Non-standard Analysis

Petra Kuřinová

MFF UK

March 22, 2012

We have bounded rational numbers:

$$\mathbf{BQ} = \{x \in \mathbf{Q}; \exists n \in \mathbf{N} : |x| < n\}$$

We have bounded rational numbers:

$$\mathbf{BQ} = \{x \in \mathbf{Q}; \exists n \in \mathbf{N} : |x| < n\}$$

and an indifferation $\sim$ on rational numbers:

$$x \sim y \iff \forall n \in \mathbb{N} : |x - y| < 2^{-n}$$

which is compatible with arithmetical operations $+, -, ., /$, and also with $|.|$ and relation $<$

We have bounded rational numbers:

$$\mathbf{BQ} = \{x \in \mathbf{Q}; \exists n \in \mathbf{N} : |x| < n\}$$

and an indifferation $\sim$ on rational numbers:

$$x \sim y \iff \forall n \in \mathbb{N} : |x - y| < 2^{-n}$$

which is compatible with arithmetical operations $+$, $-$, $.$, $/$, and also with $|.|$ and relation $<$
(for unary operation f it means $x \sim x'$ then $f(x) \sim f(x')$).

We have bounded rational numbers:

$$\mathbf{BQ} = \{x \in \mathbf{Q}; \exists n \in \mathbf{N} : |x| < n\}$$

and an indifferation $\sim$ on rational numbers:

$$x \sim y \iff \forall n \in \mathbb{N} : |x - y| < 2^{-n}$$

which is compatible with arithmetical operations $+, -, ., /,$ and also with $|.|$ and relation $<$
(for unary operation f it means $x \sim x'$ then $f(x) \sim f(x')$).
Then it is possible to make the quotient ring

$$\mathcal{R} = (\mathbf{BQ}, +, -, ., 0, 1, <)/ \sim$$

We have bounded rational numbers:

$$\mathbf{BQ} = \{x \in \mathbf{Q}; \exists n \in \mathbf{N} : |x| < n\}$$

and an indifferation $\sim$ on rational numbers:

$$x \sim y \iff \forall n \in \mathbb{N} : |x - y| < 2^{-n}$$

which is compatible with arithmetical operations $+, -, ., /$, and also with $|.|$ and relation $<$
(for unary operation f it means $x \sim x'$ then $f(x) \sim f(x')$).
Then it is possible to make the quotient ring

$$\mathcal{R} = (\mathbf{BQ}, +, -, ., 0, 1, <)/\sim$$

We can prove that $\mathcal{R}$ is the field of real numbers, therefore we label it $\mathbb{R}$.

Theorem

The structure $\mathcal{R} = (\mathbf{BQ}, +, -, ., 0, 1, <)/\sim$ is a complete ordered field and contains $\mathcal{Q} = \{[a]_\sim; a \in \mathbb{Q}\}$ as a dense subfield. Moreover $\mathcal{Q}$ is isomorphic to $\mathbb{Q}$.

Theorem

The structure $\mathcal{R} = (\mathbf{BQ}, +, -, ., 0, 1, <)/\sim$ is a complete ordered field and contains $\mathcal{Q} = \{[a]_\sim; a \in \mathbb{Q}\}$ as a dense subfield. Moreover $\mathcal{Q}$ is isomorphic to $\mathbb{Q}$.

Idea of the proof

- Field: $\mathcal{R}$ satisfies the field axioms as commutativity, distributivity and associativity, $[0]_\sim$ is the unit for $+$, $[1]_\sim$ the unit for $\cdot$, ...

Theorem

The structure $\mathcal{R} = (\mathbf{BQ}, +, -, ., 0, 1, <)/\sim$ is a complete ordered field and contains $\mathcal{Q} = \{[a]_\sim; a \in \mathbb{Q}\}$ as a dense subfield. Moreover $\mathcal{Q}$ is isomorphic to $\mathbb{Q}$.

Idea of the proof

- Field: $\mathcal{R}$ satisfies the field axioms as commutativity, distributivity and associativity, $[0]_\sim$ is the unit for $+$, $[1]_\sim$ the unit for $\cdot$, ...
- Ordering: $[x]_\sim < [y]_\sim \iff x \not\sim y$ and $x < y$

Theorem

The structure $\mathcal{R} = (\mathbf{BQ}, +, -, ., 0, 1, <)/\sim$ is a complete ordered field and contains $\mathcal{Q} = \{[a]_\sim; a \in \mathbb{Q}\}$ as a dense subfield. Moreover $\mathcal{Q}$ is isomorphic to $\mathbb{Q}$.

Idea of the proof

- Field: $\mathcal{R}$ satisfies the field axioms as commutativity, distributivity and associativity, $[0]_\sim$ is the unit for $+$, $[1]_\sim$ the unit for $\cdot$, ...
- Ordering: $[x]_\sim < [y]_\sim \iff x \not\sim y$ and $x < y$
- Density of $\mathcal{Q}$: Prove: $x_0, x_1 \in \mathbf{BQ}$, $x_0 \not\sim x_1$, then there exists $r \in \mathbb{Q}$ such that $x_0 < r < x_1$ and $x_0 \not\sim r \not\sim x_1$.

Theorem

The structure $\mathcal{R} = (\mathbf{BQ}, +, -, ., 0, 1, <)/\sim$ is a complete ordered field and contains $\mathcal{Q} = \{[a]_\sim; a \in \mathbb{Q}\}$ as a dense subfield. Moreover $\mathcal{Q}$ is isomorphic to $\mathbb{Q}$.

Idea of the proof

- Field: $\mathcal{R}$ satisfies the field axioms as commutativity, distributivity and associativity, $[0]_\sim$ is the unit for $+$, $[1]_\sim$ the unit for $\cdot$, ...
- Ordering: $[x]_\sim < [y]_\sim \iff x \not\sim y$ and $x < y$
- Density of $\mathcal{Q}$: Prove: $x_0, x_1 \in \mathbf{BQ}$, $x_0 \not\sim x_1$, then there exists $r \in \mathbb{Q}$ such that $x_0 < r < x_1$ and $x_0 \not\sim r \not\sim x_1$.
- Completeness: We prove that every nonempty above bounded $X \subseteq \mathbf{BQ}/\sim$ has a supremum using infinitely many intervals and ω-saturation.

Denote $\mathbf{R} = {}^*\mathbb{R}$, $\mathbf{BR} = \{x \in \mathbf{R}; \exists n \in \mathbf{N} : |x| < n\}$.

Denote $\mathbf{R} = {}^*\mathbb{R}$, $\mathbf{BR} = \{x \in \mathbf{R}; \exists n \in \mathbf{N} : |x| < n\}$.
On standard real numbers we define an indifference $\doteq$ as

$$x \doteq y \iff (x, y \in \mathbf{BR} \& x \sim y) \vee (x, y \notin \mathbf{BR} \& x \cdot y > 0).$$

Denote $\mathbf{R} = {}^*\mathbb{R}$, $\mathbf{BR} = \{x \in \mathbf{R}; \exists n \in \mathbf{N} : |x| < n\}$.
On standard real numbers we define an indifference $\doteq$ as

$$x \doteq y \iff (x, y \in \mathbf{BR} \& x \sim y) \vee (x, y \notin \mathbf{BR} \& x \cdot y > 0).$$

It is equal to $\sim$ on bounded real numbers and infinite instances reduce each to one monad.

Denote $\mathbf{R} = {}^*\mathbb{R}$, $\mathbf{BR} = \{x \in \mathbf{R}; \exists n \in \mathbf{N} : |x| < n\}$.
On standard real numbers we define an indifference $\doteq$ as

$$x \doteq y \iff (x, y \in \mathbf{BR} \& x \sim y) \vee (x, y \notin \mathbf{BR} \& x \cdot y > 0).$$

It is equal to $\sim$ on bounded real numbers and infinite instances reduce each to one monad.
Denote $+\infty$ and $-\infty$ different standard instances, which are not in $\mathbf{R}$. Extended real numbers is then the set

$$\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$$

The indifference $\doteq$ and relation $<$ can be extended naturally to $\overline{\mathbf{R}}$.

Denote $\mathbf{R} = {}^*\mathbb{R}$, $\mathbf{BR} = \{x \in \mathbf{R}; \exists n \in \mathbf{N} : |x| < n\}$.
On standard real numbers we define an indifference $\doteq$ as

$$x \doteq y \iff (x, y \in \mathbf{BR} \& x \sim y) \vee (x, y \notin \mathbf{BR} \& x \cdot y > 0).$$

It is equal to $\sim$ on bounded real numbers and infinite instances reduce each to one monad.
Denote $+\infty$ and $-\infty$ different standard instances, which are not in $\mathbf{R}$. Extended real numbers is then the set

$$\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$$

The indifference $\doteq$ and relation $<$ can be extended naturally to $\overline{\mathbf{R}}$.
Extension of arithmetical operations follows from requirement of stability under $\doteq$ (for $x \in \mathbf{R}$):

$$(\pm\infty) + (\pm\infty) = x + (\pm\infty) = (\pm\infty) + x = \pm\infty$$

$$\ldots$$

Non-standard Analysis

Petra Kuřinová

Real numbers

Extremes and continuosity

Non-standard enlargement

Definitions

- Let $f : \mathbf{R} \to \mathbf{R}$ be a function, $a \in \mathbf{R}$ a standard point.

Definitions

- Let $f : \mathbf{R} \to \mathbf{R}$ be a function, $a \in \mathbf{R}$ a standard point.
- We say f *is continuous*, if for every standard $a \in \mathrm{dom}(f)$, $f[a]_{\doteq} \subseteq [f(a)]_{\doteq}$.

Definitions

- Let $f : \mathbf{R} \to \mathbf{R}$ be a function, $a \in \mathbf{R}$ a standard point.
- We say f *is continuous*, if for every standard $a \in \mathrm{dom}(f)$, $f[a]_{\doteq} \subseteq [f(a)]_{\doteq}$.
- The function f is *increasing*, resp. *decreasing*, at a, if $[a]_{\doteq} \subseteq \mathrm{dom}(f)$ and for every $x, y \in [a]_{\doteq}$, $x < a < y$, $f(x) < f(a) < f(y)$, resp. $f(x) > f(a) > f(y)$.

Definitions

- Let $f : \mathbf{R} \to \mathbf{R}$ be a function, $a \in \mathbf{R}$ a standard point.
- We say f *is continuous*, if for every standard $a \in \mathrm{dom}(f)$, $f[a]_{\doteq} \subseteq [f(a)]_{\doteq}$.
- The function f is *increasing*, resp. *decreasing*, at a, if $[a]_{\doteq} \subseteq \mathrm{dom}(f)$ and for every $x, y \in [a]_{\doteq}$, $x < a < y$, $f(x) < f(a) < f(y)$, resp. $f(x) > f(a) > f(y)$.
- The function f has *a local maximum at* a, if $[a]_{\doteq} \subseteq \mathrm{dom}(f)$ and for every $x \in [a]_{\doteq}$, $f(x) \le f(a)$. Similarly we define *a local minimum.*

Definitions

- Let $f : \mathbf{R} \to \mathbf{R}$ be a function, $a \in \mathbf{R}$ a standard point.
- We say f *is continuous*, if for every standard $a \in \mathrm{dom}(f)$, $f[a]_{\doteq} \subseteq [f(a)]_{\doteq}$.
- The function f is *increasing*, resp. *decreasing*, at a, if $[a]_{\doteq} \subseteq \mathrm{dom}(f)$ and for every $x, y \in [a]_{\doteq}$, $x < a < y$, $f(x) < f(a) < f(y)$, resp. $f(x) > f(a) > f(y)$.
- The function f has *a local maximum at a*, if $[a]_{\doteq} \subseteq \mathrm{dom}(f)$ and for every $x \in [a]_{\doteq}$, $f(x) \leq f(a)$. Similarly we define *a local minimum.*
- A standard function $f : \mathbf{R} \to \mathbf{R}$ has a *derivative* $b \in \overline{\mathbf{R}}$ at a standard point $a \in \mathrm{dom}(f)$, if $[a]_{\sim} \cap \mathrm{dom}(f) \neq \{a\}$ and $a \neq x \in [a]_{\sim} \implies \frac{f(a)-f(x)}{a-x} \doteq b$. Then we write $f'(a) = b$. Similarly we define oneside derivatives. Equivalent definition: $f'(a) = \frac{f(a+\Delta)-f(a)}{\Delta}$ for $\Delta \doteq 0$.

Theorem [Continuous function on closed interval]

Let $a < b$ be real, $f : [a, b] \to \mathbf{R}$ continuous function. Then f acquires its minimum M_0 and maximum M_1 and f maps onto $[M_0, M_1]$.

Theorem [Continuous function on closed interval]

Let $a < b$ be real, $f : [a, b] \to \mathbf{R}$ continuous function. Then f acquires its minimum M_0 and maximum M_1 and f maps onto $[M_0, M_1]$.

Proof.

Let $K \in \mathbf{N}$, $K \doteq +\infty$, $\Delta = (b-a)/K$ and $x_i = a + i\Delta$ for every $0 \le i \le K$.

Theorem [Continuous function on closed interval]

Let $a < b$ be real, $f : [a, b] \to \mathbf{R}$ continuous function. Then f acquires its minimum M_0 and maximum M_1 and f maps onto $[M_0, M_1]$.

Proof.

Let $K \in \mathbf{N}$, $K \doteq +\infty$, $\Delta = (b-a)/K$ and $x_i = a + i\Delta$ for every $0 \leq i \leq K$.
Denote $y = \min\{f(x_i); i \leq K\}$ and fix i_0: $f(x_{i_0}) = y$.

Theorem [Continuous function on closed interval]

Let $a < b$ be real, $f : [a, b] \to \mathbf{R}$ continuous function. Then f acquires its minimum M_0 and maximum M_1 and f maps onto $[M_0, M_1]$.

Proof.

Let $K \in \mathbf{N}$, $K \doteq +\infty$, $\Delta = (b - a)/K$ and $x_i = a + i\Delta$ for every $0 \leq i \leq K$.
Denote $y = \min\{f(x_i); i \leq K\}$ and fix i_0: $f(x_{i_0}) = y$.
Then there exists a standard $c \sim x_{i_0}$, $c \in [a, b]$ and certainly $M_0 = f(c)$. Similarly we have $f(x_{i_1}) \doteq M_1$.

Theorem [Continuous function on closed interval]

Let $a < b$ be real, $f : [a, b] \to \mathbf{R}$ continuous function. Then f acquires its minimum M_0 and maximum M_1 and f maps onto $[M_0, M_1]$.

Proof.

Let $K \in \mathbf{N}$, $K \doteq +\infty$, $\Delta = (b-a)/K$ and $x_i = a + i\Delta$ for every $0 \leq i \leq K$.
Denote $y = \min\{f(x_i); i \leq K\}$ and fix i_0: $f(x_{i_0}) = y$.
Then there exists a standard $c \sim x_{i_0}$, $c \in [a, b]$ and certainly $M_0 = f(c)$. Similarly we have $f(x_{i_1}) \doteq M_1$.
WLOG $x_{i_0} < x_{i_1}$. If we have $M_0 < y < M_1$ a standard point, denote $j = \min\{i; i_0 < i < i_1, y \leq f(x_i)\}$. Then $f(x_{j-1}) < y \leq f(x_j)$, so from the continuosity: $f(x_j) \doteq y$.
So there must be a standard point $d \doteq x_j$ satisfying $f(d) = y$. □

Theorem [Local extrema]

Let $f : \mathbf{R} \to \mathbf{R}$ be a standard function, $a \in \mathbf{R}$.

1. If $f'(a) > 0$, respectively $f'(a) < 0$, then f is increasing, resp. decreasing, in a.
2. If $f'(a) \neq 0$ then at a is no local extremum of f.

Theorem [Local extrema]

Let $f : \mathbf{R} \to \mathbf{R}$ be a standard function, $a \in \mathbf{R}$.

1. If $f'(a) > 0$, respectively $f'(a) < 0$, then f is increasing, resp. decreasing, in a.
2. If $f'(a) \neq 0$ then at a is no local extremum of f.

Proof.

1. Let $f'(a) > 0$ and $a < x_1 \doteq a$.

Theorem [Local extrema]

Let $f : \mathbf{R} \to \mathbf{R}$ be a standard function, $a \in \mathbf{R}$.

1. If $f'(a) > 0$, respectively $f'(a) < 0$, then f is increasing, resp. decreasing, in a.
2. If $f'(a) \neq 0$ then at a is no local extremum of f.

Proof.

1. Let $f'(a) > 0$ and $a < x_1 \doteq a$.
   Then $\frac{f(x_1)-f(a)}{x_1-a} \doteq f'(a)$ implies
   $f(x_1) - f(a) \doteq f'(a)(x_1 - a) > 0$, and so $f(x_1) > f(a)$.

Non-standard Analysis

Petra Kuřinová

Real numbers

Extremes and continuosity

Non-standard enlargement

Theorem [Local extrema]

Let $f : \mathbf{R} \to \mathbf{R}$ be a standard function, $a \in \mathbf{R}$.

1. If $f'(a) > 0$, respectively $f'(a) < 0$, then f is increasing, resp. decreasing, in a.
2. If $f'(a) \neq 0$ then at a is no local extremum of f.

Proof.

1. Let $f'(a) > 0$ and $a < x_1 \doteq a$.
   Then $\frac{f(x_1)-f(a)}{x_1-a} = f'(a)$ implies
   $f(x_1) - f(a) = f'(a)(x_1 - a) > 0$, and so $f(x_1) > f(a)$.
   Similarly $a \doteq x_2 < a$ implies $f(x_2) < f(a)$ and thus f is increasing at a. Analogically for $f'(a) < 0$.

Theorem [Local extrema]

Let $f : \mathbf{R} \to \mathbf{R}$ be a standard function, $a \in \mathbf{R}$.

1. If $f'(a) > 0$, respectively $f'(a) < 0$, then f is increasing, resp. decreasing, in a.
2. If $f'(a) \neq 0$ then at a is no local extremum of f.

Proof.

1. Let $f'(a) > 0$ and $a < x_1 \doteq a$.
   Then $\frac{f(x_1)-f(a)}{x_1-a} = f'(a)$ implies
   $f(x_1) - f(a) = f'(a)(x_1 - a) > 0$, and so $f(x_1) > f(a)$.
   Similarly $a \doteq x_2 < a$ implies $f(x_2) < f(a)$ and thus f is increasing at a. Analogically for $f'(a) < 0$.
2. Immediately from 1).

$\square$

Some other helpful Lemmas:

Lemma

Let $f : \mathbf{R} \rightarrow \mathbf{R}$, $a \in \mathrm{dom}(f)$ a standard point.

1. If there exists finite $f'(a)$, then f is continuous in a.

Some other helpful Lemmas:

Lemma

Let $f : \mathbf{R} \to \mathbf{R}$, $a \in \mathrm{dom}(f)$ a standard point.

1. If there exists finite $f'(a)$, then f is continuous in a.
2. $f'(a) = b \iff$ standardly: $\forall U$ neighbourhood of b in $\overline{\mathbf{R}}$ there exists $\delta > 0$ $\forall a \neq x \in \mathrm{dom}(f) : |a - x| < \delta \implies \frac{f(a)-f(x)}{a-x} \in U$.

Some other helpful Lemmas:

Lemma

Let $f : \mathbf{R} \to \mathbf{R}$, $a \in \mathrm{dom}(f)$ a standard point.

1. If there exists finite $f'(a)$, then f is continuous in a.
2. $f'(a) = b \iff$ standardly: $\forall U$ neighbourhood of b in $\overline{\mathbf{R}}$ there exists $\delta > 0$ $\forall a \neq x \in \mathrm{dom}(f) : |a - x| < \delta \implies \frac{f(a)-f(x)}{a-x} \in U$.

Lemma [Mean value theorem]

For real $a < b$ and a continuous function $f : [a, b] \to \mathbf{R}$, which has a derivative in each $x \in (a, b)$ there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Some typical non-standard definitions

Non-standard Analysis
Petra Kuřinová
Real numbers
Extremes and continuosity
Non-standard enlargement

Some typical non-standard definitions

- Let $F : \mathbf{R} \to \mathbf{R}$ be a function, $X \subseteq \mathrm{dom}(F)$. Then F is *pointwise stabilized for* X, if for every standard $x \in X$ there exists $y \in \mathbf{BR}$ which *stabilizes* F *in* x, it means that $F[x]_\sim \subseteq [y]_\sim$.

Some typical non-standard definitions

- Let $F : \mathbf{R} \to \mathbf{R}$ be a function, $X \subseteq \mathrm{dom}(F)$. Then F is *pointwise stabilized for* X, if for every standard $x \in X$ there exists $y \in \mathbf{BR}$ which *stabilizes* F *in* x, it means that $F[x]_\sim \subseteq [y]_\sim$.
  If for every $x \in X$ there exists $y \in \mathbf{BR}$, which stabilizes F in x, we say that F *is stabilized for* X.

Some typical non-standard definitions

- Let $F : \mathbf{R} \to \mathbf{R}$ be a function, $X \subseteq \mathrm{dom}(F)$. Then F is *pointwise stabilized for* X, if for every standard $x \in X$ there exists $y \in \mathbf{BR}$ which *stabilizes* F *in* x, it means that $F[x]_\sim \subseteq [y]_\sim$.
  If for every $x \in X$ there exists $y \in \mathbf{BR}$, which stabilizes F in x, we say that F *is stabilized for* X.
- Let $F : \mathbf{R} \to \mathbf{R}$ be a pointwise stabilized function for $\mathrm{dom}(F)$. Then there exists an unique standard function f defined on $\mathrm{dom}(F)$ such that for every standard $a \in \mathrm{dom}(f)$, $F[a]_\sim = [f(a)]_\sim$. We call f *the standard trace of* F.

Some typical non-standard definitions

- Let $F : \mathbf{R} \to \mathbf{R}$ be a function, $X \subseteq \mathrm{dom}(F)$. Then F is *pointwise stabilized for* X, if for every standard $x \in X$ there exists $y \in \mathbf{BR}$ which *stabilizes* F *in* x, it means that $F[x]_\sim \subseteq [y]_\sim$.
  If for every $x \in X$ there exists $y \in \mathbf{BR}$, which stabilizes F in x, we say that F *is stabilized for* X.
- Let $F : \mathbf{R} \to \mathbf{R}$ be a pointwise stabilized function for $\mathrm{dom}(F)$. Then there exists an unique standard function f defined on $\mathrm{dom}(F)$ such that for every standard $a \in \mathrm{dom}(f)$, $F[a]_\sim = [f(a)]_\sim$. We call f *the standard trace of* F.

Lemma

The standard trace of pointwise stabilized function is continuous.

Theorem [Standard trace of the derivative]

Let $I \subseteq \mathbf{R}$ be a standard bounded interval, $F : I \to \mathbf{R}$ an internal function, which has a value from $\mathbf{BR}$ in some point from $[{}^{\sigma}I]_{\doteq}$. Let F' be pointwise stabilized for I. Then

1. F is pointwise stabilized for I and so it has a standard trace f on I which is continuous.
2. $f'(a) \doteq F'(a)$ for every standard $a \in I$.
3. f' is standard trace of F' on I and f' is continuous.

Theorem [Standard trace of the derivative]

Let $I \subseteq \mathbf{R}$ be a standard bounded interval, $F : I \to \mathbf{R}$ an internal function, which has a value from $\mathbf{BR}$ in some point from $[^{\sigma}I]_{\doteq}$. Let F' be pointwise stabilized for I. Then

1. F is pointwise stabilized for I and so it has a standard trace f on I which is continuous.
2. $f'(a) \doteq F'(a)$ for every standard $a \in I$.
3. f' is standard trace of F' on I and f' is continuous.

Proof.

1) Denote $x \in [^{\sigma}I]_{\doteq}$ such that $F(x) \in \mathbf{BR}$ and let $a \in I$ be some different standard point, $a < x \doteq a$. The function F is internally continuous on $[a, x]$, because it has a derivative on I, so from the mean value theorem there exists $y \in (a, x)$ such that $F(x) - F(a) = F'(y)(x - a)$. Due to $F'(y) \in \mathbf{BR}$ it is $F(x) \doteq F(a)$, so $F(a) \in \mathbf{BR}$. The rest from Lemmas. □

Theorem [Standard trace of the derivative]

Let $I \subseteq \mathbf{R}$ be a standard bounded interval, $F : I \to \mathbf{R}$ an internal function, which has a value from $\mathbf{BR}$ in some point from $[{}^{\sigma}I]_{\doteq}$. Let F' be pointwise stabilized for I. Then

1. F is pointwise stabilized for I and so it has a standard trace f on I which is continuous.
2. $f'(a) \doteq F'(a)$ for every standard $a \in I$.
3. f' is standard trace of F' on I and f' is continuous.

Proof.

2) We prove: $f'_+(a) \doteq F'(a)$ for $a \in I$ except the last point of I. There exists a standard point $x \in I$, $a < x$ and $f \doteq F$ on $[a, x]$, it means $|F(y) - f(y)| < \eta \doteq 0$ for every $y \in [a, x]$. Then for arbitrary $0 < \Delta \doteq 0$ such that $\eta/\Delta \doteq 0$ it is
$f'(a) = \frac{f(a+\Delta)-f(a)}{\Delta} = \frac{F(a+\Delta)+\eta_1-F(a)-\eta_2}{\Delta} =$
$\frac{F(a+\Delta)-F(a)}{\Delta} + \frac{\eta_1-\eta_2}{\Delta} \doteq F'(a).$ $\square$

Thanks

for your attention.