Group cohomology and wallpaper groups - Part I

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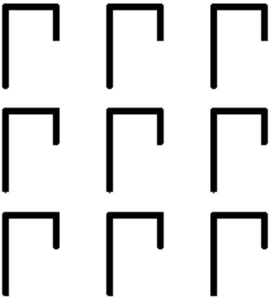
What is a wallpaper pattern?



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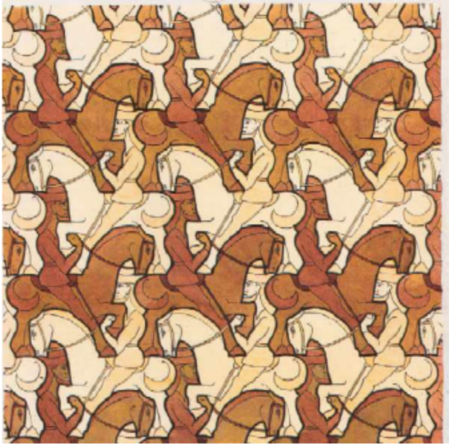

Symmetry group

Let W be a subset of $\mathbb{R}^n$, then the *symmetry group of* W is defined as

$$\mathrm{Sym}(\mathrm{W}) = \{\varphi \in \mathrm{Isom}(\mathbb{R}^n) \, : \, \varphi(\mathrm{W}) = \mathrm{W}\}$$

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Translation subgroup

For a $v \in \mathbb{R}^n$ we denote the map $\tau_v(x) = x + v$. The set of all such isometries forms a subgroup $\mathbb{T}$.

$$\mathrm{Sym}(W) \cap \mathbb{T} = \{\tau \in \mathrm{Sym}(W) \,:\, \tau \text{ is a translation.}\}$$

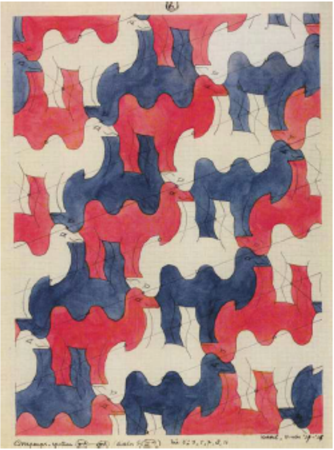
is called the *translation subgroup* of $\mathrm{Sym}(W)$

Wallpaper group

Definition (Wallpaper pattern)

A subset W of $\mathbb{R}^2$ is a wallpaper pattern if the translation subgroup of the symmetry group $\mathrm{Sym(W)}$ is a two-dimensional lattice. The symmetry group of a wallpaper pattern is then a *wallpaper group.*

Patterns with isomorphic wallpaper groups



Types of isometries of the plane $\mathbb{R}^2$

- Translations
- Reflections
- Rotations
- Glide reflexions

Properties of isometries

Defining property

Isometry f in coordinates: $\|f(u) - f(v)\| = \|u - v\|$ for all $u, v \in \mathbb{R}^{\mathrm{n}}$. Special isometries g that fix the origin: $g(0) = 0$.

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- $g(u) \cdot g(v) = u \cdot v$ for all $u, v \in \mathbb{R}^n$
- g is a linear transformation.
- $g(x) = Ax$ where A an $n \times n$ matrix satisfying $A^T A = I_n$

The Orthogonal Group

Definition

The set of $n \times n$ matrices A satisfying $A^T A = I_n$ is the orthogonal group denoted $O_n(\mathbb{R}^n)$.

It is isomorphic to the subgroup H of isometries fixing the origin via the map

$$\sigma : O_n(\mathbb{R}^n) \to H$$

$$\sigma(A) = x \mapsto Ax$$

The Orthogonal Group

Special orthogonal group

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The Group Structure of Isometries

Special isometries in $G = \mathrm{Isom}(\mathbb{R}^n)$

- Translations $\tau \in \mathbb{T}$.
- Isometries fixing the origin $g \in H$
- Every isometry can be written as a composition of a isometry fixing the origin and a translation.
- f linear isometry, $\tau_v(x) = x + v$ a translation. Then $f \circ \tau_v \circ f^{-1} = \tau_{f(v)}$

The Group Structure of Isometries

Group theoretic perspective

- $G = H \cdot \mathbb{T}$.
- $\mathbb{T}$ is normal in G.
- $H \cap \mathbb{T} = \{\mathrm{id}\}$
- G is a semi-direct product of $\mathbb{T}$ and H.

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- G is a semi-direct product of $\mathbb{T}$ and H.

G is isomorphic to $(\mathbb{R}^n \times O_n(\mathbb{R}), \cdot)$ where the operation is defined as

$$(u, A) \cdot (v, B) = (u + Av, AB)$$

Finite subgroups of $\mathrm{O}_2(\mathbb{R})$

Proposition

Let G be a finite subgroup of $\mathrm{O}_2(\mathbb{R})$. Then it is isomorphic to either a cyclic group of order n, or to a dihedral group of order $2n$.

Proof.

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- First equality has to hold, as otherwise $r' r^{-m}$ is a rotation by an angle $0 < \phi - m\theta < \theta$.
- Either $[G : N] = 1$ and $G = N$, or $[G : N] = 2$ and $G \cong D_n$.

□

The Point Group

Definition

Let G be a wallpaper group. The point group of G_0 of G is the set

$$\left\{A \in \mathrm{O}_2(\mathbb{R}) \,:\, (A, b) \in G \textit{ for some } b \in \mathbb{R}^2\right\}$$

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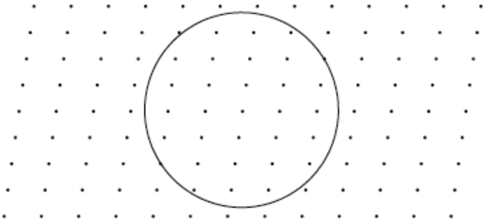
Proposition

Let G be a wallpaper group with a translation lattice T and a point group G_0. Then $G_0 \cong G/T$

The Point Group



Lemma (Finiteness of the Point Group)

The Point Group G_0 of a Wallpaper group G is finite.

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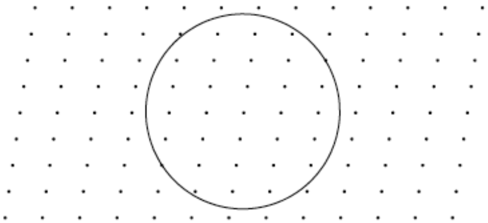

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Proof.

There is only a finite number of points of a lattice in a circle. Hence only finite number of pairs is possible as images under the action of G_0. An element of G_0 is determined by its action on the lattice. □

The Point Group

Proposition

Let G_0 be a point group of a wallpaper group G. Then G_0 is isomorphic to one of the following groups:

$$\{C_1, C_2, C_3, C_4, C_6, D_1, D_2, D_3, D_4, D_6\}$$

Proof.

- $N = G \cap \mathrm{SO}_2(\mathbb{R})$ is a cyclic group generated by an element rotation r of minimal angle.

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- We express the rotation in two basis

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  for $a, b, c, d \in \mathbb{Z}$.
- A necessary condition $2\cos(\theta) = a + d$ gives us the only possible $n = 1, 2, 3, 4, 6$.

Action of G_0 on T

Action and Group Representation of G_0

- We have 9 possible non-isomorphic groups for the point group.
- Even though $C_2 \cong D_1$, they will be distinguished by their action.
- By fixing a basis for $\{t_1, t_2\}$ we have an isomorphism $T \cong \mathbb{Z}^2$.
- This action induces a homomorphism $G_0 \to \mathrm{Aut}(\mathbb{Z}^2) \cong \mathsf{Gl}_2(\mathbb{Z})$.

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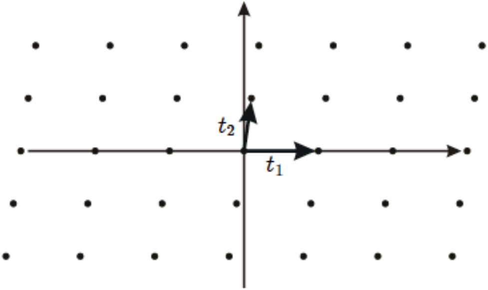
Types of lattices

Our task now is to determine possible types of lattices with respect to the G_0-action. There are five types of lattices: parallelogram, rectangular, square, rhombus, and hexagonal.

The Parallelogram Lattice



$$C_1 = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \qquad C_2 = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

Lemma

Suppose G_0 contains a rotation r about an angle $2\pi/n$ for $n \geq 3$. If t is a vector of minimal length, then $\{t, r(r)\}$ is a basis for the lattice T.

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Proof.

Let $\{t_1, t_2\}$ be a basis for T. It is possible to express $t_1 = at + br(t)$ where $a, b \in \mathbb{Q}$. We round the numbers to obtain an element of the lattice $s = \alpha t + \beta r(t)$. We have rounded to the nearest integer, so $|\epsilon| = |a - \alpha| \leq 0.5$ as well as $|\epsilon'| = |b - \beta| \leq 0.5$

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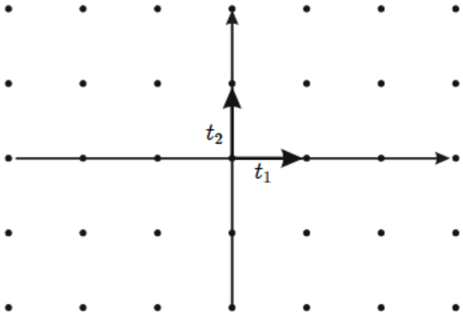
$$\|t_1 - s\| = \|\epsilon t + \epsilon' r(t)\| < \|\epsilon t\| + \|\epsilon' r(t)\| \leq \frac{1}{2}(\|t\| + \|r(t)\|) = \|t\|$$

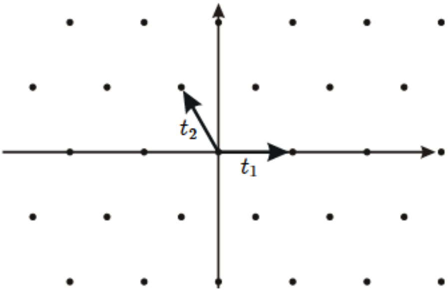
$\square$

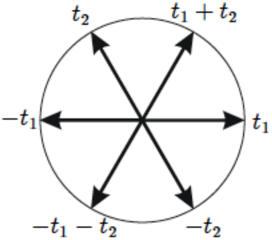
Square Lattice



$$C_4 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \qquad D_4 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

Hexagonal Lattice





$$C_3 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle \qquad C_6 = \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

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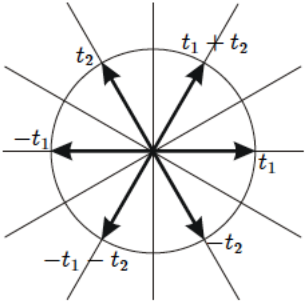

Two different actions for D_3

$$D_{3,l} = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \right\rangle \qquad D_{3,s} = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \right\rangle$$

Later on, we will see that two wallpaper groups having different point group actions *cannot* be isomorphic.

A suitable basis for D_1, D_2

Problem: Lemma is not applicable, there is no rotation by less than π. We obtain a suitable basis in a different way: we have a reflection f.

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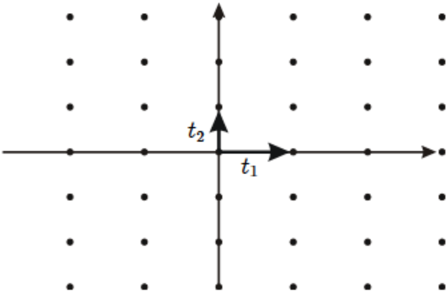
Problem: Lemma is not applicable, there is no rotation by less than π. We obtain a suitable basis in a different way: we have a reflection f.

- Take a nonzero vector t from the lattice not parallel with the line of reflection.
- $t + f(t)$ is fixed by f, hence lies on the line of reflexion.
- $t - f(t)$ is multiplied by -1, hence perpendicular to the line of reflection.
- Take minimal vectors s_1, s_2 in these directions . There exist $m_t, n_t \in \mathbb{Z}$ such that:

$$t + f(t) = m_t s_1 \qquad \text{and} \qquad t - f(t) = n_t s_2$$

- Solving these two equations yields: $t = \frac{m_t}{2} s_1 + \frac{n_t}{2} s_2$

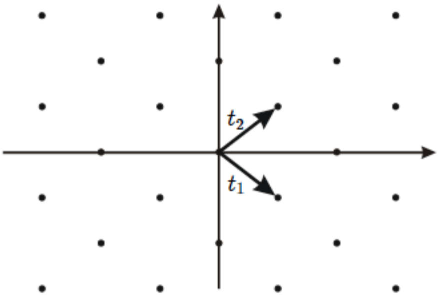
Rectangular Lattice



If n_t, m_t both even, $\{s_1, s_2\}$ is a basis of the lattice with a very good property: s_1 is fixed by f and s_2 is perpendicular.

$$D_{1,p} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

Rhombic Lattice



Otherwise n_t, m_t both odd and we can set: $t_1 = \frac{1}{2}(s_1 + s_2)$ and $t_1 = \frac{1}{2}(s_1 - s_2)$. $\{t_1, t_2\}$ is a basis whose elements are interchanged by f.

$$D_{1,c} = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

$D_{1,c}$ and $D_{1,p}$ are not conjugate in $\mathrm{Gl}_2(\mathbb{Z})$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

- a, b, c, d integers satisfying $ad - bc = \pm 1$
- Multiplying we obtain $c = a$ and $d = -b$
- $-2ab = \pm 1$

Thank you!