Non-standard Methods II – Non-standard Analysis

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Non-standard methods introduced in the previous part have large usage in calculus. They allow us to work in accordance with our intuition and we can simplify proofs of main theorems in real and complex analysis. Moreover non-standard methods reveal some new phenomena non-accessible via the standard approach to the subject of analysis.

We prove elementary theorems about *local extremes* and *continuousity* using non-standard calculus instead of (ε, δ)-approach. Concepts of *stability*, *pointwise stability* and also the idea of *standard track of a function* will be shown on the example of the derivation.

Definition 1. From the previous part:

- $\mathbf{Q} = {}^*\mathbb{Q}$, $\mathbf{R} = {}^*\mathbb{R}$
- ${}^\sigma X = X \cap S$ for every set X
- $x \sim y \iff (\forall n \in \mathbb{N})\, |x - y| < 2^{-n}$
- $[x]_\sim = \{y \,|\, y \sim x\}$ is the *monad of* x

Definition 2. Let x, $y \in \mathbf{R}$. Then we define *indifference* $\doteq$ this way:

$$x \doteq y \iff (x, y \in \mathbf{BR} \text{ and } x \sim y) \vee (x, y \notin \mathbf{BR} \text{ and } x \cdot y > 0).$$

Let $\infty, -\infty \notin \mathbf{R}$ be two standard elements. Then the *extended real numbers* is the set $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty, -\infty\}$.

Definition 3. Let $F\colon \mathbf{R} \to \mathbf{R}$ be a function, $X \subseteq \mathrm{dom}(F)$. Then F is *pointwise stabilized for* X, if for every standard $x \in X$ there exists $y \in \mathbf{BR}$ which *stabilizes* F *in* x, it means that $F''[x]_\sim \subseteq [y]_\sim$. If for every $x \in X$ there exists $y \in \mathbf{BR}$, which stabilizes F in x, we say that F *is stabilized for* X.

Definition 4. Let $F\colon \mathbf{R} \to \mathbf{R}$ be pointwise stabilized for X. Then there exists unique standard function f such that for every standard $a \in {}^\sigma X$ it is $F''[a]_\sim \subseteq [f(a)]_\sim$. We call f the *standard trace of* F *on* X.

Theorem 5 (Standard trace of derivation)**.** *Let $I \subseteq \mathbf{R}$ be a standard bounded interval, $F\colon I \to \mathbf{R}$ an internal function, which has a value from $\mathbf{BR}$ in some point from $[{}^\sigma I]_{\doteq}$. Let F' be pointwise stabilized for I. Then*

(1) F is pointwise stabilized for I and so it has a standard trace f on I which is continuous.
(2) $f'(a) \doteq F'(a)$ for every standard $a \in I$.
(3) f' is standard trace of F' on I and f' is continuous.