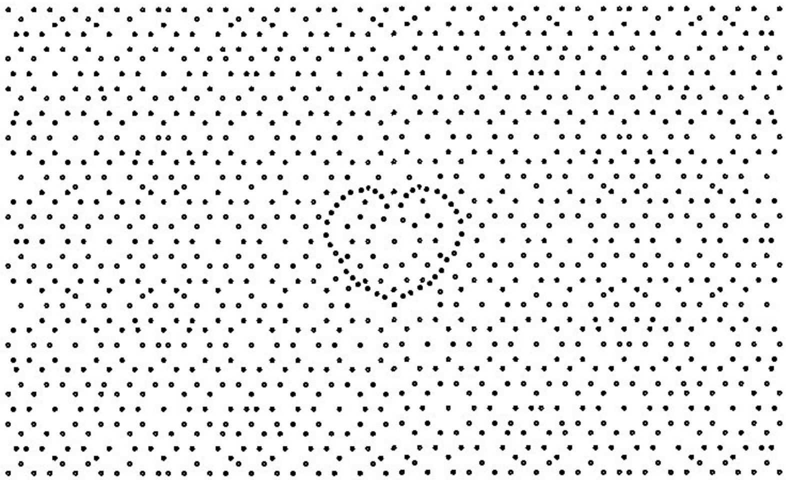

Quasicrystals

Pepa Dvořák

Katedra Algebry
MFF-UK

Spring school of Algebra
13.5.2011

What is a crystal?

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- crystal structure = pattern + lattice
- lattice ⇒ periodicity, symmetry

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Crystallographic Restriction Theorem

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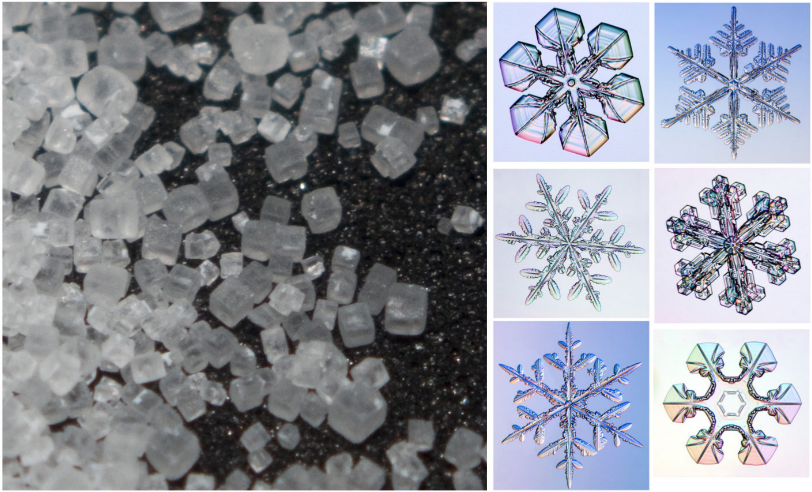
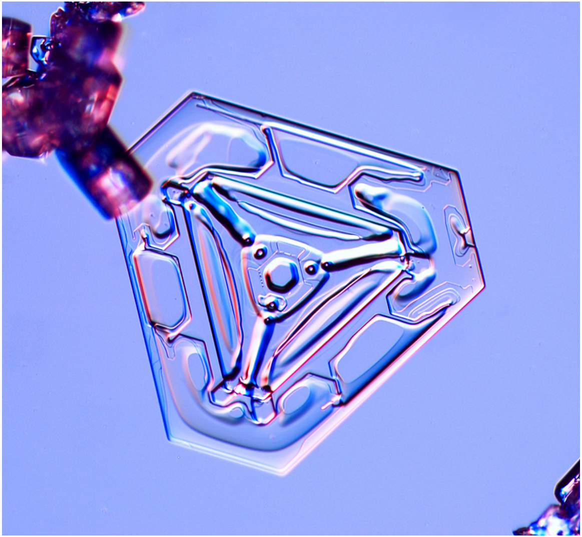
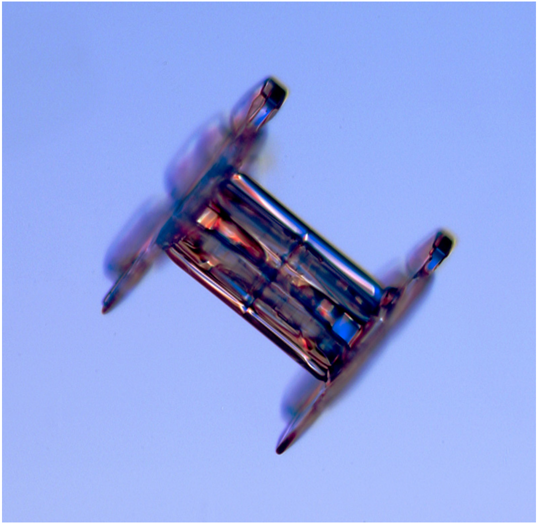
examples:



(Source: http://writescience.wordpress.com/2012/11/10/a-personal-voyage/)

3.jpg 

(Source: web Scientific American)



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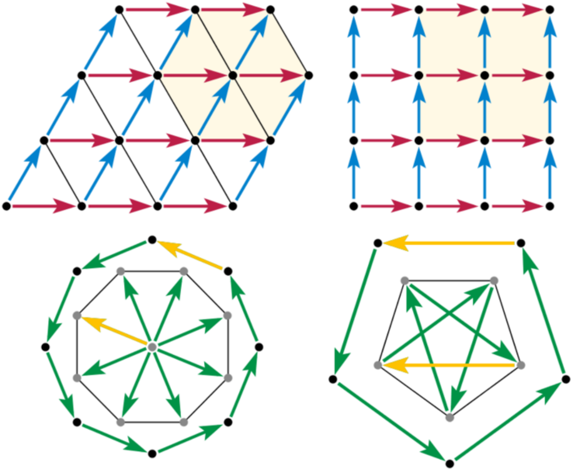
Crystallographic Restriction Theorem, the proof



(Source: wikipedia)

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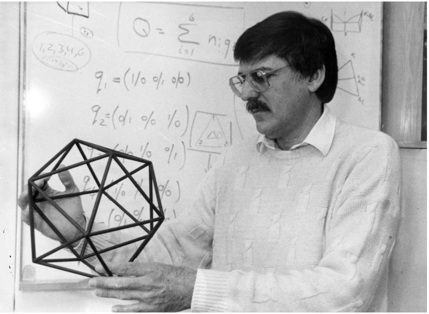

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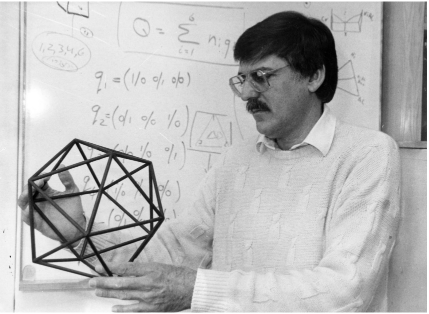

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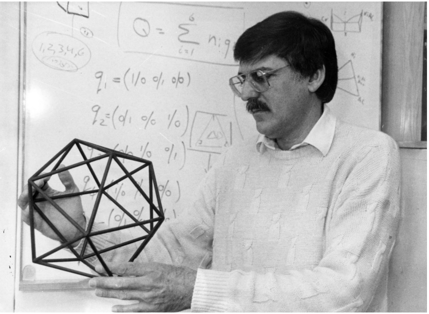

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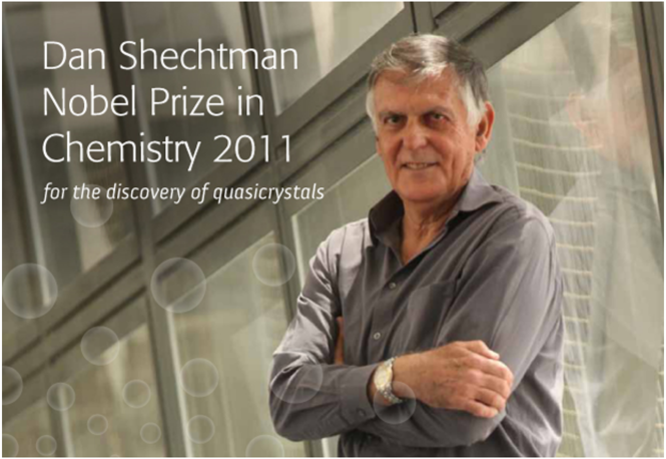
Dan Shechtman
Nobel Prize in
Chemistry 2011
for the discovery of quasicrystals

Quasicrystal modelling sets, properties

- the set $\Sigma \subseteq \mathbb{R}^d$ is a *Delone* set, i.e.
  1. (uniform discreteness) there exists $r_1 > 0$ s.t. each ball of radius r_1 contains at most one element of Σ
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- the set $\Sigma \subseteq \mathbb{R}^d$ is a *Meyer* set, i.e.
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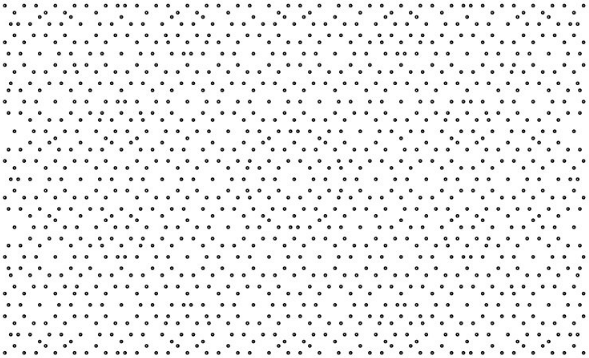
Cut & Project sets, a first exaple

- a C&P set $\mathbb{R}^2$ with 10-fold symmetry:



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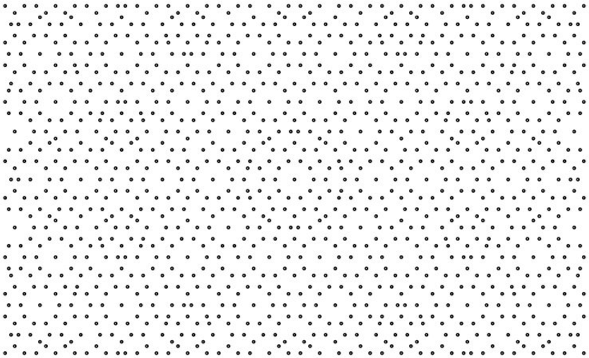

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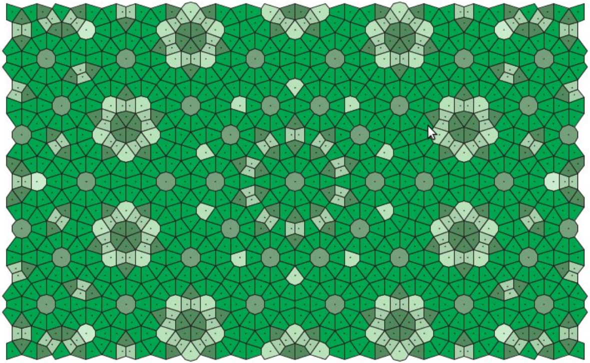

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Cut & Project sets in general

1. $X = \{x_1, \ldots, x_d\} \subseteq \mathbb{R}^d$ a full rank lattice
2. write $\mathbb{R}^d = V_1 \oplus V_2$
3. projections $\pi_1 : \mathbb{R}^d \to V_1$, $\pi_2 : \mathbb{R}^d \to V_2$
4. conditions: π_1 monic on L, $\pi_2(L)$ dense in V_2
5. schematically:

$$\begin{tikzcd} V_1 & \mathbb{R}^d \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & V_2 \\ & L \arrow[u] & \end{tikzcd}$$

6. *acceptance window*: a bounded set $\Omega \subseteq V_2$
7. C&P-set $\Sigma(\Omega) := \{\pi_1(x) \mid x \in L \text{ and } \pi_2(x) \in \Omega\}$

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one-dimensional Cut & Project sets

- lattice $L = \mathbb{Z}^2$
- subspaces $V_1 : y = \varepsilon x$, $V_2 : y = \eta x$, $\varepsilon \neq \eta$, ε, η irrational
- set $\mathbf{x}_1 := \dfrac{1}{\varepsilon - \eta}(1, \varepsilon)$, $\mathbf{x}_2 := \dfrac{1}{\eta - \varepsilon}(1, \eta)$
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- the images of projections are

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Let ε, η be distinct irrational real numbers and $\Omega \subseteq \mathbb{R}$ be a bounded interval. The set

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For each $\Sigma_{\varepsilon,\eta}(\Omega)$ there exist positive numbers $\Delta_1,\ \Delta_2 \in \mathbb{Z}[\eta]$ depending only on $\eta, \varepsilon, |\Omega|$ such that the distances between adjacent points of $\Sigma_{\varepsilon,\eta}(\Omega)$ take values in $\{\Delta_1, \Delta_2, \Delta_1 + \Delta_2\}$.

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$$\overline{\varepsilon} \in (-1, 0), \overline{\eta} > 0, max\left(1 + \overline{\varepsilon}, -\overline{\varepsilon}\right) < |\overline{\Omega}| \leq 1 \qquad (1)$$

such that $\Sigma_{\varepsilon,\eta}(\Omega) = s\Sigma_{\overline{\varepsilon},\overline{\eta}}\left(\overline{\Omega}\right)$ for some $s \in \mathbb{R}$.

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A C&P sequence $\Sigma_{\varepsilon,\eta}(\Omega)$ is self-similar (i.e. $\gamma\Sigma_{\varepsilon,\eta}(\Omega) \subseteq \Sigma_{\varepsilon,\eta}(\Omega)$ for some $\gamma > 1$) iff ε is a quadratic number, ε is its algebraic conjugate and the closure of Ω contains the origin.

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- If Ω is semi-open interval, then there are only two types of distances
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- Given a C&P set $\Sigma_{\epsilon,\eta}(\Omega)$. If $\Delta_1, \Delta_2, \Delta_1+\Delta_2$ are the three types of distances between the adjacent points of $\Sigma_{\epsilon,\eta}(\Omega)$, a bidirectional word $u_{\epsilon,\eta}(\Omega)$ in the alphabet A, B, C $u_{\epsilon,\eta}(\Omega)$ is a C&P word iff $u_n = \begin{cases} A \text{ if } x_{n+1}-x_n=\Delta_1 \\ B \text{ if } x_{n+1}-x_n=\Delta_1+\Delta_2 \\ C \text{ if } x_{n+1}-x_n=\Delta_2. \end{cases}$
- if $\eta_1, \varepsilon_1, \Omega_1$ and $\eta_2, \varepsilon_2, \Omega_2$ satisfy the condition (1), then $u_{\epsilon_1,\eta_1}(\Omega_1) = u_{\epsilon_2,\eta_2}(\Omega_2)$
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- let $u = \ldots u_{-2}u_{-1}u_0u_1u_2\ldots$ be a bidirectional infinite word
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Let $u_{\epsilon_1,\eta_1}(\Omega_1)$ be a C&P word with $\Omega = [c, c+\ell)$, then

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If $\ell \in \mathbb{Z}[\varepsilon]$, then there exists a sturmian word $v = \dots v_{-2}v_{-1}v_0v_1v_2\dots$ in alphabet $0, 1$ and finite words W_0, W_1 in the alphabet A, B, C such that

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- a word w is *palindrome*, if $w = \overline{w}$
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- in general, for a non-periodic word we have $\mathcal{P}_n \leq \frac{16}{n} \mathcal{C}_{n+\lfloor \frac{n}{4} \rfloor}$

For C&P words with $\Omega = [c, c+\ell)$ and $y_n \in \Omega$ satisfying the condition (1), the palindromic complexity is

$$\mathcal{P}_n = \begin{cases} 1 & \text{for } n \text{ even} \\ 2 & \text{for } n \text{ odd and } \ell = 1 \\ 3 & \text{for } n \text{ odd and } \ell \neq 1 \end{cases}$$

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C&P sequences and uncommon numeration systems

- β-integers are elements of $\mathbb{Z}_\beta = \left\{ \pm \sum_{i=0}^{k} x_i \beta_i \mid \sum_{i=0}^{k} x_i \beta^i \text{ is a } \beta - \text{expansion of some } x \geq 0 \right\}$
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The positive part of the set $\mathbb{Z}_\beta$ coincides with the positive part of a C&P set $\Sigma_{\varepsilon,\eta}(\Omega)$ iff β is a quadratic Pisot unit.

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The positive part of the set $\mathbb{Z}_\beta$ coincides with the positive part of a C&P set $\Sigma_{\varepsilon,\eta}(\Omega)$ iff β is a quadratic Pisot unit.

C&P sequences and uncommon numeration systems

- β-integers are elements of $\mathbb{Z}_\beta = \left\{ \pm \sum_{i=0}^{k} x_i \beta_i \mid \sum_{i=0}^{k} x_i \beta^i \textit{ is a } \beta - \textit{expansion of some } x \geq 0 \right\}$
- We can prove the following:

Theorem

The positive part of the set $\mathbb{Z}_\beta$ coincides with the positive part of a C&P set $\Sigma_{\varepsilon,\eta}(\Omega)$ iff β is a quadratic Pisot unit.

Thank you.