Group cohomology and wallpaper groups - Part II

Pavel Čoupek

MFF UK

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Question

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Answer: Via the *second cohomology group* $H^2(G_0, T)$.

Definition

- A sequence of groups and group homomorphisms

$$\cdots \xrightarrow{f^{n-2}} A^{n-1} \xrightarrow{f^{n-1}} A^{n} \xrightarrow{f^{n}} A^{n+1} \xrightarrow{f^{n+1}} \cdots$$

is called an *exact sequence* if for every integer n, $\operatorname{Ker} f^{n+1} = \operatorname{Im} f^{n}$.

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$$\begin{array}{ccccccccccccccc} \cdots & \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & \cdots \\ & & & & \| & & \| & & \| & & \| & & \| & & & & \\ & & \cdots & & A^{-2} & & A^{-1} & & A^0 & & A^1 & & A^2 & & \cdots & & \end{array}$$

  is called a *short exact sequence*, or an *extension of* A *by* C.

(picture)

Definition

Two extensions of A by C

$$1 \longrightarrow A \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C \longrightarrow 1, \qquad i = 1, 2$$

are *equivalent* if there exists a group isomorphism $g : B_1 \longrightarrow B_2$ such that the diagram

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r, "\alpha_1"] \arrow[d, equal] & B_1 \arrow[r, "\beta_1"] \arrow[d, "g"] & C \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & A \arrow[r, "\alpha_2"] & B_2 \arrow[r, "\beta_2"] & C \arrow[r] & 1
\end{tikzcd}$$

is commutative, that is, $g \circ \alpha_1 = \alpha_2$ and $\beta_2 \circ g = \beta_1$.

Example

Consider two extensions

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Claim: *These are not equivalent.*

$$\begin{tikzcd}
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Proposition

Consider a group extension

$$1 \longrightarrow T \xrightarrow{\subseteq} G \xrightarrow{\pi} G_0 \longrightarrow 1$$

with T Abelian. For each $g \in G_0$, choose $x_g \in G$ such that $\pi(x_g) = g$. Then

$$g * t := x_g t x_g^{-1}, \quad g \in G_0, t \in T$$

is a correctly and uniquely defined action of G_0 on T.

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$$g * (h * t)$$

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(Then $G = T \rtimes_\varphi G_0$, where $\varphi : g \mapsto (g * -)$.)
In particular, this "trivial" extension always exists.

Definition

- A sequence of Abelian groups

$$(\mathcal{A}) \qquad \cdots \xrightarrow{d^{n-2}} A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} \cdots$$

is called a *chain complex* if for every $n \in \mathbb{Z}$,
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  is called a *chain complex* if for every $n \in \mathbb{Z}$, $\operatorname{Im} d^n \subseteq \operatorname{Ker} d^{n+1}$, or, equivalently, $d^{n+1} \circ d^n = 0$.
- For a chain complex $(\mathcal{A})$ and $n \in \mathbb{Z}$, we define the *n-th cohomology group of $(\mathcal{A})$* by

$$\mathbb{H}^n(\mathcal{A}) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}.$$

(picture)

Proposition (± Snake lemma)

Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be a short exact sequence of chain complexes - that is, a commutative diagram

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{n-2} & \longrightarrow & A^{n-1} & \longrightarrow & A^{n} & \longrightarrow & A^{n+1} & \longrightarrow & \cdots \qquad (\mathcal{A}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & B^{n-2} & \longrightarrow & B^{n-1} & \longrightarrow & B^{n} & \longrightarrow & B^{n+1} & \longrightarrow & \cdots \qquad (\mathcal{B}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C^{n-2} & \longrightarrow & C^{n-1} & \longrightarrow & C^{n} & \longrightarrow & C^{n+1} & \longrightarrow & \cdots \qquad (\mathcal{C}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

such that the rows are complexes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and the columns are exact. Then there is an exact sequence of the following form:

$$\cdots \longrightarrow \mathbb{H}^{n-1}(\mathcal{C}) \longrightarrow \mathbb{H}^{n}(\mathcal{A}) \longrightarrow \mathbb{H}^{n}(\mathcal{B}) \longrightarrow \mathbb{H}^{n}(\mathcal{C}) \longrightarrow \mathbb{H}^{n+1}(\mathcal{A}) \longrightarrow \cdots$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^n \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
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 & 0 & 0 & 0 & {} &
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 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^{n} \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r] \arrow[d, two heads] & B^{n} \arrow[r] \arrow[d, two heads, "\beta_n"] & B^{n+1} \arrow[r] \arrow[d, two heads] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r] \arrow[d] & C^{n} \arrow[r] \arrow[d] & C^{n+1} \arrow[r] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & {} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \overline{a} & & & & \\
\cdots \arrow[r] & \mathbb{H}^n(\mathcal{A}) \arrow[r] & \mathbb{H}^n(\mathcal{B}) \arrow[r] & \mathbb{H}^n(\mathcal{C}) \arrow[r] & \mathbb{H}^{n+1}(\mathcal{A}) \arrow[r] & \cdots
\end{tikzcd}
$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^n \arrow[r] \arrow[d, hook, "\alpha_n"] \arrow[dr, phantom, "\color{purple}a" near start] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r] \arrow[d, two heads] & B^n \arrow[r] \arrow[d, two heads, "\beta_n"] & B^{n+1} \arrow[r] \arrow[d, two heads] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r] \arrow[d] & C^n \arrow[r] \arrow[d] & C^{n+1} \arrow[r] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & &
\end{tikzcd}
$$

$$
\cdots \longrightarrow \overset{\color{purple}\overline{a}}{\mathbb{H}^n(\mathcal{A})} \color{purple}\longrightarrow \color{black} \mathbb{H}^n(\mathcal{B}) \longrightarrow \mathbb{H}^n(\mathcal{C}) \longrightarrow \mathbb{H}^{n+1}(\mathcal{A}) \longrightarrow \cdots
$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^n \arrow[r] \arrow[d, hook, "\alpha_n"] \arrow[d, phantom, shift left=3, "a"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r] \arrow[d, two heads] & B^n \arrow[r] \arrow[d, two heads, "\beta_n"] \arrow[d, phantom, shift left=4, "\alpha_n(a)"] & B^{n+1} \arrow[r] \arrow[d, two heads] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r] \arrow[d] & C^n \arrow[r] \arrow[d] & C^{n+1} \arrow[r] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & {} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdots \arrow[r] & \overset{\overline{a}}{\mathbb{H}^n(\mathcal{A})} \arrow[r] & \mathbb{H}^n(\mathcal{B}) \arrow[r] & \mathbb{H}^n(\mathcal{C}) \arrow[r] & \mathbb{H}^{n+1}(\mathcal{A}) \arrow[r] & \cdots
\end{tikzcd}
$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^n \arrow[r] \arrow[d, hook, "\alpha_n"] \arrow[d, bend right, red] \arrow[d, phantom, shift left=4ex, "\color{red}a"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r] \arrow[d, two heads] & B^n \arrow[r] \arrow[d, two heads, "\beta_n"] \arrow[d, phantom, shift left=5ex, "\color{red}\alpha_n(a)"] & B^{n+1} \arrow[r] \arrow[d, two heads] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r] \arrow[d] & C^n \arrow[r] \arrow[d] & C^{n+1} \arrow[r] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & {} & \\
 & \color{red}\overline{a} \arrow[r, mapsto, red] & \color{red}\overline{\alpha_n(a)} & & & \\
\cdots \arrow[r] & \mathbb{H}^n(\mathcal{A}) \arrow[r, red] & \mathbb{H}^n(\mathcal{B}) \arrow[r] & \mathbb{H}^n(\mathcal{C}) \arrow[r] & \mathbb{H}^{n+1}(\mathcal{A}) \arrow[r] & \cdots
\end{tikzcd}
$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^{n} \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r] \arrow[d, two heads] & B^{n} \arrow[r] \arrow[d, two heads, "\beta_n"] & B^{n+1} \arrow[r] \arrow[d, two heads] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r] \arrow[d] & C^{n} \arrow[r] \arrow[d] & C^{n+1} \arrow[r] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & {} &
\end{tikzcd}
$$

$$
\cdots \longrightarrow \mathbb{H}^n(\mathcal{A}) \longrightarrow \mathbb{H}^n(\mathcal{B}) \longrightarrow \mathbb{H}^n(\mathcal{C}) \longrightarrow \mathbb{H}^{n+1}(\mathcal{A}) \longrightarrow \cdots
$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^{n} \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r] \arrow[d, two heads] & B^{n} \arrow[r] \arrow[d, two heads, "\beta_n"] & B^{n+1} \arrow[r] \arrow[d, two heads] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r] \arrow[d] & C^{n} \arrow[r] \arrow[d] & C^{n+1} \arrow[r] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & &
\end{tikzcd}
$$

$b \mapsto \beta_n(b)$

$$
\cdots \longrightarrow \mathbb{H}^n(\mathcal{A}) \longrightarrow \mathbb{H}^n(\mathcal{B}) \longrightarrow \mathbb{H}^n(\mathcal{C}) \longrightarrow \mathbb{H}^{n+1}(\mathcal{A}) \longrightarrow \cdots
$$

$\overline{b} \longmapsto \overline{\beta_n(b)}$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^{n} \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r] \arrow[d, two heads] & B^{n} \arrow[r] \arrow[d, two heads, "\beta_n"] & B^{n+1} \arrow[r] \arrow[d, two heads] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r] \arrow[d] & C^{n} \arrow[r] \arrow[d] & C^{n+1} \arrow[r] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & {} &
\end{tikzcd}
$$

$$
\cdots \longrightarrow \mathbb{H}^n(\mathcal{A}) \longrightarrow \mathbb{H}^n(\mathcal{B}) \longrightarrow \mathbb{H}^n(\mathcal{C}) \longrightarrow \mathbb{H}^{n+1}(\mathcal{A}) \longrightarrow \cdots
$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^{n} \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r] \arrow[d, two heads] & B^{n} \arrow[r] \arrow[d, two heads, "\beta_n"] & B^{n+1} \arrow[r] \arrow[d, two heads] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r] \arrow[d] & C^{n} \arrow[r] \arrow[d] & C^{n+1} \arrow[r] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & {} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & \overline{c} & & \\
\cdots \arrow[r] & \mathbb{H}^n(\mathcal{A}) \arrow[r] & \mathbb{H}^n(\mathcal{B}) \arrow[r] & \mathbb{H}^n(\mathcal{C}) \arrow[r] & \mathbb{H}^{n+1}(\mathcal{A}) \arrow[r] & \cdots
\end{tikzcd}
$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^{n} \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r] \arrow[d, two heads] & B^{n} \arrow[r] \arrow[d, two heads, "\beta_n"] & B^{n+1} \arrow[r] \arrow[d, two heads] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r, "c"'] \arrow[d] & C^{n} \arrow[r] \arrow[d] & C^{n+1} \arrow[r] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & {} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & \bar{c} & & \\
\cdots \arrow[r] & \mathbb{H}^n(\mathcal{A}) \arrow[r] & \mathbb{H}^n(\mathcal{B}) \arrow[r] & \mathbb{H}^n(\mathcal{C}) \arrow[r] & \mathbb{H}^{n+1}(\mathcal{A}) \arrow[r] & \cdots
\end{tikzcd}
$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^{n} \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "c'"] \arrow[d, two heads] & B^{n} \arrow[r] \arrow[d, two heads, "\beta_n"] & B^{n+1} \arrow[r] \arrow[d, two heads] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r, "c"'] \arrow[d] & C^{n} \arrow[r] \arrow[d] \arrow[u, bend left] & C^{n+1} \arrow[r] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & {} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathbb{H}^n(\mathcal{A}) \arrow[r] & \mathbb{H}^n(\mathcal{B}) \arrow[r] & \overset{\overline{c}}{\mathbb{H}^n(\mathcal{C})} \arrow[r] & \mathbb{H}^{n+1}(\mathcal{A}) \arrow[r] & \cdots
\end{tikzcd}
$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^{n} \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "{\color{red} c'}" near end] \arrow[d, two heads] & B^{n} \arrow[r] \arrow[d, two heads, "\beta_n"] & B^{n+1} \arrow[r] \arrow[d, two heads] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r, "{\color{red} c}"' near end] \arrow[d] & C^{n} \arrow[r] \arrow[d] \arrow[u, bend left, red] & C^{n+1} \arrow[r] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & {\color{red}\overline{c}} & & \\[-2em]
\cdots \arrow[r] & \mathbb{H}^n(\mathcal{A}) \arrow[r] & \mathbb{H}^n(\mathcal{B}) \arrow[r] & \mathbb{H}^n(\mathcal{C}) \arrow[r, red] & \mathbb{H}^{n+1}(\mathcal{A}) \arrow[r] & \cdots
\end{tikzcd}
$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^{n} \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "\color{purple}{c'}" near end] & B^{n} \arrow[r] \arrow[d, two heads, "\beta_n"] \arrow[r, bend left, purple] & B^{n+1} \arrow[r, "\color{purple}{d^n(c')}" near start] \arrow[d, two heads] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r, "\color{purple}{c}"' near end] \arrow[d] & C^{n} \arrow[r] \arrow[d] \arrow[u, bend left, purple] & C^{n+1} \arrow[r] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & {} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathbb{H}^n(\mathcal{A}) \arrow[r] & \mathbb{H}^n(\mathcal{B}) \arrow[r] & \overset{\color{purple}{\overline{c}}}{\mathbb{H}^n(\mathcal{C})} \arrow[r, purple] & \mathbb{H}^{n+1}(\mathcal{A}) \arrow[r] & \cdots
\end{tikzcd}
$$

Idea of proof

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^n \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "c'"] \arrow[d, two heads] & B^n \arrow[r] \arrow[d, two heads, "\beta_n"] \arrow[r, bend left, color=purple] & B^{n+1} \arrow[r, "d^n(c')"] \arrow[d, two heads] \arrow[d, dashed, bend left, color=blue] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r, "c"'] \arrow[d] & C^n \arrow[r] \arrow[d] \arrow[u, bend left, color=purple] & C^{n+1} \arrow[r, "??"'] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & {} &
\end{tikzcd}$$

$$\cdots \longrightarrow \mathbb{H}^n(\mathcal{A}) \longrightarrow \mathbb{H}^n(\mathcal{B}) \longrightarrow \overset{\overline{c}}{\mathbb{H}^n(\mathcal{C})} \longrightarrow \mathbb{H}^{n+1}(\mathcal{A}) \longrightarrow \cdots$$

Idea of proof

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^n \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "c'"] \arrow[d, two heads] & B^n \arrow[r] \arrow[d, two heads, "\beta_n"] \arrow[r, dashed, bend left] & B^{n+1} \arrow[r, "d^n(c')"] \arrow[d, two heads] \arrow[d, dashed, bend left] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r, "c"'] \arrow[d] & C^n \arrow[r] \arrow[d] \arrow[u, bend left] & C^{n+1} \arrow[r, "??"'] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & {} &
\end{tikzcd}$$

$$\bar{c}$$

$$\cdots \longrightarrow \mathbb{H}^n(\mathcal{A}) \longrightarrow \mathbb{H}^n(\mathcal{B}) \longrightarrow \mathbb{H}^n(\mathcal{C}) \longrightarrow \mathbb{H}^{n+1}(\mathcal{A}) \longrightarrow \cdots$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^{n} \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, phantom, "c'" near end, shift left=3ex] \arrow[r] \arrow[d, two heads] & B^{n} \arrow[r] \arrow[d, two heads, "\beta_n"] \arrow[d, dashed, bend right] \arrow[r, dashed, bend left] \arrow[dr, phantom, "\circlearrowleft"] & B^{n+1} \arrow[r] \arrow[d, two heads] \arrow[d, dashed, bend left] \arrow[r, phantom, "d^n(c')" near start, shift left=3ex] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r] \arrow[r, phantom, "c" near end, shift right=3ex] \arrow[d] & C^{n} \arrow[r] \arrow[d] \arrow[r, dashed, bend right] & C^{n+1} \arrow[r] \arrow[d] \arrow[r, phantom, "??" near start, shift right=3ex] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathbb{H}^n(\mathcal{A}) \arrow[r] & \mathbb{H}^n(\mathcal{B}) \arrow[r] & \overset{\bar{c}}{\mathbb{H}^n(\mathcal{C})} \arrow[r] & \mathbb{H}^{n+1}(\mathcal{A}) \arrow[r] & \cdots
\end{tikzcd}
$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^n \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "c'"] \arrow[d, two heads] & B^n \arrow[r] \arrow[d, two heads, "\beta_n"] \arrow[d, dashed, bend right] & B^{n+1} \arrow[r, "d^n(c')"] \arrow[d, two heads] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r, "c"'] \arrow[d] & C^n \arrow[r] \arrow[d] \arrow[r, dashed, bend right] & C^{n+1} \arrow[r, "??"'] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & {} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathbb{H}^n(\mathcal{A}) \arrow[r] & \mathbb{H}^n(\mathcal{B}) \arrow[r] & \overset{\bar{c}}{\mathbb{H}^n(\mathcal{C})} \arrow[r] & \mathbb{H}^{n+1}(\mathcal{A}) \arrow[r] & \cdots
\end{tikzcd}
$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^{n} \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "c'"] \arrow[d, two heads] & B^{n} \arrow[r] \arrow[d, two heads, "\beta_n"] & B^{n+1} \arrow[r, "d^n(c')"] \arrow[d, two heads] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r, "c"'] \arrow[d] & C^{n} \arrow[r] \arrow[r, dashed, bend right] \arrow[d] & C^{n+1} \arrow[r, "0"'] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & {} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathbb{H}^n(\mathcal{A}) \arrow[r] & \mathbb{H}^n(\mathcal{B}) \arrow[r] & \mathbb{H}^n(\mathcal{C}) \arrow[r, "\overline{c}"] & \mathbb{H}^{n+1}(\mathcal{A}) \arrow[r] & \cdots
\end{tikzcd}
$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^{n} \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "c'"] \arrow[d, two heads] & B^{n} \arrow[r] \arrow[r, bend left] \arrow[d, two heads, "\beta_n"] & B^{n+1} \arrow[r, "d^n(c')"] \arrow[d, two heads] \arrow[d, dashed, bend left] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r, "c"'] \arrow[d] & C^{n} \arrow[r] \arrow[d] \arrow[u, bend left] & C^{n+1} \arrow[r, "0"'] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & &
\end{tikzcd}
$$

$$
\cdots \longrightarrow \mathbb{H}^n(\mathcal{A}) \longrightarrow \mathbb{H}^n(\mathcal{B}) \longrightarrow \overset{\overline{c}}{\mathbb{H}^n(\mathcal{C})} \longrightarrow \mathbb{H}^{n+1}(\mathcal{A}) \longrightarrow \cdots
$$

Idea of proof

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, hook] & A^{n} \arrow[r] \arrow[d, hook, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, hook] & A^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r] \arrow[d, two heads] & B^{n} \arrow[r] \arrow[d, two heads, "\beta_n"] \arrow[r, bend left, "d^n(c')" near end] & B^{n+1} \arrow[r] \arrow[d, two heads] \arrow[u, bend left, "c''" near end] & B^{n+2} \arrow[r] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r] \arrow[d] & C^{n} \arrow[r] \arrow[d] \arrow[u, bend left, "c" near start, "c'" near end] & C^{n+1} \arrow[r] \arrow[d] & C^{n+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & {} &
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 & & & \overline{c} \arrow[r, mapsto] & \overline{c''} &
\end{tikzcd}
$$

Definition

Let T be an Abelian group and G_0 a group with action on T.
For a positive integer n, denote $C^n(G_0, T)$ the group of all maps $f : (G_0)^n \to T$, and define group homomorphisms $d^n : C^n(G_0, T) \to C^{n+1}(G_0, T)$ by

$$\begin{aligned}[d^n(f)](g_1, \dots, g_{n+1}) = & g_1 * f(g_2, \dots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1, g_2, \dots, (g_i g_{i+1}), \dots, g_{n+1}) \\ & + (-1)^{n+1} f(g_1, g_2, \dots, g_n).\end{aligned}$$

Further set $C^0(G_0, T) = T$ and
$d^0(g) : t \mapsto g * t - t, \quad g \in G_0, t \in T.$

Proposition

For every n, $d^{n+1} \circ d^n = 0$. *That is,* $C^n(G_0, T)$ *together with* $d^n, n \in \mathbb{N}_0$, *form a chain complex.*

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(Proof is omitted.)

Definition

Given a non-negative integer n, we define

$$H^n(G_0, T) = \mathbb{H}^n(C^\bullet(G_0, T)).$$

We call $H^n(G_0, T)$ the *n-th cohomology group of G_0 with coefficients in T.*

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2-cocycles: Maps $c : G_0 \times G_0 \to T$ satisfying

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2-coboundaries: Maps $b : G_0 \times G_0 \to T$ satisfying

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for some map $f : G_0 \to T$.

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where $b(g,h) = [d^1(f)](g,h)$ is a 2-coboundary obtained from the map $f(g) = t_g$.

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where $b(g,h) = [d^1(f)](g,h)$ is a 2-coboundary obtained from the map $f(g) = t_g$.
Thus, the 2-cocycle obtained is unique modulo 2-coboundaries.

Example

Consider the point group $G_0 = D_{1,p} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$ acting naturally on $T = \mathbb{Z}^2$. Then

$$H^2(G_0, T) \simeq \mathbb{Z}_2.$$

That is, there are two non-equivalent group extension of T by G_0 inducing the given action.

Proposition ($\pm$ Snake lemma)

Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be a short exact sequence of chain complexes - that is, a commutative diagram

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
\cdots & \longrightarrow & A^{n-2} & \longrightarrow & A^{n-1} & \longrightarrow & A^{n} & \longrightarrow & A^{n+1} & \longrightarrow & \cdots & \qquad (\mathcal{A}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
\cdots & \longrightarrow & B^{n-2} & \longrightarrow & B^{n-1} & \longrightarrow & B^{n} & \longrightarrow & B^{n+1} & \longrightarrow & \cdots & \qquad (\mathcal{B}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
\cdots & \longrightarrow & C^{n-2} & \longrightarrow & C^{n-1} & \longrightarrow & C^{n} & \longrightarrow & C^{n+1} & \longrightarrow & \cdots & \qquad (\mathcal{C}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
 & & 0 & & 0 & & 0 & & 0 & & &
\end{array}
$$

such that the rows are complexes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and the columns are exact. Then there is an exact sequence of the following form:

$$\cdots \longrightarrow \mathbb{H}^{n-1}(\mathcal{C}) \longrightarrow \mathbb{H}^{n}(\mathcal{A}) \longrightarrow \mathbb{H}^{n}(\mathcal{B}) \longrightarrow \mathbb{H}^{n}(\mathcal{C}) \longrightarrow \mathbb{H}^{n+1}(\mathcal{A}) \longrightarrow \cdots$$

Fact: There is a short exact sequence of chain complexes

$$
\begin{array}{ccccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^2 & \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}} & \mathbb{Z}^2 & \xrightarrow{\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}} & \mathbb{Z}^2 & \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}} & \mathbb{Z}^2 & \xrightarrow{\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}} & \cdots & & (\mathcal{A}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & C^0(G_0,T) & \longrightarrow & C^1(G_0,T) & \longrightarrow & C^2(G_0,T) & \longrightarrow & C^3(G_0,T) & \longrightarrow & \cdots & & (\mathcal{B}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & C^3 & \longrightarrow & \cdots & & (\mathcal{C}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & & &
\end{array}
$$

where the sequence $(\mathcal{C})$ is also exact.

Thus, we obtain a long exact sequence

$$0 \longrightarrow \mathbb{H}^1(\mathcal{A}) \longrightarrow H^1(G_0, T) \longrightarrow 0 \longrightarrow \mathbb{H}^2(\mathcal{A}) \longrightarrow H^2(G_0, T) \longrightarrow 0$$

$$\mathbb{H}^0(\mathcal{C}) \longrightarrow \mathbb{H}^1(\mathcal{A}) \longrightarrow \mathbb{H}^1(\mathcal{B}) \longrightarrow \mathbb{H}^1(\mathcal{C}) \longrightarrow \mathbb{H}^2(\mathcal{A}) \longrightarrow \mathbb{H}^2(\mathcal{C}) \longrightarrow \mathbb{H}^3(\mathcal{C})$$

Thus, we obtain a long exact sequence

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$$\mathbb{H}^0(\mathcal{C}) \longrightarrow \mathbb{H}^1(\mathcal{A}) \longrightarrow \mathbb{H}^1(\mathcal{B}) \longrightarrow \mathbb{H}^1(\mathcal{C}) \longrightarrow \mathbb{H}^2(\mathcal{A}) \longrightarrow \mathbb{H}^2(\mathcal{C}) \longrightarrow \mathbb{H}^3(\mathcal{C})$$

In particular, $H^2(G_0, T) \simeq \mathbb{H}^2(\mathcal{A})$.

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In particular, $H^2(G_0,T) \simeq \mathbb{H}^2(\mathcal{A})$.
Thus,

$$H^2(G_0,T) \simeq \left\{ \begin{pmatrix} 0 \\ z \end{pmatrix} \mid z \in \mathbb{Z} \right\} / \left\{ \begin{pmatrix} 0 \\ -2z \end{pmatrix} \mid z \in \mathbb{Z} \right\} \simeq \mathbb{Z}_2.$$

We conlude this part by listing the second cohomology groups of all possible choices of the point group and the lattice:

G_0	$H^2(G_0, T)$	No. of extensions
C_1	0	1
C_2	0	1
C_3	0	1
C_4	0	1
C_6	0	1
$D_{1,p}$	$\mathbb{Z}_2$	2
$D_{1,c}$	0	1
$D_{2,p}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4
$D_{2,c}$	0	1
$D_{3,l}$	0	1
$D_{3,s}$	0	1
D_4	$\mathbb{Z}_2$	2
D_6	0	1

Thank you for your attention!