Permutation groups I

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In the first talk on permutation groups we recall the basic definitions and properties as well as a bit of history of such groups. We'll also see examples and some unexpected applications such as alternative proofs of theorems from different branches of mathematics.

Definition 1. A *symmetric group on a nonempty set* Ω consists the set of all bijections from Ω to Ω together with the operation of composition. A *permutation group* is a subgroup of a symmetric group.

Definition 2. Let G be a group and Ω a nonempty set. We say that G *acts on* Ω if for each $\alpha \in \Omega$ and each $g \in G$ we have an element $\alpha^g \in \Omega$ and we have also

- $\alpha^1 = \alpha$ for each $\alpha \in \Omega$ (where 1 is the nautral element of G);
- $(\alpha^g)\, h = \alpha^{xy}$ for all $g, h \in G$ and all $\alpha \in \Omega$.

Definition 3. If group G acts on the set Ω, we call the set $\alpha^G := \{a^g \mid g \in G\}$ the *orbit of* α *under* G and the set $G_\alpha := \{g \in G \mid \alpha^g e \alpha\}$ the *stabilizer of* α *under* G.

The following theorem connects the objects in the previous definition:

Theorem 4. *Suppose a group G acts on a set Ω, $g, h \in G$ and $\alpha, \beta \in \Omega$. Then*

(1) the orbits α^G and β^G are either disjoint or equal;
(2) the stabilizer G_α is a subgroup of G and $G_\beta = g^{-1} G_\alpha g$ whenever $\beta = \alpha^g$. Moreover, $\alpha^g = \alpha^h \leftrightarrow G_\alpha g = G_\alpha h$;
(3) $\left|\alpha^G\right| = |G : G_\alpha|$.

Some more definitions:

Definition 5. A group acting on a set Ω is said to be *transitive on* Ω if the action has only one orbit, otherwise it is said to be *intransitive.*

Definition 6. A group G acting transitively on Ω is said to act regularly, if $G_\alpha = 1$ for each $\alpha \in \Omega$.

As a corollary of the previous theorem now we can formulate the following theorem on transitively acting groups:

Theorem 7. *Suppose G ants transitively on the set Ω.Then:*

(1) the stabilizers G_α form a single conjugacy class of subgroups;
(2) the indix of each stabilizer G_α in G is equal to the cardinality of Ω;
(3) if G is finite, its action is regular iff $|G| = |\Omega|$.

And the following useful lemma.

Lemma 8. *Let G be a group acting transitively on the set Ω then for each $\alpha \in \Omega$, the only transitive subgroup of G containing the stabilizer of G_α is G itself.*