Permutation groups III

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In the third talk of the cycle, we explore some more advanced topics in permutation group theory. We introduce Frobenius groups and k-transitive groups.

We also mention (but not prove) the structure theorem for finite Frobenius groups and a theorem classifying 8-transitive groups of finite order.

Definition 1. A group G is a *Frobenius group* if G is transitive, not regular and every non-trivial element of G has at most one fixed point.

Examples 2. The following groups are Frobenius groups:

(1) The group S_3,
(2) the dihedral group of order $2n$ for odd n,
(3) for a field F, the group of invertible affine transformations $x \mapsto ax + b$, $a \in F^\star, b \in F$,
(4) in the previous example, we can restrict a to some subgroup $U \leq F^\star$.

Lemma 3. *Let G be a finite Frobenius group acting on an n-point set. Then $|G| = dn$ for some $d \mid n - 1$.*

Definition 4. For a Frobenius group G we define its *Frobenius kernel*

$$K = \{g \in G : |\mathrm{Fix}(g)| \neq 1\}.$$

Theorem 5 (Frobenius, Zassenhaus, Thompson)**.** *Let K be a Frobenius kernel of a finite Frobenius group G. Then:*

(i) K is a normal subgroup of G.
(ii) For each odd prime p, the Sylow p-subgroups of G_α are cyclic, and the Sylow 2-subgroups are either cyclic or quaternion. If G_α is not solvable, then it has exactly one composition factor, namely A_5.
(iii) K is a nilpotent group.

Definition 6. A permutation group G acting on Ω is said to be *k-transitive* if G acts transitively on k-point subsets of Ω.

Theorem 7. *Let $G \leq \mathrm{Sym}(\Omega)$ be an 8-transitive group of finite order. Then $G \geq Alt(\Omega)$.*