu : A^* \to A^*$ be a morphism (i.e., mapping satisfying $\mu(xy) = \mu(x)\mu(y)$ for all $x, y \in A^*$) such that

$$\mu(0) = 01, \quad \mu(1) = 10.$$

Then the Thue-Morse word is $\mathbf{t} = lim_{n\to\infty}\mu^n(0)$.

$\mu(0)$	= 01
$\mu^2(0) = \mu(\mu(0)) = \mu(01)$	= 0110
$\mu^3(0) = \mu(\mu^2(0))$	= 01101001
$\mu^4(0) = \mu(\mu^3(0))$	= 0110100110010110

Power Series

Definition (5)

Let

$$\begin{aligned}\prod_{i=0}^{\infty}(1-X^{2^i}) &= (1-X)(1-X^2)(1-X^4)\dots\\ &= 1-X-X^2+X^3-X^4+\dots\\ &= \sum_{n=0}^{\infty}(-1)^{t_n}X^n.\end{aligned}$$

Then $\mathbf{t} = t_0 t_1 t_2 \dots$ is the Thue-Morse word.

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Overlap-free

Proposition

The Thue-Morse word is overlap-free.

Proof.

- for contradiction, let $wxuxux$ be a left factor of $\mathbf{t} = t_0 t_1 t_2 \ldots$ where $x \in A,\ u, w \in A^* : |\underbrace{\ w\ }_{k}|\underbrace{x|\ u\ }_{m}|\underbrace{x|\ u\ }_{m}|\underbrace{\ x\ }_{1}|$
- let $|w| = k$ and $|xu| = m$, then $t_{k+j} = t_{k+m+j}$ for $0 \leq j \leq m$
- assume $m \geq 1$ is as small as possible
- $m = 2m'$ is even:
  - $k = 2k'$ is even:
    - $t_{k+j} = t_{k+m+j}$, $0 \leq j \leq m$, implies $t_{k+2j'} = t_{k+m+2j'}$, $0 \leq j' \leq m'$
    - hence, $t_{2k'+2j'} = t_{2k'+2m'+2j'}$ for $0 \leq j' \leq m'$
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Observation.

For $n \geq 1$, we define $b_n = t_n \oplus t_{n-1}$.
Firstly, observe that $b_{4n+2} = 0$ since, in binary, $4n+1$ and $4n+2$ have only switched last two bits, and thus $t_{4n+2} = t_{4n+1}$. Secondly, $b_{2n+1} = 1$ since $2n+1$ and $2n$ differs in binary exactly in one bit.

Proof: (a) $m \geq 5$, odd.

- $b_{k+j} = b_{k+m+j}$ for $1 \leq j \leq m$
- since $m \geq 5$, we can choose j such that $k + j \equiv 2 \mod 4$
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For $n \geq 1$, we define $b_n = t_n \oplus t_{n-1}$.
Firstly, observe that $b_{4n+2} = 0$ since, in binary, $4n+1$ and $4n+2$ have only switched last two bits, and thus $t_{4n+2} = t_{4n+1}$.
Secondly, $b_{2n+1} = 1$ since $2n+1$ and $2n$ differs in binary exactly in one bit.

Proof: (a) $m \geq 5$, odd.

- $b_{k+j} = b_{k+m+j}$ for $1 \leq j \leq m$
- since $m \geq 5$, we can choose j such that $k+j \equiv 2 \mod 4$
- for such j then $b_{k+j} = 0$
- but $b_{k+j+m} = 1$ since $k+j+m$ is odd $\Rightarrow$ contradiction

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Proof: (b) $m = 3$.

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- choose j such that $k + j \equiv 2$ or $3 \mod 4$
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- if $k + j \equiv 3 \mod 4$, then $b_{k+j} = 1$, while $b_{k+j+3} = 0 \Rightarrow$ contradiction

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Outline

Length of Squares

Proposition

Squares of the Thue-Morse word have length either 2^k or $3 \cdot 2^k$.

Proof.

- suppose wuu is a left factor of $\mathbf{t}$ where $u, w \in A^*$
- let $|w|$ be odd
- let $w = v\bar{x}$ where $x \in A$ is a letter
- $|u|$ is even:
  - $u = u_1 u_2 \dots u_{2n}$
  - then $\bar{x} u_1, u_2 u_3, \dots, u_{2n} u_1 \in \{ab, ba\}$
  - therefore $u_1 = x$ and $u_{2n} = \bar{x}$ so that
    $wuu = \underbrace{v\bar{x}}_{w} \underbrace{x u_2 \dots u_{2n-1} \bar{x}}_{u} \underbrace{x u_2 \dots u_{2n-1} \bar{x}}_{u}$
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  - let $\mu(w'u'u') = wuu$
  - repeat until $|w'|$ is odd (k-times)
  - thus, $u = \mu^k(x)$ or $u = \mu^k(x\bar{x}x)$, $x \in \{a, b\}$
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Other Properties of the Thue-Morse Word

The Thue-Morse word $\mathbf{t} = t_0 t_1 t_2 \ldots$ satisfies the following. Let

$$I = \{0 \le i < 2^N : t_i = 0\},$$
$$J = \{0 \le j < 2^N : t_j = 1\}.$$

Then for $0 \le k < N$, we have

$$\sum_{i \in I} i^k = \sum_{j \in J} j^k.$$

For example, let $N = 3$. Then

$$0^k + 3^k + 5^k + 6^k = 1^k + 2^k + 4^k + 7^k$$

for $k = 0, 1, 2$.

Other Properties of the Thue-Morse Word

The magic square of order T can be obtained from the Thue-Morse word as follows:
Firstly, denote (lexicographically) cells of the magic square by numbers 0 to $T^2 - 1$. Secondly, insert number $n + 1$ into the cell n if $t_n = 1$ for every $0 \leq n \leq T^2 - 1$. Thirdly, insert number $T^2 - n$ into the cell $T^2 - 1 - n$ if $t_n = 0$ for every $0 \leq n \leq T^2$.

Example: For $T = 4$, we take $t_0 \dots t_{15} =$

$$0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0$$

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Bibliography

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Thank you for your attention!