Octonions

and Octonionic Projective Geometry

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Introduction

- William Rowan Hamilton - *quaternions*
- John T. Graves - *octaves*
- Arthur Cayley - *Cayley's numbers*
- Élie Cartan - *geometrical relevance*
- Jordan, von Neumann and Wigner - *the foundations of quantum mechanics*

Inventing the Quaternions

Brougham Bridge - the bridge where Hamilton carved his definition of the quaternions

Octonions: if 4, why not more?

Graves: If you're allowed make up a way of multiplying lists of 4 numbers, why not more?

Hamilton: Just because it's bigger I don't know if it is better. I have a horse with four legs, I don't know if your horse with eight legs will run twice as fast!

Basic Definitions

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- An algebra $\mathcal{A}$ is a ***division algebra*** if given $a, b \in \mathcal{A}$ with $ab = 0$, then either $a = 0$ or $b = 0$.
- A ***normed division algebra*** is an algebra $\mathcal{A}$ that is also a normed vector space with $\|ab\| = \|a\|\|b\|$.

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Commutator and Associator

- ***Commutator*** is an alternating bilinear map $[\cdot,\cdot] : \mathcal{A}^2 \to \mathcal{A}$ given by $[a,b] = ab - ba$. It switches sign whenever the two arguments are exchanged $[a,b] = -[b,a]$ or equivalently, that it vanishes when they are equal $[a,a] = 0$.

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- ***Associator*** is a trilinear map $[\cdot,\cdot,\cdot] : \mathcal{A}^3 \to \mathcal{A}$ given by $[a,b,c] = (ab)c - a(bc)$.
- The associator measures the failure of associativity just as the commutator measures the failure of commutativity.

Alternative Algebra

Theorem:
An algebra $\mathcal{A}$ is alternative if and only if for all $a, b \in \mathcal{A}$ we have
$(aa)b = a(ab),\ (ab)a = a(ba),\ (ba)a = b(aa)$
[*Emil Artin, 1930s*]

Proposition:
Associator is alternating precisely when $\mathcal{A}$ is alternative.

Basic Theorems

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Theorem 3: All division algebras have dimension 1, 2, 4, or 8. [*Kervaire and Bott-Milnor, 1958.*]

Constructing the Octonions

- The octonions are an 8-dimensional algebra with basis $1, e_1, e_2, e_3, e_4, e_5, e_6, e_7$.

- The most elementary way to construct the octonions is to give their multiplication table.

Octonion Multiplication Table

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	e_3	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Properties

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- nontrivial product: $e_1 e_2 = e_4$

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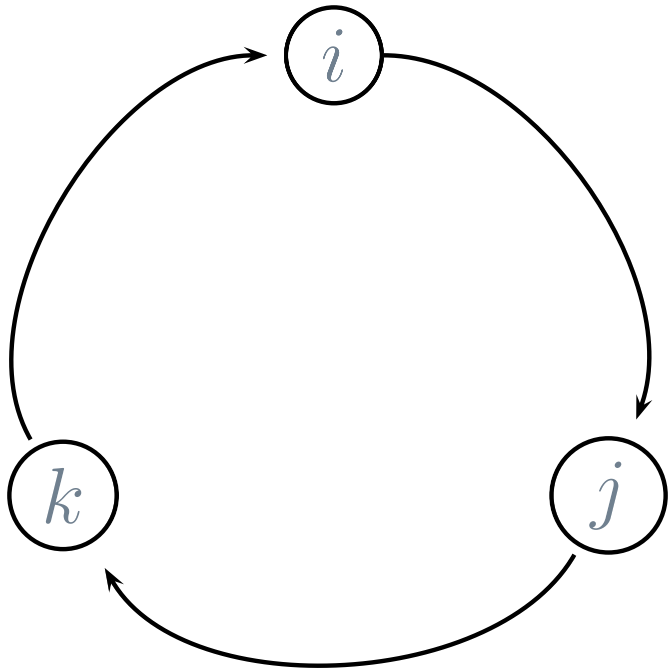
- we have $ij = k$, $ji = -k$, and all identities obtained from these by cyclic permutations of (i, j, k)

Multiplication of Quaternions



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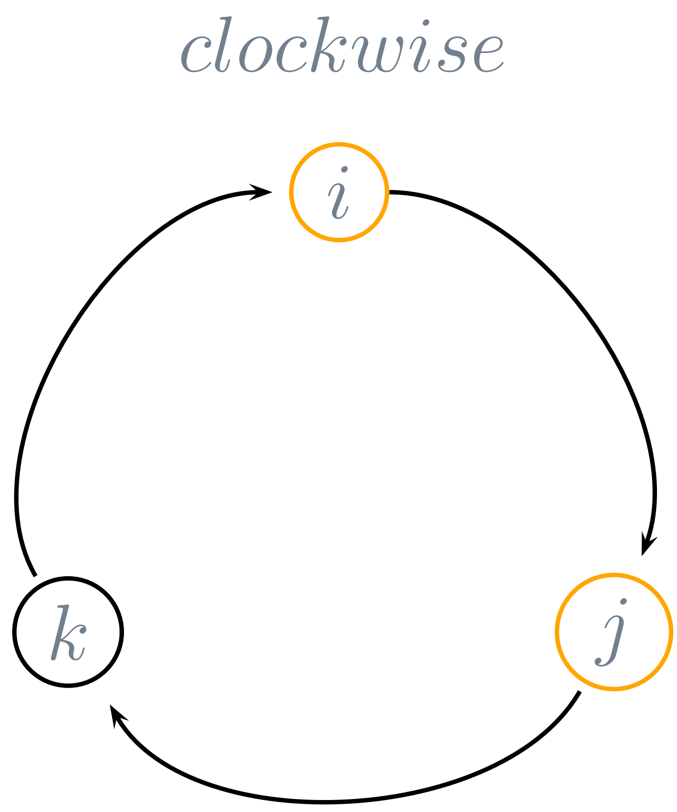
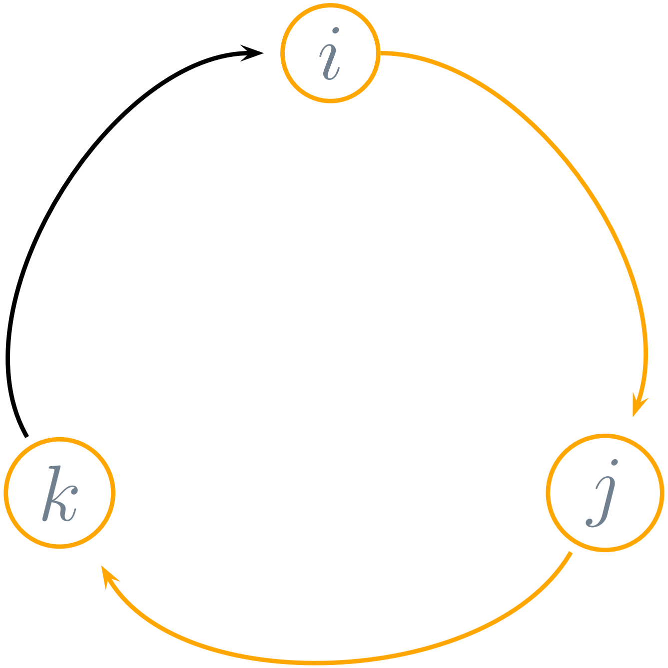


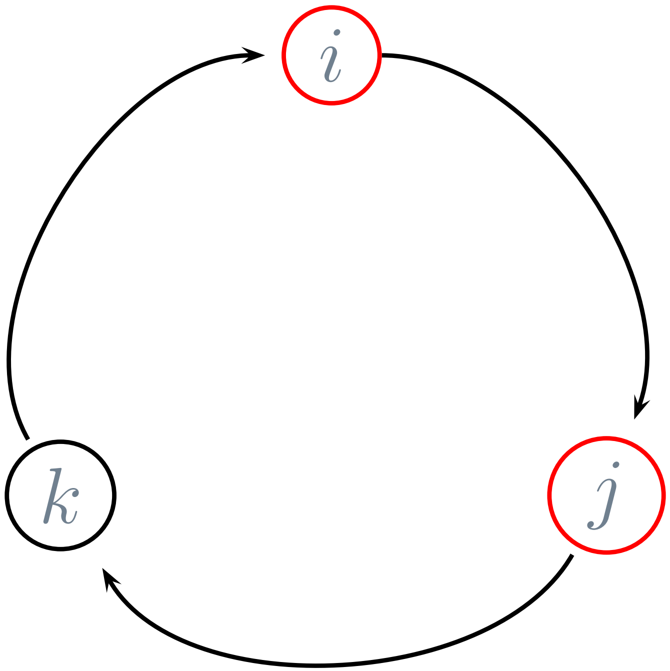
Multiplication of Quaternions

$$ij = k$$



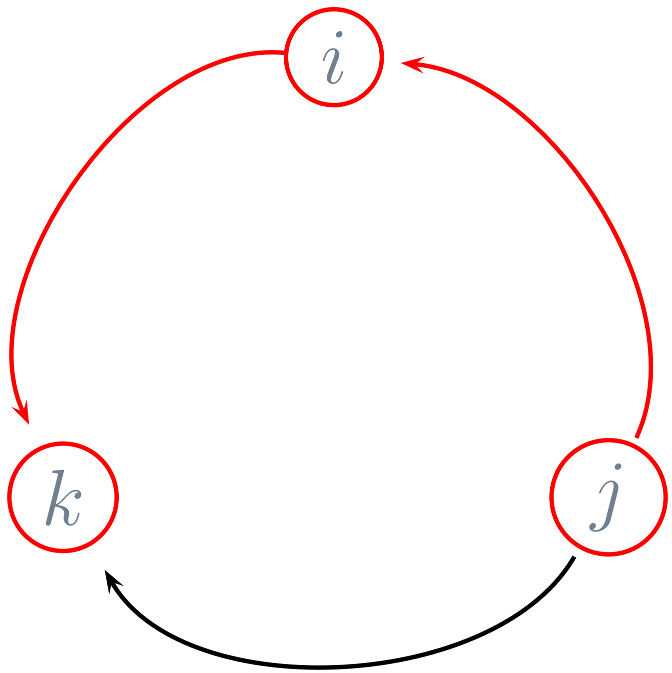
Multiplication of Quaternions

$$counterclockwise$$

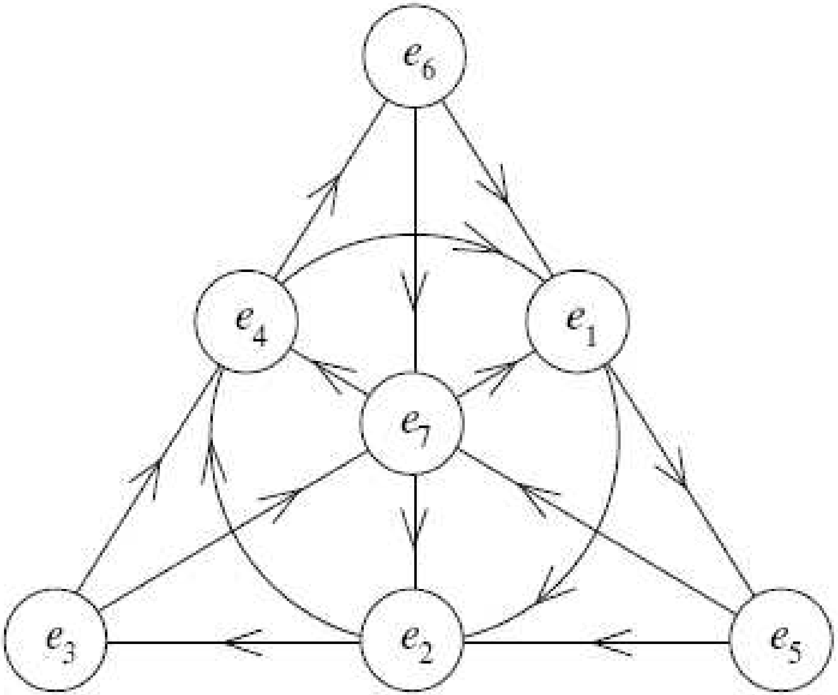


Multiplication of Quaternions

$$ji = -k$$



How to multiply octonions?



Fano plane

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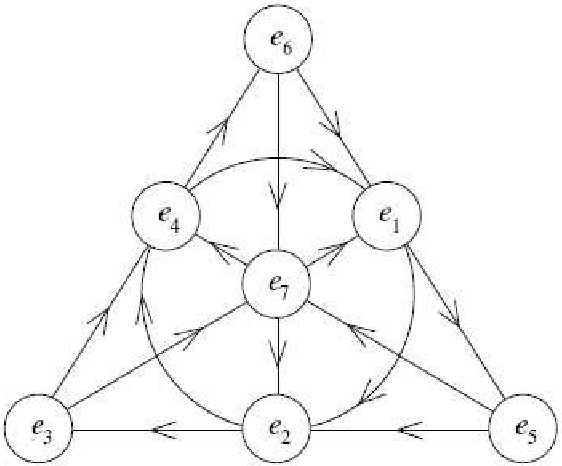
- If e_i, e_j, and e_k are cyclically ordered in this way then $e_i e_j = e_k,\ e_j e_i = -e_k$.
- the ***index doubling*** corresponds to rotating the picture a third of a turn

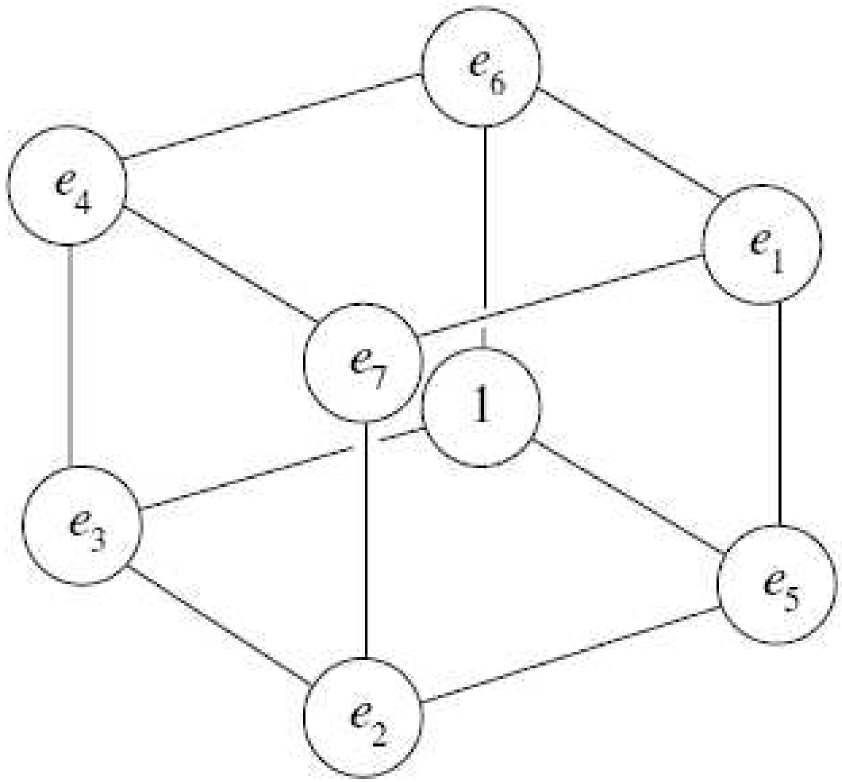
The Fano plane

- The Fano plane consists of 7 points and 7 lines.
- The 'lines' are the sides of the triangle, its altitudes, and the circle containing all the midpoints of the sides.
- It completely describes the algebra structure of the octonions.



Octonions $\mathbb{O}$

The Fano plane is the projective plane over the 2-element field $\mathbb{Z}_2$.



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- Lines through the origin give subalgebras isomorphic the complex numbers.
- The origin itself gives a subalgebra isomorphic to the real numbers.

The Cayley-Dickson Construction

The Cayley-Dickson construction gives an infinite sequence of algebras, doubling in dimension each time, with the normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ as the first four. Also, it explaines why each one fits neatly inside the next.

Complex numbers

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- Multiplication and conjugate are defined in the same manner as for the quaternions.
- Why isn't there an *infinite* sequence of division algebras, each one obtained from the preceding one by the Cayley-Dickson construction?

$*$-algebra

- An algebra $\mathcal{A}$ equipped with a conjugation, that is, a real-linear map $* : \mathcal{A} \longrightarrow \mathcal{A}$ with $a^{**} = a, \ (ab)^* = b^* a^*$ for all $a, b \in \mathcal{A}$ is called ***$*$-algebra.***
- $*$-algebra $\mathcal{A}$ is:
  - ***real*** if $a = a^*$ for every element $a \in \mathcal{A}$.
  - ***nicely normed*** if $a + a^* \in \mathbb{R}$ and $aa^* = a^* a > 0$ for all nonzero $a \in \mathcal{A}$.
- If $\mathcal{A}$ is nicely normed and alternative, $\mathcal{A}$ is a normed division algebra.

The General Construction

- Starting from any $*$-algebra $\mathcal{A}$, the Cayley-Dickson construction gives a new $*$-algebra $\mathcal{A}'$.
- Elements of $\mathcal{A}'$ are pairs $(a, b) \in \mathcal{A}^2$.
- Multipication:

$$(a, b)(c, d) = (ac - db^*, a^*d + cb)$$

- Conjugate:

$$(a, b)^* = (a^*, -b)$$

Propositions

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P5: $\mathcal{A}$ is nicely normed $\iff$ $\mathcal{A}'$ is nicely normed.

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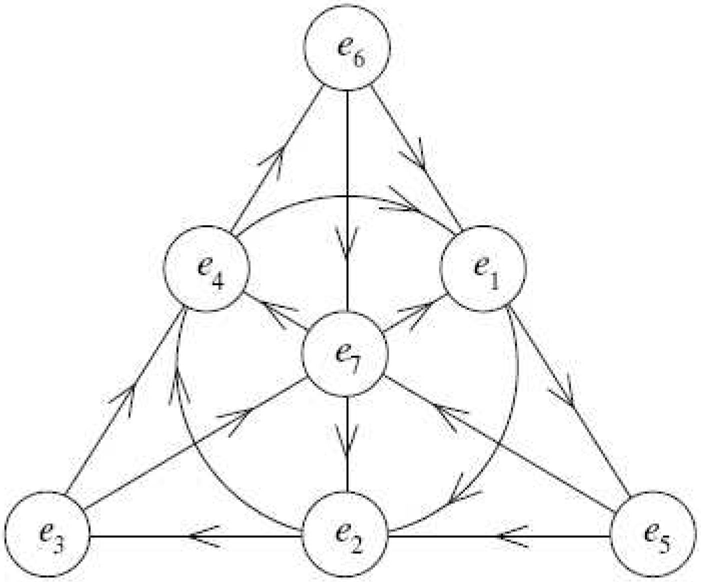
$\mathbb{O}$ is an alternative nicely normed $*$-algebra

Axioms of Projective Plane

- For any two distinct points, there is a unique line on which they both lie.
- For any two distinct lines, there is a unique point which lies on both of them.
- There exist four points, no three of which lie on the same line.
- There exist four lines, no three of which have the same point lying on them.

Examples of Projective Plane

Fano plane



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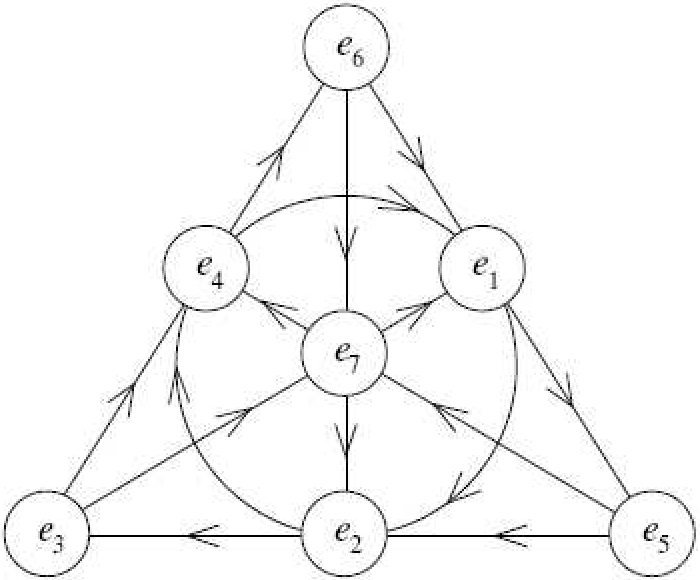

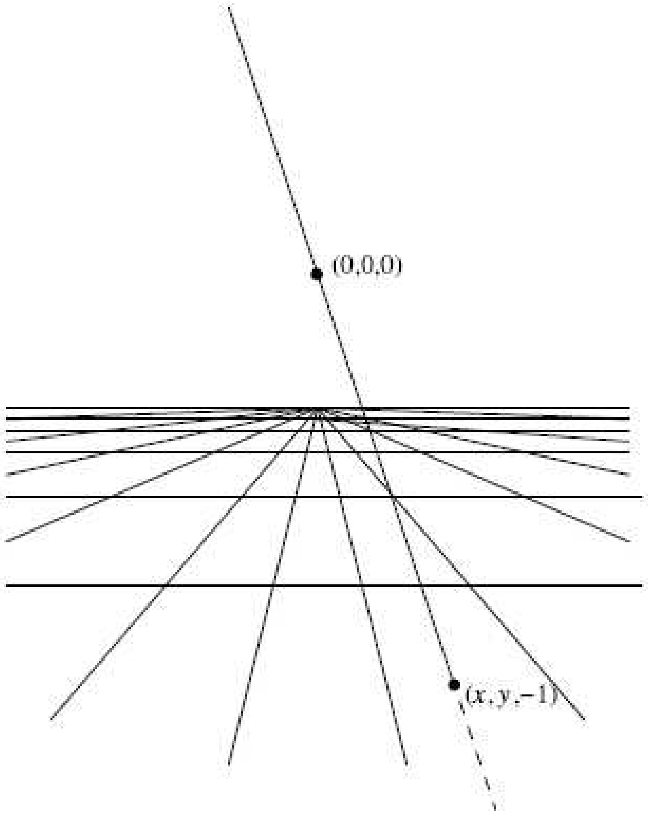

the real projective plane $\mathbb{R}P^2$

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- the lines are planes through the origin in $\mathbb{R}^3$

Axioms of Projective Space

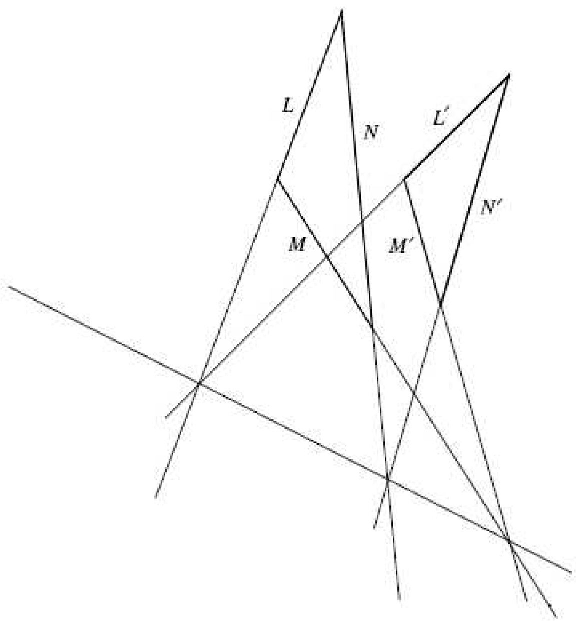
- For any two distinct points p, q, there is a unique line pq on which they both lie.
- For any line, there are at least three points lying on this line.
- If a, b, c, d are distinct points and there is a point lying on both ab and cd, then there is a point lying on both ac and bd.

n-dimensional Projective Space

- $\mathbb{K}$ - ***skew field***: a ring such that every nonzero element has a left and right multiplicative inverse
- line through the origin: $L = \{\alpha x : \alpha \in \mathbb{K}\}$, where $x \in \mathbb{K}^{n+1}$ is nonzero
- plane through the origin: $P = \{\alpha x + \beta y : \alpha, \beta \in \mathbb{K}\}$, where $x, y \in \mathbb{K}^{n+1}$ are elements such that $\alpha x + \beta y = 0$ implies $\alpha, \beta = 0$
- projective n-space is of the form $\mathbb{K}P^n$ for some skew field $\mathbb{K}$ only if $n > 2$

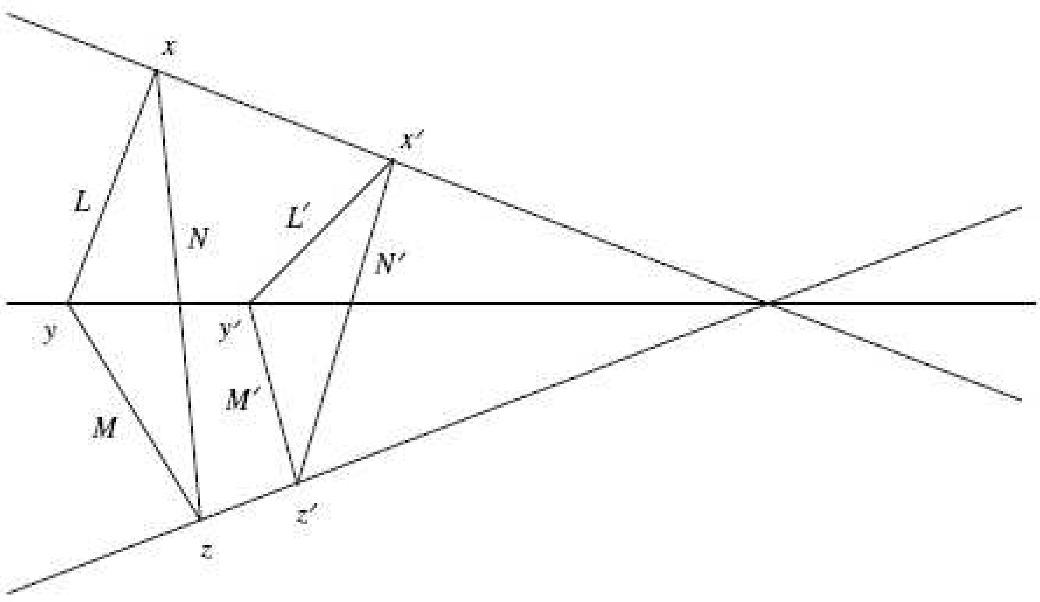
Axial Perspectivity

Axial perspectivity is the condition satisfied if and only if the points of intersection $L \cap L'$, $N \cap N'$ and $M \cap M'$, are collinear, on a line called *the axis of perspectivity*.



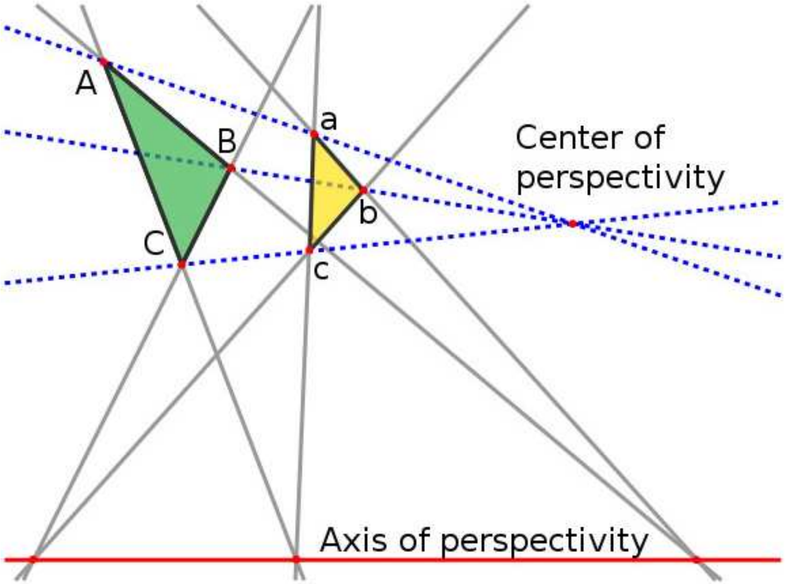
Central Perspectivity

Central perspectivity is the condition satisfied if and only if the three lines xx', yy' and zz' are concurrent, at a point called ***the center of perspectivity***.



The Axiom of Desargues

In a projective space, two triangles are in perspective axially if and only if they are in perspective centrally.



A projective plane satisfying this axiom is called ***Desarguesian***.

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- In 1933 *Ruth Moufang* constructed a remarkable example of a non-Desarguesian projective plane using the octonions.
- The 1930s also saw the rise of another reason for interest in projective geometry: quantum mechanics!
  However, in quantum mechanics, observables do not form an associative algebra.

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- Projective geometry was very fashionable in the 1800s, with such worthies as *Poncelet, Brianchon, Steiner* and *von Staudt* making important contributions.
- In 1933 *Ruth Moufang* constructed a remarkable example of a non-Desarguesian projective plane using the octonions.
- The 1930s also saw the rise of another reason for interest in projective geometry: quantum mechanics!
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- In 1932, *Pascual Jordan* attempted to understand this situation better by isolating the bare minimum axioms that an ‘algebra of observables’ should satisfy.

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- In 1932, *Pascual Jordan* attempted to understand this situation better by isolating the bare minimum axioms that an ‘algebra of observables’ should satisfy.
- In 1934, *Jordan* published a paper with *von Neumann* and *Wigner* classifying all formally real Jordan algebras

Formally Real Jordan Algebra

- ***A formally real Jordan algebra*** is a commutative and power-associative algebra satisfying

$$a_1^2 + \cdots + a_n^2 = 0 \quad \Longrightarrow \quad a_1 = \cdots = a_n = 0$$

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- The last condition gives the algebra a partial ordering: if we write $a \leq b$ when the element $b - a$ is a sum of squares, it says that $a \leq b$ and $b \leq a$ imply $a = b$.

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- One can construct a commutative but nonassociative product:

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- Any commutative algebra satisfying this identity is called a ***Jordan algebra***.
  Jordan algebras are automatically power-associative.

Classification

The simple formally real Jordan algebras consist of 4 infinite families and one exception:

- The algebra $\mathfrak{h}_n(\mathbb{R})$ with the product $a \circ b = \frac{1}{2}(ab + ba)$.
- The algebra $\mathfrak{h}_n(\mathbb{C})$ with the product $a \circ b = \frac{1}{2}(ab + ba)$.
- The algebra $\mathfrak{h}_n(\mathbb{H})$ with the product $a \circ b = \frac{1}{2}(ab + ba)$.
- The algebra $\mathbb{R}^n \oplus \mathbb{R}$ with the product $(v, \alpha) \circ (w, \beta) = (\alpha w + \beta v, \langle v, w \rangle + \alpha\beta)$.
- The algebra $\mathfrak{h}_3(\mathbb{O})$ with the product $a \circ b = \frac{1}{2}(ab + ba)$.

Relation between $\mathfrak{h}_n(\mathbb{C})$ and $\mathbb{C}P^n$

- projections in $\mathfrak{h}_n(\mathbb{C})$ correspond to subspaces of $\mathbb{C}^n$
- the projections onto 1-dimensional subspaces correspond to points in $\mathbb{C}P^n$
- the projections onto 2-dimensional subspaces correspond to lines in $\mathbb{C}P^n$

Constructing Projective Space

We can then try to construct a projective space whose points are the rank-1 projections and whose lines are the rank-2 projections, with the relation of ‘lying on’ given by the partial order $\leq$.

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- from *the spin factor* $\mathbb{R}^n \oplus \mathbb{R}$, for $n \geq 2$, we obtain a series of 1-dimensional projective spaces related to Lorentzian geometry.
- starting with *the exceptional Jordan algebra* $\mathfrak{h}_3(\mathbb{O})$, we get the projective plane discovered by Moufang $\mathbb{O}P^2$.

Projective Lines

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- A square matrix with entries in the $*$-algebra $\mathcal{A}$ is ***hermitian*** if it equals its conjugate transpose.
- Let $\mathfrak{h}_n(\mathcal{A})$ stand for the hermitian $n \times n$ matrices with entries in $\mathcal{A}$.

Projections in $\mathfrak{h}_2(\mathbb{O})$

- Besides the trivial projections 0 and 1, they are all of the form:

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x^*x & x^*y \\ y^*x & y^*y \end{pmatrix}$$

  where $(x, y) \in \mathbb{O}^2$ has $\|x\|^2 + \|y\|^2 = 1$.

- These nontrivial projections all have rank 1, so they are the points of one line
- It is easy to check that the axioms for a projective space hold.

$\mathbb{O}P^1$

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- Given any nonzero element $(x, y) \in \mathbb{O}^2$, we can normalize it and then get a point in $\mathbb{O}P^1$, which we call $[(x, y)]$.
- Define an equivalence relation on nonzero elements of $\mathbb{O}^2$ by $(x, y) \sim (x', y') \iff [(x, y)] = [(x', y')]$.
- We call an equivalence class for this relation a ***line through the origin*** in $\mathbb{O}^2$.

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- $\{(\alpha x, \alpha y) : \alpha \in \mathbb{O}\}$ is **not** line through the origin containing (x, y)!!!
- $(x, y) \sim (y^{-1}x, 1)$ when $y \neq 0$ and $(x, y) \sim (1, x^{-1}y)$ when $x \neq 0$, so the corresponding lines are $\{(\alpha, \alpha(x^{-1}y)) : \alpha \in \mathbb{O}\}$ and $\{(\alpha(y^{-1}x), \alpha) : \alpha \in \mathbb{O}\}$

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- In particular, the line through the origin containing (x, y) is always a real vector space isomorphic to $\mathbb{O}$.

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- It is easy to check that $[(x, 1)] = [(1, y)] \iff y = x^{-1}$, so the transition function from the first chart to the second is the map $x \mapsto x^{-1}$.
- Since this transition function and its inverse are smooth on the intersection of the two charts, $\mathbb{K}P^1$ becomes a smooth manifold.

Complex Case

- $\mathbb{C}P^1$ is just the *Riemann sphere*.

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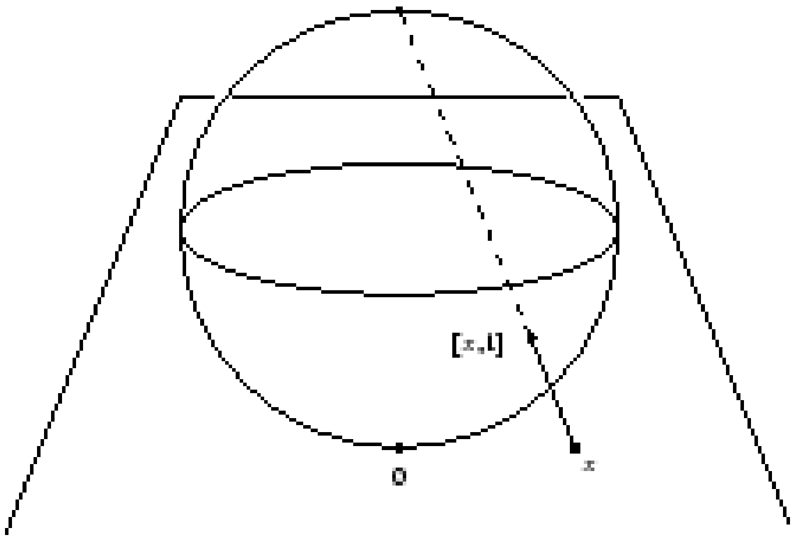
- $\mathbb{C}P^1$ is just the *Riemann sphere*.
- the map $x \mapsto [(x, 1)]$ is given by stereographic projection:



where we choose the sphere to have diameter 1.

Conformal Inversion

- This map from $\mathbb{C}$ to $\mathbb{C}P^1$ is one-to-one and almost onto, missing only the point at infinity, or *north pole*.

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- This map from $\mathbb{C}$ to $\mathbb{C}P^1$ is one-to-one and almost onto, missing only the point at infinity, or *north pole*.
- Similarly, the map $y \mapsto [(1, y)]$ misses only the *south pole*.
- Composing the first map with the inverse of the second, we get the map $x \mapsto x^{-1}$, which goes by the name of ***conformal inversion***.

Riemann Sphere

- The southern hemisphere of the Riemann sphere consists of all points $[(x,1)]$ with $\|x\| \leq 1$, while the northern hemisphere consists of all $[(1,y)]$ with $\|y\| \leq 1$.

- Unit complex numbers x give points $[(x,1)] = [(1,x^{-1})]$ on the equator.

Generalizing the Idea

Smooth manifold $\mathbb{K}P^1$ is just a sphere with dimension equal to that of $\mathbb{K}$:

$$\mathbb{R}P^1 \cong S^1$$
$$\mathbb{C}P^1 \cong S^2$$
$$\mathbb{H}P^1 \cong S^4$$
$$\mathbb{O}P^1 \cong S^8$$

Sphere Description

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- The *southern hemisphere*, *northern hemisphere* and *equator* of $\mathbb{K}$ have descriptions exactly like those given above for the complex case.
- Also, as in the complex case, the maps $x \mapsto [(x, 1)]$ and $y \mapsto [(1, y)]$ are angle-preserving with respect to the usual Euclidean metric on $\mathbb{K}$ and the round metric on the sphere.

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- $\mathbb{K}P^1$ is equipped with a vector bundle whose fiber over the point $[(x, y)]$ is the line through the origin corresponding to this point.
- This bundle is called ***the canonical line bundle,*** $L_{\mathbb{K}}$.
- When we are working with a particular division algebra, 'line' means a copy of this division algebra, so if we think of them as real vector bundles, $L_{\mathbb{R}}, L_{\mathbb{C}}, L_{\mathbb{H}}$ and $L_{\mathbb{O}}$ have dimensions 1,2,4 and 8, respectively.

Building Canonical Line Bundles

- Any k-dimensional real vector bundle over S^n can be formed by taking trivial bundles over the northern and southern hemispheres and gluing them together along the equator via a map $f : S^{n-1} \to \mathbf{O}(n)$

Building Canonical Line Bundles

- Any k-dimensional real vector bundle over S^n can be formed by taking trivial bundles over the northern and southern hemispheres and gluing them together along the equator via a map $f : S^{n-1} \to \mathbf{O}(n)$
- We must therefore be able to build the canonical line bundles $L_{\mathbb{R}}, L_{\mathbb{C}}, L_{\mathbb{H}}$ and $L_{\mathbb{O}}$ using maps:

$$f_{\mathbb{R}} : S^0 \to \mathbf{O}(1)$$
$$f_{\mathbb{C}} : S^1 \to \mathbf{O}(2)$$
$$f_{\mathbb{H}} : S^3 \to \mathbf{O}(4)$$
$$f_{\mathbb{O}} : S^7 \to \mathbf{O}(8)$$

What Are These Maps?

- In the southern hemisphere of $\mathbb{K}P^1$, we can identify any fiber of $L_{\mathbb{K}}$ with $\mathbb{K}$ by mapping the point $(\alpha x, \alpha)$ in the line $[(x, 1)]$ to the element $\alpha \in \mathbb{K}$.

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- If $x \in \mathbb{K}$ has norm one, $[(x, 1)] = [(1, x^{-1})]$ is a point on the equator, so we get two trivializations of the fiber over this point: if $(\alpha x, \alpha) = (\beta, \beta x^{-1})$ then $\beta = \alpha x$.

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- The map $\alpha \mapsto \beta$ is thus right multiplication by x.

The Importance of $f_{\mathbb{K}}$

- $f_{\mathbb{K}} : S^{n-1} \to \mathbf{O}(n)$ is just the map sending any norm-one element $x \in \mathbb{K}$ to the operation of right multiplication by x.

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- Using the obvious inclusions $\mathbf{O}(n) \hookrightarrow \mathbf{O}(n+1)$, we obtain a topological group called $\mathbf{O}(\infty)$
- Since $\mathbf{O}(n)$ is included in $\mathbf{O}(\infty)$, we can think of $f_{\mathbb{K}}$ as a map from S^{n-1} to $\mathbf{O}(\infty)$.

The Importance of $f_{\mathbb{K}}$

Homotopy class $[f_{\mathbb{K}}]$ has the following marvelous property, calculated by *Raoul Bott* in 1957:

$[f_{\mathbb{R}}]$ generates $\pi_0(\mathbf{O}(\infty)) \cong \mathbb{Z}_2$

$[f_{\mathbb{C}}]$ generates $\pi_1(\mathbf{O}(\infty)) \cong \mathbb{Z}_2$

$[f_{\mathbb{H}}]$ generates $\pi_3(\mathbf{O}(\infty)) \cong \mathbb{Z}$

$[f_{\mathbb{O}}]$ generates $\pi_7(\mathbf{O}(\infty)) \cong \mathbb{Z}$

Bott Periodicity

Bott proved that the homotopy groups of the topological group $\mathbf{O}(\infty)$ repeat with period 8:

$$\pi_{i+8}(\mathbf{O}(\infty)) \cong \pi_i(\mathbf{O}(\infty)).$$

He also computed the first 8:

$$\pi_0(\mathbf{O}(\infty)) \cong \mathbb{Z}_2 \qquad \pi_4(\mathbf{O}(\infty)) \cong 0$$

$$\pi_1(\mathbf{O}(\infty)) \cong \mathbb{Z}_2 \qquad \pi_5(\mathbf{O}(\infty)) \cong 0$$

$$\pi_2(\mathbf{O}(\infty)) \cong 0 \qquad \pi_6(\mathbf{O}(\infty)) \cong 0$$

$$\pi_3(\mathbf{O}(\infty)) \cong \mathbb{Z} \qquad \pi_7(\mathbf{O}(\infty)) \cong \mathbb{Z}$$

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Thank you for the attention!

Questions?