Nonstandard topology

Adam Ráž

Charles University in Prague
Faculty of Mathematics and Physics

23 March 2012

Uniform space

Definition

Let A be a set. *Basis of uniformity* on the set A is nonempty set V which satisfies following conditions:

- V is a set of reflexive and symmetric relations on A
- $(\forall u, v \in V)(\exists w \in V): \ w \subseteq u \wedge w \subseteq v$
- $(\forall v \in V)(\exists w \in V): \ w \circ w \subseteq v$

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The pair $\langle A, V \rangle$ is called *uniform space with basis of uniformity* V (shortly *uniform space*) and V is also called *uniform structure*.

Two bases of uniformity are *equivalent*, if for every element v of the first basis there exists an element w of the second basis that $w \subseteq v$, and this applies vice versa.

Topology induced by the uniformity

Definition

Let $\langle A, V \rangle$ be a uniform space. $X \subseteq A$ is *an open set* if for every $x \in X$ there exists $v \in V$ that $v[x] \subseteq X$.

Topology induced by the uniformity

Definition

Let $\langle A, V \rangle$ be a uniform space. $X \subseteq A$ is *an open set* if for every $x \in X$ there exists $v \in V$ that $v[x] \subseteq X$.

The set of all open sets on $\langle A, V \rangle$ is clearly the topology on A.

We call it a *topology induced by uniformity* V. We denote it $\tau(V)$.

Uniformizable space

Definition

A uniform structure on a topological space is *compatible* with the topology if the topology induced by the uniform structure coincides with the original topology.

A topological space is called *uniformizable* if there is a uniform structure compatible with the topology.

Canonical basis of uniformity

Definition

Let $\langle A, \phi\rangle$ be a uniformizable topological space. The *canonical basis of uniformity* of $\langle A, \phi\rangle$ is basis W_ϕ of uniformity on A, where

$$W_\phi = \{v_\omega;\ \omega \subseteq \phi \text{ is finite}\},$$

$$v_\omega = \{\langle x, y\rangle \in A^2;\ (\forall X \in \omega)(x \in X \leftrightarrow y \in X)\}$$

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Theorem

Let $\langle A, \phi \rangle$ be a uniformizable topological space. Then $\phi = \tau(W_\phi)$.

Quasiuniformity

Definition

Let A be a set. If we replace in the definition of basis of uniformity V the condition that says

- V is set of reflexive and symmetric relations on A

with this one

- V is set of reflexive relations on A,

we get a *basis of quasiuniformity*.

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Then we define *quasiuniform space*, its *induced topology* and their *equivalence* exactly the same way as with basis of uniformity.

We get similar results as above, but we can omit the condition of uniformizable topological space.

Canonical basis of quasiuniformity

Definition

Let $\langle A, \phi\rangle$ be a topological space. The *canonical basis of quasiuniformity* of $\langle A, \phi\rangle$ is basis W_ϕ of quasiuniformity on A, where

$$W_\phi = \{v_\omega;\ \omega \subseteq \phi \text{ is finite}\},$$

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Theorem

Let $\langle A, \phi\rangle$ be a topological space. Then $\phi = \tau(W_\phi)$.

Indifference space

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Let A be a standard set. *Indifference on A* is an equivalence Q on A, where Q is intersection of a set V of standard sets with $|V| <$ $\textcircled{\kappa}$.

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The indifference space $\langle B, R \rangle$ is its *subspace* if $B \subseteq A$ and $R = Q \cap B^2$.

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The indifference space $\langle B, R\rangle$ is its *subspace* if $B \subseteq A$ and $R = Q \cap B^2$.

Basis of an indifference space $\langle A, Q\rangle$ is any standard basis V of uniformity on A that $Q = \bigcap {}^\sigma V$ with $|{}^\sigma V| < \textcircled{\kappa}$.

Indifference space induced by uniform space

Definition

Let $\langle A, V\rangle$ be a uniform space with basis V which satisfies the condition $|{}^{\sigma}V| < \textcircled{\kappa}$. Then $\langle A, \bigcap{}^{\sigma}V\rangle$ is an indifference space with basis V.

We say that it is *an indifference space induced by uniform space* $\langle A, V\rangle$ and that $\bigcap{}^{\sigma}V$ is *indifference induced by* V.

Uniform space from indifference space

Theorem

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Corollary

Let $\langle A, Q\rangle$ be a indifference space. Then there exists unique uniform space (up to equivalence of bases) $\langle A, V_Q\rangle$ with $|{}^{\sigma}V_Q| < \kappa$ that Q is indifference induced by V_Q.

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Definition

The pair $\langle A, V_Q \rangle$ from the corollary above is called *uniform space induced by indifference space* $\langle A, Q \rangle$. Shortly the set V_Q is called *uniformity induced by indifference* Q.

Standard topology of indifference space

Definition

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Let $\langle A, Q\rangle$ be an indifference space. *Standard open set* in $\langle A, Q\rangle$ is any standard set $X \subseteq A$ which satisfies $Q[^{\sigma}X] \subseteq X$.

The standardization of set of all standard open sets on $\langle A, V\rangle$ is clearly the topology on A. It is called *standard topology* of space $\langle A, Q\rangle$ and we denote it as $\tau(\langle A, Q\rangle)$ (shortly $\tau(Q)$).

We say that $\langle A, \tau(Q)\rangle$ is a *standard topological space of indifference space* $\langle A, Q\rangle$.

Standard topology correctness

Theorem

Let $\langle A, Q\rangle$ be a indifference space and let V_Q be a uniformity induced by indifference Q.
Then standard topology of space $\langle A, Q\rangle$ is equal to topology induced by uniformity V_Q standardly, i.e.

$$[\tau(Q) = \tau(V_Q)]^{\mathsf{S}}.$$

Canonical indifference space for topological space

Definition

Let $\langle A, \phi\rangle$ be a uniformizable topological space with $|^{\sigma}\phi| < ⓚ$.
The *canonical indifference space for* $\langle A, \phi\rangle$ is an indifference space $\langle A, Q_\phi\rangle$, where

$$Q_\phi = \{\langle x, y\rangle \in A^2;\ (\forall X \in {}^{\sigma}\phi)(x \in X \leftrightarrow y \in X)\}.$$

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Let, standardly, $\langle A, \phi \rangle$ be a uniformizable topological space with $|^{\sigma}\phi| < \textcircled{\kappa}$. Then it holds:

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- $\phi = \tau(Q_\phi)$,
- $Q_\phi = \bigcap {}^{\sigma} W_\phi$.

Quasiindifference

Definition

Let A be a standard set. We can define *quasiindifference* similarly as an indifference, we just omit the condition of symmetry, so the quasiindifference would be just reflexive and transitive relation, called a quasiorder.

Equivalently we have *quasibasis of quasiindifference space*, *quasiindifference induced by quasiuniformity* and *standard topology of quasiindifference space*.

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Then we get similar results as above, but again we can omit the condition of uniformizable topological space.

Canonical quasiindifference space for topological space

Definition

Let $\langle A, \phi \rangle$ be a topological space with $|^{\sigma}\phi| < ⓚ$. The *canonical quasiindifference space for* $\langle A, \phi \rangle$ is a quasiindifference space $\langle A, Q_{\phi} \rangle$, where

$$Q_{\phi} = \{\langle x, y \rangle \in A^2;\ (\forall X \in {}^{\sigma}\phi)(x \in X \rightarrow y \in X)\}.$$

Theorem

Let, standardly, $\langle A, \phi \rangle$ *be a topological space with* $|^{\sigma}\phi| < ⓚ$*. Then it holds:*

- $\phi = \tau(Q_{\phi})$,
- $Q_{\phi} = \bigcap {}^{\sigma} W_{\phi}$.

Monad

Definition

Let $\langle A, Q \rangle$ be an indifference space. For $x \in A$ the set $Q[x] = \{y \in A;\ xQy\}$ is called *monad of point* x in space $\langle A, Q \rangle$. *Monad neighbourhood of point* x is any internal set $X \subseteq A$ which contains the monad of x.

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The point $x \in A$ is called *nearstandard* if it is in the monad of some standard point. The set $Q[{}^\sigma A] = \{x \in A;\ (\exists a \in {}^\sigma A)\ aQx\}$ of all nearstandard points we denote $\mathfrak{n}(A, Q)$.

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The point $x \in A$ is called *accessible* in $\langle A, Q\rangle$ if in every its monad neighbourhood there is some standard point. The set of all accessible points is denoted $\mathfrak{a}(A, Q)$.

Basic observations

Theorem

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Let $\langle A, Q \rangle$ be an indifference space. Then it holds:

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Theorem

Let $\langle A, Q\rangle$ be an indifference space. Then it holds:

- $Q[x] = \bigcap\{X;\ X \text{ is monad neighbourhood of } x\}$,
- *If $x \in A$ is nearstandard then x is also accessible.*
- $\mathfrak{n}(A, Q) \subseteq \mathfrak{a}(A, Q) \subseteq A$

Closure and interior of a set

Theorem

Let $\langle A, Q \rangle$ be an indifference space, $X \subseteq A$ standard, $\overline{X}$ the closure of X and X° the interior of X in $\langle A, \tau(Q) \rangle$ standardly. Then

$$\overline{X} = {}^{\mathrm{ST}}\{a \in {}^\sigma A;\ Q[a] \cap X \neq \emptyset\},$$

$$X^\circ = {}^{\mathrm{ST}}\{a \in {}^\sigma A;\ Q[a] \subseteq X\}.$$

Remark

By $\overline{X}$ we mean the smallest closed superset of X.
Equivalently X° is the biggest open subset of X.

Closure and interior of a set

Proof.

We will prove just the first equation, because the second is easy from the standard topology lemma $X^\circ = A - \overline{A - X}$ and the first one.

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Let denote the right side X'. Firstly $X' \subseteq \overline{X}$, because for $a \in {}^\sigma X'$ and $Y \supseteq X$ standard closed is $a \in Y$, because otherwise $Q[a] \cap Y = \emptyset$ and the more $Q[a] \cap X = \emptyset$.

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Let denote the right side X'. Firstly $X' \subseteq \overline{X}$, because for $a \in {}^\sigma X'$ and $Y \supseteq X$ standard closed is $a \in Y$, because otherwise $Q[a] \cap Y = \emptyset$ and the more $Q[a] \cap X = \emptyset$. Finally we just want to prove that X' is closed.

By contradiction: Let $a \in A - X'$ be standard point and let there exist a $y \in Q[a] \cap X'$. By definition of X' for every $v \in V : v[y] \cap X \neq \emptyset$. Let $v \in V$ be that $v \subseteq Q$ and $x \in v[y] \cap X$. Then $x \in Q[a] \cap X$, so $a \in X'$. $\square$

Monad in topology

Theorem

Let $\langle A, Q\rangle$ be an indifference space. Then for $a \in {}^{\sigma}A$ it holds

$$Q[a] = \bigcap\{X \in {}^{\sigma}\tau(Q);\ a \in X\}.$$

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$$Q[a] = \bigcap\{X \in {}^{\sigma}\tau(Q);\ a \in X\}.$$

Proof.

The inclusion $\subseteq$ is trivial by the definition. We will prove the other one.

Let $x \in A - Q[a]$. Then there exists a standard set Y that $Q[a] \subseteq Y \subseteq A, x \notin Y$. It holds that $a \in Y^{\circ} \in {}^{\sigma}\tau(Q)$ and $x \notin Y^{\circ}$, so $x \notin \bigcap\{X \in {}^{\sigma}\tau(Q);\ a \in X\}$.

$\square$

Hausdorff space

Definition

An indifference space is called *Hausdorff space* if any two distinct standard points have disjoint monads. Then every its monad contains at most one standard point.

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Theorem

Let $\langle A, Q\rangle$ be an indifference space with basis V.
Then $\langle A, Q\rangle$ is Hausdorff space if and only if $\bigcap V = id_A$.

Condensed space

Definition

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An indifference space $\langle A, Q \rangle$ is called *condensed* if all $a \in A$ are nearstandard, i.e. $\mathfrak{n}(A, Q) = A$.

Theorem

Canonical indifference space for standardly compact uniformizable topological space is condensed.

Theorem

Let A be a standard set which satisfies $|{}^{\sigma}A| < \textcircled{\kappa}$. Then standard topological space of condensed indifference space $\langle A, Q \rangle$ is standardly compact.

Condensed space

Like before, we can generalize the definition for quasiindifference and we get the following.

Theorem

Canonical indifference space for standardly compact topological space is condensed.

We see that except for technical issues around $\textcircled{\kappa}$ we got an equivalent term for compactness. Now we can prove theorems related to compact spaces with different point of view.

Theorem

Closed subspace of condensed quasiindifference space is condensed.

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Proof.

Let $\langle A, Q\rangle$ be condensed, $B \subseteq A$ standard and closed. Let $x \in B$. Then there exists $a \in {}^{\sigma}A$, aQx. Because B is closed, $a \in B$. □

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Let $\langle A, Q\rangle$ be condensed, $B \subseteq A$ standard and closed. Let $x \in B$. Then there exists $a \in {}^{\sigma}A,\ aQx$. Because B is closed, $a \in B$. $\square$

Theorem

Condensed subspace of Hausdorff condensed quasiindifference space is closed.

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Closed subspace of condensed quasiindifference space is condensed.

Proof.

Let $\langle A, Q\rangle$ be condensed, $B \subseteq A$ standard and closed. Let $x \in B$. Then there exists $a \in {}^{\sigma}A,\ aQx$. Because B is closed, $a \in B$. □

Theorem

Condensed subspace of Hausdorff condensed quasiindifference space is closed.

Proof.

Let $\langle A, Q\rangle$ be Hausdorff condensed, $\langle B, Q \cap B^2\rangle$ condensed. Let $x \in B$. There exists $a \in {}^{\sigma}A,\ aQx$. There also exists $b \in {}^{\sigma}B,\ bQx$. Because of Hausdorff condition, it is $a = b$, so $a \in B$. □

Bounded space

Definition

An indifference space $\langle A, Q\rangle$ is called *bounded* if all $a \in A$ are accessible, i.e. $\mathfrak{a}(A, Q) = A$.

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Theorem

Subspace of bounded indifference space is bounded.

Complete space

Definition

An indifference space $\langle A, Q \rangle$ is called *complete* if every accessible point is also nearstandard, i.e. $\mathfrak{a}(A, Q) = \mathfrak{n}(A, Q)$.

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An indifference space $\langle A, Q\rangle$ is called *complete* if every accessible point is also nearstandard, i.e. $\mathfrak{a}(A, Q) = \mathfrak{n}(A, Q)$.

Theorem

An indifference space $\langle A, Q\rangle$ with basis V_Q is complete, if and only if uniform space $\langle A, V_Q\rangle$ is complete standardly. (It is for countable basis of uniformity iff every cauchy sequence converges in it.)

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An indifference space is condensed, if and only if it is complete and bounded.

Proof.

By the definitions, it is trivial. □

Theorem

Closed subspace of complete indifference space is complete.

Proof.

Let $\langle A, Q\rangle$ be complete indifference space with basis V, $B \subseteq A$ standard and closed. Let $x \in B$ be an accessible in $\langle B, Q \cap B^2\rangle$. Then it is also accessible in $\langle A, Q\rangle$, so it is nearstandard to some standard point in A, which lies also in B, because B is closed. □

Thank you for attention