Hadamard codes II – Hadamard transform

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This lecture is concerned in an application of Hadamard matrices in modern telecommunications and signal data processing.

At first, we will focus on a representation of a piecewise constant function (digital signal) as a sum of more simple orthogonal components called Walsh functions. The presented approach is analogous to the more known Fourier transform that approximates a periodic function (a carrier of an analogous signal) by sines and cosines functions.

In the second part of the lecture we will use this decomposition to model a digital signal and transform an information in an image compression and image encoding.

1 Hadamard matrices and Hadamard transform

As was defined in the previous lectures, the Hadamard matrix of an order n is a $n \times n$ square matrix with elements $+1$ and -1 such that any distinct row or column vectors are mutually orthogonal.

The Hadamard matrices of order 2^m, $m \geq 0$, that are constructed using Sylvester method (see the lecture "Hadamard matrices") are called Sylvester-Hadamard matrices.

Since every Hadamard matrix H_n is an orthogonal matrix (i.e. $H_n H_n^T = nI_n$), it represents an orthogonal transformation.

Definition 1. Let H_n be a Hadamard matrix of an order n. The transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$T(f) = f \cdot H \tag{1}$$

is called a Hadamard transform.

In the literature, the previous function is also called Walsh-Hadamard or Walsh transform.

2 Walsh functions and their connection to Hadamard matrices

Walsh functions form a complete set of orthogonal functions. In the literature a couple of formal definitions of these functions are introduced. The most frequent method construct the Walsh functions $\mathit{Wal}(i,t)$, $i = 0, 1, \ldots$, on the interval $[0,1]$ by recursive formulas.

But we will show that the Walsh functions can be easily generated using the Hadamard matrices. If we consider a Sylvester-Hadamard matrix H_{2^m} of order

2^m we can define the functions $Wal(0,t)$ to $Wal(2^{m-1},t)$ in the following way. At first, we divide the interval $[0,1]$ into 2^m parts

$$\left[0, \frac{1}{2^m}\right), \left[\frac{1}{2^m}, \frac{2}{2^m}\right), \ldots, \left[\frac{2^m-1}{2^m}, 1\right). \tag{2}$$

Then the Walsh function defined by the i^{th} row of a matrix H_{2^m} takes the value 1 in the interval $[k/2^m, (k+1)/2^m]$ if

$$H_{2^m}(i, k+1) = 1, \tag{3}$$

otherwise it takes the value -1.

3 Application of Hadamard transform

The last part of the lecture will focus on practical applications of a Hadamard transform.

The first application is in the method for transmission multiple signals over one channel called CDMA (Code Division Multiple Access). The basic idea of CDMA lies in the existence of a unique signature for each sender that allows to identify his data in the final transmited signal.

The second application that will be mentioned is related to image compression. We will show that a binary image can be represented by coefficients that corresponds to a set of basis images. This correspondence is given by two dimensional Hadamard transform.