Hovadina

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Theme of the talk:

- Moore graphs:

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- Existence

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- Unicity

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Definition

Let $d(u, v)$ be the length of the shortest path from node u to node v in graph G. The length of the longest shortest path between any two nodes in a graph is called its *diameter*.
Formally, $diameter(G) = maxd(u, v), u, v \in G$

Let one node of the graph be distinguished, let call it 0. Let $n_i, i = 0, \dots k$ be the number of nodes nodes at distance i from 0. Then:

$$\begin{aligned} &n_0 = 1 \\ &n_i \leq d(d-1)^{i-1} \text{ for } i \geq 1 \end{aligned} \tag{1}$$

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$$n := \sum_{i=0}^{k} n_i \leq 1 + d \sum_{i=1}^{k} (d-1)^{i-1} \tag{2}$$

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$$n := \sum_{i=0}^{k} n_i \leq 1 + d \sum_{i=1}^{k} (d-1)^{i-1} \quad (2)$$

Definition

A graph is called a Moore graph if equality holds in (2), we denote it (d, k).

For $k = 2$ *do* exist:

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- $(2, 2)$
- $(3, 2)$
- $(7, 2)$
- $(57, 2)$ (maybe)

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- **0** is zero matrix
- **J** is the unity matrix
- **K** is the matrix of size $d(d-1)$ with **J**'s of order $d \times d$ on diagonal

$$K = \begin{bmatrix} J & 0 & \dots & 0 \\ 0 & J & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & J \end{bmatrix}$$

- u is the unity vector

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- S_i the subset of nodes of tier k which are joined to i'th node of tier $k - 1$

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Adjacency matrix of a graph is a matrix $\mathbf{A}$, where:

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Observation:
In A^p on each place a_{ij} is the number of paths of length p from i to j.

Holds: $1 + d^2 = n$ and:

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Therefore u is an eigenvector of **A**:

$Ju = nu$ and $Au = du$
$(1 + d^2)u = nu$

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using (3):

$$r^2 + r - (d - 1) = 0$$
$$r_1 = (-1 + \sqrt{4d - 3})/2$$
$$r_2 = (-1 - \sqrt{4d - 3})/2$$

If $r_1, r_2 \notin \mathbb{Q}$, then:

$$d + \frac{n-1}{2}(r_1 + r_2) = d - \frac{d^2}{2} = 0$$

$d_1 = 0$ and $d_2 = 2$

If $r_1, r_2 \in \mathbb{Q}$, if $4d - 3 = s^2$ for some $s \in \mathbb{N}$. Then:

$$d + m\frac{s-1}{2} + (n-1-m)\frac{-s-1}{2} = 0$$

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The solutions are:
1,3,5 and 15.

After calculating corresponding degrees and numbers of vertices we obtain:

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$$\begin{array}{llll} s = 1, & m = 0, & d = 1, & n = 2 \\ s = 3, & m = 5, & d = 3, & n = 10 \\ s = 5, & m = 28, & d = 7, & n = 50 \\ s = 15, & m = 1729, & d = 57, & n = 3250 \end{array}$$

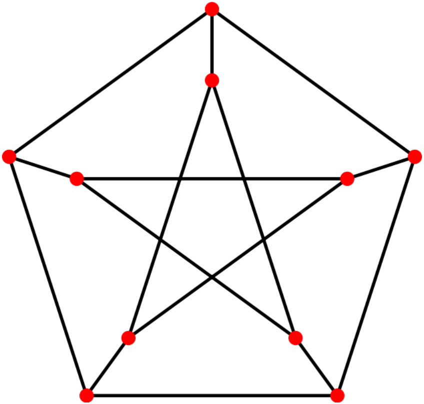

Figure: The case $d = 3$ is the Petersen graph

Let nodes be numbered as follows:

- 0 any (fixed) node
- $1 \dots d$ nodes adjacent to 0 in arbitrary order
- $d + 1 \dots 2d - 1$ nodes from S_1 in arbitrary order

  ...
- $(i(d-1)+2) \dots (i(d-1)+d$ nodes from S_i in arbitrary order

Then the adjacency matrix of the graph looks like this:

0	11...1	00...0	00...0	...	00...0	1
1 1 ⋮ 1	00...0 00...0 ⋮ 00...0	11...1 00...0 ⋮ 00...0	00...0 11...1 ⋮ 00...0	... ... ⋮ ...	00...0 00...0 ⋮ 11...1	d
0 0 ⋮ 0	10...0 10...0 ⋮ 10...0	0	P_{12}	...	P_{1d}	d-1
0 0 ⋮ 0	01...0 01...0 ⋮ 01...0	P_{21}	0	...	P_{2d}	d-1
⋮	⋮	⋮	⋮	⋮	⋮	
0 0 ⋮ 0	00...1 00...1 ⋮ 00...1	P_{d1}	P_{d2}	...	0	d-1

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0 0 ⋮ 0	01...0 01...0 ⋮ 01...0	P_{21}	0	...	P_{2d}	d-1
⋮	⋮	⋮	⋮	⋮	⋮	
0 0 ⋮ 0	00...1 00...1 ⋮ 00...1	P_{d1}	P_{d2}	...	0	d-1

We denote adjacency submatrix of nodes from tier 2 **B**.

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Theorem

The nodes may be so numbered, that $\mathbf{P_{1j}} = \mathbf{P_{j1}} = \mathbf{I}$.
(canonical form of $\mathbf{A}$*)*

By using the canonical form of **A**) in equation (3) we gain:

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For $j \neq 1$,

$\sum_i P_{ij} = J$ and

$$P_{ik} + \sum j P_{ij} P_{jk} = J \text{ if } i \neq k \tag{6}$$

Repeating the the steps we have made for **A**, we obtain:

	eigenvalue	multiplicity
$d = 3$	2	1
	-1	2
	1	2
	-2	1
$d = 7$	6	1
	-1	6
	2	21
	-3	14
$d = 57$	56	1
	-1	56
	7	1672
	-8	1463

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$P \neq I$ therefore must hold

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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  if it exists, it *is not* vertex transitive

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THANK YOU! =)