Non-standard Methods III – Non-standard Topology

Adam Ráž

Definition 1. *Indifference* is an equivalence Q on a standard set A, where Q is intersection of a set V of standard sets with $|V| < ⓚ$. The pair $\langle A, Q\rangle$ is then called *indifference space.*

For $x \in A$ the set $Q[x] = \{y \in A;\ xQy\}$ is called the *monad of point x* in space $\langle A, Q\rangle$ and for $X \subseteq A$ the set $Q[X] = \{y \in A;\ (\exists x \in X)xQy\}$ is called the *figure of set X* in space $\langle A, Q\rangle$. *Monad neighbourhood of point x* is any internal set $X \subseteq A$ which contains the monad of x.

Definition 2. An indifference space is called *Hausdorff space* if any two distinct standard points have disjoint monads. Then every monad contains at most one standard point.

Definition 3. *Basis of uniformity of an indifference space* $\langle A, Q\rangle$ is any nonempty standard set V which satisfies the following conditions:

- V is a set of reflexive and symmetric relations on A
- $(\forall u, v \in V)(\exists w \in V):\ w \subseteq u \wedge w \subseteq v$
- $(\forall v \in V)(\exists w \in V):\ w \circ w \subseteq v$
- $Q = \bigcap {}^{\sigma}V$ and $|{}^{\sigma}V| < ⓚ$.

Two bases of uniformity of an indifference space are *equivalent*, if for each element v of the first basis there exists an element w of the second basis such that $w \subseteq v$, and this applies vice versa.

Theorem 4. *Let $\langle A, Q\rangle$ be an indifference space. Then there exists a basis of $\langle A, Q\rangle$ and any two such bases are equivalent.*

Definition 5. *Standard open set* in $\langle A, Q\rangle$ is any standard set $X \subseteq A$ which contains the whole monad of each its standard point, i.e. $Q[{}^{\sigma}X] \subseteq X$. *Standard closed set* in $\langle A, Q\rangle$ is any standard set $Y \subseteq A$ such that its complement $X = A \setminus Y$ is a standard open set.

Standard topology of space $\langle A, Q\rangle$ is standardization of the set of all standard open sets of $\langle A, Q\rangle$; we denote it as $\tau(\langle A, Q\rangle)$. We say that $\langle A, \tau(\langle A, Q\rangle)\rangle$ is the *standard topological space* of indifference space $\langle A, Q\rangle$.

Theorem 6. *Let $\langle A, Q\rangle$ be an indifference space, $X \subseteq A$ standard, $\overline{X}$ the closure of X and X° the interior of X in $\langle A, \tau(\langle A, Q\rangle)\rangle$ standardly. Then*

$$\overline{X} = {}^{\mathrm{ST}}\{a \in {}^{\sigma}A;\ Q[a] \cap X \neq \emptyset\},$$

$$X^\circ = {}^{\mathrm{ST}}\{a \in {}^{\sigma}A;\ Q[a] \subseteq X\}.$$

Theorem 7. *Let $\langle A, Q\rangle$ be an indifference space. Then for $a \in {}^\sigma A$ it holds*

$$Q[a] = \bigcap \{X \in {}^\sigma\tau(\langle A, Q\rangle);\ a \in X\}.$$

Definition 8. Let $\langle A, Q\rangle$ be an indifference space. The point $x \in A$ is called *nearstandard* if it is in the monad of some standard point. We denote the set $Q[{}^\sigma A] = \{x \in A;\ (\exists a \in {}^\sigma A)\colon aQx\}$ of all nearstandard points as $\mathfrak{n}(A, Q)$.

The point $x \in A$ is called *accessible* in $\langle A, Q\rangle$ if there is a standard point in every its monad neighbourhood. The set of all accessible points is denoted $\mathfrak{a}(A, Q)$.

Definition 9. An indifference space $\langle A, Q\rangle$ is *condensed* if $\mathfrak{n}(A, Q) = A$.

An indifference space $\langle A, Q\rangle$ is *complete* if $\mathfrak{n}(A, Q) = \mathfrak{a}(A, Q)$.

An indifference space $\langle A, Q\rangle$ is *bounded* if $\mathfrak{a}(A, Q) = A$.