Geometric algebra I

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Geometric algebra – Part I should introduce you to the theory of geometric lattices. At first we will sum up basic definitions of lattices, geometric lattices and geometries, and then we will investigate some attributes of them and the relation between geometric lattices and geometries.

Definitions of key notions are written down for the listener's convenience:

Definition 1. A partially ordered set—poset $\langle L; \leq \rangle$ is a *lattice* if $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all a, $b \in L$.

Equivalent definition is the following: A poset $\langle L; \leq \rangle$ is a *lattice* iff $\sup H$ and $\inf H$ exist for any finite nonvoid subset H of L.

Notation:

- *meet:* $a \wedge b = \inf\{a, b\}$.
- *join:* $a \vee b = \inf\{a, b\}$.
- $\sup H = \bigvee H$, $\inf H = \bigwedge H$.

Definition 2. A lattice L is called *complete* if $\bigvee H$ and $\bigwedge H$ exist for any subset $H \subseteq L$.

Definition 3. An element a of lattice $\langle L; \leq \rangle$ is an *atom* if $a \succ 0$, i.e., if a covers 0. It means that $a > 0$ and for no $x \in L$, $a > x > 0$. A lattice L is *atomic* iff it has 0 and for every $b \in L, b \neq 0$, there is an atom $a \leq b$.

Definition 4. Let L be a complete lattice and let a be an element of L. Then a is called *compact* iff $a \leq \bigvee X$ for some $X \subseteq L$ implies that $a \subseteq \bigvee X_1$ for some finite $X_1 \subseteq X$. A complete lattice L is called *algebraic* iff every element of L is a join of compact elements.

Definition 5. A lattice is called *semimodular* iff it satisfies the Upper Covering Condition, that is, if $a \prec b \Rightarrow a \vee c \prec b \vee c$ or $a \vee c = b \vee c$.

Definition 6. *Height function*:

$$h(a) = \begin{cases} \text{the length of longest maximal chain in } [0, a], \text{if there is a finite one} \\ \infty \text{ otherwise} \end{cases}$$

Theorem 7. *Let L be a lattice of finite length. Then*

$$L \textit{ is semimodular } \iff h(a) + h(b) \geq h(a \wedge b) + h(a \vee b) \textit{ for all } a, b \in L.$$

Definition 8. A lattice L is called *geometric* iff L is semimodular, algebraic and the compact elements of L are exactly the finite joins of atoms of L.

Definition 9. *Modular lattice* is lattice satisfying condition

$$x \geq z \Rightarrow (x \wedge y) \vee z = x \wedge (y \vee z),$$

which is equivalent to the following identity:

$$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee (x \wedge z)).$$

Definition 10. A geometry $\langle A, ^{-} \rangle$ is a set A and a function $^{-}\colon P(A) \to P(A)$, satisfying the following properties:

(i) a) $X \subseteq \overline{X}$;
   b) if $X \subseteq Y$, then $\overline{X} \subseteq \overline{Y}$;
   c) $\overline{\overline{X}} = \overline{X}$.

(ii) $\overline{\varnothing} = \varnothing$, and $\overline{\{x\}} = \{x\}$ for all $x \in A$.

(iii) If $x \in \overline{X \cup \{y\}}$, but $x \notin \overline{X}$, then $y \in \overline{X \cup \{x\}}$.

(iv) If $x \in \overline{X}$, then $x \in \overline{X_1}$ for some finite $X_1 \subseteq X$.

Theorem 11. *Let $\langle A, ^{-} \rangle$ be a geometry. Then $L = L\langle A, ^{-} \rangle$ (i.e., the lattice of all closed subsets of A) is a geometric lattice. Conversely, if L is a geometric lattice, A is the set of atoms of L, and for every $X \subseteq A$, $\overline{X}$ is the set of atoms spanned by X, then $\langle A, ^{-} \rangle$ is a geometry and $L \cong L\langle A, ^{-} \rangle$.*