AUTOMATA, LANGUAGES AND MONOIDS I

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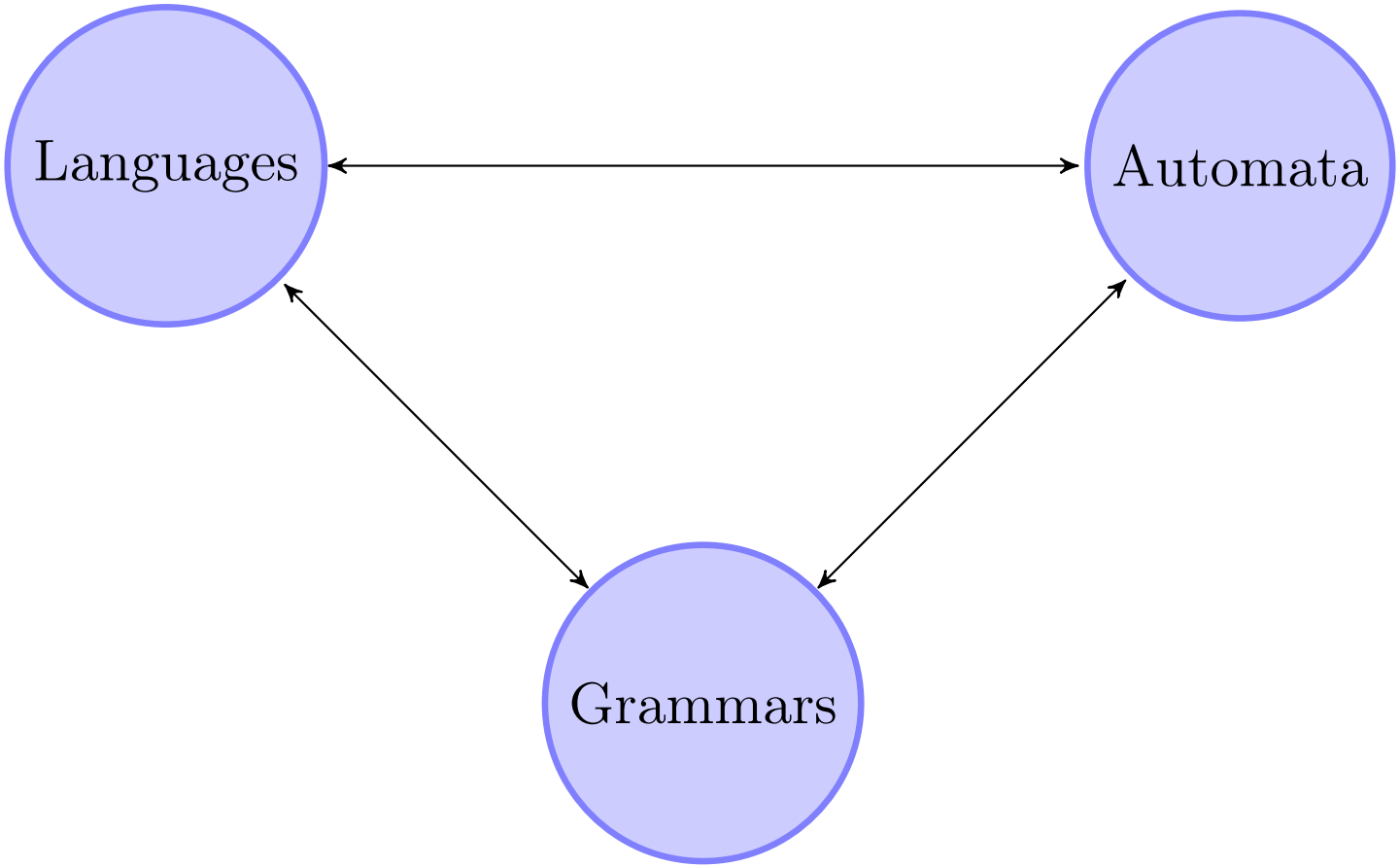
Motivation



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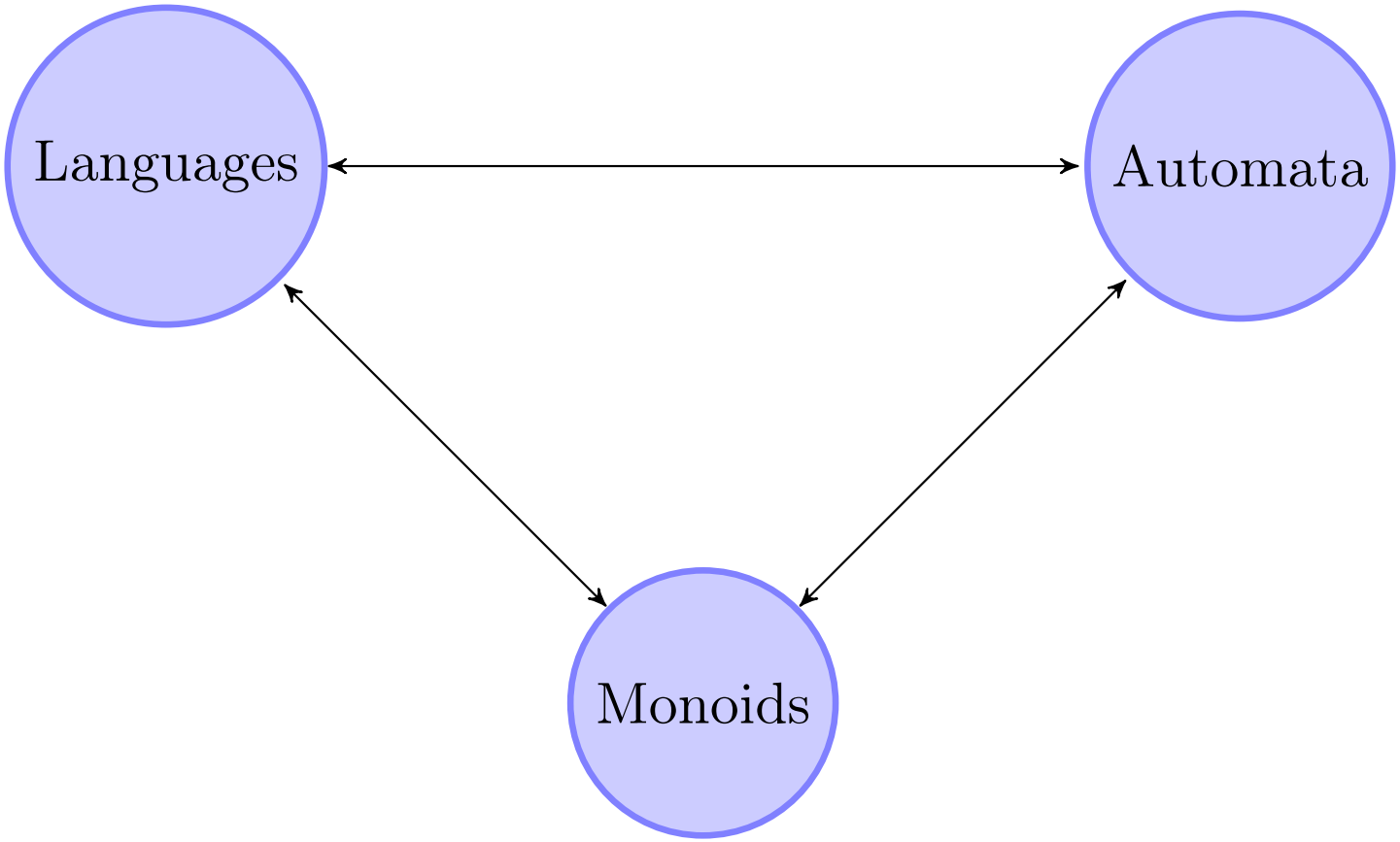

Outline

- First part: introduction, basic definitions, properties etc.

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- Second part (Adéla): piecewise testable languages

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- Second part (Adéla): piecewise testable languages
- Third part (Miloš): Simon's theorem and its consequences

Outline

- First part: Useless
- Second part (Adéla): Useless
- Third part (Miloš): Cancelled

Outline

- First part: Not completely useless
- Second part (Adéla): Not completely useless
- Third part: Short summary

Semigroups and monoids

Definition

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Definition

A *congruence* on a semigroup S is an equivalence relation $\sim$ on S satisfying for each $a, b, c \in S$: $a \sim b \Rightarrow (a \cdot c \sim b \cdot c \wedge c \cdot a \sim c \cdot b)$.

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The set $A^+ = A^* \setminus \{\lambda\}$ with the same operation of concatenation is a *free semigroup* on the set A.

Subwords, factors, etc.

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Let $w \in A^*$ be a word and let $a \in A$ be a letter. The number of occurences of a in w is denoted $|w|_a$.

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A word $u = a_1 \dots a_n$, $a_i \in A$ is a *subword* of a word w if there exist words $v_0, v_1, \dots, v_n$ such that $w = v_0 a_1 v_1 \dots a_n v_n$.

Examples

Example

Take a word $u = abacbacb$.

$$abacbacb$$

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$$a\boldsymbol{bacb}acb$$

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Take a word $u = abacbacb$. Then aba is a prefix of u, acb is a suffix of u, $bacb$ is a factor of u and $bcbb$ is a subword of u.

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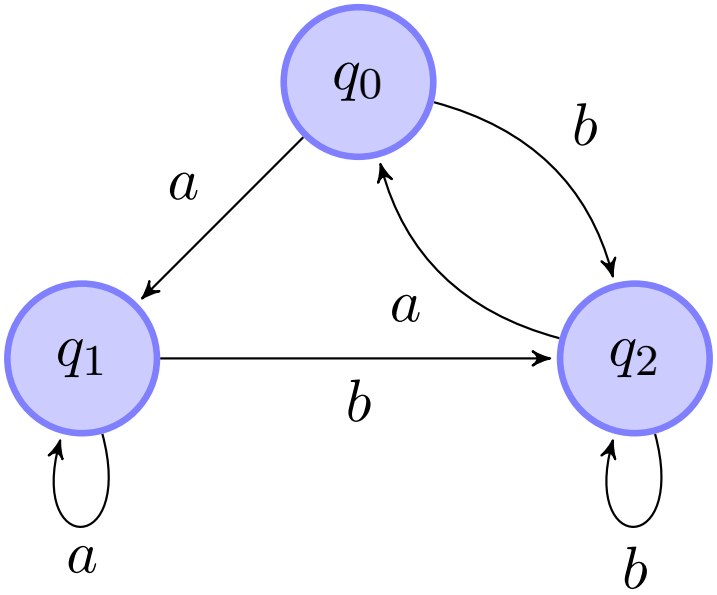
Example

$L = \{a^k \mid k \geq 1\} \subset \{a, b\}^*$ is a language.



Definition

An *automaton* is a triplet $\mathscr{A} = (Q, A, \cdot)$, where Q is a (finite) set of states, A is a finite alphabet and $\cdot$ is a function from $Q \times A$ to Q.

Definition

We say that a language L is *recognized* by an automaton $\mathscr{A} = (Q, A, \cdot)$ if there exists a state $q_0 \in Q$ and a set of states F such that $u \in L$ iff $q_0 \cdot u \in F$.

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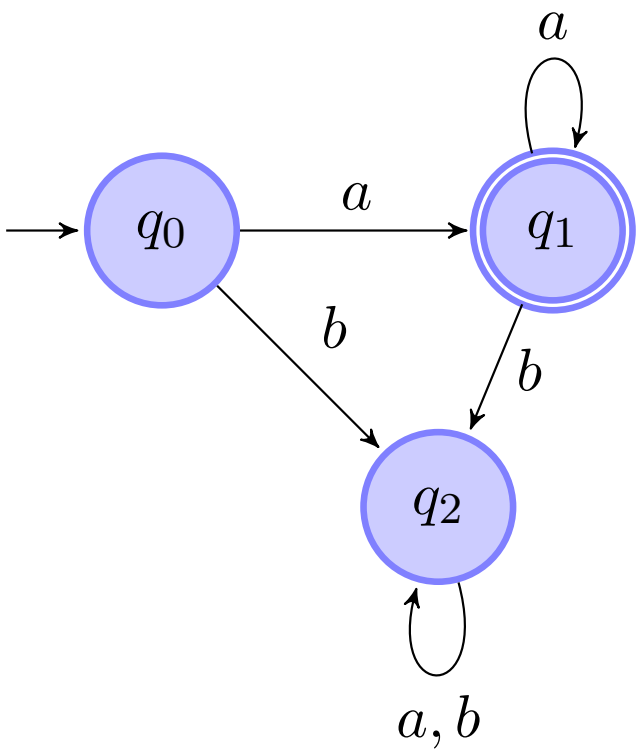

Regular languages

Definition

Let A be a finite alphabet. The set of *regular languages* (also *rational languages*) over A is the smallest set of languages of A^* such that

1. the empty language $\varnothing$ is regular,
2. for every word $u \in A^*$, the language $\{u\}$ is regular,
3. if L_1 and L_2 are regular languages, then $L_1 \cup L_2$, $L_1 \cdot L_2$ and L_1^* are also regular.

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Example

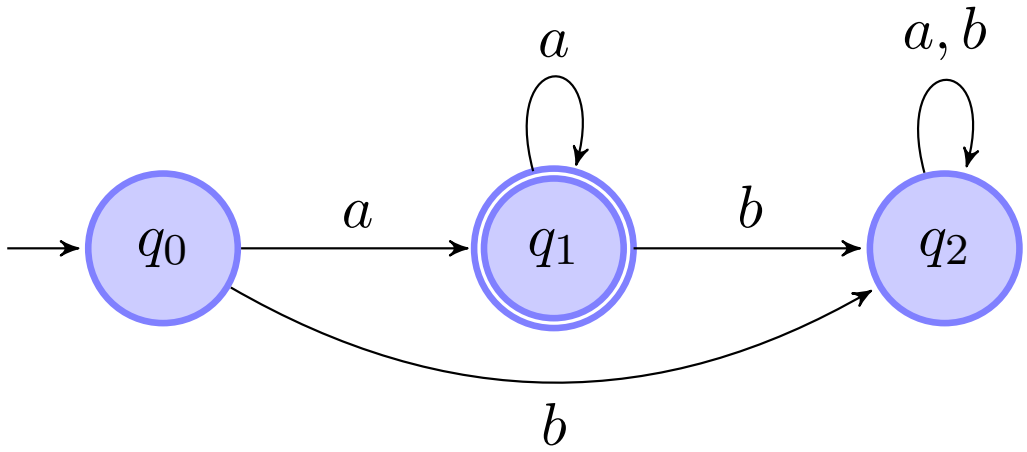
$L = \{a^k \mid k \geq 1\} = \{a\} \cdot \{a\}^*$ is a regular language.

Transition monoid

Definition

Let $\mathscr{A} = (Q, A, \cdot)$ be an automaton. We can extend the operation $\cdot$ to a function from $Q \times A^*$ to Q by the following rules: $q \cdot \lambda = q$ and $q \cdot (wa) = (q \cdot w) \cdot a$ where $q \in Q, w \in A^*$ and $a \in A$. Each word from A^* thus defines a function from Q to Q. The monoid generated by all these functions (w varying over A^*) is called the *transition monoid* of the automaton $\mathscr{A}$. It is denoted $M(\mathscr{A})$.

Examples



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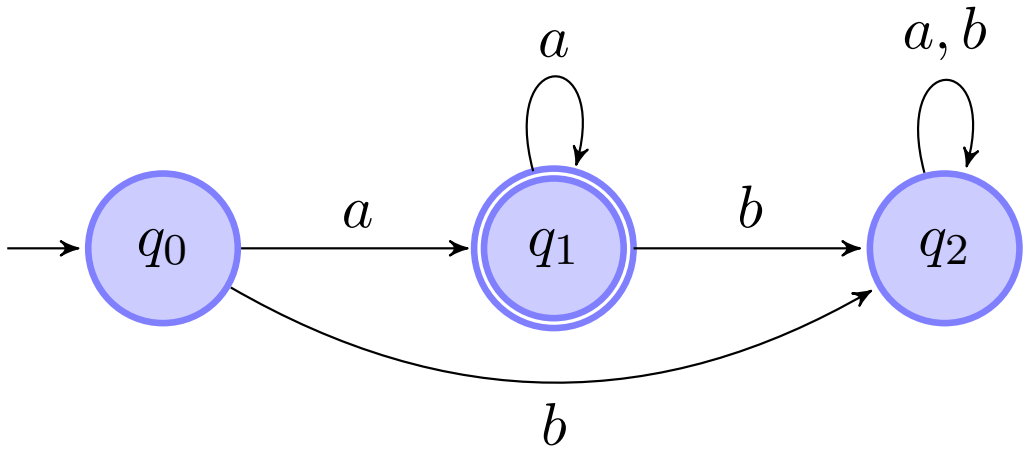

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$$q_0 \cdot a = q_1 \quad q_1 \cdot a = q_1 \quad q_2 \cdot a = q_2$$
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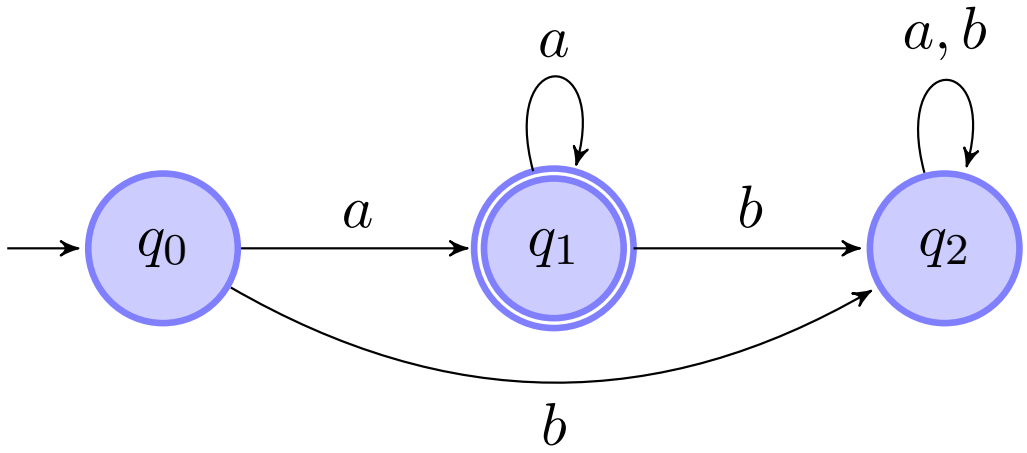

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So we have: $f_a(q_0) = q_1$, $f_a(q_1) = q_1$, $f_a(q_2) = q_2$ and $f_b(q_0) = q_2$, $f_b(q_1) = q_2$, $f_b(q_2) = q_2$.

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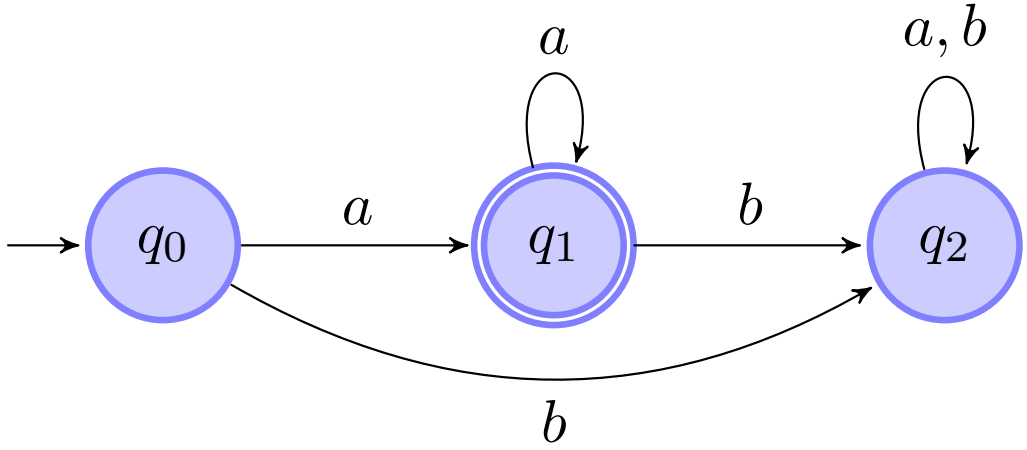

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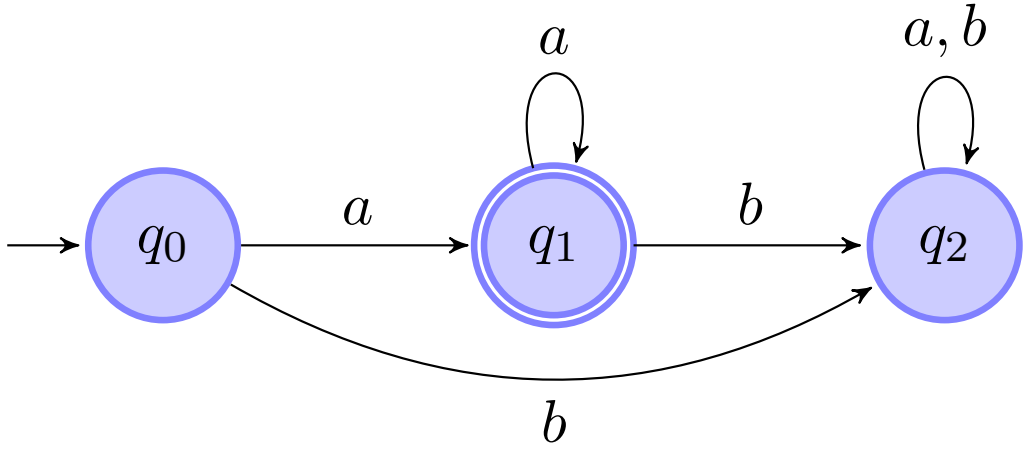

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Recognizable languages

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A language L is called *recognizable* if it is recognized by a finite monoid, i.e. there exists a finite monoid M and a morphism $\alpha : A^* \to M$ such that $L = \alpha^{-1}(P)$ for some $P \subseteq M$.

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Theorem

A language $L \subseteq A^*$ is regular iff it is recognizable.

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Regular $\Leftrightarrow$ *recognizable* $\Leftrightarrow$ *recognized by a finite automaton.*

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Proof.

Suppose there exists a finite automaton which recognizes the language. This automaton has n states. Take a word $a^n b^n$. Then, when reading a's, there must be some loop. We can repeat this path once again, i.e. the automaton would accept the word $a^{n+\ell} b^n$ for some $\ell \geq 1$. However, this word does not belong to the language. $\square$

Properties of regular languages

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Syntactic monoid

Definition

Let $L \subseteq A^*$ be a language. We define the *syntactic* congruence of L (denoted $\sim_L$) on A^* by $u \sim_L v$ iff $xuy \in L \Leftrightarrow xvy \in L$ for every $x, y \in A^*$. The *syntactic monoid* of L is then defined as $M(L) = A^*/\sim_L$.

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Example

Let $L = \{a^k \mid k \geq 1\}$. Then there are three equivalence classes of $\sim_L$, namely $[a]_{\sim_L} = \{u \in A^* \mid u \neq \lambda, |u|_b = 0\}$, $[b]_{\sim_L} = \{u \in A^* \mid |u|_b \geq 1\}$ and $[\lambda]_{\sim_L} = \{\lambda\}$.

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$\cdot$	$[\lambda]_{\sim_L}$	$[a]_{\sim_L}$	$[b]_{\sim_L}$
$[\lambda]_{\sim_L}$	$[\lambda]_{\sim_L}$	$[a]_{\sim_L}$	$[b]_{\sim_L}$
$[a]_{\sim_L}$	$[a]_{\sim_L}$	$[b]_{\sim_L}$	$[b]_{\sim_L}$
$[b]_{\sim_L}$	$[b]_{\sim_L}$	$[b]_{\sim_L}$	$[b]_{\sim_L}$

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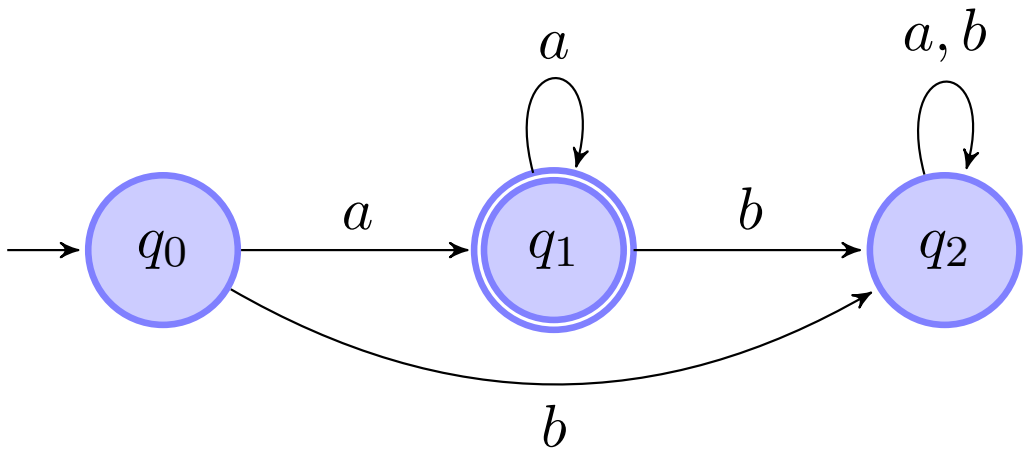

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Example

Take $L = \{a^k \mid k \geq 1\}$ and its minimal automaton $\mathscr{A}$. We know that $M(\mathscr{A}) = \{\mathrm{id}, f_a, f_b\}$ and $M(L) = \{[\lambda]_{\sim_L}, [a]_{\sim_L}, [b]_{\sim_L}\}$. It is obvious that φ defined by
$\varphi(\mathrm{id}) = [\lambda]_{\sim_L}$, $\varphi(f_a) = [a]_{\sim_L}$, $\varphi(f_b) = [b]_{\sim_L}$ is a monoid isomorphism.

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Definition

We say that a semigroup S satisfies the equation $u = v$, $u, v \in A^+$ if $\varphi(u) = \varphi(v)$ for every morphism $\varphi : A^+ \to S$.

Definition

We say that a variety $\mathbf{V}$ is defined (ultimately defined) by equations $u_n = v_n$, $n > 0$ if S lies in $\mathbf{V}$ iff S satisfies the equations $u_n = v_n$ for every $n > 0$ (for every n large enough).

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Example

The variety of finite commutative semigroups is defined by the equation $xy = yx$.

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We say that M is $\mathscr{J}$-trivial if $a \mathscr{J} b \Rightarrow a = b$.

Theorem

The variety $\boldsymbol{J}$ of finite $\mathscr{J}$-trivial monoids is ultimately defined by the equations $(xy)^n x = (xy)^n = y(xy)^n$.

Thank you for your attention!