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Lambda-modules

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Motivation

Theorem

Let R be a principle ideals domain, M finitely generated R-module. Then M is isomorphic to a sum

$$M \cong R^r \oplus R/(p_1^{\alpha_1}) \oplus \cdots \oplus R/(p_m^{\alpha_m})$$

where $r \in \mathbb{N}_0$, $p_1, \ldots, p_m$ irreducible elements (not necessarily different) and $\alpha_1, \ldots, \alpha_m \in \mathbb{N}$.

Unavoidable definitions

Definition

Set of all infinite sequences $\mathbb{Z}_p^{\mathbb{N}_0}$, together with operations of summing and multiplying defined as follows

$$\begin{aligned}(a_i)_{i=0}^{\infty} + (b_i)_{i=0}^{\infty} &= (a_i + b_i)_{i=0}^{\infty}, \\ (a_i)_{i=0}^{\infty} \cdot (b_i)_{i=0}^{\infty} &= (\sum_{j=0}^{i} a_j b_{i-j})_{i=0}^{\infty},\end{aligned}$$

is called the ring of formal power series over the ring $\mathbb{Z}_p$ or Λ.

Definition (2)

Polynomial $P \in \mathbb{Z}_p[T]$ is called distinguished if
$P = T^n + a_{n-1}T^{n-1} + \cdots + a_0$, where $a_0, \ldots, a_{n-1} \in (p)$.

Properties of Λ

We will need to know, that Λ is

- noetherian,
- unique factorization domain (irreducible elements are p and all irreducible distinguished polynomial),
- its prime ideals are only of following form: $\{0\}$, (p), (p, T) a ideals (P), where P is an irreducible distinguished polynomial and (p, T) is the only maximal ideal.

Pseudoisomorphism

Definition (10)

We call two Λ-modules M, M' pseudoisomorphic, denote $M \sim M'$, if there is a homomorphism $\varphi : M \to M'$ with a finite kernel and cokernel.

Obviously it is reflexive, it is NOT symetric (counterexample $(p, T) \sim \Lambda$, but $\Lambda \not\sim (p, T)$)

Lemma (12)

Let M, M', M'' be modules such that $M \sim M'$, $M' \sim M''$. Then $M \sim M''$.

Lemma (13)

Let M, M', N, N' be Λ-modules such that $M \sim M'$, $N \sim N'$. Then $M \oplus N \sim M' \oplus N'$.

Some useful lemmas

Lemma (14)

Let R be a noetherian commutative ring, M finitely generated R-module. Then every submodule $N \subseteq M$ is finitely generated.

Lemma (9)

Let M be a finitely generated Λ-module and $f, g \in \Lambda$ are relatively prime. If the ideal (f, g) annihilates M, then M is finite.

Lemma (11)

Assume that $f, g \in \Lambda$ are relatively prime. Then

1. *the natural homomorphism $\Lambda/(fg) \to \Lambda/(f) \oplus \Lambda/(g)$ is an injection with finite kernel*
2. *there exists an injective homomorphism $\Lambda/(f) \oplus \Lambda/(g) \to \Lambda/(fg)$ with a finite cokernel.*

Notation

Every finitely generated Λ-module $M \cong \Lambda^n/N$. The submodule $N \subseteq \Lambda^n$ is also finitely generated by $(\lambda_{11}, \dots, \lambda_{1n}), \dots, (\lambda_{m1}, \dots, \lambda_{mn}) \in \Lambda^n$ (Λ is noetherian). We will denote $\mathrm{r}(M) = (\lambda_{ij})_{m \times n}$.
On the other hand we will denote
$\mathrm{m}(R) = \Lambda^n/((\lambda_{11}, \dots, \lambda_{1n}), \dots, (\lambda_{m1}, \dots, \lambda_{mn}))$ for each $m \times n$ matrix $R = (\lambda_{ij})_{m \times n}$.

Lemma (15)

Let A, B be matrices over the ring Λ. Then

$$\mathrm{m}\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \cong \mathrm{m}(A) \oplus \mathrm{m}(B).$$

Structure theorem

Theorem (16)

Let M be a finitely generated Λ-module. Then $M \sim \Lambda^r \oplus (\bigoplus_{i=1}^{s} \Lambda/(p^{n_i})) \oplus \left(\bigoplus_{j=1}^{t} \Lambda/(P_j^{m_j})\right)$*, where* $r, s, t, n_i, m_j \in \mathbb{Z}$ *are non-negative integers and* P_j *are irreducible distinguished polynomials.*

Proof.

Let $R = \mathrm{r}(M)$. we will later show 6 operations which can change R to R' so that

$$\mathrm{m}(R) \sim \mathrm{m}(R')$$

or, in the case of operation 5

$$\mathrm{m}(R) \sim \mathrm{m}(R') \oplus \Lambda/(p^k).$$

Structure theorem

We will also show that R can be changed to R' of the form

$$\begin{pmatrix} \lambda_1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & & & & \vdots \\ 0 & 0 & \dots & \lambda_t & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \dots & 0 & \dots & 0 \end{pmatrix},$$

using finitely many of these operations, where λ_i are distinguished polynomials.

Structure theorem

Denote $M' = \mathrm{m}(R')$. Pseudoisomorphism is transitive and is preserved by sums, we have

$$\begin{aligned} M &\sim M' \oplus \bigoplus_{j=1}^{s} \Lambda/(p^{n_j}) = \\ &= \Lambda^n/((\lambda_1, 0, \ldots, 0), \ldots, (0, \ldots, \lambda_t, \ldots, 0)) \oplus \bigoplus_{j=1}^{s} \Lambda/(p^{n_j}), \end{aligned}$$

where $\bigoplus_{j=1}^{s} \Lambda/(p^{n_j})$ comes from using operation 6. Moreover, by lemma 15

$$\Lambda^n/((\lambda_1, 0, \ldots, 0), \ldots, (0, \ldots, \lambda_t, \ldots, 0)) \cong \Lambda^{n-t} \oplus \bigoplus_{i=1}^{t} \Lambda/(\lambda_i),$$

Structure theorem

Let $\lambda_1 = \prod_{i=1}^{k} P_i^{e_i}$. By lemma 11

$$\Lambda/(\lambda_1) = \Lambda/\left(\prod_{i=1}^{k} P_i^{e_i}\right) \sim (\Lambda/(P_1^{e_1})) \oplus \cdots \oplus (\Lambda/(P_k^{e_k})).$$

Do the same for the rest of λ_i and we are done.

Allowed operations

Operation 1. We may interchange two rows (columns).

Operation 2. We may add a multiple of a row (column) to another row (column).

Operation 3. We may multiply a row (column) by $\lambda \in \Lambda^{\times}$.

Operation 4. If R contains a row $(\lambda_1, p\lambda_2, \dots, p\lambda_n)$, $p \nmid \lambda_1$, then we may change R to R' which contains a row $(\lambda_1, \lambda_2, \dots, \lambda_n)$ and all elements of the first column except for λ_1 are multiplied by p.

Operation 5. If all elements in the first column of R are divisible by p^k and if there is a row $(p^k\lambda_1, p^k\lambda_2, \dots, p^k\lambda_n)$, $p \nmid \lambda_1$, then we may change R to R' which contains a row $(\lambda_1, \lambda_2, \dots, \lambda_n)$ and otherwise is the same as R.

Operation 6. If R contains a row $(p^k\lambda_1, p^k\lambda_2, \dots, p^k\lambda_n)$ and $(\lambda\lambda_1, \lambda\lambda_2, \dots, \lambda\lambda_n)$ is also a relation for some $\lambda \in \Lambda$, $p \nmid \lambda$ then we may change R to R' which contains a row $(\lambda_1, \lambda_2, \dots, \lambda_n)$ and otherwise is the same as R.

Operations 1-3

Operation 1. We may interchange two rows (columns).
Operation 2. We may add a multiple of a row (column) to another row (column).
Operation 3. We may multiply a row (column) by $\lambda \in \Lambda^{\times}$.

Proof.

These row operations does not change the generated submodule and the matrix changed by column operations coresponds to a module, that is isomorphic to the module that corresponds to the original matrix by following isomorphisms

- (interchange ith and jth column)

$$\begin{aligned} e_k + N &\mapsto e'_k + N' \text{ pro } k \neq i, j, \\ e_i + N &\mapsto e'_j + N', \\ e_j + N &\mapsto e'_i + N'. \end{aligned}$$

Operations 1-3

- (add λ-times jth column to ith column)

$$
\begin{aligned}
e_k + N &\mapsto e'_k + N' \text{ pro } k \neq j, \\
e_j + N &\mapsto e'_j + \lambda e'_i + N'.
\end{aligned}
$$

- (multiply ith column by λ)

$$
\begin{aligned}
e_k + N &\mapsto e'_k + N' \text{ pro } k \neq i, \\
e_i + N &\mapsto \lambda e'_i + N',
\end{aligned}
$$

Operation 4

Operation 4. If R contains a row $(\lambda_1, p\lambda_2, \ldots, p\lambda_n)$, $p \nmid \lambda_1$, then we may change R to R' which contains a row $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and all elements of the first column except for λ_1 are multiplied by p.

$$\mathrm{m}\begin{pmatrix} \lambda_1 & p\lambda_2 & \cdots & p\lambda_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & & & \\ \alpha_{m1} & \alpha_{n2} & \cdots & \alpha_{mn} \end{pmatrix} \sim \mathrm{m}\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ p\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & & & \\ p\alpha_{m1} & \alpha_{n2} & \cdots & \alpha_{mn} \end{pmatrix}.$$

Operation 4

Proof.

Let $M = \mathrm{m}(R) = \Lambda^n/N$ and $M_1 = \mathrm{m}(R_1) = \Lambda^{n+1}/N_1$, where

$$R_1 = \begin{pmatrix} \lambda_1 & p\lambda_2 & \cdots & p\lambda_n & 0 \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} & 0 \\ \vdots & & & & \vdots \\ \alpha_{m1} & \alpha_{n2} & \cdots & \alpha_{mn} & 0 \\ 1 & 0 & \cdots & 0 & -p \\ 0 & \lambda_2 & \cdots & \lambda_n & \lambda_1 \end{pmatrix}$$

Operation 4

Using operations 1-3 and lemma, we know that

$$\mathrm{m}(R_1) \cong \mathrm{m}\begin{pmatrix} 0 & p\lambda_2 & \cdots & p\lambda_n & p\lambda_1 \\ 0 & \alpha_{12} & \cdots & \alpha_{1n} & p\alpha_{11} \\ \vdots & & & & \vdots \\ 0 & \alpha_{n2} & \cdots & \alpha_{mn} & p\alpha_{m1} \\ 0 & \lambda_2 & \cdots & \lambda_n & \lambda_1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \cong \mathrm{m}\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ p\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & & & \\ p\alpha_{m1} & \alpha_{n2} & \cdots & \alpha_{mn} \end{pmatrix}.$$

Claim: $M \sim M_1$.
Let $\varphi : \Lambda^n \to M_1$ be a homomorphism given as

$$\varphi((x_1, \dots, x_n)) = (x_1, \dots, x_n, 0) + N_1.$$

Operation 4

Element $(x_1, \ldots, x_n) \in \mathrm{Ker}(\varphi)$ if there are $a_1, \ldots, a_m, b, c \in \Lambda$ such that

$$\begin{aligned}(x_1, \ldots, x_n, 0) &= a_1(\lambda_1, p\lambda_2, \ldots, p\lambda_n, 0) + \cdots + a_m(\alpha_{m1}, \ldots, \alpha_{mn}, 0) + \\ &+ b(1, 0, \ldots, 0, -p) + c(0, \lambda_2, \ldots, \lambda_n, \lambda_1).\end{aligned}$$

Therefore $c\lambda_1 = bp$ and $p \mid c$ nad $b = \lambda_1 \frac{c}{p}$.

$$\begin{aligned}(x_1, \ldots, x_n, 0) &= a_1(\lambda_1, p\lambda_2, \ldots, p\lambda_n, 0) + \cdots + a_m(\alpha_{m1}, \ldots, \alpha_{mn}, 0) + \\ &+ \frac{c}{p}(\lambda_1, p\lambda_2, \ldots, p\lambda_n, 0),\end{aligned}$$

therefore $(x_1, \ldots, x_n) \in N$ and φ is injective.

Operation 4

On the other hand the cokernel is annihilated by (p, λ_1), because for $(x_1, \ldots, x_{n+1}) \in M$

$$
\begin{aligned}
& p(x_1, \ldots, x_{n+1}) + N_1 = \\
& = (px_1, \ldots, px_{n+1}) + (x_{n+1}, 0, \ldots, 0, -px_{n+1}) + N_1 = \\
& = (px_1 + x_{n+1}, \ldots, px_n, 0) + N_1 \in \mathrm{Im}(\bar{\varphi}), \\
& \lambda_1(x_1, \ldots, x_{n+1}) + N_1 = \\
& = (\lambda_1 x_1, \ldots, \lambda_1 x_{n+1}) - (0, x_{n+1}\lambda_2, \ldots, x_{n+1}\lambda_n, x_{n+1}\lambda_1) + N_1 = \\
& = (\lambda_1 x_1, \lambda_1 x_2 - x_{n+1}\lambda_2, \ldots, \lambda_1 x_n - x_{n+1}\lambda_n, 0) + N_1 \in \mathrm{Im}(\bar{\varphi}).
\end{aligned}
$$

and is by lemma 9 finite.

Operation 5

Operation 5. If all elements in the first column of R are divisible by p^k and if there is a row $(p^k\lambda_1, p^k\lambda_2, \dots, p^k\lambda_n)$, $p \nmid \lambda_1$, then we may change R to R' which contains a row $(\lambda_1, \lambda_2, \dots, \lambda_n)$ and otherwise is the same as R.

$$\mathrm{m}\begin{pmatrix} p^k\lambda_1 & p^k\lambda_2 & \cdots & p^k\lambda_n \\ p^k\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & & & \\ p^k\alpha_{m1} & \alpha_{n2} & \cdots & \alpha_{mn} \end{pmatrix} \sim \mathrm{m}\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ p^k\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & & & \\ p^k\alpha_{m1} & \alpha_{n2} & \cdots & \alpha_{mn} \end{pmatrix} \oplus \Lambda/(p^k).$$

Operation 5

Proof.

Let $M = \mathrm{m}(R) = \Lambda^n/N$ and $M_1 = \mathrm{m}(R_1) = \Lambda^{n+1}/N_1$, where

$$R_1 = \begin{pmatrix} p^k\lambda_1 & p^k\lambda_2 & \cdots & p^k\lambda_n & 0 \\ p^k\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} & 0 \\ \vdots & & & & \vdots \\ p^k\alpha_{m1} & \alpha_{n2} & \cdots & \alpha_{mn} & 0 \\ 0 & \lambda_2 & \cdots & \lambda_n & \lambda_1 \\ p^k & 0 & \cdots & 0 & -p^k \end{pmatrix}$$

Operation 5

Using operations 1-3 and lemma 15, we know that

$$\mathrm{m}\begin{pmatrix} 0 & p^k\lambda_2 & \cdots & p^k\lambda_n & p^k\lambda_1 \\ 0 & \alpha_{12} & \cdots & \alpha_{1n} & p^k\alpha_{11} \\ \vdots & & & & \vdots \\ 0 & \alpha_{n2} & \cdots & \alpha_{mn} & p^k\alpha_{m1} \\ 0 & \lambda_2 & \cdots & \lambda_n & \lambda_1 \\ p^k & 0 & \cdots & 0 & 0 \end{pmatrix} \cong \mathrm{m}\begin{pmatrix} \lambda_1 & \cdots & \lambda_n \\ p^k\alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \\ p^k\alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix} \oplus \Lambda/(p^k).$$

Claim: $M \sim M_1$. Let $\varphi : \Lambda^n \to M_1$ be a homomorphism given as

$$\varphi((x_1, \ldots, x_n)) = (x_1, \ldots, x_n, 0) + N_1.$$

Operation 5

Element $(x_1, \dots, x_n) \in \mathrm{Ker}(\varphi)$ if there are $a_1, \dots, a_m, b, c \in \Lambda$ such that

$$
\begin{aligned}
(x_1, \dots, x_n, 0) &= a_1(p^k\lambda_1, \dots, p^k\lambda_n, 0) + \dots + a_m(p^k\alpha_{m1}, \dots, \alpha_{mn}, 0) + \\
&\quad + b(p^k, 0, \dots, 0, -p^k) + c(0, \lambda_2, \dots, \lambda_n, \lambda_1),
\end{aligned}
$$

Therefore $c\lambda_1 = bp^k$ and $p^k \mid c$ nad $b = \lambda_1 \frac{c}{p^k}$.

$$
\begin{aligned}
(x_1, \dots, x_n, 0) &= a_1(p^k\lambda_1, \dots, p^k\lambda_n, 0) + \dots + a_m(p^k\alpha_{m1}, \dots, \alpha_{mn}, 0) + \\
&\quad + \frac{c}{p^k}(p^k\lambda_1, p^k\lambda_2, \dots, p^k\lambda_n, 0),
\end{aligned}
$$

therefore $(x_1, \dots, x_n, 0) \in N$ and φ is injective.

Operation 5

On the other hand the cokernel is annihilated by (p, λ_1), because for $(x_1, \dots, x_{n+1}) \in M$

$$\begin{aligned}
&p(x_1, \dots, x_{n+1}) + N_1 = \\
&= (px_1, \dots, px_{n+1}) + (x_{n+1}, 0, \dots, 0, -px_{n+1}) + N_1 = \\
&= (px_1 + x_{n+1}, \dots, px_n, 0) + N_1 \in \mathrm{Im}(\bar{\varphi}), \\
&\lambda_1(x_1, \dots, x_{n+1}) + N_1 = \\
&= (\lambda_1 x_1, \dots, \lambda_1 x_{n+1}) - (0, x_{n+1}\lambda_2, \dots, x_{n+1}\lambda_n, x_{n+1}\lambda_1) + N_1 = \\
&= (\lambda_1 x_1, \lambda_1 x_2 - x_{n+1}\lambda_2, \dots, \lambda_1 x_n - x_{n+1}\lambda_n, 0) + N_1 \in \mathrm{Im}(\bar{\varphi}).
\end{aligned}$$

and is by lemma 9 finite.

Operation 6

Operation 6. If R contains a row $(p^k\lambda_1, p^k\lambda_2, \dots, p^k\lambda_n)$ and $(\lambda\lambda_1, \lambda\lambda_2, \dots, \lambda\lambda_n)$ is also a relation for some $\lambda \in \Lambda$, $p \nmid \lambda$ then we may change R to R' which contains a row $(\lambda_1, \lambda_2, \dots, \lambda_n)$ and otherwise is the same as R.

$$\mathrm{m}\begin{pmatrix} p^k\lambda_1 & p^k\lambda_2 & \cdots & p^k\lambda_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & & & \\ \alpha_{m1} & \alpha_{n2} & \cdots & \alpha_{mn} \end{pmatrix} \sim \mathrm{m}\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & & & \\ \alpha_{m1} & \alpha_{n2} & \cdots & \alpha_{mn} \end{pmatrix}.$$

Operation 6

Proof.

Let $M = \mathrm{m}(R) = \Lambda^n/N$, $M_1 = \Lambda^n/N_1$, where $N_1 = N + (\lambda_1, \dots, \lambda_n)\Lambda$.

$$\pi((x_1, \dots, x_n) + N) = (x_1, \dots, x_n) + N_1.$$

This is clearly a surjective homomorpshism so the cokernel is finite.
The kernel is

$$\mathrm{Ker}(\pi) = (N + (\lambda_1, \dots, \lambda_n)\Lambda)/N \cong (\lambda_1, \dots, \lambda_n)\Lambda/((\lambda_1, \dots, \lambda_n)\Lambda \cap N).$$

The kernel is clearly annihilated by (λ, p^k)

$$\begin{aligned} \lambda(\lambda_1, \dots, \lambda_n) &= (\lambda\lambda_1, \dots, \lambda\lambda_n) \in (\lambda_1, \dots, \lambda_n)\Lambda \cap N \\ p^k(\lambda_1, \dots, \lambda_n) &= (p^k\lambda_1, \dots, p^k\lambda_n) \in (\lambda_1, \dots, \lambda_n)\Lambda \cap N. \end{aligned}$$

and therefore finite by lemma 9.

The last useful lemma

Lemma (17)

Using the six operations, every matrix R over Λ can be changed to a diagonal matrix, which has only zeros or distinguished polynomials on the diagonal.

Proof.

Define Weierstrass degree of element as follows: if $x \neq 0$, then $x = p^n PU$ (see Theorem 3)

$$\deg_w(x) = \begin{cases} \infty & \text{jestliže } x = 0 \text{ nebo } n > 0; \\ \deg(P) & \text{jestliže } n = 0. \end{cases}$$

The last useful lemma

For R define $\deg_w^{(k)}(R)$: Take all matrices $(a'_{ij})_{m\times n}$ that can be a result of applying finitely many of operations 1-6 on R that do not change (but may use) the first $k-1$ rows of R. We have $\min(\deg_w(a'_{ij}) \mid i \geq k, j \geq k)$ for all these matrices. Then take $\deg_w^{(k)}(R)$ minimum of these minimums. Assume the matrix is of form

$$\begin{pmatrix} D_{r-1} & 0 \\ A & B \end{pmatrix},$$

where

$$D_{r-1} = \begin{pmatrix} \lambda_{11} & 0 & \dots & 0 \\ 0 & \lambda_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \lambda_{r-1r-1} \end{pmatrix},$$

where $\deg_w^{(i)}(R) = \deg(\lambda_{ii})$ and $r \geq 1$.

The last useful lemma

$$\begin{pmatrix} \lambda_{1,1} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \lambda_{2,2} & & \vdots & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots & 0 & \dots & 0 \\ 0 & \dots & \dots & \lambda_{r-1,r-1} & 0 & \dots & 0 \\ a_{1,1} & \dots & \dots & a_{1,r-1} & b_{1,1} & \dots & b_{1,n-r+1} \\ \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{m-r+1,1} & \dots & \dots & a_{m-r+1,r-1} & b_{m-r+1} & \dots & b_{m-r+1,n-r+1} \end{pmatrix},$$

The last useful lemma

We will show that:

1. if $B \neq 0$ then using operations 1-6 we can change the matrix to a matrix, which is of form
$$\begin{pmatrix} D_r & 0 \\ A' & B' \end{pmatrix},$$
2. if $B = 0$ then $A = 0$.

The last useful lemma

Assume $B \neq 0$. Then $\deg_w^{(r)}(R) < \infty$ (otherwise we can use this procedure)

- we use operation 4, until A is divisible by greater power of p then B.
- we use operation 5 on the row containing element divisible by lowest power of p.

Let us assume that at $b_{1,1}$ is distinguished polynomial and $\deg(b_{1,1}) = \deg_w^{(r)}(R) < \infty$. We can divide the rest of elements on the rth row with remainder (theorem 1) and subtract a suitable multiple of rth column from the rest of columns so all elements on rth row are polynomials of degree less then $\deg(b_{1,1})$. Do the same for all $\lambda_{i,i}$, $1 \leq i \leq r-1$ so $\deg(a_{1,i}) < \deg(\lambda_{i,i})$ pro $i \leq r-1$.

The last useful lemma

Assume $b_{1,s} \neq 0$ for some s and it is divisible by lower power of p then $b_{1,t}$ for all t. Now use the procedure

- use operation 4 until A is divisible by larger power of p then $b_{1,s}$
- use operation 4 on rth row with respect to $b_{1,1}$ until $p \nmid b_{rs}$,

which gives us contradiction.
Assume $a_{1,s} \neq 0$ for some s and it is divisible by lower power of p then $a_{1,t}$ for all t. Use operation 4 with respect to element $b_{1,1}$. Thus we get contradiction with $\deg_w^{(s)}(R) = \deg(\lambda_{s,s}) > \deg(a_{1,s}) = \deg_w(a_{1,s})$.

The last useful lemma

Now let $B = 0$. By subtracting a suitable multiple of ith row, $1 \leq r - 1$, we can have all $a_{i,j}$ polynomials and $\deg(a_{i,j}) < \deg(\lambda_{j,j})$. The all $a_{i,j}$ are divisible by p else we get contradiction. Assume $a_{i,j} \neq 0$ for some i, j and it is divisible by lowest power of p on its row, suppose p^k. Denote $\lambda = \prod_{i=1}^{r-1} \lambda_{ii}$. Surely $p \nmid \lambda$. Now

- $(\lambda \frac{a_{i,1}}{p^k}, \ldots, \lambda \frac{a_{i,r-1}}{p^k}) \in N$,
- $(p^k \frac{a_{i,1}}{p^k}, \ldots, p^k \frac{a_{i,r-1}}{p^k}) \in N$.

If we then use operation 6, we get contradiction.