Non-standard Methods I – Three Worlds

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Non-standard methods are based on the existence of an extension of the (set-theoretical) mathematical universe by ideal elements (among others infinitely large and small real numbers, limits, completions, closures, ...). This universal extension generalizes many particular classical extension constructions (such as Dedekind cuts or topological compactifications) and also contains new extensions unnoticed by the classical mathematics.

Non-standard approach enables to reformulate whole disciplines (e.g. analysis) in a conceptually simpler and intuition-respecting way and reveals connections among classical concepts hidden to the classical view. As an important example—in the non-standard perspective the relations of finite-infinite or discrete-continuous can be seen as not sharply separated but rather smoothly connected and even reversible.

1 Three Universes

S standard universe
I internal universe
V external universe

$\mathbf{S} \subset \mathbf{I} \subset \mathbf{V}$,

$$\begin{tikzcd} \mathbf{S} \arrow[r, phantom, "\prec", "\mathit{elementary}"'] & \mathbf{I} \arrow[r, phantom, "\subseteq_t", "\mathit{transitive}"'] & \mathbf{V} \arrow[ll, bend right, "*"', "\mathit{isomorphism}"] \end{tikzcd}$$

I is ⓚ-*saturated* and *almost universal*.

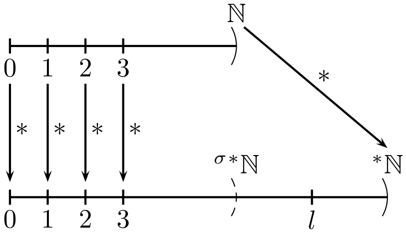
Definition 1. For φ an $\in$-formula $\varphi^{\mathbf{X}}$ is the *relativization of* φ *to* **X** obtained by replacing all quantifiers $\forall x$, $\exists x$ in φ by $\forall x \in \mathbf{X}$, $\exists x \in \mathbf{X}$.

$^*: \mathbf{V} \to \mathbf{S}$ isomorphism	means	$\varphi^{\mathbf{V}}(\overline{a}) \Leftrightarrow \varphi^{\mathbf{S}}(^*\overline{a})$ for $\in$-formula φ and $\overline{a} \in \mathbf{V}$, * is injective and onto $\mathbf{S} = \mathrm{rng}(^*) = {}^*[\mathbf{V}]$,
$\mathbf{S} \prec \mathbf{I}$ (elementary)	means	$\varphi^{\mathbf{S}}(\overline{a}) \Leftrightarrow \varphi^{\mathbf{I}}(\overline{a})$ for $\in$-formula φ and $\overline{a} \in \mathbf{S}$,
$\mathbf{I} \subseteq_t \mathbf{V}$ (transitive)	means	$y \in x \in \mathbf{I} \to y \in \mathbf{I}$,
I almost universal	means	$x \subseteq \mathbf{I} \to (\exists y \in \mathbf{I})(x \subseteq y)$,
I ⓚ-saturated	means	$(\mathcal{C} \subseteq \mathbf{I}$ centered system & $\lvert\mathcal{C}\rvert <$ ⓚ$) \to \bigcap \mathcal{C} \neq \emptyset$ where $\mathcal{C}$ is centered system if for any $\mathcal{C}' \subseteq \mathcal{C}$ finite it is $\bigcap \mathcal{C}' \neq \emptyset$

Definition 2. An $\in$-formula $\varphi(\overline{a})$ *holds standardly/internally/externally* if it holds $\varphi^{\mathbf{S}}(\overline{a})/\varphi^{\mathbf{I}}(\overline{a})/\varphi(\overline{a})$ and $\overline{a} \in \mathbf{S}/\mathbf{I}/\mathbf{V}$.

Definition 3. *Standard extension* of a set x is the set ${}^\sigma x = x \cap \mathbf{S}$. *Standardization* ${}^{ST}x$ of a set $x \subseteq \mathbf{S}$ is the unique standard set y with ${}^\sigma y = x$.

2 Non-standard Extension of $\mathbb{N}$



$^*n = n \in \mathbf{S}$ for $n \in \mathbb{N}$

$l \in \mathbf{I} - \mathbf{S}$ infinite natural number

$^*\mathbb{N} \in \mathbf{S}$ standard set of nat. numbers

$^{\sigma *}\mathbb{N} = \mathbb{N} \notin \mathbf{I}$ set of standard natural numbers

$^*\mathbb{N} - \mathbb{N} \notin \mathbf{I}$ set of non-standard natural numbers

3 Non-standard Principles

Standardization principle:	$(\forall x)(\exists! y \in \mathbf{S})(^\sigma x = {}^\sigma y)$,
Overflow principle:	$\mathbb{N} \subseteq x \in \mathbf{I} \to x \cap (^*\mathbb{N} - \mathbb{N}) \neq \emptyset$,
or equivalently:	$(\forall x \in \mathbb{N})\varphi^{\mathbf{I}}(x, \overline{a}) \to (\exists x \in {}^*\mathbb{N} - \mathbb{N})\varphi^{\mathbf{I}}(x, \overline{a})$
	for $\overline{a} \in \mathbf{I}$ and φ an $\in$-formula,
κ-finitarization principle:	$(x \subseteq \mathbf{I}\ \&\ \lvert x \rvert < \kappa) \to (\exists y \in \mathbf{I})([y \text{ is finite}]^{\mathbf{I}}\ \&\ x \subseteq y)$.

4 Real Numbers and Functions

Definition 4. $\mathbf{R} = {}^*\mathbb{R}$ is the *standard set of real numbers.*
$\mathbf{BR} = \{r \in \mathbf{R}; \lvert r \rvert < s \text{ for some } s \in {}^\sigma\mathbf{R}\}$ is the set of *bounded real numbers.*
For $x, y \in \mathbf{R}$ we define *indifference* $\sim$ on $\mathbf{R}$ as follows:

$$x \sim y \Leftrightarrow \lvert x - y \rvert < 2^{-n} \text{ for all } n \in \mathbb{N}.$$

If $x \sim y$, we say that x and y are *infinitely close* to each other.

Theorem 5 (Fundamental about $\mathbf{R}$). *For every $r \in \mathbf{BR}$ there is standard $s \in \mathbf{R}$ such that $r \sim s$.*

Definition 6. We say that a standard (partial) function $f \subseteq \mathbf{R} \times \mathbf{R}$ is continuous in a standard point $a \in \mathrm{dom}(f)$ if for every $x \in \mathrm{dom}(f)$ it is

$$a \sim x \to f(a) \sim f(x). \tag{1}$$

Standard function $f \subseteq \mathbf{R} \times \mathbf{R}$ is continuous if (1) holds for every standard $a \in \mathrm{dom}(f)$ and it is uniformly continuous if (1) holds for every $a \in \mathrm{dom}(f)$.

Definition 7. For a standard function $f \subseteq \mathbf{R} \times \mathbf{R}$ and a such that $[a]_\sim \subseteq \mathrm{dom}(f)$ we say that $b \in \mathbf{R}$ is the derivative of f in a if

$$0 \neq \delta \sim 0 \to \frac{f(a+\delta) - f(a)}{\delta} \sim b.$$