Gabriel's Theorem

Part I

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Introduction

- Goal: Gabriel's theorem
  - Formulate
  - Partially prove (in part II)

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- Goal: Gabriel's theorem
  - Formulate
  - Partially prove (in part II)
- In this part:
  - Basic definitions - quivers, representations, underlying graphs, ...
  - Dynkin diagrams, Euclidean diagrams
  - Lemmas and theorems needed for the proof of Gabriel's theorem

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- Let V be a vector space
- Let $U, W \subseteq V$ be subspaces of V

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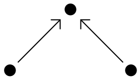

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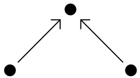

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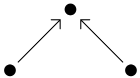

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  - more subspaces - for three still finitely many; for four infinitely many
  - not just subspaces
- Gabriel's theorem - "When do we have only finitely many indecomposable possibilities up an isomorphism"

Quivers

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Definition (Quiver)

A quiver is a quadruple (Q_0, Q_1, s, t), where

- Q_0 is a finite set of vertices,
- Q_1 is a finite set of arrows,
- $s : Q_0 \to Q_1$ is a map which denotes where does an arrow start,
- $t : Q_0 \to Q_1$ is a map which denotes where does an arrow terminate.

For an arrow $\alpha \in Q_1$ we sometimes write $\alpha : s(\alpha) \to t(\alpha)$.

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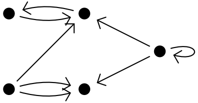
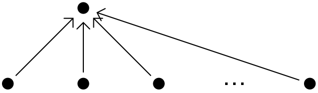
- We can naturally define oriented paths and oriented cycles.

Quivers

Examples



an arbitrary quiver



n-subspace quiver

Jordan quiver

Kronecker quiver

Representations

- Let K be a field. We fix that field throughout the presentation.
- When we talk about vector space, we mean vector space over K.

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Definition (Representation)

Let Q be a quiver. A representation of Q is a collection

$$X = (X_i, X_\alpha)_{i \in Q_0,\, \alpha \in Q_1},$$

where:

- X_i is a vector space for each vertex $i \in Q_0$,
- $X_\alpha : X_{s(\alpha)} \to X_{t(\alpha)}$ is a linear map for each arrow $\alpha \in Q_1$.

A representation is finite dimensional if each vector space X_i is finite dimensional.

Morphisms

Definition (Morphism)

Let Q be a quiver. Let X, Y be representations of Q.
A morphism $\phi : X \to Y$ of these representations is a collection

$$\phi = (\phi_i)_{i \in Q_0}$$

of linear maps $\phi_i : X_i \to Y_i$ for each vertex i, such that for each arrow α the following diagram commutes:

$$\begin{tikzcd} X_{s(\alpha)} \arrow[r, "\phi_{s(\alpha)}"] \arrow[d, "X_\alpha"] & Y_{s(\alpha)} \arrow[d, "Y_\alpha"] \\ X_{t(\alpha)} \arrow[r, "\phi_{t(\alpha)}"] & Y_{t(\alpha)} \end{tikzcd}$$

(That is such that $Y_\alpha \phi_{s(\alpha)} = \phi_{t(\alpha)} X_\alpha$)

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$$\begin{tikzcd}
X_i \arrow[r, "\phi_i"] \arrow[d, "X_\alpha"] & Y_i \arrow[r, "\psi_i"] \arrow[d, "Y_\alpha"] & Z_i \arrow[d, "Z_\alpha"] \\
X_j \arrow[r, "\phi_j"] & Y_j \arrow[r, "\psi_j"] & Z_j
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Morphisms

- Let X, Y be representations of a quiver Q
- The composition of two morphisms $\phi, \psi : X \to Y$ is given by

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- There is an identity morphism $\mathrm{id}_X : X \to X$ given by

$$(\mathrm{id}_X)_i = \mathrm{id}_{X_i}$$

The category of representations

- For a given quiver Q we have
  - representations of Q
  - morphisms between representations of Q
  - morphisms can be composed and an identity morphism exists

Definition (The category of representations)

For a quiver Q we denote by $\mathrm{Rep}_K(Q)$ a category of representations of Q where

- objects are representations of Q,
- morphisms are the morphisms we have defined before.

We denote by $\mathrm{rep}_K(Q)$ the full subcategory with objects the finite dimensional representations.

Point-wise definitions for representations and morphisms

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Definition

- Let X, Y be two representations of a quiver Q.
- We call X a subrepresentation of Y and write $X \subseteq Y$ if
  - X_i is a subspace of Y_i for each vertex i
  - $X_\alpha(x) = Y_\alpha(x)$ for each arrow α and $x \in X_{s(\alpha)}$.
  - (i. e. subrepresentation is a morphism of inclusions)

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- Kernel of a morphism $\phi : X \to Y$ is a subrepresentation $\mathrm{Ker}\,\phi \subseteq X$ given by $(\mathrm{Ker}\,\phi)_i = \mathrm{Ker}\,\phi_i$ for each vertex i.
- The dimension vector of a finite dimensional representation X is a vector $\dim X$ in $\mathbb{Z}^{Q_0}$ with $(\dim X)_i = \dim X_i$.

Simple representations

Definition (Simple representation)

A representation X is simple if it is non-zero and has no proper subrepresentations (i. e. if $Y \subsetneq X$, then Y is zero representation).

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Definition ($S(i)$)

Let Q be a quiver and $i \in Q_0$ be a vertex. We define a representation $S(i)$ by

$$S(i)_j = \begin{cases} K & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases} \quad \text{and} \quad S(i)_\alpha = 0$$

for $j \in Q_0$ and $\alpha \in Q_1$. This representation is simple.

Simple representations

Lemma

- *Let X be a representation of a quiver Q and $i \in Q_0$ is a vertex.*
- *Suppose that $X_i \neq 0$ and $X_\alpha = 0$ for each arrow $\alpha \in Q_1$ starting at i.*

Then representation $S(i)$ is a subrepresentation of X.

Proof.

We need to verify that

- $S(i)_j$ is a subspace of X_j for each vertex j
  - $0 \subseteq X_j$ for $j \neq i$
  - $K \subseteq X_i$
- $0 = S(i)_\alpha(x) = X_\alpha(x)$ for each $\alpha \in Q_1$ and $x \in S(i)_{s(\alpha)}$
  - if $s(\alpha) = i$, we have $X_\alpha = 0$ by assumption
  - for $s(\alpha) \neq i$ we have $S(i)_{s(\alpha)} = 0$

□

Simple representations

Lemma

Let

- Q *be a quiver with no oriented cycles,*
- S *be a simple representation of* Q*.*

Then there exits a unique vertex $i \in Q_0$ *such that* $S \cong S(i)$*.*

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Take the longest "path of non-zero linear maps" in S. The last vertex of that path is the desired vertex i.

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- By the previous lemma $S(i)$ is a subrepresentation of S.
- S is simple, so $S \cong S(i)$.
- The uniqueness of the vertex i is obvious.

$\square$

Direct sums

Definition

Let $X^1, \dots, X^r$ be representations of a quiver Q. A direct sum

$$X = X^1 \oplus X^2 \oplus \cdots \oplus X^r$$

is a representation given by direct summation in each vertex. That is

$$X_i = X_i^1 \oplus \cdots \oplus X_i^r \text{ for each vertex } i.$$

The linear maps are defined componentwise, that is

$$X_\alpha = (X_\alpha^1, \dots, X_\alpha^r) \text{ for each arrow } \alpha.$$

Definition

- If a representation X can be written as a direct sum of $X^1, \dots, X^r$, we call $X^1 \oplus \cdots \oplus X^r$ a decomposition of X.
- If X can not be written as a direct sum of non-zero representations, we say X is indecomposable.

Krull-Schmidt theorem

Theorem (Krull-Schmidt)

Let X be a finite dimensional representation. Then there exists a decomposition

$$X = (X^1)^{a_1} \oplus (X^2)^{a_2} \oplus \cdots \oplus (X^r)^{a_r}$$

with X^i pairwise non-isomorphic and indecomposable.

The decomposition is unique, i. e. if

$$X = (Y^1)^{b_1} \oplus (Y^2)^{b_2} \oplus \cdots \oplus (Y^r)^{b_s}$$

is another decomposition, then $r = s$ and after reordering $X^i \cong Y^i$, $a_i = b_i$.

Finite representation type

Definition

Let Q be a quiver. We say Q is of finite representation type if there exists up to an isomorphism only finitely many indecomposable representations of Q.

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- Gabriel's theorem classifies the quivers which are of finite representation type.

Example - decomposition

- Let X be a representation of the n-subspace quiver.

$$\begin{tikzcd} & 0 & & & \\ 1 \arrow[ur] & 2 \arrow[u] & 3 \arrow[ul] & \cdots & n \arrow[ulll] \end{tikzcd}$$

- Then X admits a unique decomposition $X = X' \oplus X(1) \oplus X(2) \oplus \cdots \oplus X(n)$ such that
  - $X(i)$ is a direct sum of copies of $S(i)$ for $1 \leq i \leq n$,
  - X' is a subspace representation, i. e. each map $X_{\alpha_i} : X'_i \to X'_0$ is injective.

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  - X' is a subspace representation, i. e. each map $X_{\alpha_i} : X'_i \to X'_0$ is injective.
- $X(1) \oplus X(2) \oplus \cdots \oplus X(n)$ is a representation which has $\mathrm{Ker}\, X_{\alpha_i}$ in a vertex i.

Underlying graph

- By a finite graph we mean a quiver without an orientation, i. e. a non-oriented graph with multiple edges and loops.
- Let Q be a quiver. By $\overline{Q}$ we denote a graph created from Q by forgetting the orientation. We call $\overline{Q}$ an underlying graph of Q.

Underlying graph

- By a finite graph we mean a quiver without an orientation, i. e. a non-oriented graph with multiple edges and loops.
- Let Q be a quiver. By $\overline{Q}$ we denote a graph created from Q by forgetting the orientation. We call $\overline{Q}$ an underlying graph of Q.
- We will now work with an arbitrary graph Γ.
- We will define a symmetric bilinear form and a quadratic form for Γ.

Finite graphs

Definition

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- We define a symmetric bilinear form $(-,-) : \mathbb{Z}^n \to \mathbb{Z}$ for Γ by

$$(e_i, e_j) = \begin{cases} -d_{ij} & \text{if } i \neq j, \\ 2 - d_{ii} & \text{if } i = j. \end{cases}$$

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$$(x, y) = (x_1, x_2, \ldots, x_n) \begin{pmatrix} 2 - d_{11} & -d_{12} & -d_{13} & \ldots & -d_{1n} \\ -d_{12} & 2 - d_{22} & -d_{23} & \ldots & -d_{2n} \\ -d_{13} & -d_{23} & 2 - d_{33} & \ldots & -d_{3n} \\ \vdots & & & \ddots & \vdots \\ -d_{1n} & -d_{2n} & -d_{3n} & \ldots & 2 - d_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

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- We also define a quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$ for Γ by

$$q(x) = \sum_{i=1}^{n} x_i^2 - \sum_{i \leq j}^{n} d_{ij} x_i x_j = \sum_{i=1}^{n} (1 - d_{ii}) x_i^2 - \sum_{i<j}^{n} d_{ij} x_i x_j$$

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- Note that $q(x) = \frac{1}{2}(x, x)$. Also $(x, y) = q(x + y) - q(x) - q(y)$.
- That means Γ, q and $(-,-)$ determine each other.

Finite graphs

Definition

- We define radical of the form $q : \mathbb{Z}^n \to \mathbb{Z}$ by

$$\operatorname{rad} q = \{x \in \mathbb{Z}^n \mid (x, -) = 0\}$$

A vector $x \in \mathbb{Z}^n$ is radical if $x \in \operatorname{rad} q$

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  A vector $x \in \mathbb{Z}^n$ is radical if $x \in \operatorname{rad} q$
- A vector $x \in \mathbb{Z}^n$ is sincere if $x_i = 0$ for all i

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  A vector $x \in \mathbb{Z}^n$ is radical if $x \in \operatorname{rad} q$
- A vector $x \in \mathbb{Z}^n$ is sincere if $x_i = 0$ for all i
- We say that q is
  - positive definite if $q(x) > 0$ for all non-zero $x \in \mathbb{Z}^n$
  - positive semi-definite if $q(x) \geq 0$ for all $x \in \mathbb{Z}^n$

Finite graphs

Lemma

Let

- Γ *be connected finite graph (and* $(-,-)$*,* q *are defined as before)*
- $y \in \mathbb{Z}^n$ *be a positive radical vector, i. e.*
  - $y_i \geq 0$ *for all* i
  - $(x, y) = 0$ *for all* $x \in \mathbb{Z}^n$

Then

- y *is sincere (i. e.* $y_i \neq 0$ *for all* i*)*
- q *is positive semi-definite*
- $q(x) = 0 \iff x \in \mathbb{Q}y \iff x \in \operatorname{rad} q \qquad$ *(for all* $x \in \mathbb{Z}^n$*)*

Finite graphs

proof of the lemma

- We know that Γ is connected and y is positive radical vector.
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- So y has to be sincere.

- We know that y is sincere and positive and $0 = (e_i, y) = (2 - 2d_{ii})y_i - \sum_{j \neq i} d_{ij} y_j$ for every i
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$$q(x) = \sum_i (1 - d_{ii}) x_i^2 - \sum_{i<j} d_{ij} x_i x_j$$

Finite graphs

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$$q(x) = \sum_i (1 - d_{ii})x_i^2 - \sum_{i<j} d_{ij} x_i x_j = \sum_i (2 - 2d_{ii}) y_i \frac{1}{2y_i} x_i^2 - \sum_{i<j} d_{ij} x_i x_j$$

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Finite graphs
proof of the lemma

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$$
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&= \sum_{i \neq j} \frac{d_{ij} y_j}{2y_i} x_i^2 - \sum_{i<j} d_{ij} x_i x_j = \sum_{i<j} d_{ij} \frac{y_j}{2y_i} x_i^2 - \sum_{i<j} d_{ij} x_i x_j + \sum_{i<j} d_{ij} \frac{y_i}{2y_j} x_j^2 \\
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Finite graphs

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\end{aligned}
$$

- We know that $q(x) = \sum_{i<j} d_{ij} \frac{y_i y_j}{2} \left(\frac{x_i}{y_i} - \frac{x_j}{y_j} \right)^2 \geq 0$, Γ is connected and y is radical
- We want to prove that $q(x) = 0 \iff x \in \mathbb{Q}y \iff x \in \operatorname{rad} q$

Finite graphs

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- That means $x \in \mathbb{Q}y$.

Finite graphs
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- That means $x \in \mathbb{Q}y$.
- If $x \in \mathbb{Q}y$ then $x \in \operatorname{rad} q$ since $y \in \operatorname{rad} q$.

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- That means $x \in \mathbb{Q}y$.
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- If $x \in \operatorname{rad} q$ then $q(x) = \frac{1}{2}(x, x) = 0$.

Finite graphs

We proved the lemma:

Lemma

Let

- Γ *be connected finite graph (and* $(-,-)$*,* q *are defined as before)*
- $y \in \mathbb{Z}^n$ *be a positive radical vector, i. e.*
  - $y_i \geq 0$ *for all* i
  - $(x, y) = 0$ *for all* $x \in \mathbb{Z}^n$

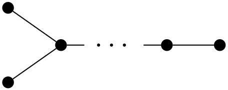
Then

- y *is sincere (i. e.* $y_i \neq 0$ *for all* i*)*
- q *is positive semi-definite*
- $q(x) = 0 \iff x \in \mathbb{Q}y \iff x \in \operatorname{rad} q$ *(for all* $x \in \mathbb{Z}^n$*)*

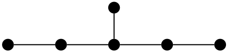
Dynkin diagrams

A_n

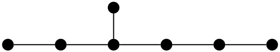
D_n



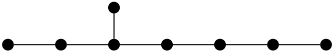
E_6



E_7

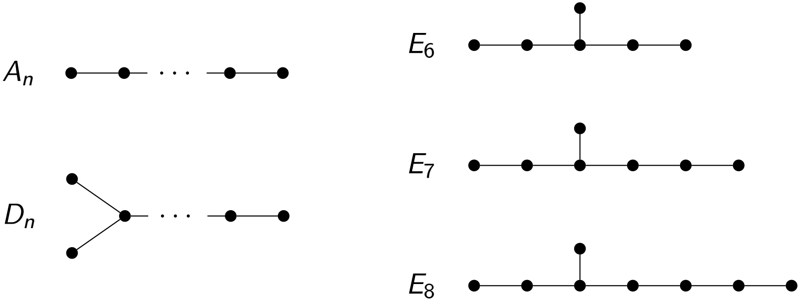


E_8



- A_n and D_n have n vertices.

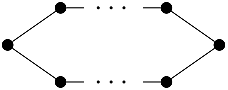
Dynkin diagrams



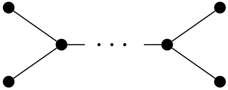
- A_n and D_n have n vertices.
- Gabriel's theorem: A quiver Q is of finite representation type $\iff$ The underlying graph $\overline{Q}$ is a Dynkin diagram.

Euclidean diagrams

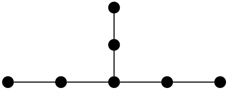
$\tilde{A}_m$



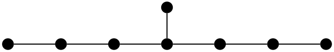
$\tilde{D}_m$



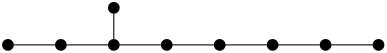
$\tilde{E}_6$



$\tilde{E}_7$



$\tilde{E}_8$



- $\tilde{A}_m$ and $\tilde{D}_m$ have $m + 1$ vertices.
- Euclidean diagrams are the smallest graphs that are not Dynkin, i. e. every non-Dynkin graph has some Euclidean subgraph.

Finite graphs classification

Theorem

Let Γ be a connected graph and q the corresponding quadratic form.

(1) *Γ is a Dynkin diagram $\iff$ q is positive definite*

(2) *Γ is an Euclidean diagram $\iff$ q is positive semi-definite but not positive definite. In that case there is a unique positive vector δ such that $\operatorname{rad} q = \mathbb{Z}\delta$*

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Theorem

Let Γ be a connected graph and q the corresponding quadratic form.

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- We will prove the theorem in three steps.
  - a) We prove " $\Longrightarrow$ " in (2).
  - b) We prove " $\Longrightarrow$ " in (1).
  - c) We prove that if Γ is neither Dynkin nor Euclidean then $q(x) < 0$ for some $x \in \mathbb{Z}^n$.

Finite graphs classification

proof of the theorem

a) If Γ is Euclidean then q is positive semi-definite but not positive and there exists $\delta \in \mathbb{Z}^n$ such that $\operatorname{rad} q = \mathbb{Z}^n \delta$.

- Each vertex i is now marked with the value δ_i of δ:

$$\tilde{A}_m \quad \begin{array}{ccccc} & & 1 - \cdots - 1 & & \\ & \diagup & & \diagdown & \\ 1 & & & & 1 \\ & \diagdown & & \diagup & \\ & & 1 - \cdots - 1 & & \end{array}$$

$$\tilde{D}_m \quad \begin{array}{ccccc} 1 & & & & 1 \\ & \diagdown & & \diagup & \\ & & 2 - \cdots - 2 & & \\ & \diagup & & \diagdown & \\ 1 & & & & 1 \end{array}$$

$$\tilde{E}_6 \quad \begin{array}{ccccc} & & 1 & & \\ & & | & & \\ & & 2 & & \\ & & | & & \\ 1 - & 2 - & 3 & - 2 & - 1 \end{array}$$

$$\tilde{E}_7 \quad \begin{array}{ccccccc} & & & 2 & & & \\ & & & | & & & \\ 1 - & 2 - & 3 - & 4 & - 3 & - 2 & - 1 \end{array}$$

$$\tilde{E}_8 \quad \begin{array}{cccccccc} & & 3 & & & & & \\ & & | & & & & & \\ 2 - & 4 - & 6 & - 5 & - 4 & - 3 & - 2 & - 1 \end{array}$$

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$\tilde{A}_m$
$$\begin{tikzcd}
& 1 \arrow[r, dash] & \cdots \arrow[r, dash] & 1 \arrow[dr, dash] & \\
1 \arrow[ur, dash] \arrow[dr, dash] & & & & 1 \\
& 1 \arrow[r, dash] & \cdots \arrow[r, dash] & 1 \arrow[ur, dash] &
\end{tikzcd}$$

$\tilde{E}_6$
$$\begin{tikzcd}
& & 1 & & \\
& & 2 \arrow[u, dash] & & \\
1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[u, dash] \arrow[r, dash] & 2 \arrow[r, dash] & 1
\end{tikzcd}$$

$\tilde{D}_m$
$$\begin{tikzcd}
1 \arrow[dr, dash] & & & & 1 \\
& 2 \arrow[r, dash] & \cdots \arrow[r, dash] & 2 \arrow[ur, dash] \arrow[dr, dash] & \\
1 \arrow[ur, dash] & & & & 1
\end{tikzcd}$$

$\tilde{E}_7$
$$\begin{tikzcd}
& & & 2 & & & \\
1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \arrow[u, dash] \arrow[r, dash] & 3 \arrow[r, dash] & 2 \arrow[r, dash] & 1
\end{tikzcd}$$

$\tilde{E}_8$
$$\begin{tikzcd}
& & 3 & & & & & \\
2 \arrow[r, dash] & 4 \arrow[r, dash] & 6 \arrow[u, dash] \arrow[r, dash] & 5 \arrow[r, dash] & 4 \arrow[r, dash] & 3 \arrow[r, dash] & 2 \arrow[r, dash] & 1
\end{tikzcd}$$

- We check $\delta \in \operatorname{rad} q$: $0 \stackrel{?}{=} (e_i, \delta) = (2 - 2d_{ii})\delta_i - \sum_{j \neq i} d_{ij}\delta_j = 2\delta_i - \sum_{j: d_{ij} \neq 0} \delta_j$

Finite graphs classification

proof of the theorem

a) If Γ is Euclidean then q is positive semi-definite but not positive and there exists $\delta \in \mathbb{Z}^n$ such that $\operatorname{rad} q = \mathbb{Z}^n\delta$.

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$$\tilde{A}_m \quad \begin{tikzcd}[column sep=small, row sep=small]
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 & 1 \arrow[r, dash] & \cdots \arrow[r, dash] & 1 \arrow[ur, dash] &
\end{tikzcd}$$

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1 \arrow[ur, dash] & & & & 1
\end{tikzcd}$$

$$\tilde{E}_6 \quad \begin{tikzcd}[column sep=small, row sep=small]
 & & 1 \arrow[d, dash] & & \\
 & & 2 \arrow[d, dash] & & \\
1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 2 \arrow[r, dash] & 1
\end{tikzcd}$$

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 & & & 2 \arrow[d, dash] & & & \\
1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \arrow[r, dash] & 3 \arrow[r, dash] & 2 \arrow[r, dash] & 1
\end{tikzcd}$$

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 & & 3 \arrow[d, dash] & & & & & \\
2 \arrow[r, dash] & 4 \arrow[r, dash] & 6 \arrow[r, dash] & 5 \arrow[r, dash] & 4 \arrow[r, dash] & 3 \arrow[r, dash] & 2 \arrow[r, dash] & 1
\end{tikzcd}$$

- We check: $0 \overset{?}{=} 2\delta_i - \sum\limits_{j:d_{ij}\neq 0} \delta_j \quad$ (for every i)

a) If Γ is Euclidean then q is positive semi-definite but not positive and there exists $\delta \in \mathbb{Z}^n$ such that $\operatorname{rad} q = \mathbb{Z}^n \delta$.

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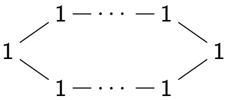
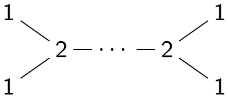
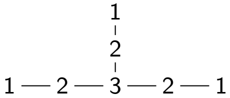
$\tilde{A}_m$



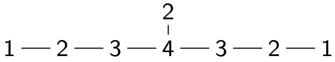
$\tilde{D}_m$



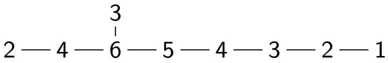
$\tilde{E}_6$



$\tilde{E}_7$



$\tilde{E}_8$



- We check: $0 = 2\delta_i - \sum_{j : d_{ij} \neq 0} \delta_j \quad$ (for every i)
- For $\tilde{A}_2$ ($1 = 1$): $(2 - 2d_{ii})\delta_i - \sum_{j \neq i} d_{ij}\delta_j = 2\delta_i - 2\delta_j = 0$, where $i \neq j$

Finite graphs classification

proof of the theorem

a) If Γ is Euclidean then q is positive semi-definite but not positive and there exists $\delta \in \mathbb{Z}^n$ such that $\operatorname{rad} q = \mathbb{Z}^n \delta$.

- Each vertex i is now marked with the value δ_i of δ:

$$\tilde{A}_m \quad \begin{array}{ccccc} & 1 - \cdots - 1 & \\ 1 & & 1 \\ & 1 - \cdots - 1 & \end{array}$$

$$\tilde{D}_m \quad \begin{array}{ccc} 1 & & 1 \\ & 2 - \cdots - 2 & \\ 1 & & 1 \end{array}$$

$$\tilde{E}_6 \quad \begin{array}{ccccc} & & 1 & & \\ & & | & & \\ & & 2 & & \\ & & | & & \\ 1 - & 2 - & 3 & - 2 & - 1 \end{array}$$

$$\tilde{E}_7 \quad \begin{array}{ccccccc} & & & 2 & & & \\ & & & | & & & \\ 1 - & 2 - & 3 - & 4 & - 3 & - 2 & - 1 \end{array}$$

$$\tilde{E}_8 \quad \begin{array}{cccccccc} & & 3 & & & & & \\ & & | & & & & & \\ 2 - & 4 - & 6 & - 5 & - 4 & - 3 & - 2 & - 1 \end{array}$$

- We check: $0 = 2\delta_i - \sum_{j : d_{ij} \neq 0} \delta_j \qquad$ (for every i)
- So we have positive $\delta \in \operatorname{rad} q$. Claim now follows from the lemma.

Finite graphs classification

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- $x_0 = 0$, so we have

$$q_\Gamma(x) = \sum_{i=1}^{n} x_i^2 - \sum_{\substack{i,j=1 \\ i \leq j}}^{n} d_{ij}x_ix_j = \sum_{i=0}^{n} x_i^2 - \sum_{\substack{i,j=0 \\ i \leq j}}^{n} d_{ij}x_ix_j = q_{\tilde{\Gamma}}(x) > 0$$

c) If Γ is neither Dynkin nor Euclidean then $q(x) < 0$ for some $x \in \mathbb{Z}^n$

- We find a Euclidean subgraph Γ' with radical vector δ.

$$q(x) = \sum_{i=1}^{n} (1 - d_{ii}) x_i^2 - \sum_{i<j}^{n} d_{ij} x_i x_j$$

Finite graphs classification

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$$q(x) = \sum_{i=1}^{n} (1 - d_{ii})x_i^2 - \sum_{i<j}^{n} d_{ij}x_ix_j \qquad (x, e_i) = (2 - 2d_{ii})x_i + \sum_{j \neq i} -d_{ij}x_j$$

$$q(2\delta + e_i) = q(2\delta) + (2\delta, e_i) + q(e_i) \leq 0 + \underbrace{(2\sum_{j\neq i} -d_{ij}\delta_j)}_{\leq -1} + \underbrace{(1 - d_{ii})}_{\leq 1} < 0$$

Roots

Definition

Let Γ be Dynkin or Euclidean. We define

$$\Delta = \{x \in \mathbb{Z}^n \mid q(x) \leq 1\}.$$

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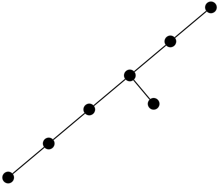
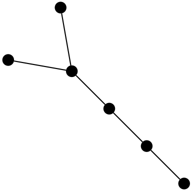
Lemma

Let Γ be Dynkin or Euclidean. Then

- *Each e_i is a root.*
- *If $x \in \Delta$, then also $-x \in \Delta$.*
- *If $x \in \Delta$ and $y \in \operatorname{rad} q$, then also $x + y \in \Delta$*
- *Every root is either positive or negative.*
- *For Euclidean diagram $\Delta/\operatorname{rad} q$isfinite*
- *For Dynkin diagram Δ is finite.*





Thank you for your attention!