Numeration systems irrational basis

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Outline

Non-rational basis

Definition

Let $\beta > 1$ an irrational number. The *set of β-integers* with a finite representation is

$$\mathbb{Z}_\beta = \{\pm x | x \in \mathbb{R}, x \geq 0, (x)_\beta = x_k x_{k-1} \dots x_0 \bullet\} .$$

The set of all finite numbers with basis β will be defined as

$$\text{Fin}(\beta) = \bigcup_{n \in \mathbb{N}} \frac{1}{\beta^n} \mathbb{Z}_\beta .$$

Greedy β expansion

Let's have $\beta > 1$ and $x \in \mathbb{R}$, $x > 0$.

1. Find maximal $k \in \mathbb{Z}$ such as $\beta^k \leq x$.
2. Put $x_k = \lfloor \frac{x}{\beta^k} \rfloor$.
3. $x := x - x_k \beta^k$

Repeat until the value x is equal 0.
The result is $x = x_k \beta^k + x_{k+1} \beta^{k+1} + x_{k+2} \beta^{k+2} + \ldots$.

Example

Let us have the golden ratio $\beta = \frac{1+\sqrt{5}}{2}$.
The digits of our system will be 0 and 1 because $\lfloor \beta \rfloor = 1$.
Observe that:

- $(1)_\beta = 1\bullet$
- $(2)_\beta = 10 \bullet 01$
- $(3)_\beta = 11 \bullet 01$
- $(4)_\beta = 101 \bullet 01$
- $(5)_\beta = 110 \bullet 1001$

Addition

Lemma

If $\beta \notin \mathbb{Z}$ then $\mathbb{Z}_\beta + \mathbb{Z}_\beta \not\subset \mathbb{Z}_\beta$.

$$\mathbb{Z}_\beta = \{\pm x | x \in \mathbb{R}, x \geq 0, (x)_\beta = x_k x_{k-1} \dots x_0 \bullet\}.$$

$$(1)_\beta + (1)_\beta = (2)_\beta$$

$$1 \bullet + 1 \bullet = 10 \bullet 01$$

Algebraic integer

Definition

Algebraic integer α is a root of some monic $P(x) \in \mathbb{Z}[x]$.
The *degree* of α is a the degree of its minimal polynomial.
The other roots of $P(x)$ are called the *conjugates* of α.

Frougny and Solomyak's theorem

Theorem (Frougny and Solomyak)

Let $\beta > 1$ be such that $\mathrm{Fin}(\beta) + \mathrm{Fin}(\beta) \subset \mathrm{Fin}(\beta)$. Then β is an algebraic integer and all of its conjugates have absolute value less than 1.

$1 \in \mathrm{Fin}(\beta) \Rightarrow \mathbb{Z}_+ \subseteq \mathrm{Fin}(\beta)$.
Consider the expansion of $x = \lfloor \beta \rfloor + 1$.
We get that β is an algebraic integer.

Definition of the Pisot number

Definition

If $\beta > 1$ is a root of a rational polynomial $P(x)$ whose all other roots have absolute value less than 1, then β is called a *Pisot number*.

Definition of Tiling

Definition

Tiling of a space $\mathbb{R}^n$ is given by finite set D of tiles, which can fill the space without gaps and overlapping.

Finite chains

Let R be the set of all finite binary sequences.

Definition

A set $T_{\bullet w}$ of positive integers with finite chains after $\bullet$, for every $w \in R$ we can note it in a following manner

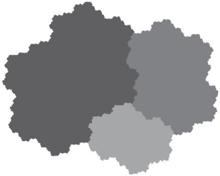
$$T_{\bullet w} = \{x \geq 0 | (x)_\beta = x_k x_{k-1} \dots x_0 \bullet w\} .$$

In particular, if w is an empty sequence, then $T_\bullet = \mathbb{Z}_\beta \cap [0, \infty)$.

Tiling

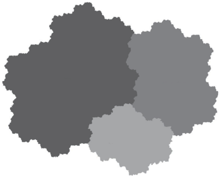
- Let γ be a root of $P(x)$ conjugate to β.
- Then the fields $\mathbb{Q}(\beta)$ and $\mathbb{Q}(\gamma)$ are isomorphic via some σ.
- σ changes basis from $\beta > 1$ to γ, the norm of γ is less than 1, and is an identity on $\mathbb{Q}$.

Tiling



- For basis γ is $(T_\bullet)' = \{x' = \sigma(x) | x \geq 0, x \in \mathbb{Z}_\beta\}$ bounded in $\mathbb{C}$.
- A closure of this set is called a *central tile* $D_\bullet$ belonging to Pisot number β.
- $D_{\bullet w} = \overline{\{x' = \sigma(x) | x \in T_{\bullet w}\}}$ is a tile given by $w \in R$

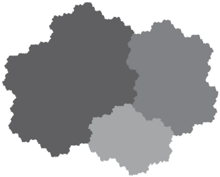
Tiling



This tiling has following properties:

- Any tile $D_{\bullet w}$ is a copy of one of tiles $D_{\bullet}$, $D_{\bullet 1}$ or $D_{\bullet 11}$.
- Any tile multiplied by $\frac{1}{\gamma}$ could be composed of tiles $D_{\bullet}$, $D_{\bullet 1}$ and $D_{\bullet 11}$.

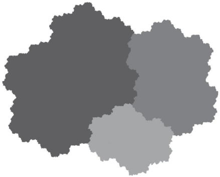
Tiling



For a case $\bullet w = x$, where $(x)_\beta = \bullet w$ we have

- $T_{\bullet 0w} = T_\bullet + \bullet 0w$,
- $T_{\bullet 10w} = T_{\bullet 1} + \bullet 00w$ and
- $T_{\bullet 110w} = T_{\bullet 11} + \bullet 000w$
- $D_{\bullet 0w}$ is a shifted copy of $D_\bullet$,
- $D_{\bullet 10w}$ is a shifted copy of $D_{\bullet 1}$ and
- $D_{\bullet 110w}$ is a shifted copy of $D_{\bullet 11}$

Tiling - instructions by scaling



- A set $\frac{1}{\beta} T_{\bullet}$ contains only numbers with maximally one cipher following $\bullet$
- $\frac{1}{\beta} T_{\bullet} = T_{\bullet} \cup T_{\bullet 1}$
- $\frac{1}{\gamma} D_{\bullet} = D_{\bullet} \cup D_{\bullet 1}$

It gives instructions for tilling all plane by scaling by factor $\frac{1}{\gamma}$.
Pisot number of degree d gives tiling for a space $\mathbb{R}^{d-1}$.

Pisot unit

Pisot unit is a Pisot number which minimal polynomial has an absolute coefficient ± 1.

Theorem (Tiling)

Let $\beta > 1$ be a Pisot unit of degree $d \geq 2$. Then sets $D_{\bullet w}$ form tiling of a space $\mathbb{R}^{d-1}$ if and only if $\mathrm{Fin}(\beta) + \mathrm{Fin}(\beta) \subset \mathrm{Fin}(\beta)$.

Fractal 1 from hal.archives-ouvertes.fr/docs

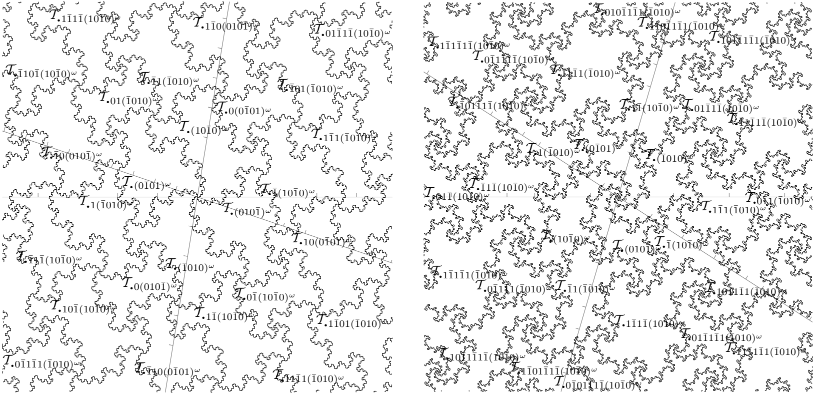


Figure: Tilings from the symmetric β-transformation, $\beta^3 = 2\beta^2 - \beta + 1$ (left) and $\beta^3 = \beta^2 + 1$ (right).

Fractal 2 from hal.archives-ouvertes.fr/docs

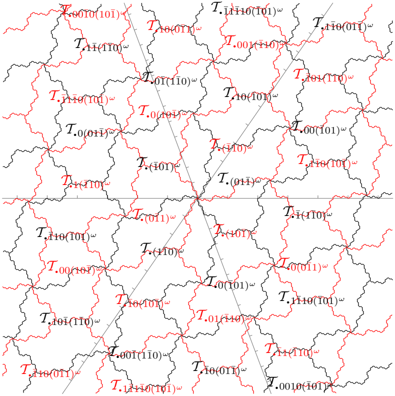


Figure: Double tiling from the symmetric β-transformation, $\beta^3 = \beta^2 + \beta + 1$.

Epilogue

Thank for your attention.