CANTOR'S DIAGONAL METHOD - PART II

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Outline

1. Preliminaries
2. Diagonal lemma and its consequences
3. Gödel's first theorem
4. Undecidability of arithmetic
5. Non-standard model of arithmetic

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Preliminaries

- *Numerical language*: $\langle S, 0 \rangle$ where S is a unary function symbol (successor) and 0 is a constant symbol.
- *Numeral* $\underline{n}$ is defined as $S^n(0)$.
- *Language of arithmetic*: $\langle S, +, \cdot, 0, \leq \rangle$.

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Preliminaries

Robinson arithmetic (denoted Q) is a theory in the language of arithmetic (we also use the symbol " $=$") with the following axioms:

1. $0 \neq S(x)$,
2. $x \neq 0 \to (\exists y)(x = S(y))$,
3. $S(x) = S(y) \to x = y$,
4. $x + 0 = x$,
5. $x + S(y) = S(x + y)$,
6. $x \cdot 0 = 0$,
7. $x \cdot S(y) = x \cdot y + x$,
8. $x \leq y \leftrightarrow (\exists z)(z + x = y)$.

Standard model of Robinson arithmetic is $\mathcal{N} = \langle \mathbb{N}, S, +, \cdot, 0, \leq \rangle$.

Preliminaries

- We can assign a natural number to each formula (it is called Gödel's number) so it makes sense to write $\varphi(\underline{\varphi})$.
- We call a (total) function $f : \mathbb{N}^n \to \mathbb{N}$ *computable* if there exists a Σ_1-formula $\delta(\overline{x}, y)$ that defines the function f.
- A set is *computable* (or *recursive*) if its characteristic function is computable.
- A function $F : \mathbb{N}^n \to \mathbb{N}$ is represented in a numerical theory T by a formula φ if

$$T \vdash \varphi(\underline{a_1}, \ldots, \underline{a_n}, y) \leftrightarrow y = \underline{F(a_1, \ldots, a_n)}$$

for all $a_1, \ldots, a_n \in \mathbb{N}$.

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Diagonal lemma

Lemma

Let T be an extension of the theory Q and let $\varphi(v_0)$ be a formula of T. Then there exists a sentence φ^* such that $T \vdash \varphi^* \leftrightarrow \varphi(\underline{\varphi^*})$.

Proof.

Let $D(x) = \mathrm{Sub}(x, \mathrm{Vr}(0), \mathrm{Num}(x))$ be a function that for each formula $\alpha(x)$ returns $\alpha(\underline{\alpha})$. D is computable. Let $\delta(v_0, v_1)$ be a formula representing D in Q. Then

$$Q \vdash (\forall v_1)(\delta(\underline{\beta}, v_1) \leftrightarrow v_1 = \underline{\beta(\underline{\beta})})$$

for each formula $\beta(v_0)$. Define

$$\psi(v_0) \leftrightarrow (\exists v_1)(\delta(v_0, v_1) \mathbin{\&} \varphi(v_1)).$$

Then $T \vdash \psi(\underline{\beta}) \leftrightarrow \varphi(\underline{\beta(\underline{\beta})})$ and we can choose φ^* as $\psi(\underline{\psi})$. $\square$

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Theory T proves:

$$\begin{aligned} \varphi^* \leftrightarrow \psi(\underline{\psi}) \leftrightarrow (\exists v_1)(\delta(\underline{\psi}, v_1) \ \&\ \varphi(v_1)) \leftrightarrow \\ \leftrightarrow (\exists v_1)(v_1 = \underline{D(\psi)} \ \&\ \varphi(v_1)) \leftrightarrow \varphi(\underline{D(\psi)}) \leftrightarrow \\ \leftrightarrow \varphi(\underline{\psi(\underline{\psi})}) \leftrightarrow \varphi(\underline{\varphi^*}). \end{aligned}$$

Definition

Formula $\tau(x)$ of a numerical theory T is a *definition of truth in* T if for each sentence φ of T the following statement holds: $T \vdash \varphi \leftrightarrow \tau(\underline{\varphi})$.

Definition

Theory T is consistent if there is no formula φ such that $T \vdash \varphi$ and $T \vdash \neg\varphi$.

Definition

A set X of natural numbers is *arithmetical* if there is a formula $\varphi(n)$ in the language of arithmetic such that each number n is in X iff $\varphi(\underline{n})$ holds in the standard model of arithmetic.

Definition

Let L be a language and $\mathcal{M}$ an L-structure. Then $\mathrm{Th}(\mathcal{M}) = \{\varphi : \varphi \text{ is a sentence in } L \text{ and } \mathcal{M} \models \varphi\}$.

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Theorem

1. *There is no definition of truth in a consistent extension of the theory Q.*
2. $\mathrm{Th}(\mathcal{N})$ *is not an arithmetical set.*

Proof.

1) For a formula $\tau(x)$ in the language of T, there exists a sentence φ such that $T \vdash \varphi \leftrightarrow \neg\tau(\underline{\varphi})$. Thus, τ cannot be a definition of truth in T.
2) Let $T = \mathrm{Th}(\mathcal{N})$ and let $\tau(x)$ be a formula defining $\mathrm{Th}(\mathcal{N})$. Then, for each sentence φ in the language of arithmetic, we have $T \vdash \varphi \Leftrightarrow \varphi \in T \Leftrightarrow \mathcal{N} \models \tau(\underline{\varphi}) \Leftrightarrow \tau(\underline{\varphi}) \in T$. This means that $T \vdash \varphi \leftrightarrow \tau(\underline{\varphi})$, i.e. τ is a definiton of truth in T – a contradiction with 1). $\square$

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Definition

A theory is *recursively axiomatized* if its set of axioms is recursive.

Definition

$\mathrm{Prf}_T(x, y)$ is a formula that holds iff "y is a proof of x in T".

Fact

Q is Σ_1-complete, i.e.

$$Q \vdash \varphi(\underline{m_1}, \dots, \underline{m_k}) \Leftrightarrow \mathcal{N} \models \varphi[m_1, \dots, m_k]$$

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Gödel's first theorem

Theorem

Let T be a consistent and recursively axiomatized extension of the theory Q. Then there exists a Π_1-sentence in the language of arithmetic which is true in $\mathcal{N}$ and unprovable in T.

Precisely: Let $\Theta(x, y)$ be a Σ_1-formula that defines Prf_T and let ν be a sentence such that $Q \vdash \nu \leftrightarrow \neg(\exists y)\Theta(\underline{\nu}, y)$. Then $T \nvdash \nu$ and $\mathcal{N} \models \nu$.

Proof.

Suppose $T \vdash \nu$. Then $\mathrm{Prf}_T(\underline{\nu}, \underline{d})$ holds for some $d \in \mathbb{N}$, i.e. $Q \vdash (\exists y)\Theta(\underline{\nu}, y)$ (from Σ_1-completeness). However, $T \vdash \neg(\exists y)\Theta(\underline{\nu}, y)$, which is a contradiction.
Let us prove $\mathcal{N} \models \nu$. Suppose $\mathcal{N} \models \neg\nu$. Then $\mathcal{N} \models \Theta(\underline{\nu}, \underline{d})$ for some $d \in \mathbb{N}$. Thus, $Q \vdash \Theta(\underline{\nu}, \underline{d})$ so $\mathrm{Prf}_T(\underline{\nu}, \underline{d})$ holds, i.e. $T \vdash \nu$ and we obtain a contradiction. □

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Definition

Th_T is the set of all sentences that are provable in T.
nTh_T is the set of all sentences such that their negation is provable in T.

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A theory is *decidable* if Th_T is recursive.

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A theory T is *complete* if T is consistent and for each sentence φ either $T \models \varphi$ or $T \models \neg\varphi$.

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Δ_1-inseparability

Theorem

Let T be a consistent numerical theory and let every Δ_1-subset of $\mathbb{N}$ be represented in T by some formula.

1. *Suppose $P \subseteq \mathbb{N}$ separates Th_T and nTh_T, i.e. P contains one of the sets and is disjoint from the other one. Let $E_P = \{\langle a, b\rangle \in \mathbb{N}^2;\ P(\mathrm{Sub}(a, \mathrm{Vr}(0), \mathrm{Num}(b)))\}$ be a relation. Then for each Δ_1-set $A \subseteq \mathbb{N}$, there exists $a \in \mathbb{N}$ such that $A = E_P[a]$.*
2. *Th_T and nTh_T cannot be separated by any Δ_1-set. In particular, T is undecidable.*

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2. *Th_T and nTh_T cannot be separated by any Δ_1-set. In particular, T is undecidable.*

Proof.

1) Th_T and nTh_T are disjoint because T is consistent. Denote $\mathrm{Sub}(a, \mathrm{Vr}(0), \mathrm{Num}(b))$ by $Sb(a, b)$. Then $E_P = \{\langle a, b\rangle \in \mathbb{N}^2;\ P(Sb(a, b))\}$. Let $P \subseteq \mathbb{N}$ be a set separating Th_T and nTh_T; because of symmetry, we may suppose that $\mathrm{Th}_T \subseteq P$. For a Δ_1-set $A \subseteq \mathbb{N}$ there exists a formula a with one free variable $\mathrm{Vr}(0)$ such that

$$\begin{aligned} b \in A \quad &\Rightarrow \quad \mathrm{Th}_T(Sb(a, b)) \Rightarrow P(Sb(a, b)), \\ b \notin A \quad &\Rightarrow \quad \mathrm{nTh}_T(Sb(a, b)) \Rightarrow \neg P(Sb(a, b)). \end{aligned}$$

Therefore, $b \in A \Leftrightarrow E_P(a, b)$, i.e. $E_P[a] = A$.

2) If a Δ_1-set $P \subseteq \mathbb{N}$ separates Th_T and nTh_T, then also $A = \{a \in \mathbb{N};\ \neg E_P(a, a)\}$ is a Δ_1-set. From 1), there exists $a \in \mathbb{N}$ such that $A = E_P[a]$. Then we have $\neg E_P(a, a) \Leftrightarrow a \in A \Leftrightarrow E_P(a, a)$ – a contradiction. $\square$

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$\neg E_P(a, a) \Leftrightarrow a \in A \Leftrightarrow E_P(a, a)$ – a contradiction. $\square$

Corollary

Let T be a consistent extension of the theory Q. Then T is undecidable. Moreover, if T is recursively axiomatized, then T is not complete.

Proof.

Direct consequence of the previous theorem and the fact that every Δ_1-relation can be represented in Q by a Σ_1-formula. $\square$

Corollary

Let T be a consistent extension of the theory Q. Then T is undecidable. Moreover, if T is recursively axiomatized, then T is not complete.

Proof.

Direct consequence of the previous theorem and the fact that every Δ_1-relation can be represented in Q by a Σ_1-formula. $\square$

Definition

An *ultrafilter* over a set X is a set $\mathscr{U} \subseteq \mathcal{P}(X)$ such that

1. if $A \in \mathscr{U}$ and $A \subseteq B$ then $B \in \mathscr{U}$,
2. if $A, B \in \mathscr{U}$ then $A \cap B \in \mathscr{U}$,
3. $\emptyset \notin \mathscr{U}$, and
4. for each subset $A \subseteq X$, exactly one of $A, X \setminus A$ is in $\mathscr{U}$.

Definition

Let $\mathscr{U}$ be an ultrafilter over I. For two elements f, g of the cartesian product $\prod_{i \in I} A_i$, we define an equivalence by $f \equiv_{\mathscr{U}} g$ iff $\{i \in I : f(i) = g(i)\} \in \mathscr{U}$. We denote the equivalence class of f by $f_{\mathscr{U}}$. The *ultraproduct* is then defined as

$$\prod_{i \in I} A_i / \mathscr{U} = \{f_{\mathscr{U}} : f \in \prod_{i \in I} A_i\}.$$

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Definition

Let L be a first-order language, I a non-empty set and $(A_i : i \in I)$ a family of non-empty L-structures. Let $\phi(\overline{x})$ be a formula of L and $\overline{a}$ a tuple of elements of the product $\prod_I A_i$. We define the *Boolean value* of $\phi(\overline{a})$, denoted $\|\phi(\overline{a})\|$, to be the set $\{i \in I : A_i \models \phi(\overline{a}(i))\}$.

Theorem

Let L be a first-order language, $(A_i : i \in I)$ a non-empty family of non-empty L-structures and $\mathscr{U}$ an ultrafilter over I. Then for any formula $\phi(\bar{x})$ of L and tuple $\bar{a}$ of elements of $\prod_I A_i$,

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Ultrapowers

Definition

We call the ultraproduct $\prod_I A_i/\mathscr{U}$ the *ultrapower* of A modulo $\mathscr{U}$ if $A_i = A$ for each $i \in I$. We denote the ultrapower by $A^I/\mathscr{U}$.

Definition

The *diagonal map* $e : A \to A^I/\mathscr{U}$ is defined by $e(b) = a_{\mathscr{U}}$ where $a(i) = b$ for all $i \in I$.

Corollary

If $A^I/\mathscr{U}$ is an ultrapower of A, then the diagonal map $e : A \to A^I/\mathscr{U}$ is an elementary embedding.

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Non-standard model of arithmetic

Definition

An ultrafilter $\mathscr{U}$ over X is *principal* if there exists $x \in X$ such that $\mathscr{U} = \{A \subseteq X : x \in A\}$.

Remark

If $\mathscr{U}$ is a principal ultrafilter, then the ultraproduct $\prod_I A_i/\mathscr{U}$ is isomorphic to one of the A_i.

Corollary

There is a model A of the theory of natural numbers and $a \in A$ such that $A \models a > n$ for every natural number n.

Proof.

Let $\mathscr{U}$ be a non-principal ultrafilter over $\mathbb{N}$. Then $A = \mathbb{N}^{\mathbb{N}}/\mathscr{U}$ is a model of the theory of natural numbers. Take $a = b_{\mathscr{U}}$ where $b(i) = i$ for each $i \in \mathbb{N}$. $\square$

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Thank you for your attention!