The Book proof of Bertrand's postulate

Jakub Opršal

Charles University in Prague

Autumn School 2014, Libuň

Introduction

Introduction

Joseph Bertrand

Bertrand's postulate

Postulate

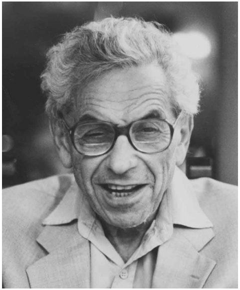
For every $n \geq 1$, there is some prime number p with $n < p \leq 2n$.

$$2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631, 1259, 2503, 4001, \ldots$$

Bertrand verified it for $n < 3\,000\,000$.

More people





The proof, part 1

Claim

$$\prod_{p \leq n} p \leq 4^{n-1}$$

We'll proof this claim by induction. For $n = 2$, it holds. Further it suffices to prove just for odd n since there is no bigger even prime. So, let $n = 2m + 1$.

First, observe that every prime p, $m + 1 < p \leq 2m + 1$ divides $\binom{2m+1}{m+1}$ exactly once. Hence,

$$\prod_{p \leq 2m+1} p = \prod_{p \leq m+1} p \cdot \prod_{m+1 < p \leq 2m+1} p \leq 4^m \binom{2m+1}{m+1}$$

The proof, part 1.1

We have

$$\prod_{p \le 2m+1} p \le 4^m \binom{2m+1}{m+1}.$$

For the upper bound on the right-hand-side observe that

$$\binom{2m+1}{m} + \binom{2m+1}{m+1} \le 2^{2m+1}$$

Hence

$$\binom{2m+1}{m+1} \le 2^{2m},$$

and this completes the proof of the claim: $\prod_{p \le n} p \le 4^{x-1}$

Intermezzo

The binomial coeficient $\binom{2n}{n}$ is the largest of $2n+1$ values

$$\binom{2n}{2n} + \binom{2n}{0}, \binom{2n}{1}, \ldots, \binom{2n}{2n-1}, \binom{2n}{2n}$$

Hence

$$\binom{2n}{n} \geq \frac{2^{2n}}{2n} = \frac{4^n}{2n}.$$

The proof, part 2

What is so great about $\binom{2n}{n}$?

It's prime decomposition contains every prime p, $n < p \leq 2n$ exactly once, and it does not contain primes $p > 2$, $\frac{2}{3}n < p \leq n$ at all!

Indeed, if $\frac{2}{3}n < p \leq n$ then the denomitor contains factor p twice, while the numerator contains factors p and $2p$ and no others.

The proof, part 2.1

Theorem (Legendre's theorem)

The number $n!$ contains the prime factor p exactly

$$\sum_{k\geq 1}\left\lfloor \frac{n}{p^k}\right\rfloor$$

times.

As a corollary $\binom{2n}{n}$ contains prime factor p exactly

$$\sum_{k\geq 1}\left\lfloor \frac{2n}{p^k}\right\rfloor - 2\left\lfloor \frac{n}{p^k}\right\rfloor$$

times. While each summand is at most 1 since

$$\left\lfloor \frac{2n}{p^k}\right\rfloor - 2\left\lfloor \frac{n}{p^k}\right\rfloor < \frac{2n}{p^k} - 2\left(\frac{n}{p^k} - 1\right) = 2.$$

In particular, if $p > \sqrt{2n}$ then the factor p apears at most once.

Louis Legendre

The proof, part 3

Further, let $p \leq \sqrt{2n}$. Then $\binom{2n}{n}$ contains factor p at most

$$\sum_{k \geq 1} \left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \leq \max\{r : p^r \leq 2n\}$$

times.

Altogether,

$$\begin{aligned} \binom{2n}{n} &\leq \prod_{p \leq \sqrt{2n}} 2n \cdot \prod_{\sqrt{2n} < p \leq \frac{2}{3} n} p \cdot \prod_{n < p \leq 2n} p \\ \frac{4^n}{2n} &\leq (2n)^{\sqrt{2n}} \cdot 4^{2n/3} \cdot \prod_{n < p \leq 2n} p \\ 4^{n/3} &\leq (2n)^{1+\sqrt{2n}} \cdot \prod_{n < p \leq 2n} p \end{aligned}$$

The proof, part 3.1

Suppose there is no prime between n and $2n$, hence $\prod_{n<p\leq 2n} p = 1$, and consequently

$$4^{n/3} \leq (2n)^{1+\sqrt{2n}}$$

that is eventually (for large enough n) not true!

What does eventually mean?

$$4^{n/3} \leq (2n)^{1+\sqrt{2n}}$$

We will use a famous inequality $a+1 < 2^a$ for $a \geq 2$ to get

$$2n = \left(\sqrt[6]{2n}\right)^6 < \left(\sqrt[6]{2n}+1\right)^6 < 2^{6\sqrt[6]{2n}}$$

Then for $n \leq 50$, hence $18 < 2\sqrt{2n}$.

$$2^{2n} \leq (2n)^{3(1+\sqrt{2n})} < 2^{\sqrt[6]{2n}(18+18\sqrt{2n})} < 2^{\sqrt[6]{2n}20\sqrt{2n}} = 2^{20(2n)^{2/3}}$$

So...

$$(2n)^{1/3} < 20 \quad \text{, and thus} \quad n < 4000.$$

Hence we know that the postulate is true for all $n \geq 4000$, but

$$2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631, 1259, 2503, 4001 \qquad \square$$

Thank you for your attention!