Permutation Groups: Frobenius Groups

Michal Szabados

Katedra Algebry
MFF-UK

Spring school of Algebra
25.3.2012

Overview

What we already know

- Nothing, we are really dumb.

Definition

A permutation group G is a Frobenius group if it is

- transitive
- non-regular
- every non-trivial element fixes at most one point

Particularly

$$G = \{id\} \cup \{g \in G : |Fix(g)| = 0\} \cup \{g \in G : |Fix(g)| = 1\}$$

Definition

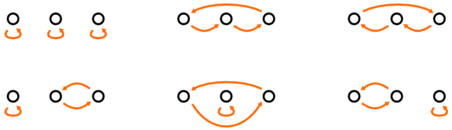
A permutation group G is a Frobenius group if it is

- transitive
- non-regular
- every non-trivial element fixes at most one point

Particularly

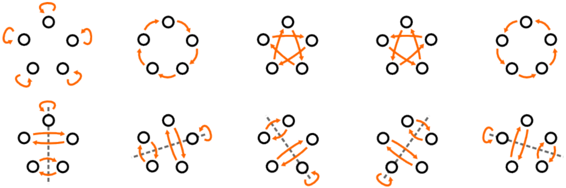
$$G = \{id\} \cup \{g \in G : |Fix(g)| = 0\} \cup \{g \in G : |Fix(g)| = 1\}$$

The smallest example: S_3



More examples

- The dihedral group D_n of size $2n$ for odd n



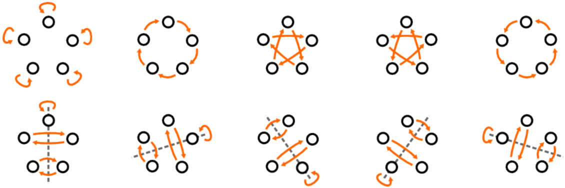
- For a field F the group of invertible affine transformations of F

$$x \mapsto ax + b, \quad a \in F^*, b \in F$$

In the finite case the size of this group is $|U| \cdot |F|$, which is dn for some $n = p^k$, $d \mid n - 1$.

More examples

- The dihedral group D_n of size $2n$ for odd n



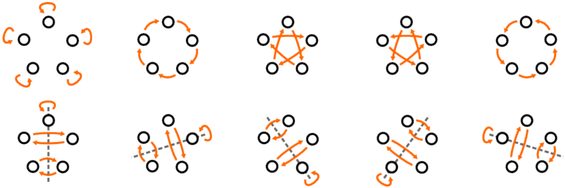
- For a field F the group of invertible affine transformations of F

$$x \mapsto ax + b, \quad a \in F^*, b \in F$$

In the finite case the size of this group is $|U| \cdot |F|$, which is dn for some $n = p^k$, $d \mid n - 1$.

More examples

- The dihedral group D_n of size $2n$ for odd n



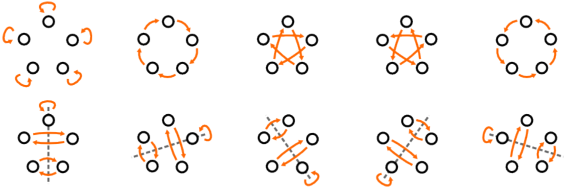
- For a field F the group of invertible affine transformations of F

$$x \mapsto ax + b, \quad a \in U \leq F^*, b \in F$$

In the finite case the size of this group is $|U| \cdot |F|$, which is dn for some $n = p^k$, $d \mid n - 1$.

More examples

- The dihedral group D_n of size $2n$ for odd n



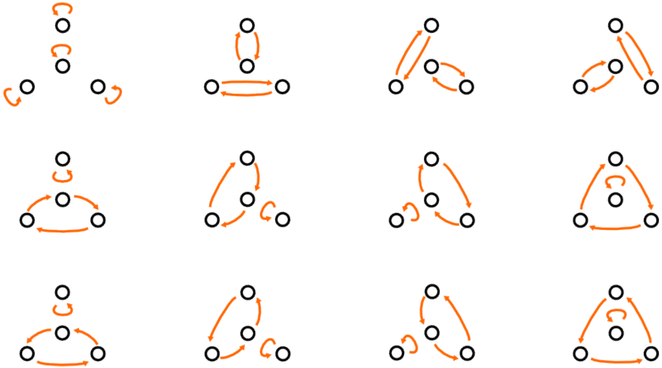
- For a field F the group of invertible affine transformations of F

$$x \mapsto ax + b, \quad a \in U \leq F^*, b \in F$$

In the finite case the size of this group is $|U| \cdot |F|$, which is dn for some $n = p^k$, $d \mid n - 1$.

More examples

Direct isometries of a tetrahedron:



Properties

Lemma

Let G be a finite Frobenius group acting on an n-point set X. Then $|G| = dn$ for some $d \mid n-1$.

Recall: $|G| = |G_x| \cdot |Orb(x)|$

Proof:

- The stabilizer G_x of $x \in X$ acts regularly on every orbit on $Y := X \setminus \{x\}$.
- $\Rightarrow$ Every orbit on Y has size $|G_x|$.
- $\Rightarrow |G_x| \mid |Y| = n-1$, denote $d = |G_x|$.
- $|G| = |G_x| \cdot |Orb(x)| = dn$.

Properties

Lemma

Let G be a finite Frobenius group acting on an n-point set X. Then $|G| = dn$ for some $d \mid n-1$.

Recall: $|G| = |G_x| \cdot |Orb(x)|$

Proof:

- The stabilizer G_x of $x \in X$ acts regularly on every orbit on $Y := X \setminus \{x\}$.
- $\Rightarrow$ Every orbit on Y has size $|G_x|$.
- $\Rightarrow |G_x| \mid |Y| = n - 1$, denote $d = |G_x|$.
- $|G| = |G_x| \cdot |Orb(x)| = dn$.

$\square$

Properties

Lemma

Let G be a finite Frobenius group acting on an n-point set X. Then $|G| = dn$ for some $d \mid n-1$.

Recall: $|G| = |G_x| \cdot |Orb(x)|$

Proof:

- The stabilizer G_x of $x \in X$ acts regularly on every orbit on $Y := X \setminus \{x\}$.
- $\Rightarrow$ Every orbit on Y has size $|G_x|$.
- $\Rightarrow |G_x| \mid |Y| = n-1$, denote $d = |G_x|$.
- $|G| = |G_x| \cdot |Orb(x)| = dn$.

$\square$

Properties

Lemma

Let G be a finite Frobenius group acting on an n-point set X. Then $|G| = dn$ for some $d \mid n-1$.

Recall: $|G| = |G_x| \cdot |Orb(x)|$

Proof:

- The stabilizer G_x of $x \in X$ acts regularly on every orbit on $Y := X \setminus \{x\}$.
- $\Rightarrow$ Every orbit on Y has size $|G_x|$.
- $\Rightarrow |G_x| \mid |Y| = n - 1$, denote $d = |G_x|$.
- $|G| = |G_x| \cdot |Orb(x)| = dn$.

□

Properties

Lemma

Let G be a finite Frobenius group acting on an n-point set X. Then $|G| = dn$ for some $d \mid n-1$.

Recall: $|G| = |G_x| \cdot |Orb(x)|$

Proof:

- The stabilizer G_x of $x \in X$ acts regularly on every orbit on $Y := X \setminus \{x\}$.
- $\Rightarrow$ Every orbit on Y has size $|G_x|$.
- $\Rightarrow |G_x| \mid |Y| = n - 1$, denote $d = |G_x|$.
- $|G| = |G_x| \cdot |Orb(x)| = dn$.

□

Properties

Lemma

Let G be a finite group with order pq, where $p < q$ are primes. Then either G is abelian or $p \mid q - 1$ and $G \cong F_{p,q}$.

That means:

- either $G \cong \mathbb{Z}_p \times \mathbb{Z}_q$,
- or G is isomorphic to a group of affine transformations of F_q

$$x \mapsto ax + b, \quad a \in U, b \in \mathbb{F}_q$$

with $U \leq F_q^*$, $|U| = p$.

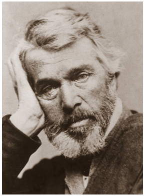
Example

Every group of order 15 is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_5$ and every group of order 14 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_7$ or D_7.

Ferdinand Georg Frobenius

Definition

Ferdinand Georg Frobenius (October 26, 1849 – August 3, 1917) was a German mathematician, best known for his contributions to the theory of differential equations and to group theory. He also gave the first full proof for the Cayley–Hamilton theorem.



Ferdinand Georg Frobenius

Definition

Ferdinand Georg Frobenius (October 26, 1849 – August 3, 1917) was a German mathematician, best known for his contributions to the theory of differential equations and to group theory. He also gave the first full proof for the Cayley–Hamilton theorem.

Structure Theorem for Finite Frobenius Groups

Definition

For a Frobenius group G we define its Frobenius kernel

$$K = \{g \in G : |Fix(g)| \neq 1\}$$

Theorem

Let K be a Frobenius kernel of a finite Frobenius group G. Then:

(i) *K is a normal subgroup of G.*

(ii) *For each odd prime p, the Sylow p-subgroups of G_α are cyclic, and the Sylow 2-subgroups are either cyclic or quaternion. If G_α is not solvable, then it has exactly one composition factor, namely A_5.*

(iii) *K is a nilpotent group.*

Structure Theorem for Finite Frobenius Groups

Definition

For a Frobenius group G we define its Frobenius kernel

$$K = \{g \in G : |Fix(g)| \neq 1\}$$

Theorem

Let K be a Frobenius kernel of a finite Frobenius group G. Then:

(i) *K is a normal subgroup of G.*

(ii) *For each odd prime p, the Sylow p-subgroups of G_α are cyclic, and the Sylow 2-subgroups are either cyclic or quaternion. If G_α is not solvable, then it has exactly one composition factor, namely A_5.*

(iii) *K is a nilpotent group.*

Exercise

Exercise

Show that a primitive permutation group G with abelian point stabilizers is either regular of prime degree or a Frobenius group.

Case 1: It is regular.

- Regular action of G on X is isomorphic to the action of right translations on G:

$$g^x = gx$$

- Suppose G has a proper subgroup H.
- Claim: right cosets of H $\{Hg \mid g \in H\}$ form blocks:
- $g \mapsto gx$, $hg \mapsto hgx$
- $\not\downarrow$ with primitivity $\Rightarrow$ G has no proper subgroups.

Exercise

Exercise

Show that a primitive permutation group G with abelian point stabilizers is either regular of prime degree or a Frobenius group.

Case 1: It is regular.

- Regular action of G on X is isomorphic to the action of right translations on G:

$$g^x = gx$$

- Suppose G has a proper subgroup H.
- Claim: right cosets of H $\{Hg \mid g \in H\}$ form blocks:
- $g \mapsto gx$, $hg \mapsto hgx$
- $\lightning$ with primitivity $\Rightarrow$ G has no proper subgroups.

Exercise

Exercise

Show that a primitive permutation group G with abelian point stabilizers is either regular of prime degree or a Frobenius group.

Case 1: It is regular.

- Regular action of G on X is isomorphic to the action of right translations on G:

$$g^x = gx$$

- Suppose G has a proper subgroup H.
- Claim: right cosets of H $\{Hg \mid g \in H\}$ form blocks:
- $g \mapsto gx$, $hg \mapsto hgx$
- $\lightning$ with primitivity $\Rightarrow$ G has no proper subgroups.

Exercise

Exercise

Show that a primitive permutation group G with abelian point stabilizers is either regular of prime degree or a Frobenius group.

Case 1: It is regular.

- Regular action of G on X is isomorphic to the action of right translations on G:

$$g^x = gx$$

- Suppose G has a proper subgroup H.
- Claim: right cosets of H $\{Hg \mid g \in H\}$ form blocks:
- $g \mapsto gx$, $hg \mapsto hgx$
- $\lightning$ with primitivity $\Rightarrow$ G has no proper subgroups.

Exercise

Exercise

Show that a primitive permutation group G with abelian point stabilizers is either regular of prime degree or a Frobenius group.

Case 1: It is regular.

- Regular action of G on X is isomorphic to the action of right translations on G:
$$g^x = gx$$
- Suppose G has a proper subgroup H.
- Claim: right cosets of H $\{Hg \mid g \in H\}$ form blocks:
- $g \mapsto gx$, $hg \mapsto hgx$
- $\lightning$ with primitivity $\Rightarrow$ G has no proper subgroups.

Exercise

Exercise

Show that a primitive permutation group G with abelian point stabilizers is either regular of prime degree or a Frobenius group.

Case 1: It is regular.

- Regular action of G on X is isomorphic to the action of right translations on G:
$$g^x = gx$$
- Suppose G has a proper subgroup H.
- Claim: right cosets of H $\{Hg \mid g \in H\}$ form blocks:
- $g \mapsto gx$, $hg \mapsto hgx$
- $\lightning$ with primitivity $\Rightarrow$ G has no proper subgroups.

Exercise

Exercise

Show that a primitive permutation group G with abelian point stabilizers is either regular of prime degree or a Frobenius group.

Case 1: It is regular. ✓

Exercise

Exercise

Show that a primitive permutation group G with abelian point stabilizers is either regular of prime degree or a Frobenius group.

Case 1: It is regular. ✓

Case 2: Non-regular: home exercise

Definition

A permutation group G acting on Ω is said to be *k-transitive* if G acts transitively on k-point subsets of Ω.

Theorem

Let $G \leq Sym(\Omega)$ be an 8-transitive group of finite order. Then $G \geq Alt(\Omega)$.

Thank you for your attention!