Hilbert's third problem: Decomposing polyhedra

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In 1900, David Hilbert presented his 23 problems for a new century. Based on two Gauss' letters from 1844, the third problem was stated:

> 'Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces which can be reassembled to yield the second?'

The third problem have been resolved already the same year by Max Dehn.

1 Dehn invariants

For polyhedron P and finite set of real numbers $M = \{m_1, \ldots, m_k\} \subseteq \mathbb{R}$, we define

$$V(M) := \left\{ \sum_{i=1}^{k} q_i m_i : q_i \in \mathbb{Q} \right\} \subseteq \mathbb{R}$$

and M_P to be the set of all angles between adjacent facets (dihedral angles), together with the number π.

Given any finite set $M \subseteq \mathbb{R}$, which contains M_P, and any $\mathbb{Q}$-linear function $f \colon V(M) \to \mathbb{Q}$, that satisfies $f(\pi)$, we define the *Dehn invariant* of P with respect to f to be

$$D_f(P) := \sum_{e \in P} \ell(e) f(\alpha(e)),$$

where e stands for an edge of the polyhedron, $\ell(e)$ denotes its length and $\alpha(e)$ is the angle between the two facets that meet in e.

2 The Dehn-Hadwinger theorem

Using Dehn invariants, we can state the principal theorem:

Theorem 1 (Dehn-Hadwiger, 1949). *Let P and Q be polyhedra with dihedral angles $\alpha_1, \ldots, \alpha_p$, resp. $\beta_1, \ldots, \beta_q$ at their edges, and let M be a finite set of real numbers with $\{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q, \pi\} \subseteq M$.*

If $f \colon V(M) \to \mathbb{Q}$ is any $\mathbb{Q}$-linear function with $f(\pi) = 0$ such that $D_f(P) \neq D_f(Q)$, then P and Q are not equicomplementable.

Using this theorem, we can prove that tetrahedra with $D_f(P) \neq 0$ (such as regular tetrahedron) and tetrahedra with $D_f(P) = 0$ (such as Schläfli orthoscheme) are never equicomplementable, thus never equidecomposable.

If we consider both tetrahedra of the same volume, we get an explicit answer to Hilbert's third problem: *No.*