Geometric algebra II

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In this talk we will examine one special case of the correspondence between geometries and geometric lattices. First we will recapitulate projective space and mention its axiomatic definition. We will talk about the one-to-one correspondence between projective spaces, projective geometries and modular geometric lattices. Then we will turn our attention to Desargues' Axiom which holds in many projective spaces and with some restrictions in affine spaces too. Finally, we will look for an equivalent condition of Desargues Axiom in a modular geometric lattice associated with projective space. This condition is known as Arguesian identity, resp. Arguesian lattice.

1 Correspondence between projective spaces and projective geometries

Definition 1 (Projective space)**.** Let A be a set and let L be a collection of subsets of A. The pair $\langle A, L\rangle$ is called a *projective space* iff the following properties hold:

(1) Every $l \in L$ has at least two elements.
(2) For any two distinct $p, q \in A$ there is exactly one $l \in L$ satisfying $p, q \in L$.
(3) For $p, q, r, x, y \in A$ and $l_1, l_2 \in L$ satisfying $p, q, x \in l_1$ and $q, r, y \in l_2$, there exist $z \in A$ and $l_3, l_4 \in L$ satisfying $p, r, z \in l_3$ and $x, y, z \in l_4$.

The members of A are called *points* and those of L are called *lines.*

Notation. Let $\langle A, L\rangle$ be a projective space. For $p, q \in A$, $p \neq q$, let $p+q$ denote the (unique) line containing p and q; if $p = q$ set $p + q = \{p\}$.

Definition 2 (Subspace of a projective space)**.** Let $\langle A, L\rangle$ be projective space. A set $X \subseteq A$ is called a *linear subspace* (of $\langle A, L\rangle$) iff $p, q \in X$ imply that $p+q \subseteq X$.

Definition 3 (Projective geometry)**.** The geometry is *projective* iff the associated geometric lattice is modular.

Theorem 4 (Correspondence between projective spaces and projective geometries)**.** *There is a one-to-one correspondence between projective spaces (defined by points and lines) and projective geometries (defined as geometries with modular subspace lattices). Under this correspondence, linear subspaces of projective spaces correspond to subspaces of projective geometries.*

2 Desargues' Axiom and Arguesian identity

Definition 5 (Collinearity, triangle, perspectivity)**.** Let $\langle A, L\rangle$ projective space, a set of points $X \in A$ is *collinear* iff $X \subseteq l$, for some line $l \in L$. A triple $\langle a_0, a_1, a_2\rangle$

of noncollinear points is called a *triangle*.

Two triangles $\langle a_0, a_1, a_2\rangle$ and $\langle b_0, b_1, b_2\rangle$ are *perspective with respect to a point* p (see Figure 1) iff the following hold:

(1) $a_i \neq b_i$ for $0 \leq i < 3$,
(2) $a_i + a_j \neq b_i + b_j$ for $0 \leq i < j < 3$,
(3) the points p, a_i, b_i are collinear, for $i = 0, 1, 2$.

The triangles are *perspective with respect to a line* l iff $c_{01}, c_{12}, c_{20} \subseteq l$, where c_{ij} is the intersection of $a_i + a_j$ and $b_i + b_j$.

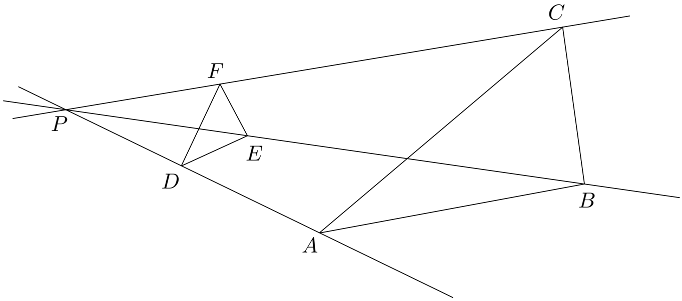

Figure 1: Triangles $\langle A, B, C\rangle$, $\langle D, E, F\rangle$ perspective with respect to the point P.

Theorem 6 (Desargues' Axiom)**.** *If two triangles are perspective with respect to a point, then they are perspective with respect to a line.*

Definition 7 (Arguesian identity, Arguesian lattice)**.** Let x_0, x_1, x_2, y_0, y_1, y_2 be variables. We define polynomials:

$$\begin{aligned} z_{ij} &= (x_i \vee x_j) \wedge (y_i \vee y_j), \quad 0 \leq i < j < 3, \\ z &= z_{01} \wedge (z_{02} \vee z_{12}). \end{aligned}$$

The *Arguesian identity* is

$$(x_0 \vee y_0) \wedge (x_1 \vee y_1) \wedge (x_2 \vee y_2) \leq ((z \vee x_1) \wedge x_0) \vee ((z \vee y_1) \wedge y_0).$$

A lattice satisfying this identity is called *Arguesian*.

Theorem 8 (Lattice formulation of Desargues' Axiom)**.** *Let L be a modular geometric lattice. Then L satisfies the Arguesian identity iff Desargues' Theorem holds in the associated projective geometry.*