Hilbert's third problem: Decomposing polyhedra

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- Problem #3 of David Hilbert's famous 23 problems (1900)
- Based on 2 Gauss' letters from 1844 (published 1900)

Hilbert's third problem

Is it true or not? There are two tetrahedra of equal bases and equal altitudes which can in no way be split into congruent tetrahedra to form two polyhedra which themselves could be split up into congruent tetrahedra.

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Formulation of Hilbert's third problem today

Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces which can be reassembled to yield the second?

- Similar holds in plane geometry (Bolyai–Gerwien theorem)
- Hilbert expected that the answer to his question is "No."
- This problem appeared to be the easiest one out of all 23
- Solved already in 1900 by Hilbert's student Max Dehn
  - Reworked and redone by M. Dehn (1902), H. Hadwiger (1954) and V.G. Boltianskii (1978) to a present scheme

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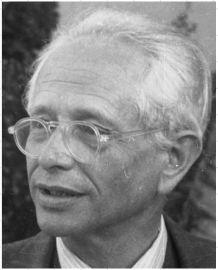

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- $\mathbb{Q}$-linear functions
- Equidecomposability and equicomplementability

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Definition

For a finite set of real numbers $M = \{m_1, \ldots, m_k\} \subseteq \mathbb{R}$, we define

$$V(M) := \left\{ \sum_{i=1}^{k} q_i m_i : q_i \in \mathbb{Q} \right\} \subseteq \mathbb{R}$$

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Properties of $V(M)$:

- $V(M)$ is a finite dimensional vector space over the field $\mathbb{Q}$
- For the dimension, it holds: $\dim_{\mathbb{Q}} V(M) \leq k = |M|$

Definition

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Key property (for every $q_i \in \mathbb{Q}$):

$$\sum_{i=1}^{k} q_i m_i = 0 \implies \sum_{i=1}^{k} q_i f(m_i) = f\left(\sum_{i=1}^{k} q_i m_i\right) = f(0) = 0$$

Lemma

For any finite subsets $M \subseteq M'$ of $\mathbb{R}$, the $\mathbb{Q}$-vector space $V(M)$ is a subspace of the $\mathbb{Q}$-vector space $V(M')$.
Thus if $f : V(M) \to \mathbb{Q}$ is a $\mathbb{Q}$-linear function, then f can be extended to a $\mathbb{Q}$-linear function $f' : V(M') \to \mathbb{Q}$ so that $f'(m) = f(m)$ for all $m \in M$.

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Proof.

Homework :-) $\square$

Let P and Q be two polyhedra.

Definition

We call P and Q *equidecomposable* iff they can be decomposed into finite sets of polyhedra $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ such that P_i and Q_i are congruent for all $i \in \{1, \ldots, n\}$.

Definition

We call P and Q *equicomplementable* iff there are polyhedra $P_1, \ldots, P_m$ and $Q_1, \ldots, Q_m$ so that the interiors of the P_i are disjoint from each other and from P (and similarly for Q_i and Q), such that P_i is for all $i \in \{1, \ldots, n\}$ congruent to Q_i and such that $\tilde{P} := P \cup P_1, \cup \ldots, \cup P_m$ and $\tilde{Q} := Q \cup Q_1, \cup \ldots, \cup Q_m$ are equidecomposable.

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Obviously: to be equidecomposable $\Rightarrow$ to be equicomplementable

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Definition

Let P be a 3–dimensional polyhedron and M_P the set of all angles between adjacent its facets (dihedral angles), together with the number π. Considering any finite set $M \subseteq \mathbb{R}$, which contains M_P, and any $\mathbb{Q}$-linear function $f : V(M) \to \mathbb{Q}$, that satisfies $f(\pi) = 0$, we define the *Dehn invariant* of P with respect to f to be

$$D_f(P) := \sum_{e \in P} \ell(e) f(\alpha(e)),$$

where e stands for an edge of the polyhedron, $\ell(e)$ denotes its length and $\alpha(e)$ is the angle between the two facets that meet in e.

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Theorem (Dehn–Hadwiger, 1949)

Let P and Q be polyhedra with dihedral angles $\alpha_1, \dots, \alpha_p$, resp. $\beta_1, \dots, \beta_q$ at their edges, and let M be a finite set of real numbers with $\{\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q, \pi\} \subseteq M$.
If $f : V(M) \to \mathbb{Q}$ is any $\mathbb{Q}$-linear function with $f(\pi) = 0$ such that

$$D_f(P) \neq D_f(Q),$$

then P and Q are not equicomplementable.

Proof.

Let a polyhedron P have a decomposition into polyhedral pieces $P_1, \ldots, P_n$ with edges $e_1, \ldots, e_m$ and dihedral angles (from now on only angles) $\alpha_1, \ldots, \alpha_m \in M'$ for a finite set M' which holds $M \subseteq M'$.
Let f is a restriction of $\mathbb{Q}$-linear function $f' : V(M') \to \mathbb{Q}$. Such f' exists due to a before-stated lemma.
Then we define a *mass* to a part of any edge $\widetilde{e} \subseteq e_i$ as

$$m_{f'}(\widetilde{e}) := \ell(\widetilde{e}) f'(\alpha(\widetilde{e})).$$

Proof cont. 1.

$$m_{f'}(\widetilde{e}) := \ell(\widetilde{e}) f'(\alpha(\widetilde{e}))$$

- Along the edges of P, the angles of the pieces add up to the angle of P. Hence the masses are also equal.
- Inside a face of P, the angles of the pieces add up to π. Considering linearity of f, constant length of the edge and a property $f'(\pi) = 0$, we get that the mass of any such edge equals to 0.
- In the interior of P, the angles of the pieces add up to 2π. Using $f'(2\pi) = 2f'(\pi) = 0$, the mass of the edge is again 0.

Proof cont. 2.

From the previous, we get

$$D_{f'}(P) = D_{f'}(P_1) + \ldots + D_{f'}(P_n).$$

Now, for a contradiction, assume that P and Q are equicomplementable. Then

$$D_{f'}(P)+D_{f'}(P_1)+\ldots+D_{f'}(P_k) = D_{f'}(Q)+D_{f'}(Q_1)+\ldots+D_{f'}(Q_k),$$

where $D_{f'}(P_i) = D_{f'}(Q_i)$ since P_i and Q_i are congruent.
Thus $D_{f'}(P) = D_{f'}(Q)$, which is a contradiction.
Hence P and Q are not equicomplementable. □

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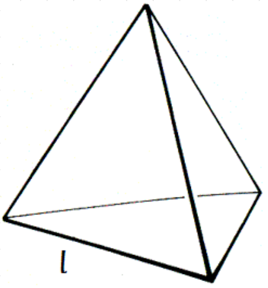
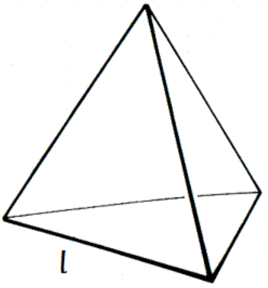
Regular tetrahedron P

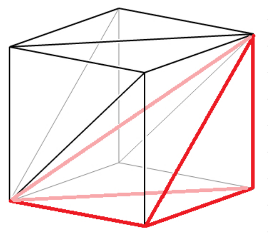


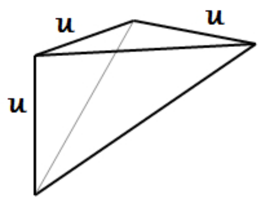
Regular tetrahedron P



$$D_f(P) = f\left(\arccos\frac{1}{3}\right) 6l \neq 0$$

Schläfli orthoscheme Q





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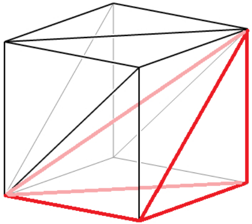

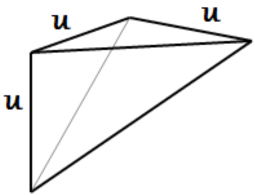

$$D_f(Q) = 0$$

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Last two examples showed us that there exist 2 tetrahedra of equal bases and equal altitudes which can in no way be split into congruent tetrahedra and which meet Gauss' description.
Thus the answer to Hilbert's third problem is:

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Thus the answer to Hilbert's third problem is:

No

Questions, comments?

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Comments of comments?

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Thank you for attention! :-)