Permutation groups

Pepa Dvořák

March 24, 2012

History, pioneers

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- Alexandre-Théophile Vandermonde (1735-1796)
- Joseph-Louis Lagrange (1736–1813)
- Paolo Ruffini (1765–1822)
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History, E. Galois

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- discovered the notion of *normal subgroup*
- "some men will find it profitable to sort out this mess"

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Definitions, which all of us know

Let Ω be a nonempty set. A *permutation* is any bijection α of the set Ω on itself, the *symmetric group on* Ω is the set of **all** permutations of Ω together with the operation of composition of mappings, a *permutation group* is just a **subgroup** of a symmetric group.

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Group actions

Let G be a group and Ω a nonempty set. We say that G *acts on* Ω if for each $\alpha \in \Omega$ and each $g \in G$ we have an element $\alpha^g \in \Omega$ and we have also

- $\alpha^1 = \alpha$ for each $\alpha \in \Omega$ (where 1 is the nautral element of G);
- $(\alpha^g)^h = \alpha^{gh}$ for all $g, h \in G$ and all $\alpha \in \Omega$.

Examples

- symmteries of cube
- G on itself
- group of bijections on V acts on the set of functions $x : V \to W$ by $(gx)(v) = x\left(g^{-1}(v)\right)$, $g \in G$

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Group action as a homomorphism

If a group G acts on a set Ω, then to each element $g \in G$ we can associate a mapping from $Sym(\Omega)$, namely $\alpha \mapsto \alpha^g$. We get therefore a group homomorphism $G \to Sym(\Omega)$.
We call such (and in fact any of this kind) homomorphism a *representation of G on* Ω. The *degree* of an action is the cardinality of Ω. The *kernel* of a an action is the kernel of the corresponding represenation. The action is *faithful*, if $ker = 1$

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Some well known facts

Theorem (Cayley representation (1854))

Every group is isomorphic to a subgroup of a suitable symmetric group.

Example (Action on right cosets)

Let $H \leq G$ and define $\Gamma_H := \{Hg \,|\, g \in G\}$. We can define an action ρ_H of G on Γ_H by putting $(Hg)^h := Hgh$. We get the following results:

- $ker \rho_H$ is the largest normal subgroup of G containing H.
- If H is of finite index in G, then G has a normal subgroup K contained in H whose index in G divides $n!$ (i.e. $K \trianglelefteq H$ and $[K : G]\ n!$).

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Number theory application

Theorem (Euler, 1747)

Every prime p of the form $4n+1$ is a sum of two squares.

Proof.

Discuss the properties of the following mapping:

$$(x,y,z) \mapsto \begin{cases} (x+2z, z, y-x-z) & \text{if } x < y-z; \\ (2y-x, y, x-y+z) & \text{if } y-z < x < 2y; \\ (x-2y, x-y+z, y) & \text{if } x > 2y. \end{cases}$$

on the set $\Omega = \{(x,y,z) \in \mathbb{N}^3 \mid x^2 + 4yz = p\}$. □

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Orbits and stabilizers

If group G acts on the set Ω, we call the set $\alpha^G := \{a^g \mid g \in G\}$ the *orbit of α under G* and the set $G_\alpha := \{g \in G \mid \alpha^g = \alpha\}$ the *stabilizer of α under G*. The most important properties are summed up in the following theorem.

Theorem

Suppose the group G acts on the set Ω, $g, h \in G$ and $\alpha, \beta \in \Omega$. Then

1. *the orbits α^G and β^G are either disjoint or equal;*
2. *the stabilizer G_α is a subgroup of G and $G_\alpha = g^{-1} G_\alpha g$ whenever $\beta = \alpha^g$. Moreover, $\alpha^g = \alpha^h \Leftrightarrow G_\alpha g = G_\alpha h$;*
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Theoretical applications

Theorem (Burnside)

Let G be a finite group that acts on a set X. For each $g \in G$ let X_g denote the set of elements in X that are fixed by g. Then the number of orbits is equal to

$$\frac{1}{|G|} \sum_{g \in G} X_g .$$

Theorem

Let G be a finite p-group. Then the center $Z(G) := \{g \in G \,|\, g \text{ commutes with each } h \in G\}$ is non-trivial.

Theorem

Let G be a finite p-group and H its non-trivial normal subgroup. Then $H \cap Z(G)$ is non-trivial.

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Transitive action

A group acting on a set Ω is said to be *transitive on* Ω if the action has only one orbit, otherwise it is said to be *intransitive*.

A group G acting transitively on Ω is said to act regularly, if $G_\alpha = 1$ for each $\alpha \in \Omega$.

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Theorem

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Suppose G acts transitively on a finite set Ω and let $\Gamma \subseteq \Omega$. Then G acts "evenly", i.e. all $\alpha \in \Omega$ are in the same number of sets Γ^g, $g \in G$.

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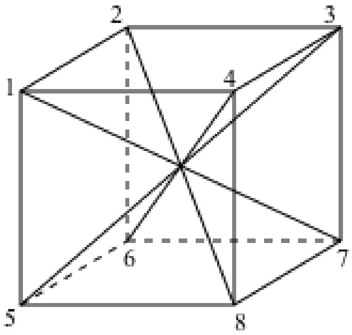
Examples - cube

The group of symmetries of cube..



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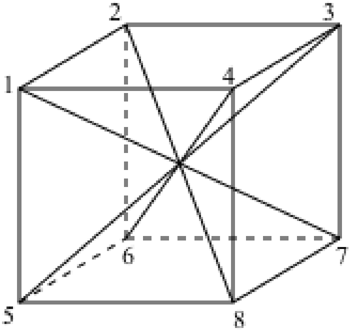

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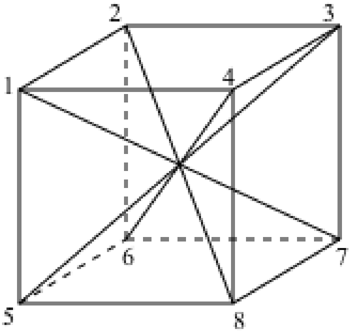

Examples - octahedron

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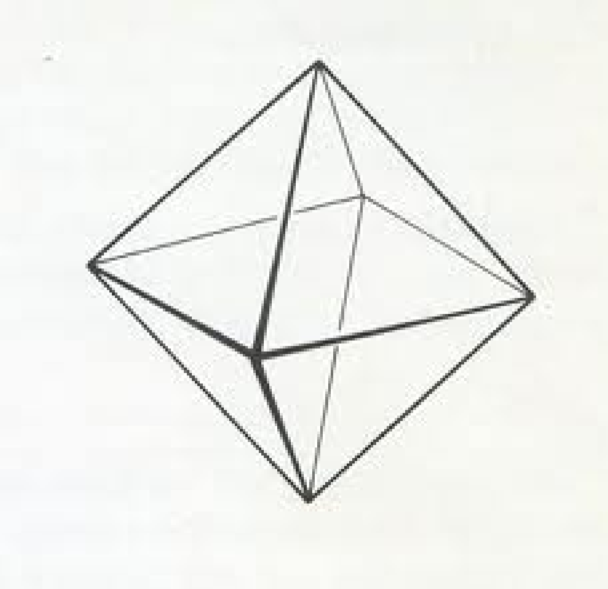

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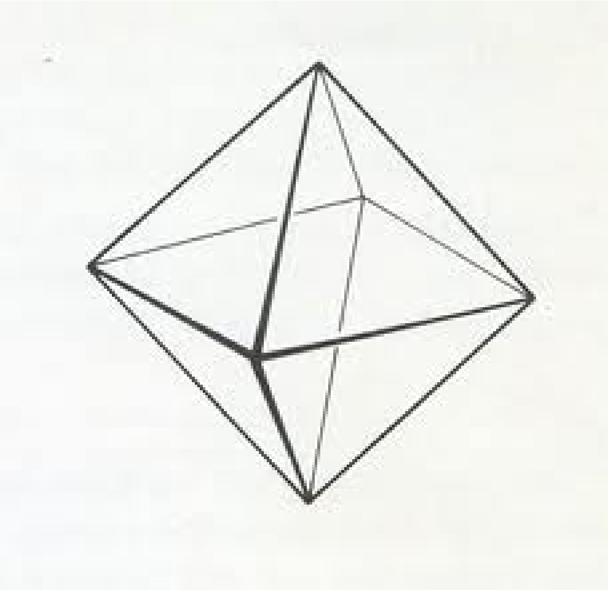
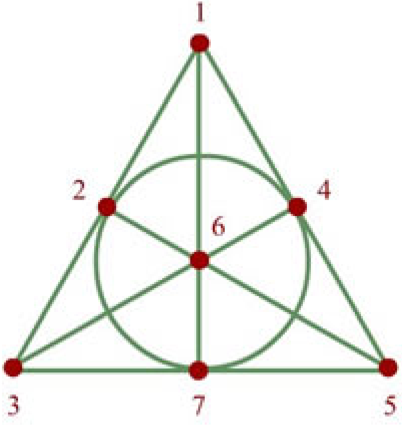


Examples - Fano plane

And what about the group of automorphisms of Fano's plane?



(an automorphism of Fano plane is a permutation of vertices which preserves collinearity)

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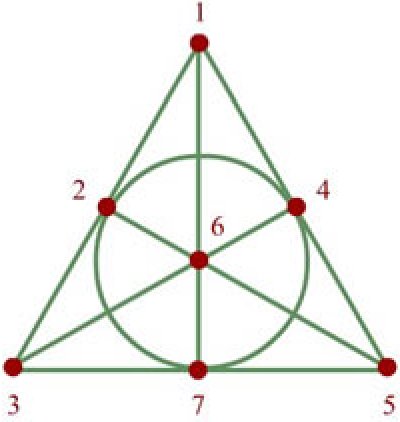

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Thank you.