Quaternions and their geometry

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Overview

- Motivation
  - $\mathbb{C}$ and 2D geometry
  - History
- Quaternions
  - Definition
  - $\mathbb{H}$ and 3D geometry
- Generalization

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  - when $|z_0| = 1$ (z_0 is a *complex unit*) the multiplication by z_0 is *isometry*

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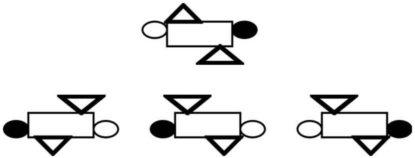

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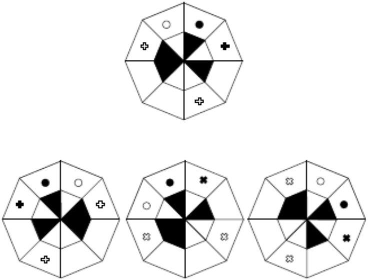

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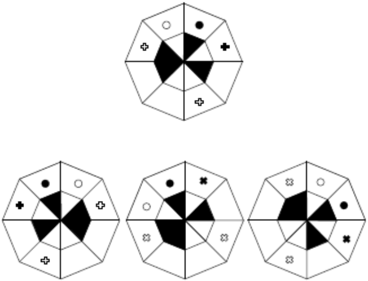

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Theorem

If u is a complex unit then the map $z \to zu$ is a rotation, while $z \to z\overline{u}$ is a reflection.

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Proof.

Let $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be the orthogonal matrix.

First orthogonality condition ($a_{11}^2 + a_{12}^2 = 1$) tells that $a_{11} = \cos\theta$ and $a_{12} = \sin\theta$ for some θ.

Second orthogonality condition ($a_{11}a_{21} + a_{12}a_{22} = 0$) is equivalent to $a_{21} = -a_{22}\tan\theta$ and

from the third condition ($a_{21}^2 + a_{22}^2 = 1$) follows that $a_{22} = \pm\cos\theta$ and $a_{21} = \mp\sin\theta$. □

GO_2, SO_2

rotation	**reflection**
$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$	$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$
determinant $+1$	determinant -1
SO_2	$GO_2 \setminus SO_2$
$z \to zu$	$z \to z\overline{u}$

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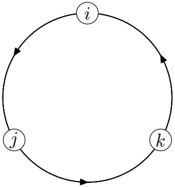
- rest of life with quaternions and their application to geometry

Quaternions - definition

$$\mathbb{H} = \{q | q = a+bi+cj+dk, a, b, c, d \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}$$



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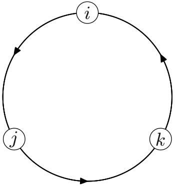

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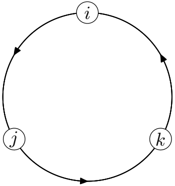

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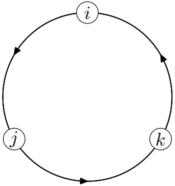

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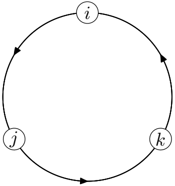

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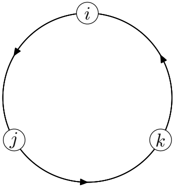

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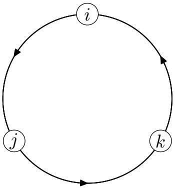

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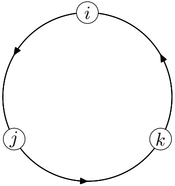
- multiplication by a scalar is commutative



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  - $q^{-1} = \dfrac{\overline{q}}{\|q\|^2}$

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- equivalently $v = 1$ is fixed iff $q_1 q_2 = 1$, i. e. $q_1 = q_2^{-1}$

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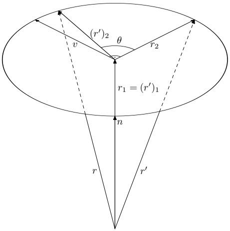
Rotation in 3D

We would like to find a formula for rotation of vector $\mathbf{r} \in \mathbb{R}^3$ about vector $\mathbf{n}$ throughout the angle θ.



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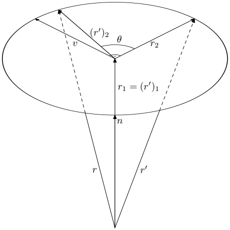

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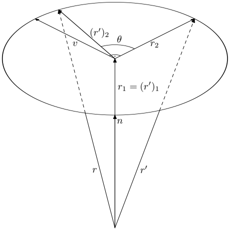
  - $\mathbf{r_2}$ perpendicular to $\mathbf{n}$, $\mathbf{r_2} = \mathbf{r} - \mathbf{r_1}$



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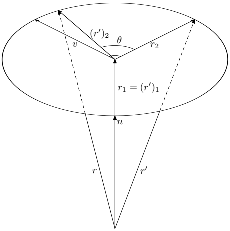

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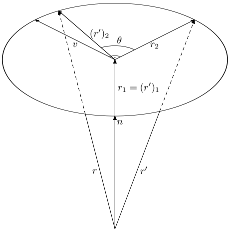
- $\mathbf{v}$ perpendicular to $\mathbf{n}, \mathbf{r_2}$, $\mathbf{v} = \mathbf{n} \times \mathbf{r_2} = \mathbf{n} \times \mathbf{r}$
- $(\mathbf{r}')_2 = \mathbf{r_2} \cos\theta + \mathbf{v} \sin\theta$
- $(\mathbf{r}')_1 = \mathbf{r_1}$



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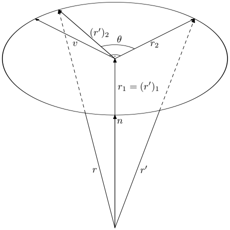
We would like to find a formula for rotation of vector $\mathbf{r} \in \mathbb{R}^3$ about vector $\mathbf{n}$ throughout the angle θ.

- $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$
  - $\mathbf{r}_1$ projection to $\mathbf{n}$, $\mathbf{r_1} = (\mathbf{r} \cdot \mathbf{n})\mathbf{n}$
  - $\mathbf{r_2}$ perpendicular to $\mathbf{n}$, $\mathbf{r_2} = \mathbf{r} - \mathbf{r_1}$
- $\mathbf{v}$ perpendicular to $\mathbf{n}, \mathbf{r_2}$, $\mathbf{v} = \mathbf{n} \times \mathbf{r_2} = \mathbf{n} \times \mathbf{r}$
- $(\mathbf{r}')_\mathbf{2} = \mathbf{r_2} \cos\theta + \mathbf{v} \sin\theta$
- $(\mathbf{r}')_\mathbf{1} = \mathbf{r_1}$
- $\mathbf{r}' = (1 - \cos\theta)(\mathbf{r} \cdot \mathbf{n})\mathbf{n} + \mathbf{r}\cos\theta + (\mathbf{n} \times \mathbf{r})\sin\theta$



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- unit quaternions is a double cover of SO_3

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Survey

Were quaternions

- discovered?
- invented?

Thank you for your attention!