

Non-standard Methods – Three Worlds

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Spring School of Algebra
Sázava, March 2012

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What is it good for?

- In the non-standard world imaginary companions of an object (limit of sequence, completion of space, . . .) are already present in the object itself. This holds also for imaginaries (such as infinitely small numbers) that defy to be correctly constructed by classical methods.
- Mathematical analysis can be reformulated using infinitesimals (correctly) instead of the (ε, δ) -machinery. This makes definitions in non-standard analysis simpler, proofs more clear and also reveals truths unnoticed by classical analysis.

Three Universes

Classically all mathematical objects (considered as sets) are elements of the universal class \mathbf{V} of all sets. The scheme of the universe is very simple:

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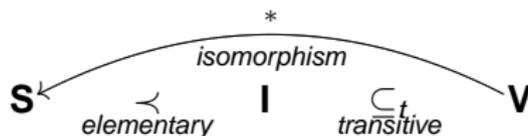
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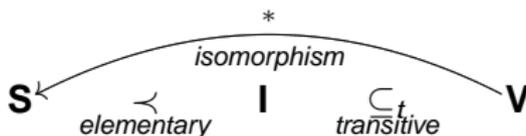


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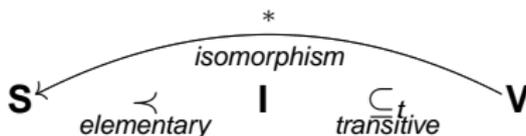
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where \mathbf{I} is *almost universal* and (κ) -saturated for some uncountable cardinal (κ) .

Here $\mathbf{S} \subseteq \mathbf{I} \subseteq \mathbf{V}$ are three classes called the **standard**, **internal** and **external universe** respectively.

Three Universes



By \in -formula we further mean a formula in the language of set theory. In particular classes \mathbf{S} , \mathbf{I} nor the mapping $*$ do not appear in \in -formulas. For φ an \in -formula $\varphi^{\mathbf{X}}$ is the **relativization of φ to \mathbf{X}** obtained by replacing all quantifiers $\forall x, \exists x$ in φ by $\forall x \in \mathbf{X}, \exists x \in \mathbf{X}$.

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Definition

$*$: $\mathbf{V} \rightarrow \mathbf{S}$ isomorphism $\Leftrightarrow \varphi^{\mathbf{V}}(\bar{a}) \Leftrightarrow \varphi^{\mathbf{S}}(*\bar{a})$ for \in -formula φ , $\bar{a} \in \mathbf{V}$,
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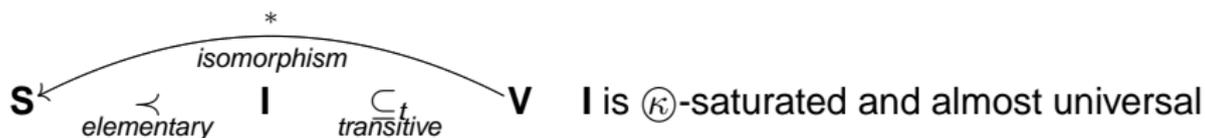


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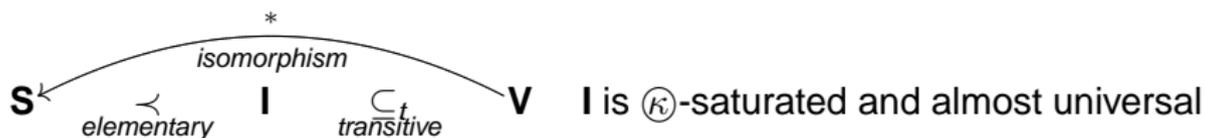
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- \mathbf{I} almost universal $\Leftrightarrow x \subseteq \mathbf{I} \rightarrow (\exists y \in \mathbf{I})(x \subseteq y)$,
- \mathbf{I} (κ) -saturated $\Leftrightarrow (\mathcal{C} \subseteq \mathbf{I}$ centered & $|\mathcal{C}| < (\kappa) \rightarrow \bigcap \mathcal{C} \neq \emptyset$
 where a system \mathcal{C} is centered if for any finite $\mathcal{C}' \subseteq \mathcal{C}$ it is $\bigcap \mathcal{C}' \neq \emptyset$

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Standard extension of a set x is the set ${}^{\sigma}x = x \cap \mathbf{S}$ of standard elements of x . **Standardization** ${}^{\text{ST}}x$ of $x \subseteq \mathbf{S}$ is the unique standard set y with ${}^{\sigma}y = x$.

Natural numbers in the non-standard perspective

For the set $\mathbb{N} \in \mathbf{V}$ of natural numbers there is its “standard instance” ${}^*\mathbb{N} \in \mathbf{S}$ and the pointwise copy ${}^*[\mathbb{N}] = \sigma^*\mathbb{N}$. It is ${}^*n = n$ for $n \in \mathbb{N}$ and therefore $\sigma^*\mathbb{N} = \mathbb{N}$.

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Since $*$ is isomorphism of \mathbf{S} and \mathbf{V} , ${}^*\mathbb{N}$ is standardly the set of all natural numbers. For $\mathbf{S} \prec \mathbf{I}$ and ${}^*\mathbb{N} \in \mathbf{S}$, ${}^*\mathbb{N}$ is also internally the set of all natural numbers.

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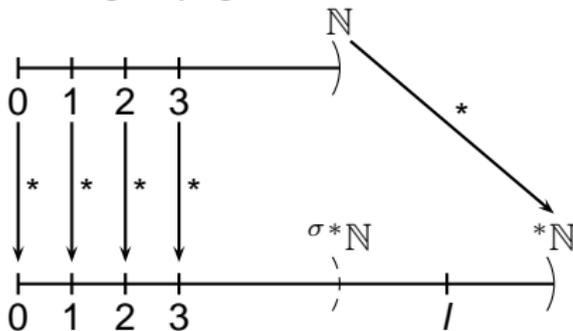
$\mathcal{C} = \{{}^*\mathbb{N} - \{n\}; n \in \mathbb{N}\}$ is a centered system of internal sets of size $\omega < \kappa$. By κ -saturation of \mathbf{I} there is $I \in \bigcap \mathcal{C}$, clearly $I \in {}^*\mathbb{N} - \mathbb{N}$.

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${}^*n = n \in \mathbf{S}$ for $n \in \mathbb{N}$

$I \in \mathbf{I} - \mathbf{S}$ infinite natural number

${}^*\mathbb{N} \in \mathbf{S}$ standard set of nat. numbers

$\sigma^*\mathbb{N} = \mathbb{N} \notin \mathbf{I}$ set of standard natural numbers

${}^*\mathbb{N} - \mathbb{N} \notin \mathbf{I}$ set of non-standard natural numbers

Universal (κ) -saturated elementary extension

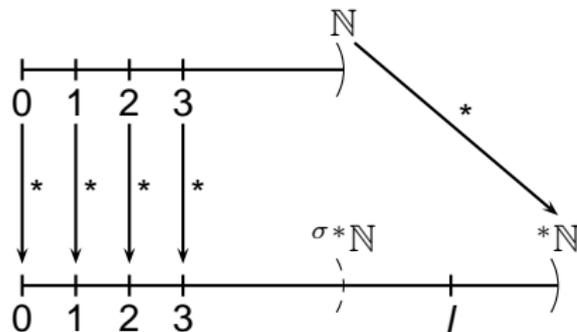
For the structure $\underline{\mathbb{N}} = \langle \mathbb{N}, 0, S, +, \cdot, \leq \rangle$ of natural numbers it is ${}^*\underline{\mathbb{N}} = \langle {}^*\mathbb{N}, {}^*0, {}^*S, {}^*+, {}^*\cdot, {}^*\leq \rangle$ and ${}^*\underline{\mathbb{N}}$ is (κ) -saturated elementary extension of $\underline{\mathbb{N}}$.

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Arbitrary structure \mathcal{A} in a finite language can be elementarily embedded into ${}^*\mathcal{A}$ in the similar way and ${}^*\mathcal{A}$ is (κ) -saturated structure. Hence * gives an universal way for construction of (κ) -saturated elementary extensions of structures.

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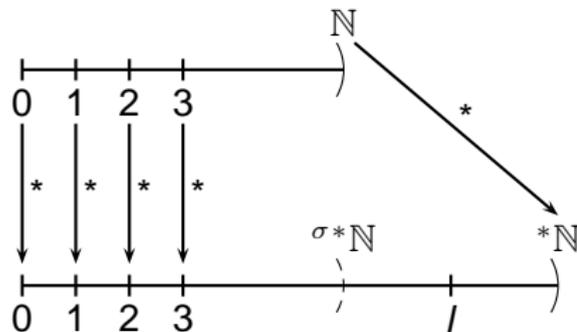
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The set $*\mathbb{N} - \mathbb{N}$ is a subset of $*\mathbb{N}$ without the least element. This is no contradiction at all.

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$*\mathbb{N}$ is not “the set of all natural numbers” externally. It is “the set of all natural numbers” standardly and internally but $*\mathbb{N} - \mathbb{N} \notin \mathbf{I}$ (actually this is the proof of that). Hence we have also $\mathbb{N} \notin \mathbf{I}$.

Non-standard Principles

The fact that $\mathbb{N} \subseteq {}^*\mathbb{N}$ but \mathbb{N} is not internal can be reformulated as follows:

Proposition (Overflow Principle)

- 1 $\mathbb{N} \subseteq \mathbf{I} \rightarrow \mathbf{x} \cap ({}^*\mathbb{N} - \mathbb{N}) \neq \emptyset$
- 2 For $\bar{a} \in \mathbf{I}$ and φ an ϵ -formula it is

$$(\forall x \in \mathbb{N})\varphi^{\mathbf{I}}(x, \bar{a}) \rightarrow (\exists x \in {}^*\mathbb{N} - \mathbb{N})\varphi^{\mathbf{I}}(x, \bar{a}).$$

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The concept of finiteness is not absolute for the nonstandard universes. For example any $I \in {}^*\mathbb{N} - \mathbb{N}$ is internally finite ($|I| = I^I$) but infinite (even $|I| \geq \aleph_1$).

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Proposition (κ -finitarization Principle)

$$(x \subseteq I \ \& \ |x| < \kappa) \rightarrow (\exists y \in I)([y \text{ is finite}]^I \ \& \ x \subseteq y)$$

Application of the Finitarization Principle

Let $G = \langle V, E \rangle$ be a graph such that each of its finite subgraphs is colorable by at most k -colors (k -colorable). We prove that G is k -colorable.

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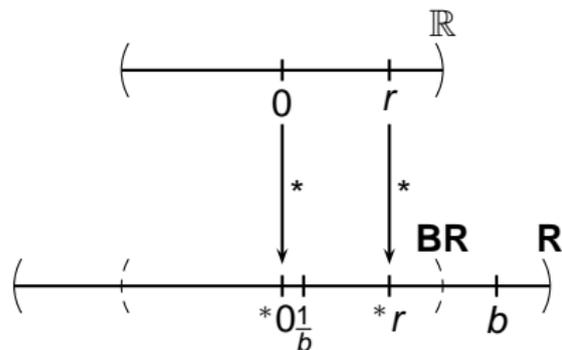
By \mathfrak{K} -finitarization principle there is an internally finite set Y with $V \subseteq Y \subseteq {}^*V$. Then internally $\langle Y, {}^*E \upharpoonright Y \rangle$ is a finite subgraph of *G and hence k -colorable. The restriction of the coloring to $V \subseteq Y$ is the k -coloring of G .

Real Numbers

Denote $\mathbf{R} = {}^*\mathbb{R}$ the **standard set of real numbers**. We will identify \mathbb{R} with the subset ${}^\sigma\mathbf{R} = \{{}^*r; r \in \mathbb{R}\} \subseteq \mathbf{R}$ of standard real numbers.

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b is infinitely big
 $1/b$ is infinitely small
 $1/b \in \mathbf{BR} - {}^\sigma\mathbf{R}$

$\mathbf{BR} = \{r \in \mathbf{R}; |r| < s \text{ for some } s \in {}^\sigma\mathbf{R}\}$ is the set of **bounded real numbers**.

Definition

The indifference \sim on \mathbf{R} is defined as follows:

$$x \sim y \Leftrightarrow |x - y| < 2^{-n} \text{ for all } n \in \mathbb{N}$$

If $x \sim y$, we say that x and y are *infinitely close* to each other.

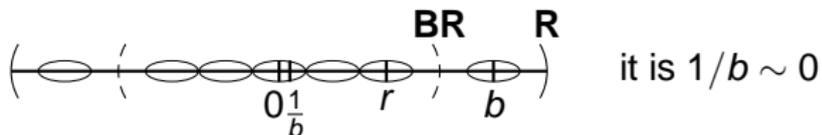
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The indifference \sim is an equivalence and hence gives a partition of \mathbf{R} . The equivalence classes $[x]_{\sim}$ are called **monads**.



Theorem (Fundamental about \mathbf{R})

For every $r \in \mathbf{BR}$ there is standard $s \in \mathbb{R}$ such that $r \sim s$.

Non-standard Analysis

Definition (continuous and uniformly continuous function)

We say that a standard (partial) function $f \subseteq \mathbf{R} \times \mathbf{R}$ is continuous in a standard point $a \in \text{dom}(f)$ if for every $x \in \text{dom}(f)$ it is

$$a \sim x \rightarrow f(a) \sim f(x). \quad (1)$$

Standard function $f \subseteq \mathbf{R} \times \mathbf{R}$ is continuous if (1) holds for every standard $a \in \text{dom}(f)$ and it is uniformly continuous if (1) holds for every $a \in \text{dom}(f)$.

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Definition (derivative)

For a standard function $f \subseteq \mathbf{R} \times \mathbf{R}$ and a such that $[a]_{\sim} \subseteq \text{dom}(f)$ we say that $b \in \mathbb{R}$ is the derivative of f in a if

$$0 \neq \delta \sim 0 \rightarrow \frac{f(a + \delta) - f(a)}{\delta} \sim b.$$

Strange functions

Let $\sin : \mathbf{R} \rightarrow \mathbf{R}$ be the standard sinus function and $I \in {}^*\mathbf{N} - \mathbf{N}$. Define $\sigma : \mathbf{R} \rightarrow \mathbf{R}$ as the “standard trace” of $\sin(Ix)$, i.e. for $r \in \mathbf{R}$ let $\sigma(r)$ be the unique standard element of the monad $[\sin(Ir)]_{\sim}$.

Strange functions

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Another “strange” function can be defined as follows.

Suppose $\kappa > |\mathbf{R}|$. Then by the κ -finitarization principle there is $y \in \mathbf{I}$ internally finite such that $\mathbf{R} \subseteq y \subseteq \mathbf{R}$. Let internally $(y)_i, i < |y|$ be the ascending enumeration of y . Let $f : \mathbf{R} \mapsto \{0, 1\}$ is defined as

$$f(y_i) = \begin{cases} 0 & \text{for } i \text{ even} \\ 1 & \text{for } i \text{ odd} \end{cases}.$$

Differential equations

Let standardly $u'(t) = f(u(t))$ be an ordinary differential equation with the condition $u(t_0) = a$ and with f bounded. We show the idea of finding the solution on an interval $[t_0, t_0 + r)$ using non-standard methods.

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Idea: First find an internal “infinitesimal difference solution”. Let $l \in {}^*\mathbb{N} - \mathbb{N}$ and $\Delta = r/l$; it is $\Delta \sim 0$. Now internally: For $i < l$ let $t_i = t_0 + i \cdot \Delta$ and (y_i) , $i < l$ is defined recursively $y_0 = a$, $y_i = y_{i-1} + \Delta \cdot f(y_{i-1})$.

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Then $U : t_i \mapsto y_i$ is a partial function. Since f is standardly bounded by some $M \in \mathbf{BR}$, for $t_i \sim t_j$, $i < j$ we have

$$|y_j - y_i| = \left| \sum_{k=i}^{j-1} \Delta \cdot f(y_k) \right| \leq (j - i) \cdot \Delta \cdot M = (t_j - t_i) \cdot M \sim 0.$$

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We define the “standard trace” of U . For $x \in [t_0, t_0 + r)$ standard let i be an index such that $t_i \sim x$ and let $y(x)$ be the unique standard element of the monad $[y_i]_{\sim}$. Then $u = {}^*y$ is the solution of the original differential equation.

Thanks

Thank you for your attention.