Cobham's Theorem

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AND PHYSICS
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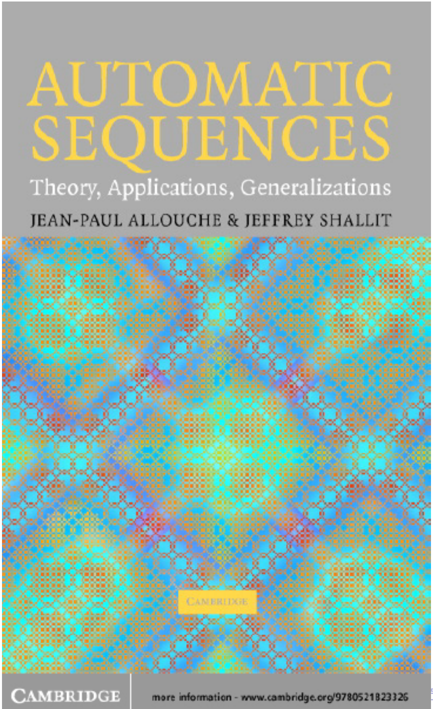
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Alphabets and Words

Definition

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$\Sigma_k := \{0, 1, \ldots, k-1\}$

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- an *infinite word* is a map from $\mathbb{N}_0$ to Σ

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Example

$\Sigma_2^* = \{\epsilon, 0, 1, 01, 10, 001, 010, \ldots\}$

Morphisms

Definition

Let Σ be an alphabet. A map $\varphi : \Sigma^* \longrightarrow \Sigma^*$ is called a *morphism* if φ satisfies $\varphi(xy) = \varphi(x)\varphi(y)\ \forall x, y \in \Sigma^*$.

If there exists a constant k such that $|\varphi(a)| = k\ \forall a \in \Sigma$, we say φ is *k-uniform*. A 1-uniform morphism is called a *coding*.

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Example - It is time to introduce the celebrity

The Thue-Morse morphism is a morphism $\mu : \Sigma_2^* \longrightarrow \Sigma_2^*$ *where* $\mu(0) = 01$ *and* $\mu(1) = 10$*.*

Let $\varphi : \Sigma^* \longrightarrow \Sigma^*$ be a morphism. A finite or infinite word satisfying $\varphi(w) = w$ is said to be a *fixed point* of φ.

Morphisms

Definition

If there exists $a \in \Sigma$ such that $\varphi(a) = ax$ for some $x \in \Sigma^*$ such that $\varphi^i(x) \neq \epsilon$ $\forall i \in \mathbb{N}_0$ we say φ is *prolongable on a*. In this case, the sequence of words $a, \varphi(a), \varphi^2(a), \ldots$ converges, in the limit, to the infinite word

$$\overrightarrow{\varphi^{\omega}}(a) := ax\varphi(x)\varphi^2(x)\varphi^3(x)\ldots$$

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Example

Since $\mu(0) = 01$ and $\mu(1) = 10$ we have that μ is prolongable on both 0 and 1, hence:

$$\begin{aligned} \mu(0) &= 01 \\ \mu^2(0) &= 0110 \\ \mu^3(0) &= 01101001 \\ \mu^4(0) &= 0110100110010110 \\ &\ldots \\ \mathbf{t} = \overrightarrow{\mu^\omega}(0) &= 0110100110010110100101 10\ldots \end{aligned}$$

Morphisms

Observation

Let $k \geq 2$, Σ an alphabet, $\varphi : \Sigma^* \longrightarrow \Sigma^*$ a k-uniform morphism and $\mathbf{w} = w_0 w_1 w_2 \dots$ an infinite word over the alphabet Σ.
Then $\mathbf{w} = \varphi(\mathbf{w}) \Leftrightarrow \varphi$ is prolongable on w_0 and $\mathbf{w} = \varphi^\omega(w_0)$.

Proof.

$\Leftarrow$ $\mathbf{w} = w_0 x \varphi(x) \varphi^2(x) \dots$
$\Rightarrow$ we have:

$$\begin{aligned} \mathbf{w} &= w_0 w_1 w_2 \dots \\ &= \varphi(w_0)\varphi(w_1)\varphi(w_2)\dots \end{aligned}$$

$\forall i \in \mathbb{N}_0$: $\varphi^i(w_0)$ is a prefix of $\mathbf{w}$
Hence $\mathbf{w} = \varphi^\omega(w_0)$. $\square$

Numeration System Notation

Definition

Let $n \in \{0, 1, 2, \ldots\}$, $k \geq 2$ an integer.

- By $(n)_k$ we understand the unique base-k expansion of n.
- More formally: $(n)_k = a_t a_{t-1} \ldots a_1 a_0$ such that $n = \sum_{i=0}^{t} a_i k^i$ with $a_t \neq 0$.

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Let $k \geq 2$ be an integer, $w \in \Sigma_k = \{0, 1, \ldots, k-1\}$;
$w = a_t a_{t-1} \ldots a_1 a_0$.

- Then we define $[w]_k := \sum_{i=0}^{t} a_i k^i$.

Example

$(42)_2 = 101010$
$[1110]_2 = 13$

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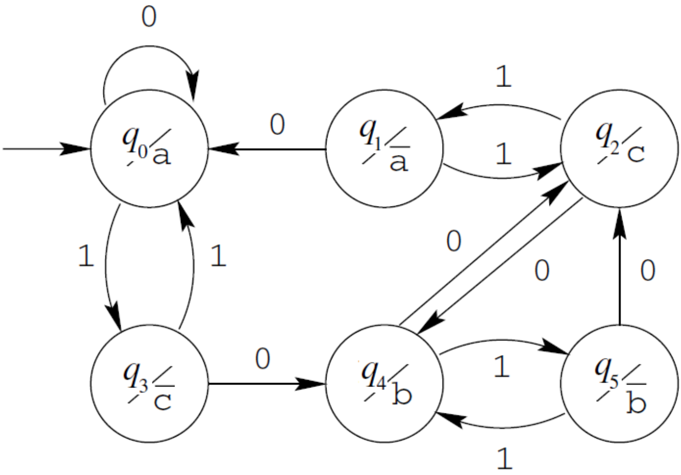
Deterministic Finite Automata with Output



More formally

Definition

A *deterministic finite automaton with output*, or DFAO is a 6-tuple $M = (\mathcal{Q}, \Sigma, \delta, q_0, \Delta, \tau)$ where

$\mathcal{Q}$ is a finite set of states
Σ is the finite input alphabet
$\delta : \mathcal{Q} \times \Sigma \to \mathcal{Q}$ is the transition function
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Moreover, when the input alphabet $\Sigma = \Sigma_k$ for an integer $k \geq 2$, we call a DFAO a k-DFAO.

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Notation

$q_i a := \delta(q_i, a)$, $a \in \Sigma$

k-automatic sequeces

Definition

We say the sequence $(a_n)_{n\geq 0}$ over a finite alphabet Δ is *k-automatic* if there exists a k-DFAO $M = (\mathcal{Q}, \Sigma_k, \delta, q_0, \Delta, \tau)$ such that $a_n = \tau(q_0 w))$ for all $n \geq 0$ and all w with $[w]_k = n$.

Example - yet another definition of the Thue-Morse word

$\mathbf{t} = (t_n)_{n\geq 0}$ *is defined as:*
$t_n = 0$ *if the number of 1's in* $(n)_2$ *is even*
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We can easily show $\mathbf{t}$ is 2-automatic.

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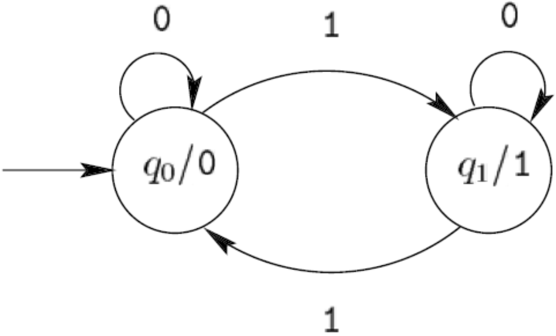

Parents and Children

Suppose we have a fixed point **a** of a k-uniform morphism $\varphi : \Sigma^* \to \Sigma^*$.

$$\begin{aligned}\mathbf{a} &= a_0 a_1 a_2 a_3 \ldots \\ &= \varphi(a_0)\varphi(a_1)\varphi(a_2)\varphi(a_3)\ldots\end{aligned}$$

Observation

$\varphi(a_i) = a_{ki} a_{ki+1} a_{ki+1} \ldots a_{ki+k-1} \; \forall i \in \mathbb{N}_0$

Definition

$p \in \mathbb{N}_0$ is a *parent* of q if the element of **a** at position q arises as an image of the element at position p under the morphism φ. We say q is a *child* of p and we put $p := par(q)$.

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Properties

- *children of p:* $kp, kp+1, \dots kp+k+1$
- $p = q \; div \; k$
- $q = par(q) + (q \; mod \; k)$

Cobham's Theorem

Theorem (Cobham's Theorem)

Let $k \geq 2$. *Then a sequence* $\mathbf{u} = (u_n)_{n \geq 0}$ *is k-automatic if and only if it is the image, under a coding, of a fixed point of a k-uniform morphism.*

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Proof

$\Leftarrow$ *We have* $\mathbf{u} = \tau(\mathbf{a})$ *for some coding* $\tau : \Sigma \to \Delta$ *and* $\mathbf{a} = \varphi(\mathbf{a})$ *for a k-uniform morphism* $\varphi : \Sigma^* \to \Sigma^*$, $\mathbf{a} = a_0 a_1 a_2 \ldots$.

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We will construct a k-DFAO $M = (\mathcal{Q}, \Sigma_k, \delta, q_0, \Delta, \tau)$ *such that:*
$q_0(n)_k = a_n \; \forall n \in \mathbb{N}_0$.

Proof

$\Leftarrow$

$\varphi(a_i) = a_{ki} a_{ki+1} a_{ki+1} \dots a_{ki+k-1}$
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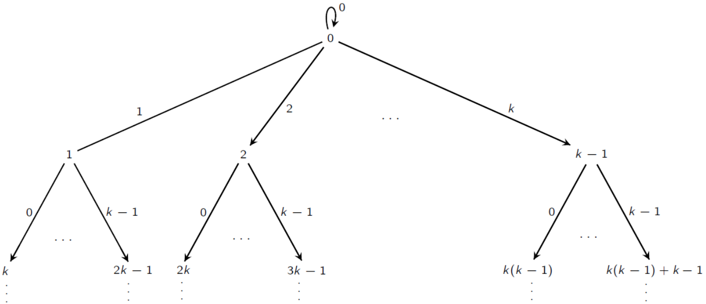

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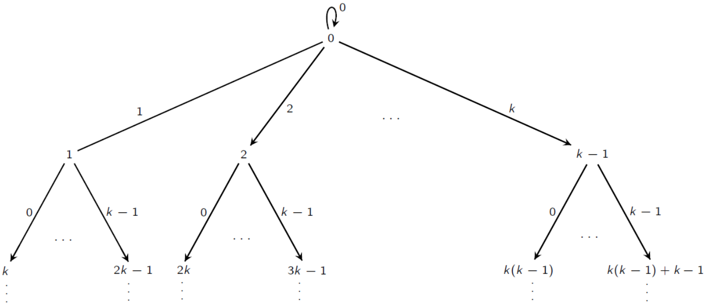

$$(n)_k = n_t n_{t-1} \ldots n_1 n_0$$

Proof

Formal construction of k-DFAO $M = (\mathcal{Q}, \Sigma_k, \delta, q_0, \Delta, \tau)$

$\mathcal{Q} := \Sigma$, $q_0 := a_0$ and $\delta : \mathcal{Q} \times \Sigma_k \to \mathcal{Q}$ is defined as: $\delta(q, b) :=$ the b-th letter of $\varphi(q)$. By induction we will show that $\delta(q_0, (n)_k) = w_n$.

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- $n = 0 : \checkmark$
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$$
\begin{aligned}
q_0(n)_k &= q_0(n_t n_{t-1} \dots n_0) \\
&= (q_0, n_t n_{t-1} \dots n_1) n_0 \\
&= (q_0(n')_k) n_0 \\
&= w_{n'} n_0 \\
&= \textit{the } {n_0}' \textit{th symbol of } \varphi(w_{n'}) \\
&= w_{kn'+n_0} \\
&= w_n
\end{aligned}
$$

Proof

$\Rightarrow$

$M = (\mathcal{Q}, \Sigma_k, \delta, q_0, \mathcal{Q}, id)$ produces an automatic sequence $\mathbf{a} = (a_n)_{n \geq 0}$

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\begin{aligned}
a_0 &= q_0 0 \\
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&\vdots \\
a_i &= q_0 (i)_k \\
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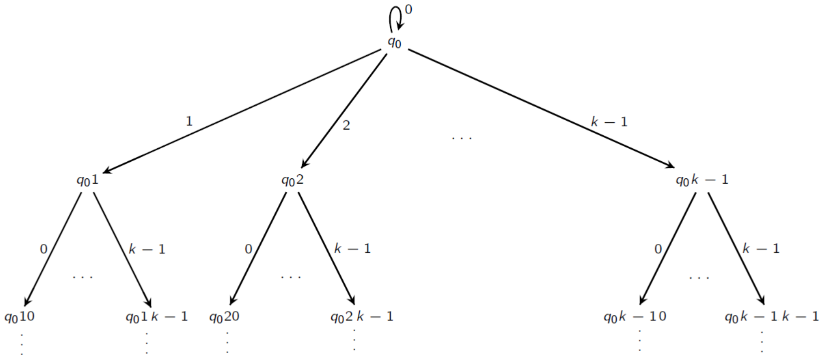

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It suffices to show that: $\varphi(a_i) = a_{ki} a_{ki+1} \ldots a_{ki+k-1}\ \forall i$
Since then: $\varphi(a_0 a_1 \ldots a_i) = a_0 a_1 \ldots a_{ki} a_{ki+1} \ldots a_{ki+k-1}\ \forall i$

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$$\begin{aligned} \varphi(a_i) &= \varphi(q_0(i)_k) \\ &= q_0(i)_k 0\, q_0(i)_k 1\ \ldots\ q_0(i)_k k - 1 \\ &= q_0(ki)_k\, q_0(ki+1)_k\ \ldots q_0(ki+k-1)_k \\ &= a_{ki} a_{ki+1} \ldots a_{ki+k-1} \end{aligned}$$

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$\Rightarrow$

$M = (\mathcal{Q}, \Sigma_k, \delta, q_0, \mathcal{Q}, id)$ produces an automatic sequence $\mathbf{a} = (a_n)_{n\geq 0}$

$$\begin{aligned} a_0 &= q_0 0 \\ a_0 &= q_0 1 \\ &\vdots \\ a_i &= q_0 (i)_k \\ &\vdots \end{aligned}$$

$\varphi(q) \coloneqq q0\, q1 \ldots qk-1\ \forall q \in \mathcal{Q}$
It suffices to show that: $\varphi(a_i) = a_{ki} a_{ki+1} \ldots a_{ki+k-1}\ \forall i$
Since then: $\varphi(a_0 a_1 \ldots a_i) = a_0 a_1 \ldots a_{ki} a_{ki+1} \ldots a_{ki+k-1}\ \forall i$

$$\begin{aligned} \varphi(a_i) &= \varphi(q_0(i)_k) \\ &= q_0(i)_k 0\, q_0(i)_k 1\ \ldots\ q_0(i)_k k-1 \\ &= q_0(ki)_k\, q_0(ki+1)_k\ \ldots q_0(ki+k-1)_k \\ &= a_{ki} a_{ki+1} \ldots a_{ki+k-1} \end{aligned}$$

$\tau(\varphi(\mathbf{a})) = \tau(\mathbf{a}) = \tau(q_0 0)\tau(q_0 1)\tau(q_0 2)\ldots$

Proof

$\Rightarrow$

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It suffices to show that: $\varphi(a_i) = a_{ki} a_{ki+1} \ldots a_{ki+k-1}\ \forall i$
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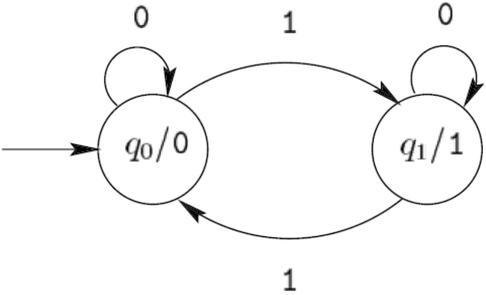
Corollary

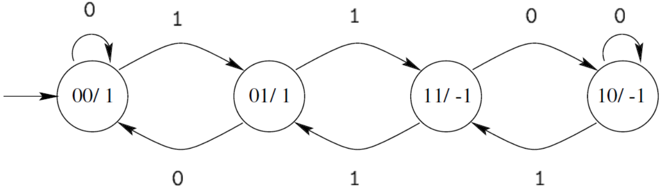
The two definitions of the Thue-Morse word are equivalent.
$\mathbf{t} = (t_n)_{n\geq 0}$ is defined as:
$t_n = 0$ if the number of 1's in $(n)_2$ is even
$t_n = 1$ if the number of 1's in $(n)_2$ is odd



$\mathbf{t} = \overrightarrow{\mu^{\omega}}(0)$ where $\mu(0) = 01$, $\mu(1) = 10$

Example

Rudin-Shapiro sequence $\mathbf{r} = (r_n)_{n \geq 0}$ is defined by the following:
- $r_n = 1$ if the number of (possibly overlapping) occurences of the block 11 in $(n)_2$ is even
- $r_n = -1$ otherwise



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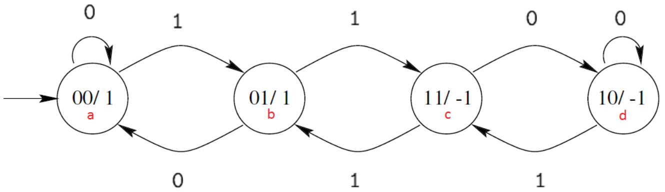

$\varphi(a) = a0\, a1 = ab$
$\varphi(b) = b0\, b1 = ac$
$\varphi(c) = c0\, c1 = db$
$\varphi(d) = d0\, d1 = dc$

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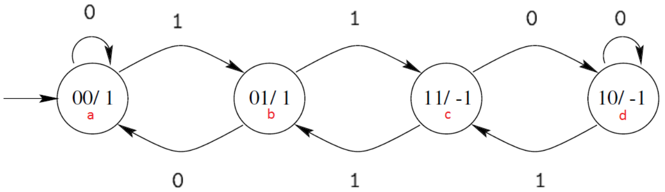

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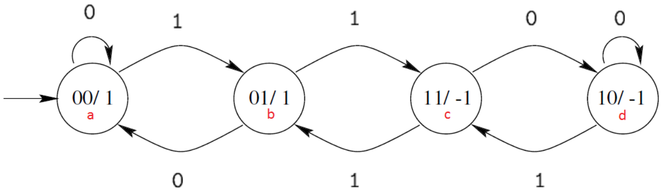

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$\varphi(d) = d0\, d1 = dc$
$\varphi^{\omega}(a)) = abacabdbabacdcac \cdots$
$\tau(\varphi^{\omega}(a))) =$1 1 1 -1 1 1 -1 1 1 1 1 -1 -1 -1 1 1 -1$\cdots$

Any Questions?