Dual AES

Jaroslav Šeděnka

Institute of Mathematics and Statistics
Faculty of Science, Masaryk University

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Overview

1. Galois Field $GF(2^8)$
2. Description of AES
3. What operations do you need to compute AES?
4. Dual AES
5. What can you do with dual AES?

Galois Field $GF(2^8)$

Consider the field $GF(2) = \{0, 1\}$.

For any irreducible polynomial $p(x) \in GF(2)[x]$, we can construct the factorring $GF(2^8) := GF(2)[x]/(p(x))$.

Then, $GF(2^8)$ is up to isomorphism the unique finite field with 2^8 elements. Trivially, $GF(2^8)$ is a vector space over $GF(2)$.

The multiplicative group of $GF(2^8)$ is cyclic.

Operations in the $GF(2^8)$

Addition

$a(x) \oplus b(x) = (a_7 \oplus b_7)x^7 + (a_6 \oplus b_6)x^6 + ... + (a_0 \oplus b_0)$, where the $a \oplus b$ denotes XOR of bits a, b.

For example $10100110 \oplus 10000011 = 00100101$.

Operations in the $GF(2^8)$

Multiplication

$$a(x) \bullet b(x) = a(x)b(x) \mod (x^8 + x^4 + x^3 + x + 1)$$

$$10100110 \bullet 10000011 = 01110110$$

Operations in the $GF(2^8)$

Multiplication

$$a(x) \bullet b(x) = a(x)b(x) \mod (x^8 + x^4 + x^3 + x + 1)$$

$10100110 \bullet 10000011 = 01110110$
$(x^7 + x^5 + x^2 + x) \bullet (x^7 + x + 1) =$
$(x^{14} + x^{12} + x^9 + x^7 + x^8 + x^6 + x^3 + x^2 + x^7 + x^5 + x^2 + x)$
$\mod (x^8 + x^4 + x^3 + x + 1) = x^6 + x^5 + x^4 + x^2 + x$

Operations in the $GF(2^8)$

Multiplicative inverse

$a(x)^{-1}$ modulo $(x^8 + x^4 + x^3 + x + 1)$ can be computed using Extended Euclidean Algorithm.

Inverse of 10100110 is ...

Squaring in $GF(2^8)$

Proposition

The function $f(x) = x^2$ is $GF(2)$-linear in $GF(2^8)$.

Proof.

The only scalar multiples are $0, 1$.
Let $a, b \in GF(2^8)$. The characteristic of $GF(2^8)$ is 2, thus
$(a \oplus b)^2 = a^2 \oplus 2ab \oplus b^2 = a^2 \oplus b^2$. $\square$

So, there exists matrix Q with boolean coefficients, such that $Qx = x^2$. Also, the matrix Q is invertible.

Advanced Encryption Standard (AES)

- symmetric block cipher
- current NIST standard
- proposed in 1999 by J. Daemen and V. Rijmen (Rijndael)
- substitution-permutation network
- block size is fixed (128 bits), key size is variable (128, 194 or 256 bits)
- in this lecture, the AES-128 will be described

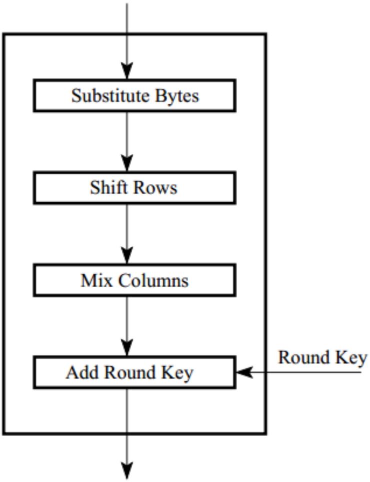
AES State Array

$S_{0,0}$	$S_{0,1}$	$S_{0,2}$	$S_{0,3}$
$S_{1,0}$	$S_{1,1}$	$S_{1,2}$	$S_{1,3}$
$S_{2,0}$	$S_{2,1}$	$S_{2,2}$	$S_{2,3}$
$S_{3,0}$	$S_{3,1}$	$S_{3,2}$	$S_{3,3}$

Data representation:

- intermediate data block is stored in a 4x4 array of bytes
- Each byte $s_{i,j}$ is interpreted as an element of $GF(2^8) = GF(2)[x]/(x^8 + x^4 + x^3 + x + 1)$
- Example: byte 10100110 is represented as $x^7 + x^5 + x^2 + x$
- Also, 10100110 can be written as $\{A6\}$

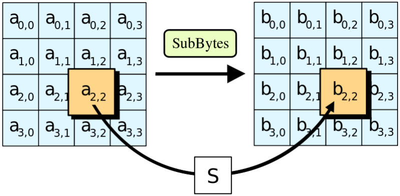
AES Algorithm overview



Encryption Round

- 10 round of AES are performed
- initial and final rounds are slightly different

AES SubBytes



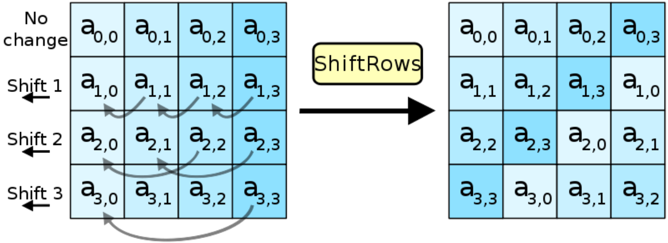
Byte-wise substitution:

- take the multiplicative inverse $b_{i,j} = a_{i,j}^{-1}$ in $GF(2^8)$
- apply an affine transformation $b'_{i,j} = Ab_{i,j} + c$

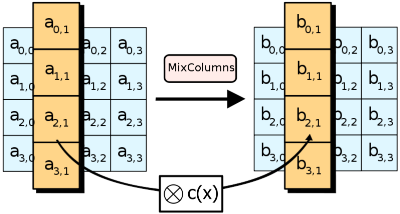
AES SubBytes

$$\begin{bmatrix} b_0' \\ b_1' \\ b_2' \\ b_3' \\ b_4' \\ b_5' \\ b_6' \\ b_7' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

AES ShiftRows



AES MixColumns

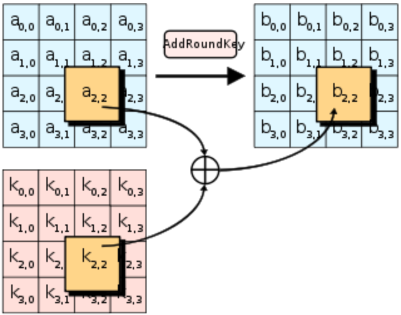


$$b(y) = a(y) \otimes c(y) \mod y^4 + 1$$

as polynomials in $GF(2^8)[y]$.

$$c(y) = \{03\}y^3 + \{01\}y^2 + \{01\}y + \{02\}$$

AES AddRoundKey



Whole state array is XORed with expanded Round key.

What operations do you need to compute AES?

$GF(2^8)$ operations

- addition $x \oplus y$
- XOR with a constant $x \oplus c$
- multiplication $x \bullet y$
- multiplication by a constant $x \bullet c$
- raise to any power in $GF(2^8)$, including to power -1

Non-$GF(2^8)$ operations

- permutation of n-tuples
- $GF(2)$-linear transformation
- table lookup

Let's call these $EGF(2^8)$ *operations*.

Dual cipher

Definition: Dual Ciphers

Two ciphers E, E' are called *dual ciphers*, if there exist invertible functions $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ such that for each plaintext P and key K

$$f(E_K(P)) = E'_{g(K)}(h(P))$$

Square cipher

Definition: Square Cipher

Given a cipher E that uses only $EGF(2^8)$ operations, we define the cipher E^2 by modifying the constants of E this way:

- whenever there is XOR with c in E, there is XOR with c^2 in E^2
- whenever there is multiplication by c in E, there is multiplication by c^2 in E^2
- whenever there is multiplication by matrix A in E, there is multiplication by QAQ^{-1} in E^2
- whenever there is table lookup $S(x)$ in E, there is $QS(Q^{-1}x)$ in E^2.

Square AES

Theorem

For any cipher E using only operations in $EGF(2^8)$, the ciphers E and E^2 are dual ciphers.

Proof

We need to show the "duality" for all operations in $EGF(2^8)$, that is, $(E_K(P))^2 = E^2_{K^2}(P^2)$ for all P, K. Note that by $(E_K(P))^2$ we mean byte-wise squaring.

- addition: $(x \oplus y)^2 = x^2 \oplus y^2$ is exactly the linearity of squaring in $GF(2^8)$.

Square AES

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- addition: $(x \oplus y)^2 = x^2 \oplus y^2$ is exactly the linearity of squaring in $GF(2^8)$.
- multiplication: $(x \bullet y)^2 = x^2 \bullet y^2$

Square AES

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- addition: $(x \oplus y)^2 = x^2 \oplus y^2$ is exactly the linearity of squaring in $GF(2^8)$.
- multiplication: $(x \bullet y)^2 = x^2 \bullet y^2$
- exponentiation: $(x^k)^2 = (x^2)^k$

Square AES

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- addition: $(x \oplus y)^2 = x^2 \oplus y^2$ is exactly the linearity of squaring in $GF(2^8)$.
- multiplication: $(x \bullet y)^2 = x^2 \bullet y^2$
- exponentiation: $(x^k)^2 = (x^2)^k$
- permutation of n-tuples is trivial

Proof.

- linear transformation: $(Ax)^2 = QAQ^{-1}x^2 = QAx = (Ax)^2$
- table lookup: $S(x)^2 = QS(Q^{-1}x)$.

By structural induction, $(E_K(P))^2 = E^2_{K^2}(P^2)$.

$\square$

In a similar way, any invertible linear transformation can be used to create dual ciphers. Mainly, change of irreducible polynomial $p(x)$ used to construct the Galois Field is also linear.

What can you do with dual ciphers?

- Different dual variants of a cipher may be faster for encryption/decryption
- When the attacker has partial or total access to the encryption process, change of bases during the computatiton can increase security
- The property of cipher being nontrivialy dual to itself can be abused for cryptoanalysis

Thank you for your attention

Questions, comments?

References

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