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Projective Space, Geometry and Geometrical Lattice

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Geometry and Projective Space

Projective Space

Definition (Projective space)

Let A be a set and let L be a collection of subsets of A. The pair $\langle A, L\rangle$ is called a **projective space** iff the following properties hold:

1. Every $l \in L$ has at least two elements.
2. For any two distinct $p, q \in A$ there is exactly one $l \in L$ satisfying $p, q \in L$.
3. For $p, q, r, x, y \in A$ and $l_1, l_2 \in L$ satisfying $p, q, x \in l_1$ and $q, r, y \in l_2$, there exist $z \in A$ and $l_3, l_4 \in L$ satisfying $p, r, z \in l_3$ and $x, y, z \in l_4$.

The members of A are called **points** and those of L are called **lines**.

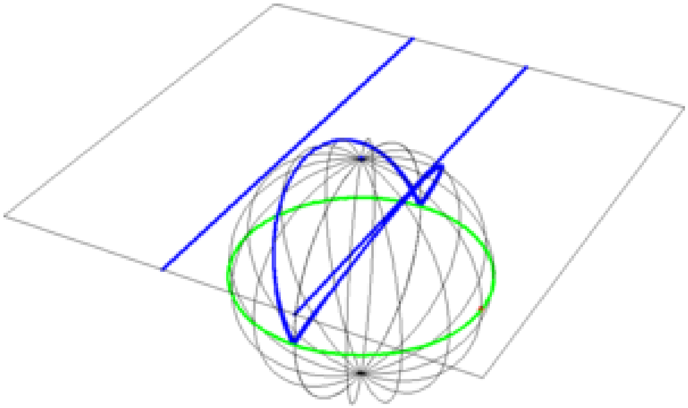

Figure: Projective and affine space.

Projective Space - Examples

Construction from vector space $\mathbb{R}^3$:
We define equivalence on $\mathbb{R}^3 \setminus \{0\}$:

$$x, y \in \mathbb{R}^3 \setminus \{0\}, x \sim y \iff x = \lambda y, \text{ for some } \lambda \in \mathbb{R}$$

Then $\langle A, L \rangle$ is a projective space where set of points (A) and set of lines (L) are defined as follows:

$$\begin{aligned} A &= \{(V \setminus \{0\})/\sim |V \text{ is a subspace of } \mathbb{R}^3, \dim V = 1\} \\ L &= \{(V \setminus \{0\})/\sim |V \text{ is a subspace of } \mathbb{R}^3, \dim V = 2\} \end{aligned}$$

Projective Space - Examples

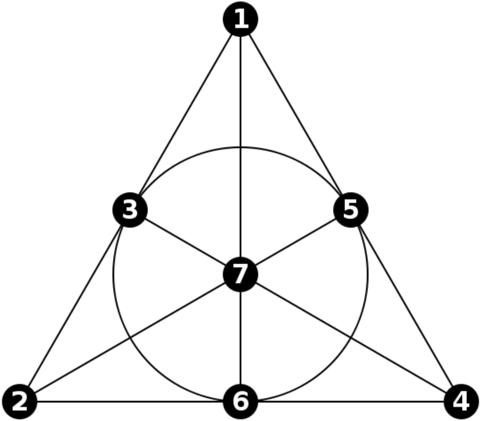

Figure: Fano plane.

Projective Space - Basic Notation

Notation

Let $\langle A, L\rangle$ be a projective space.

- *For $p, q \in A$, $p \neq q$, let $p+q$ denote the (unique) line containing p and q; if $p = q$ set $p+q = \{p\}$.*
- *$X, Y \subseteq A$:*

$$X + Y = \bigcup(x + y | x \in X, y \in Y)$$

Linear Subspace of Projective Space

Definition (Subspace of a projective space)

Let $\langle A, L \rangle$ be projective space. A set $X \subseteq A$ is called a **linear subspace** of $\langle A, L \rangle$ iff $p, q \in X$ imply that $p + q \subseteq X$.

Remark

Every point is a linear subspace.

Lemma

If X and Y are linear subspaces of a projective space, then so is $X + Y$.

Theorem

Let $\langle A, L\rangle$ *be a projective space. For every* $X \subseteq A$ *we define* $\overline{X}$*:*

$$\begin{aligned}\overline{X} &= \text{the smallest subspace of } \langle A, L\rangle \text{ that contains } X \\ &= \bigcap (Z \mid Z \text{ is a subspace of } \langle A, L\rangle, X \subseteq Z)\end{aligned}$$

Then $\langle A, \bar{\ } \rangle$ *is a geometry.*

From Projective Space to Geometry

We need to check:

1. is a closure relation, that is,
   - $X \subseteq \overline{X}$,
   - if $X \subseteq Y$, then $\overline{X} \subseteq \overline{Y}$,
   - $\overline{\overline{X}} = \overline{X}$.
2. $\overline{\varnothing} = \varnothing$, and $\overline{\{x\}} = \{x\}$, for every $x \in A$.
3. If $x \in \overline{X \cup \{y\}}$, but $x \notin \overline{X}$, then $y \in \overline{X \cup \{x\}}$.
4. If $x \in \overline{X}$, then $x \in \overline{X_1}$, for some finite $X_1 \subseteq X$.

Definition

The geometry is **projective** iff the associated geometric lattice is modular.

Modularity of geometric lattice translated into language of geometry:

- modular (geometric) lattice: x, y, z elements of lattice and $z \leq x$, then it holds

$$z \vee (x \wedge y) = x \wedge (z \vee y)$$

- projective geometry: X, Y, Z subspaces (closed sets) in geometry, $Z \subseteq X$, then it holds

$$\overline{Z \cup (X \cap Y)} = X \cap \overline{(Z \cup Y)}$$

We focus now on the question:

"Can some of the geometries that are associated with projective spaces be projective?"

The answer is that every geometry associated with projective space is a projective geometry.
We must prove that the following inclusions hold for every projective subspaces X, Y, Z where $Z \subseteq X$:

1. $Z + (X \cap Y) \subseteq X \cap (Z + Y)$
2. $Z + (X \cap Y) \supseteq X \cap (Z + Y)$

We have proved the following statement:

Lemma

Every geometry associated with projective space is projective.

The question now is if we can create projective space from every projective geometry.
Let $\langle A, \rangle$ be a projective geometry. We define

$$L = \{\overline{\{x, y\}} | \{x, y\} \subseteq A\}.$$

Is $\langle A, L\rangle$ a projective space? We try to verify 3 axioms of a projective space:

1. Every $l \in L$ has at least two elements.
2. For any two distinct $p, q \in A$ there is exactly one $l \in L$ satisfying $p, q \in L$.
3. For $p, q, r, x, y \in A$ and $l_1, l_2 \in L$ satisfying $p, q, x \in l_1$ and $q, r, y \in l_2$, there exist $z \in A$ and $l_3, l_4 \in L$ satisfying $p, r, z \in l_3$ and $x, y, z \in l_4$.

Theorem (Pasch Axiom)

Let $\langle A, ^{-} \rangle$ be a projective geometry. Then the geometry satisfies the following property:
If p, q, r, x, y are points, $x \in \overline{\{p, q\}}, y \in \overline{\{q, r\}}$, and $x \neq y$, then there is a point z such that $z \in \overline{\{p, r\}} \cap \overline{\{x, y\}}$.

Projective Spaces and Projective Geometries - Conclusion

Theorem (Correspondence between projective spaces and projective geometries)

There is a one-to-one correspondence between projective spaces (defined by points and lines) and projective geometries (defined as geometries with modular subspace lattices). Under this correspondence, linear subspaces of projective spaces correspond to subspaces of projective geometries.

Desargues' Axiom and Arguesian Lattice

Definition (Collinearity, triangle, perspectivity)

Let $\langle A, L\rangle$ projective space, a set of points $X \in A$ is **collinear** iff $X \subseteq l$, for some line $l \in L$. A triple $\langle a_0, a_1, a_2\rangle$ of noncollinear points is called a **triangle**.

Two triangles $\langle a_0, a_1, a_2\rangle$ and $\langle b_0, b_1, b_2\rangle$ are **perspective with respect to a point** $\mathbf{p}$ iff the following hold:

1. $a_i \neq b_i$ for $0 \leq i < 3$,
2. $a_i + a_j \neq b_i + b_j$ for $0 \leq i < j < 3$,
3. the points p, a_i, b_i are collinear, for $i = 0, 1, 2$.

The triangles are **perspective with respect to a line** $\mathbf{l}$ iff $\{c_{01}, c_{12}, c_{20}\} \subseteq l$, where c_{ij} is the intersection of $a_i + a_j$ and $b_i + b_j$.

Perspectivity of triangles

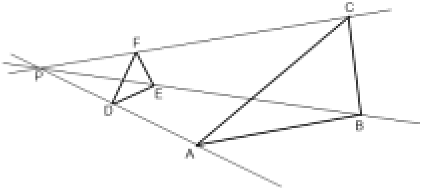

Figure: Triangles $\langle A, B, C\rangle$, $\langle D, E, F\rangle$ perspective with respect to the point P.

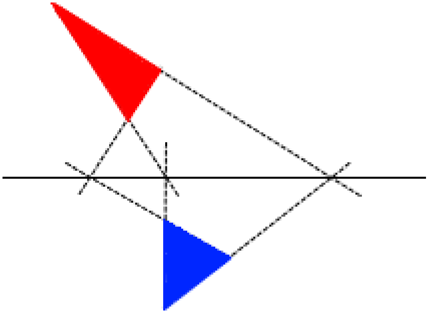

Figure: Triangles perspective with respect to a line.

Theorem (Desargues' Axiom)

If two triangles are perspective with respect to a point, then they are perspective with respect to a line.

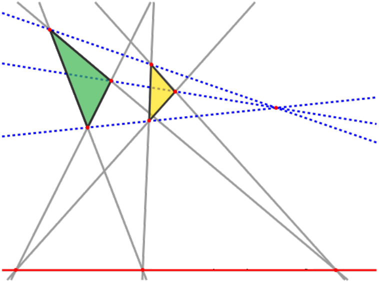

Figure: Desargues' Axiom

Desargues' Axiom - proof

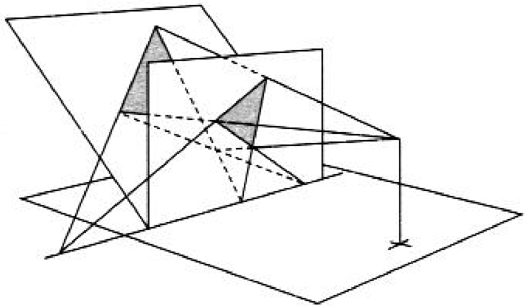

Figure: Proof of the Desargues' Axiom.

Arguesian lattice

Definition (Arguesian identity, Arguesian lattice)

Let $x_0, x_1, x_2, y_0, y_1, y_2$ be variables. We define polynomials:

$$
\begin{aligned}
z_{ij} &= (x_i \vee x_j) \wedge (y_i \vee y_j), \;\; 0 \leq i < j < 3, \\
z &= z_{01} \wedge (z_{02} \vee z_{12}).
\end{aligned}
$$

The **Arguesian identity** is

$$(x_0 \vee y_0) \wedge (x_1 \vee y_1) \wedge (x_2 \vee y_2) \leq ((z \vee x_1) \wedge x_0) \vee ((z \vee y_1) \wedge y_0).$$

A lattice satisfying this identity is called **Arguesian**.

Theorem (Lattice formulation of Desargues' Axiom)

Let L be a modular geometric lattice. Then L satisfies the Arguesian identity iff Desargues' Theorem holds in the associated projective geometry.

Thank you for your attention.