Permutation groups II

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When studying mappings, it is often useful to divide the pre-image of the mapping into smaller parts and then study how the mapping acts on these fragments. Another method is finding connection to some other, better known mapping. In this talk we shall follow both of these methods for permutation groups.

The first approach uses *blocks*, which turn out to be useful when designing algorithms (for example Luks' graph isomorphism algorithm). For the second there are two definitions of permutation group "similarity": *permutation isomorphism* and *equivalence*. These definitions seem very alike, but in fact define two different things, as will be shown in the talk.

1 Blocks

Definition 1. Let G be a group acting on a set Ω. A *block* for G is a subset $\Delta \subseteq \Omega$ such that for all $x \in G$:

$$\Delta^x = \Delta \text{ or } \Delta^x \cap \Delta = \emptyset.$$

Lemma 2. *Each block Δ for a group G acting on Ω is a union of orbits for G_α for some $\alpha \in \Delta$.*

2 Similarity

For this part we will need these two terms:

Definition 3. Let $G \leq \mathrm{Sym}(\Omega)$ and $H \leq \mathrm{Sym}(\Omega')$. Groups G and H are called *permutation isomorphic* if there exist two mappings λ and ψ such that:

- $\lambda\colon \Omega \to \Omega'$ is a bijection,
- $\psi\colon G \to H$ is a group isomorphism and
- $\lambda(\alpha^x) = \lambda(\alpha)^{\psi(x)}$ for all $\alpha \in \Omega$ and $x \in G$.

Definition 4. Two permutation representations $\rho\colon G \to \mathrm{Sym}(\Omega)$ and $\sigma\colon G \to \mathrm{Sym}(\Gamma)$ are *equivalent* if there exists a mapping λ such that:

- $\lambda\colon \Omega \to \Gamma$ is a bijection and
- $\lambda(\alpha^{\rho(x)}) = (\lambda(\alpha))^{\rho(x)}$.

We shall show the connection of blocks to permutation isomorphism. The essential fact is expressed in the following theorem.

Theorem 5. *Let G be a group acting transitively on a set Ω, and let H be a normal subgroup of G. Then:*

- *the orbits of H form a system of blocks for G;*

- *if* Δ *and* Δ' *are two* H*-orbits, then the restrictions of the action of* H *to* Δ *and* Δ' *are permutation isomorphic;*
- *if any point in* Ω *is fixed by all elements of* H*, then* H *lies in the kernel of the action on* Ω*;*
- *the group* H *has at most* $|G:H|$ *orbits, and if the index* $|G:H|$ *is finite, then the number of orbits of* H *divides* $|G:H|$*;*
- *if* G *acts primitively on* Ω*, then either* H *is transitive or* H *lies in the kernel of the action.*