Hadamard transform

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Content

Sylvester-Hadamard matrices

- **Hadamard matrix of order** n - an $n \times n$ matrix with elements $+1$ and -1 such that any distinct row or column vectors are mutually orthogonal

Definition

A (normalized) Sylvester-Hadamard matrix of size 2^m, $m \geq 0$, is a squared $2^m \times 2^m$ matrix that is defined recursively by

$$H_{2^m} = \frac{1}{\sqrt{2}} \begin{pmatrix} H_{2^{m-1}} & H_{2^{m-1}} \\ H_{2^{m-1}} & -H_{2^{m-1}} \end{pmatrix},$$

where recursion is initiated by $H_1 = (1)$.

Hadamard transform and Inverse Hadamard transform

- **Hadamard transform** - mapping $T : \mathbb{R}^{2^m} \to \mathbb{R}^{2^m}$ defined by $T(\mathbf{x}) = H_{2^m} \cdot \mathbf{x}$
- **Inverse Hadamard transform** - inverse matrix of Sylvester-Hadamard matrix is equal to its transpose $\Rightarrow$ inverse Hadamard transform is performed by applying $H_{2^m}^T$, i.e.

$$\mathbf{x} = H_{2^m}^T T(\mathbf{x}) = H_{2^m}^T H_{2^m} \mathbf{x}$$

Complexity of Hadamard transform

- Hadamard transform uses only additions and subtractions (no multiplication)
- **Fast Hadamard transform** - complexity $O(n \log n)$

Walsh functions - formal definition

- Complete set of orthogonal functions on the interval $[0, 1]$
- **Formal definition of Walsh functions** $Wal(i, t)$**:** $(i = 0, 1, \dots, 0 \leq t \leq 1)$

$$Wal(0, t) = x_0(t) = 1,\ 0 \leq t \leq 1$$

$$Wal(1, t) = x_1(t) = \begin{cases} 1, & 0 \leq t < 0,5 \\ -1, & 0,5 \leq t \leq 1 \end{cases}$$

$$Wal(2, t) = x_2(t) = \begin{cases} 1, & 0 \leq t < 0,25 \text{ and } 0,75 \leq t \leq 1 \\ -1, & 0,25 \leq t \leq 0,75 \end{cases}$$

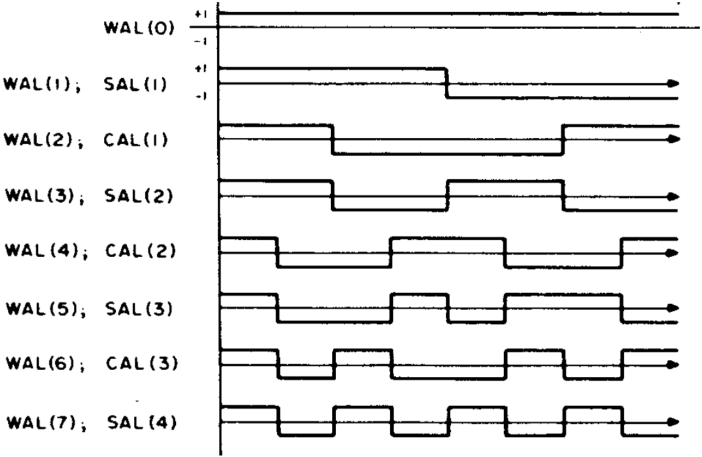
$$Wal(3, t) = x_3(t) = \begin{cases} 1, & 0 \leq t < 0,25 \text{ and } 0,5 \leq t < 0,75 \\ -1, & 0,25 \leq t < 0,5 \text{ and } 0,7f \leq t \leq 1 \end{cases}$$

and recursively for $m = 1, 2, \dots$ and $k = 1, \dots, 2^{m-1}$ we have $Wal(2^{m-1} + k - 1, t) = x_m^k(t)$

$$x_{m+1}^{2k-1}(t) = \begin{cases} x_m^k(2t), & 0 \leq t < 0,5 \\ (-1)^{k+1} x_m^k(2t-1), & 0,5 \leq t \leq 1 \end{cases}$$

$$x_{m+1}^{k}(t) = \begin{cases} x_m^k(2t), & 0 \leq t < 0,5 \\ (-1)^{k} x_m^k(2t-1), & 0,5 \leq t \leq 1 \end{cases}$$

The first 8 Walsh functions



Walsh functions and Hadamard matrices

- **Definition of Walsh functions $Wal(0, t)$ to $Wal(2^{m-1}, t)$ using Sylvester-Hadamard matrix H_{2^m}:**
  - Divide the interval $[0, 1]$ into 2^m parts

$$\left[0, \frac{1}{2^m}\right), \left[\frac{1}{2^m}, \frac{2}{2^m}\right), \ldots, \left[\frac{2^m-1}{2^m}, 1\right)$$

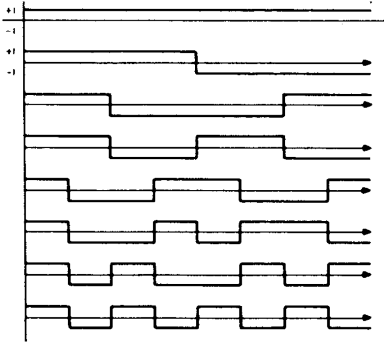
  - Walsh function defined by the i^{th} row of a matrix H_{2^m} takes the value 1 in the interval $[k/2^m, (k+1)/2^m]$ if

$$H_{2^m}(i, k+1) = 1,$$

otherwise it takes the value -1

The first 8 Walsh functions using Hadamard matrix H_8

$$H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & - & - & 1 & 1 & - & - & 1 \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & - & 1 & - & - & 1 & - & 1 \\ 1 & 1 & - & - & - & - & 1 & 1 \\ 1 & - & - & 1 & - & 1 & 1 & - \end{pmatrix} \begin{matrix} Wal(0,t) \\ Wal(7,t) \\ Wal(3,t) \\ Wal(4,t) \\ Wal(1,t) \\ Wal(6,t) \\ Wal(2,t) \\ Wal(5,t) \end{matrix}$$



Synchronous Code Division Multiple Access (CDMA)

- **Multiple access** - ability of many users to communicate with each other while sharing a common transmission medium (channel)
- **CDMA** - method of multiple access that separates different signals using unique sequence (signature) for each user
- **Synchronous CDMA** - all communication signals are synchronous

Coding in CDMA and Hadamard matrices - I.

- Find a set S of mutually orthogonal vectors (a cardinality of the set must be greater than the number of users)
  - As a set S row or column vectors in Hadamard matrix H_n of sufficient order can be considered

- Assign unique vector s_i from S to each user i (signature)
  - Each signature corresponds to given row (or column) in the matrix H_n

Coding in CDMA and Hadamard matrices - II.

- For user i, a bit 1 is represented as a sequence s_i and a bit 0 is represented as a sequence $-s_i$
  - User i wants to transmit a bit 1 $\Rightarrow$ i transmit sequence s_i
  - User i wants to transmit a bit 0 $\Rightarrow$ i transmit sequence $-s_i$
  - User i doesn't want to transmit anything $\Rightarrow$ do nothing
- In the common channel, all sequences transmitted by users are added $\Rightarrow$ we get the raw signal s (interference pattern)

Decoding in CDMA and Hadamard matrices

- Reciever extracts original signal for user i by combining the sequence s_i with an interference pattern (a dot product of the vectors s_i and s)
    - $s \cdot s_i > 0 \Rightarrow$ user i transmited bit 1
    - $s \cdot s_i < 0 \Rightarrow$ user i transmited bit 0
    - $s \cdot s_i = 0 \Rightarrow$ user i didn't transmit anything

Coding of digital images by Hadamard transform

- **Digital image** - a $n_1 \times n_2$ matrix with integral input

Image coding by two-dimensional Hadamard transform:

- Divide an image to the blocks (matrices) of size $2^m \times 2^m$ (each block is coded separately)
- Transform block B by Hadamard matrix H_2^m as

$$\bar{B} = H_{2^m} B H_{2^m}^T \tag{1}$$

Basis images

- Transform of block provides a representation in terms of orthogonal basis images $\Rightarrow$ components of $\bar{B}$ provide the coefficients for the expression of B as a linear combination of the basis images I_{ij}
- Basis images for the 4×4 case (white=1, black=-1):

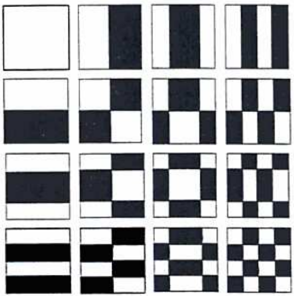


Image compression

- For images that don't have a lot of rapid intensity oscillations, energy tends to be packed into **upper left corner** of $\bar{B}$
  - Compression can be achieved by keeping only some set of upper left components

Example: B - original 4×4 block, $\bar{B}_0$ - upper left four entries in $\bar{B}$, $\bar{B}_1$ - upper left entry in $\bar{B}$

$$B = \begin{pmatrix} 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 \\ 6 & 6 & 0 & 0 \\ 6 & 6 & 0 & 0 \end{pmatrix} H_4^T \bar{B}_0 H_4 = \begin{pmatrix} 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 \\ 6 & 6 & 0 & 0 \\ 6 & 6 & 0 & 0 \end{pmatrix} H_4^T \bar{B}_1 H_4 = \begin{pmatrix} 45 & 45 & 45 & 45 \\ 45 & 45 & 45 & 45 \\ 45 & 45 & 45 & 45 \\ 45 & 45 & 45 & 45 \end{pmatrix}$$

Other applications

- Quantum information processing (Grover's algorithm and Shor's algorithm)

References

[1] J. Seberry, B.J. Wysocki, T.A. Wysocki; On some applications of Hadamard matrices
[2] D. Sinha, E.R. Dougherty: Introduction to computer-based imaging systems
[3] Hadamard transform, Wikipedia
[4] J. Peterka, lecture Počítačové sítě I.