Geometric algebra

Part – I

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26. března 2012

Partially ordered sets

1. Reflexivity: $a \leq a$.
2. Antisymmetry: $(a \leq b$ and $b \leq a) \Rightarrow a = b$.
3. Transitivity: $(a \leq b$ and $b \leq c) \Rightarrow a \leq c$.

P is a poset, $H \subseteq P$ and $a \in P$.

- upper bound of H: $h \leq a$, $\forall h \in H$.
- supremum of H $(\bigvee H)$:$a \leq b$, $\forall b$ - upper bound of H.
- lower bound of H: $h \geq a$, $\forall h \in H$.
- infimum of H $(\bigwedge H)$:$a \leq b$, $\forall b$ - lower bound of H.
- $b \in P, a \prec b \iff b < a$ and for no $x \in P, b < x < a$.

- A poset $\langle L; \leq \rangle$ is a *lattice* if $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all $a, b \in L$.
- A poset $\langle L; \leq \rangle$ is a *lattice* iff $\sup H$ and $\inf H$ exist for any finite nonvoid subset H of L.

Remark

Definitions are equivalent.

Proof

$H \subseteq L, H \neq \varnothing$, *finite.*

- $H = \{a\}$ *then* $a \leq a \Rightarrow \bigvee H = a$.
- $H = \{a, b, c\}$ *then* $\bigvee\{a, b\} = d$, $\bigvee\{c, d\} = e$.
  - $a \leq d, b \leq d, d \leq e, c \leq e \Rightarrow h \leq e \forall h \in H$.
  - $a \leq f, b \leq f \Rightarrow d \leq f$; *also* $c \leq f \Rightarrow e \leq f \Rightarrow e = \bigvee H$.
- $H = \{a_1, a_2, \ldots, a_n\}, n \geq 1 \Rightarrow \bigvee H = \bigvee\{\cdots \bigvee\{\bigvee\{a_1, a_2\}, a_3\} \ldots, a_n\}$

- A lattice L is called *complete* if $\bigvee H$ and $\bigwedge H$ exist for any subset $H \subseteq L$.
- An element a of lattice $\langle L; \leq \rangle$ is an *atom* if $a \succ 0$, i.e., if a covers 0. It means that $a > 0$ and for no $x \in L$, $a > x > 0$. A lattice L is *atomic* iff it has 0 and for every $b \in L, b \neq 0$, there is an atom $a \leq b$.
- Let L be a complete lattice and let a be an element of L. Then a is called *compact* iff $a \leq \bigvee X$ for some $X \subseteq L$ implies that $a \leq \bigvee X_1$ for some finite $X_1 \subseteq X$. A complete lattice L is called *algebraic* iff every element of L is a join of compact elements.

Height function:

$$h(a) = \begin{cases} \text{the length of longest maximal chain in } [0,a], \text{ if there is a finite one} \\ \infty \text{ otherwise} \end{cases}$$

- A lattice is called *semimodular* iff it satisfies the Upper Covering Condition, that is, if $a \prec b \Rightarrow a \vee c \prec b \vee c$ or $a \vee c = b \vee c$.
- *Modular lattice* is lattice satisfying condition

$$x \geq z \Rightarrow (x \wedge y) \vee z = x \wedge (y \vee z),$$

which is equivalent to the following identity:

$$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee (x \wedge z)).$$

Theorem

Let L be a finite lattice. Then
L is semimodular $\iff$ $h(a) + h(b) \geq h(a \wedge b) + h(a \vee b)$ for all $a, b \in L$.

Proof

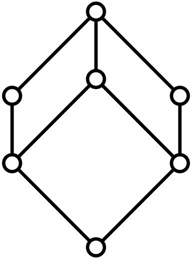
$x, y \in L$

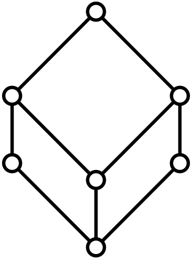
$\Rightarrow$
- $\nu_x : [x \wedge y, y] \to [y, x \vee y], \nu_x(z) = z \vee x.$
- *maximal chains to maximal chains*
- *length of $[y, x \vee y] \geq$ length of $[x \wedge y, y]$*

$\Leftarrow$

- *assume $x \wedge y \prec x$*
- $h(x) = h(x \wedge y) + 1 \Rightarrow h(x \vee y) \leq h(y) + 1$
- $x \vee y > y \Rightarrow h(x \vee y) = h(y) + 1 \Rightarrow y \prec x \vee y$

Example





- A lattice L is called *geometric* iff L is semimodular, algebraic and the compact elements of L are exactly the finite joins of atoms of L.
- A geometry $\langle A, ^{-} \rangle$ is a set A and a function $^{-}: P(A) \to P(A)$, satisfying the following properties:
  - (i)
    - a) $X \subseteq \overline{X}$;
    - b) if $X \subseteq Y$, then $\overline{X} \subseteq \overline{Y}$;
    - c) $\overline{\overline{X}} = \overline{X}$.
  - (ii) $\overline{\varnothing} = \varnothing$, and $\overline{\{x\}} = \{x\}$ for all $x \in A$.
  - (iii) If $x \in \overline{X \cup \{y\}}$, but $x \notin \overline{X}$, then $y \in \overline{X \cup \{x\}}$.
  - (iv) If $x \in \overline{X}$, then $x \in \overline{X_1}$ for some finite $X_1 \subseteq X$.

Theorem

Let $\langle A, ^- \rangle$ be a geometry. Then $L = L\langle A, ^- \rangle = \{\overline{X} | X \subseteq A\}$ (i.e., the lattice of all closed subsets of A) is a geometric lattice. Conversely, if L is a geometric lattice, A is the set of atoms of L, and for every $X \subseteq A$, $\overline{X}$ is the set of atoms spanned by X, then $\langle A, ^- \rangle$ is a geometry and $L \cong L\langle A, ^- \rangle$.

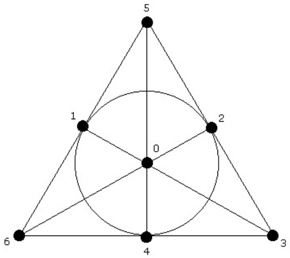
Lemma (without proof)

A lattice L is algebraic iff L is isomorphic to the lattice of closed sets of an algebraic closure space.

Proof (Theorem)

1)
- *algebraic (by Lemma).*
- *(X is compact $\Leftrightarrow X = \overline{X_1}$, finite $X_1 \subseteq X) \Rightarrow$ (X is compact $\Leftrightarrow X$ is finite join of atoms).*
- $X, Y \in L, Y = \overline{X \cup \{x\}}, x \notin \overline{X}$. *Then* $X \prec Y$.
  Let $U \in L$; $Y \vee U = \overline{Y \cup U} = \overline{X \cup x \cup U}$ *and* $X \vee U = \overline{X \cup U}$.
  Hence $X \vee U = Y \vee U$ *or* $X \vee U \prec Y \vee U \Rightarrow$ *semimodularity.*

2)
- $\langle A, ^- \rangle$ *is closure space, (ii) and (iv) from definition.*
- $x \in \overline{X \cup y}, x \notin \overline{X}$. $\overline{X \cup y} = \overline{X} \vee \overline{\{y\}} \succ \overline{X}$
  $\overline{X} \subset \overline{X \cup \{x\}} \subseteq \overline{X \cup \{y\}} \Rightarrow y \in \overline{X \cup \{x\}}$.
- $\varphi : X \to \bigvee X, X \subset A, X \in L\langle A, ^- \rangle$
  $X \subseteq Y \Leftrightarrow \bigvee X \geq \bigvee Y$, φ *is onto, one-to-one and isotone.*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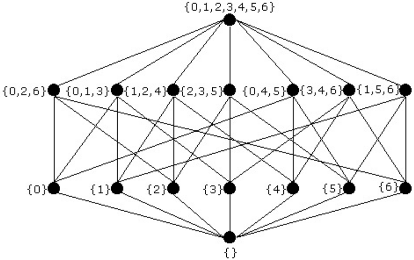
1
2
0
6
4
3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Thank you for your attention.