

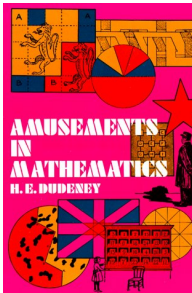
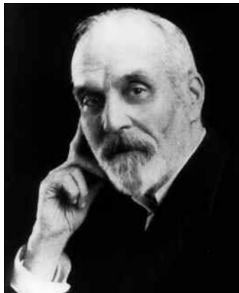
# No-three-in-line-problem on a torus

Michael Skotnica

April 2019

# History - Amusements in Mathematics

Henry E. Dudeney



## Puzzle 317

Place two pawns in the middle of the chessboard, one at Q4 and the other at K5. Now, place the remaining fourteen pawns (sixteen in all) so that no three shall be in a straight line in any possible direction.

# History - No-three-in-line-problem

## No-three-in-line-problem

How many points can be placed on an  $n \times n$  grid so that no three points are collinear.

- Still unsolved for general  $n$ .

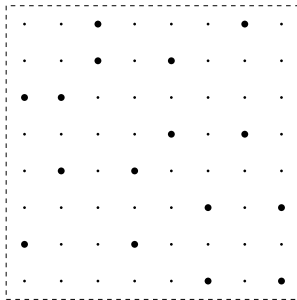


Figure: Dudeney's solution for the chessboard ( $8 \times 8$  grid).

# Discrete torus $T_{m \times n}$

Cartesian product  $\{0, \dots, m-1\} \times \{0, \dots, n-1\} \subset \mathbb{Z}^2$ .

Line on  $T_{m \times n}$  is an image of a line in  $\mathbb{Z}^2$  under a mapping which maps a point  $(x, y) \in \mathbb{Z}^2$  to the point  $(x \bmod m, y \bmod n)$ .

Line in  $\mathbb{Z}^2$   $\{(b_1, b_2) + k(v_1, v_2); k \in \mathbb{Z}\}$ , where  $\gcd(v_1, v_2) = 1$ .

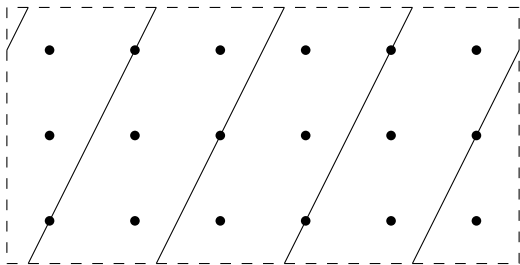
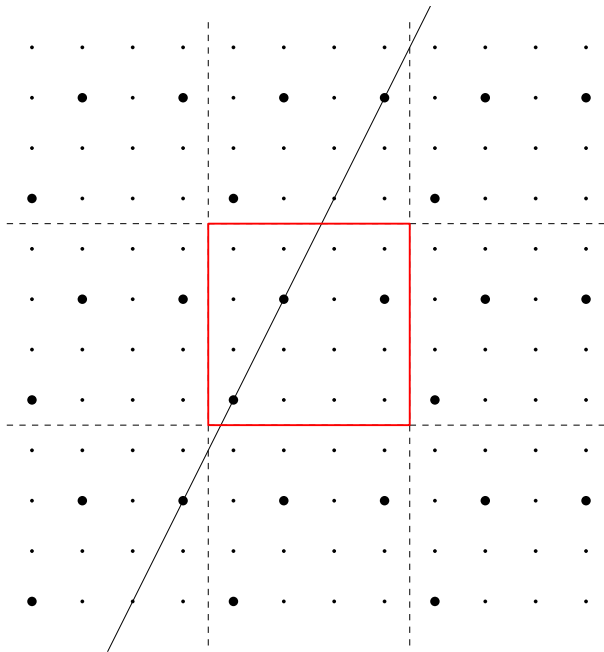
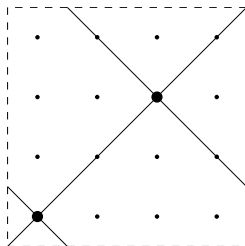


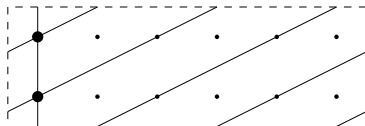
Figure:  $T_{3 \times 6}$



- More lines between two points.



- A line is a proper subset of another line.



# No-three-in-line-problem on a torus

## No-three-in-line-problem on a torus [Fowler et al. 2012]

How many points can be placed on a discrete torus  $T_{m \times n}$  of size  $m \times n$  so that no three points are collinear.

- Let  $\tau_{m,n}$  denote such maximum number of points.

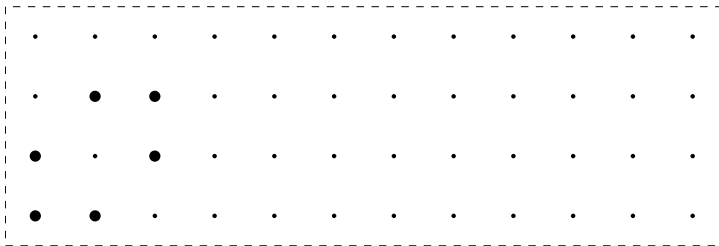


Figure:  $\tau_{4,12} = 6$ .

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## Question

What is  $\tau_{m,n}$  for relative primes  $m, n$ ?

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## Question

What is  $\tau_{m,n}$  for relative primes  $m, n$ ?

- $\tau_{m,n} = 2$  by the Chinese remainder theorem.

# General upper bound

## Theorem

$$\tau_{m,n} \leq 2 \gcd(m, n).$$

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- Then by Lagrange's theorem  $[T_{m \times n} : \ell] = \frac{|T_{m \times n}|}{|\ell|}$ .

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- Then by Lagrange's theorem  $[T_{m \times n} : \ell] = \frac{|T_{m \times n}|}{|\ell|}$ .
- Therefore we have  $[T_{m \times n} : \ell] = \frac{mn}{\text{lcm}(m, n)} = \gcd(m, n)$  disjoint lines covering the whole torus. Each of these lines may contain at most two points.



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- The image of each line from  $T_{xm \times yn}$  is a line on  $T_{m \times n}$ .
- Therefore each set of points from  $T_{m \times n}$  with no three of them collinear can be used on the bigger torus  $T_{xm \times yn}$  so that no three points are collinear.

## Theorem

*If  $\gcd(x, y) = \gcd(m, y) = \gcd(n, x) = 1$  then  $\tau_{m,n} = \tau_{xm,yn}$ .*

# Basic relations

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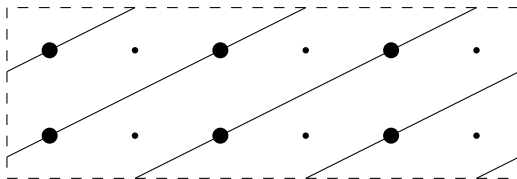
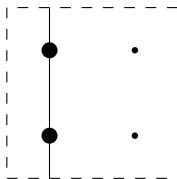
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- Therefore the image of each set of points from  $T_{xm \times yn}$  with no three of them collinear can be used on the smaller torus  $T_{m \times n}$  so that no three points are collinear.



# Other known results

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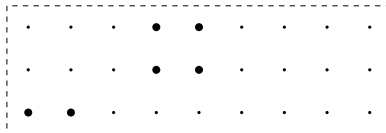


Figure: Example of the construction for  $T_{3 \times 9}$ .

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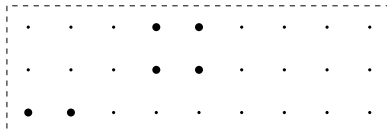


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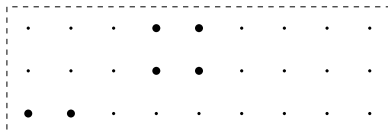


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Upper bound

- $\tau_{p^a, p^a} \leq p^a + p^{\lceil \frac{a}{2} \rceil} + 1$

# Sequences

If we fix one coordinate of a torus, we get the sequence  $\tau_{z,1}, \tau_{z,2}, \tau_{z,3}, \dots$  for  $z \geq 2$ , which we denote  $\sigma_z$ .

$z$	1	2	3	4	5	6	7	8	9	10	11	12	13	...
2	2	4	2	4	2	4	2	4	2	4	2	4	2	...
3	2	2	4	2	2	4	2	2	6	2	2	4	2	...
4	2	4	2	6	2	4	2	8	2	4	2	6	2	...
5	2	2	2	2	6	2	2	2	2	6	2	2	2	...
6	2	4	4	4	2	8	2	4	6	4	2	8	2	...

Table: Initial values of  $\tau_{z,n}$ .

- The potential maximum of the sequence is  $2z$ .  
Since  $\tau_{m,n} \leq 2 \gcd(m, n)$ .

## Question

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- Yes! For each  $z$ .
- However, the proof is purely existence.



## Theorem

*Let  $p$  be a prime and let us denote  $m := \min\{x; \sigma_{p^a}(x) = 2p^a\}$ .  
The sequence  $\sigma_{p^a}$  is periodic with the period  $m$ .*

## Strategy.

We know:

- $\sigma_z(n) \leq \sigma_z(xn)$ .

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- We want to show  $\tau_{4,12} = \tau_{4,28}$ .

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## Example for $\sigma_4$ .

- We want to show  $\tau_{4,12} = \tau_{4,28}$ .
- It is true since  $\tau_{4,12} = \tau_{4,4}$  and also  $\tau_{4,28} = \tau_{4,4}$ .

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## Proof.

We consider two cases:

- $x < p^b$ .

In this case  $x + \alpha p^b = p^l(r + \alpha p^{b-l})$ . Since  $b - l \geq 1$ , we get  $\gcd(r + \alpha p^{b-l}, p) = 1$ . Therefore  $\sigma_{p^a}(p^l(r + \alpha p^{b-l})) = \sigma_{p^a}(p^l) = \sigma_{p^a}(x)$ .

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In this case  $x + \alpha p^b = p^b(1 + \alpha)$ . Since  $\sigma_{p^a}(p^b) = 2p^a = \max \sigma_{p^a}$ , we know  $\sigma_{p^a}(p^b) = \sigma_{p^a}(hp^b)$  for any  $h > 0$  and hence also for  $h = (1 + \alpha)$ .

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Therefore  $\sigma_{p^a}(x) = \sigma_{p^a}(x + \alpha p^b)$  for any  $x \in \{1, \dots, p^b\}$ . □

## “Conjecture”

The least period for  $\sigma_6$  is 108.

- Is it true that  $\sigma_6(2 \cdot 3^j) = 6$  for all  $j \geq 2$ ?

$\gcd(n, 2^2 \cdot 3^3)$	$2^0$	$2^1$	$2^2$
$3^0$	2	4	4
$3^1$	4	8	8
$3^2$	6	10	12
$3^3$	6	12	12

**Table:** Values of  $\sigma_6(n)$  according to  $\gcd(n, 2^2 \cdot 3^3) = 2^i \cdot 3^j$ .



## Summary:

- $\tau_{m,n} = 2$  if  $\gcd(m, n) = 1$ .
- $\tau_{m,n} \leq 2 \gcd(m, n)$ .
- $\tau_{m,n} \leq \tau_{xm,yn}$ .
- $\tau_{m,n} = \tau_{xm,yn}$  if  $\gcd(x, y) = \gcd(m, y) = \gcd(n, x) = 1$ .
- $\tau_{p,p} = p + 1$ .
- $\tau_{p^a, p^{(a-1)p+2}} = 2p^a$ .
- $\tau_{2^a, 2^{2a-1}} = 2^{a+1}$ .
- $\tau_{p^a, p^a} \leq p^a + p^{\lceil \frac{a}{2} \rceil} + 1$ .
- The sequence  $\tau_{z,1}, \tau_{z,2}, \tau_{z,3}, \dots$  is periodic for all  $z$ .

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