Group cohomology and wallpaper groups III

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- The point group G_0 is finite and uniquely determined by G. Moreover, G_0 is always one of the following groups:

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- With any extension of T by G_0 comes naturally an action φ of G_0 on T defined as follows: $\varphi_g(t) = x_g t x_g^{-1}$, where x_g is an element of G which is sent to g in the projection $G \to G_0$.

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2. Choice of the action φ of G_0 on T.
3. Choice of an element of the second cohomology group $H^2_\varphi(G_0, T)$.

E.g., the zero element of $H^2_\varphi(G_0, T)$ corresponds to the trivial extension of T by G_0 with action φ, that is, the semidirect product $T \rtimes_\varphi G_0$.

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- There two representations of each of D_1, D_2, D_3.
- Let us distinguish the representations as $D_{1,p}, D_{1,c}, D_{2,p}, D_{2,c}, D_{3,l}, D_{3,s}$.

Recapitulation

Together we have 18 equivalence classes of extensions:

G_0	$H^2(G_0, T)$	No. of extensions
C_1	0	1
C_2	0	1
C_3	0	1
C_4	0	1
C_6	0	1
$D_{1,p}$	$\mathbb{Z}_2$	2
$D_{1,c}$	0	1
$D_{2,p}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4
$D_{2,c}$	0	1
$D_{3,l}$	0	1
$D_{3,s}$	0	1
D_4	$\mathbb{Z}_2$	2
D_6	0	1

How do the extensions look? Are they all non-isomorphic?

Lemma

Two extensions with different point groups and/or actions are non-isomorphic.

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It follows that we have at least 13 non-isomorphic wallpaper groups, namely the semidirect products $T \rtimes_\varphi G_0$ for each point group G_0 and each action $\varphi : G_0 \to GL(\mathbb{Z}^2)$.

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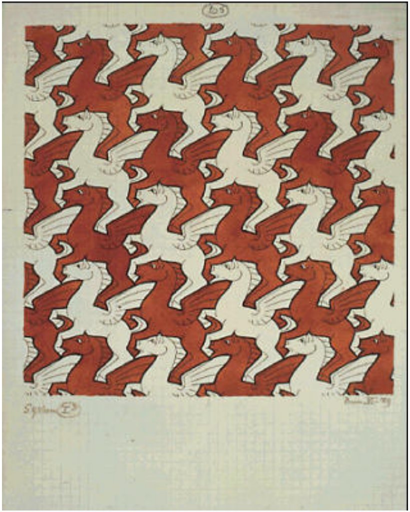
It follows that we have at least 13 non-isomorphic wallpaper groups, namely the semidirect products $T \rtimes_\varphi G_0$ for each point group G_0 and each action $\varphi : G_0 \to GL(\mathbb{Z}^2)$.
Let us see what the corresponding patterns look like.

p1 - Just translations

G_0	C_1
G	$\mathbb{Z}^2 \rtimes C_1 = \mathbb{Z}_2$

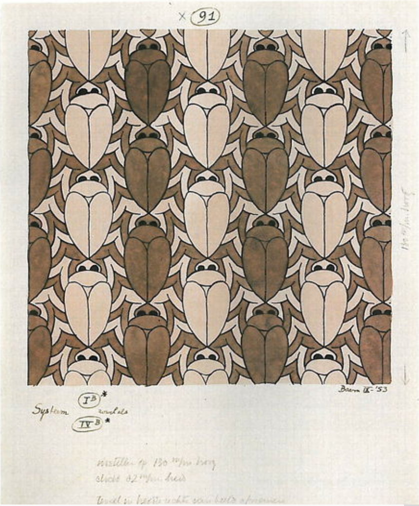


pm - 1 reflection

G_0	D_1
G	$\mathbb{Z}^2 \rtimes D_{1,p}$

cm - 1 reflection, 1 glide reflection

G_0	D_1
G	$\mathbb{Z}^2 \rtimes D_{1,c}$

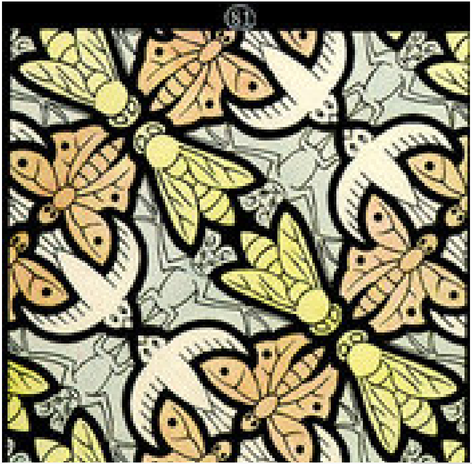


p2 - 180°rotation

G_0	C_2
G	$\mathbb{Z}^2 \rtimes C_2$

pmm - Two reflections and a 180°rotation only in the axes intersection

G_0	D_2
G	$\mathbb{Z}^2 \rtimes D_{2,p}$

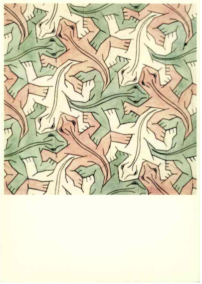


cmm - Two reflections and a 180°rotation NOT in the axes intersection

G_0	D_2
G	$\mathbb{Z}^2 \rtimes D_{2,c}$

p3 - 120°rotation

G_0	C_3
G	$\mathbb{Z}^2 \rtimes C_3$



p3m1 - 120°rotation around the intersection of 3 reflection axes

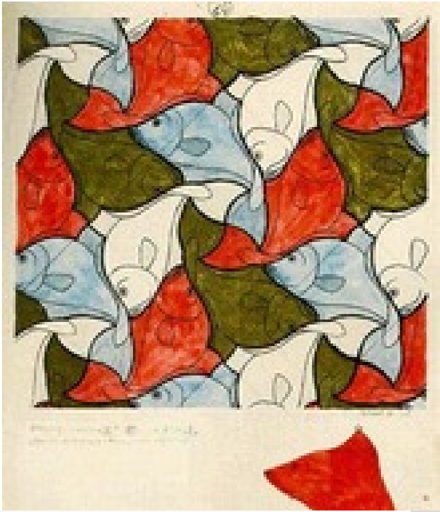
G_0	D_3
G	$\mathbb{Z}^2 \rtimes D_{3,I}$

p31m - 120°rotation around a point, which is NOT an intersection of axes

G_0	D_3
G	$\mathbb{Z}^2 \rtimes D_{3,s}$

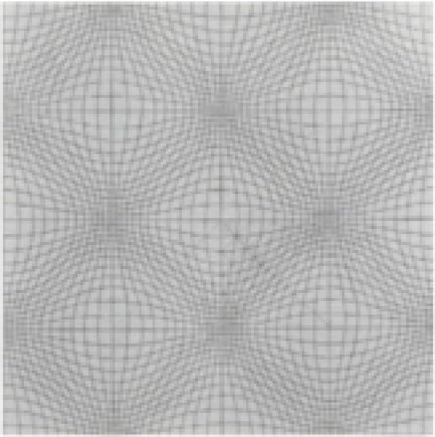
p4 - 90°rotation

G_0	C_4
G	$\mathbb{Z}^2 \rtimes C_4$



p4m - 90°rotation and 4 reflection axes

G_0	D_4
G	$\mathbb{Z}^2 \rtimes D_4$

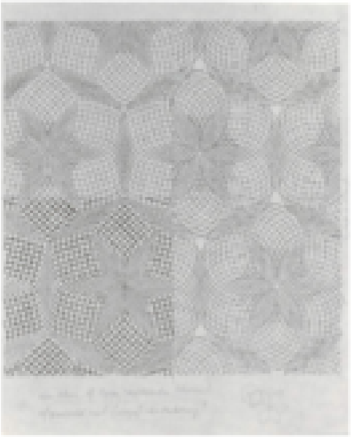


p6 - 60° rotation

G_0	C_6
G	$\mathbb{Z}^2 \rtimes C_6$

p6m - 60°rotation and 6 reflection axes

G_0	D_6
G	$\mathbb{Z}^2 \rtimes D_6$



Non-split extensions

Let our point group be $D_{1,p}$.

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As $H^2(D_{1,p}, T) = \mathbb{Z}_2$, we know that there is one non-split extension. What is it?

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$$c(g,g) = (g, t_g)(g, t_g)(1, t_1)^{-1} = (g^2, t_g + g(t_g))(1, t_1)^{-1} =$$

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- Value of β is irrelevant as $t_1 = 0$ results in $c(g,g)$ being a 2-boundary.

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- For any $t \in T$, $(g, \frac{1}{2}t_1 + t)^2 = (1, t_1 + t + g(t))$ is not zero.
- Therefore, G does not contain any non-trivial element of finite order.

pg - 1 glide reflection

G_0	D_1
G	$\mathrm{Span}(t_1, t_2, (g, \frac{1}{2}t_1)) \subseteq \mathrm{Iso}(\mathbb{R}^2)$

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- Compute $(r, 0)(f, t_f) = (rf, r(t_f))$
- Compute the 2-cocycle: $c(rf, r) = (rf, r(t_f))(r, 0)(f, t_f)$, as $rfr = f$.

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- This equals to: $(rfr, r(t_f))(f, t_f) = (1, f(t_f) + r(t_f)) = r(t_f) - t_f$, an element of T.

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- First option produces the semidirect product $T \rtimes D_4$, the second is a group generated by $t_1, t_2, r, (f, \frac{1}{2}(t_1 + t_2))$.
- These two are non-isomorphic, as the non-split extension does not contain a copy of D_4.

p4g - 90°rotation and 2 reflection axes and 2 glide reflections

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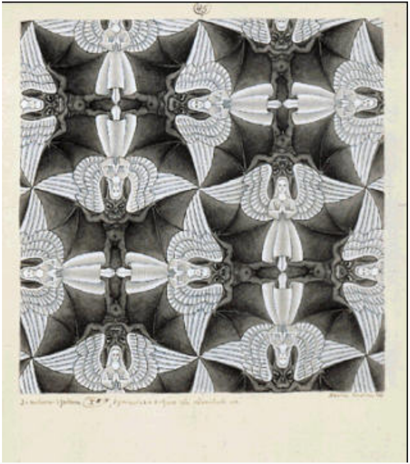
G	$\mathrm{Span}(t_1, t_2, r, (f, \frac{1}{2}(t_1 + t_2)) \subseteq \mathrm{Iso}(\mathbb{R}^2)$



Almost done

Point group $D_{2,p}$, i.e. $G_0 = D_2$ with action given by matrices:
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- Let t_1, t_2 is a basis of T satisfying $r(t_i) = -t_i$ and $f(t_1) = t_1, f(t_2) = -t_2$, and $t_f = \alpha t_1 + \beta t_2$.
- We obtain $r(t_f) - t_f = -2t_f \in T$. Modulo T, we have 4 options: $t_f = 0, t_f = \frac{1}{2}t_1, t_f = \frac{1}{2}t_2, t_f = \frac{1}{2}(t_1 + t_2)$.

Almost done

Point group $D_{2,p}$, i.e. $G_0 = D_2$ with action given by matrices:
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

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- Of the 4 obtained extensions, the two middle ones are obviously isomorphic.
- The three remaining extensions are pairwise non-isomorphic.

pmg - 180°rotation, one reflection, one glide reflection

G_0	D_2
G	$\mathrm{Span}(t_1, t_2, r, (f, \frac{1}{2}t_1)) \subseteq \mathrm{Iso}(\mathbb{R}^2)$

pgg - 180°rotation, no reflections, two glide reflections

G_0	D_2
G	$\mathrm{Span}(t_1, t_2, r, (f, \frac{1}{2}(t_1 + t_2)) \subseteq \mathrm{Iso}(\mathbb{R}^2)$

Result: 17 wallpaper patterns!

Thank you for bearing with me! :-)