Geometric algebra III

Renáta Ševčíková

As we have already seen, there are some relations among projective space, projective geometry (as special closure space) and modular geometric lattices. In this part of the talk about geometric algebra we focus on showing how one can build a field (commutative) such that the points of a given geometry (with certain axioms) can be described by coordinates from the constructed field.

The geometry in which we will work is given by the following axioms:

(1) Axiom: Given two distinct points P, Q, then there exists a unique line l, s.t. $P \in l$ & $Q \in l$. $(l = P + Q)$

(2) Axiom: Given a point P and a line l, there exists exactly one line m, s.t. $P \in m$ & $m \parallel l$.

(3) Axiom: There exist three distinct points A, B, C, s.t. $C \notin A + B$; non-collinear points.

(4) Axiom: Given any two points P, Q, there exists a translation τ_{PQ}: $\tau_{PQ}(P) = Q$.

(5) Axiom: If τ_1 and τ_2 are translations with the same direction and if $\tau_1 \neq 1$, $\tau_2 \neq 1$, $\tau_1 \neq \tau_2$, then there exists unique $\alpha \in \mathcal{K}$: $\tau_2 = \alpha(\tau_1)$;
or equivalently for a given point P: Given points Q, R, s.t. P, Q, R are distinct but line on a line, then there exists a dilatation σ: $\sigma(P) = P$ & $\sigma(Q) = R$.

1 Construction of a field

Definition 1 (Dilatation and trace)**.** The map $\sigma : P \to P$ is called *dilatation* if it has the following property:
Let us have two distinct points P and Q, s.t. $\sigma(P) = P'$ and $\sigma(Q) = Q'$ If there is some line $l' \parallel P + Q$ and $P' \in l'$ then also $Q' \in l'$.

We call a dilatation σ *degenerate* if it maps all points onto one given point. Let σ be non-degenerate dilatation and P point on line l, s.t. $\sigma(P) \in l$. If $P \neq \sigma(P)$ then we call the line l the *trace* of σ; $l = P + \sigma(P)$.

Definition 2 (Translation and direction)**.** A non-degenerate dilatation τ is called a *translation* if $\tau = 1$ (idenitity) or if it has no fixed point. In this case the trace is called the *direction*.

Lemma 3. *In given geometry all dilatations form a group $\mathcal{D}$ and all translations form its sub-group $\mathcal{T}$. If the translations are of different directions, then $\mathcal{T}$ is commutative group.*

Definition 4 (Trace-preserving homomorphism)**.** A map $\alpha\colon T \to T$ is called the *trace-preserving homomorphism* if

- α is a homomorphism: $\alpha(\tau_1 \cdot \tau_2) = \alpha(\tau_1) \cdot \alpha(\tau_2)$
- α *preserves* traces; i.e., either $\alpha(\tau) = 1$ or τ and $\alpha(\tau)$ have the same direction.

The set of all α will be denoted by $\mathcal{K}$.

Definition 5. Let α and $\beta \in \mathcal{K}$. We construct maps $\alpha + \beta$ and $\alpha \cdot \beta$:

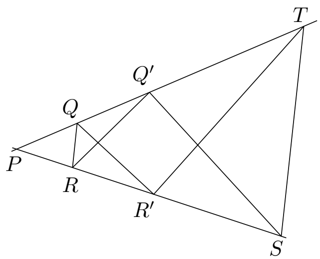
$$\alpha + \beta\colon \tau \to \alpha(\tau) \cdot \beta(\tau)$$
$$\alpha \cdot \beta\colon \tau \to \alpha(\beta(\tau))$$

Theorem 6. *If α and $\beta \in \mathcal{K}$, then $\alpha + \beta$ and $\alpha \cdot \beta \in \mathcal{K}$. Under this definition the set $\mathcal{K}$ becomes an associative ring with unit element* 1 *(identity).*

Theorem 7 (Desargues' theorem)**.** *Let l_1, l_2, l_3 be distinct lines, which are either parallel or meet in some point P. If $Q, Q' \in l_1$, $R, R' \in l_2$ and $S, S' \in l_3$ and we assume that $Q+R \parallel Q'+R'$ and $Q+S \parallel Q'+S'$ then also $S+R \parallel S'+R'$.*

Theorem 8. *Desargues' theorem holds in projective space satisfying all the five axioms above.*

We will show the condition under which the field $\mathcal{K}$ is commutative by introducing Pappus' theorem. The Pappus' theorem is the following configuration:



$Q+R'$ and $Q'+S$, $Q'+R$ and $R'+T$, $Q+R$ and $T+S$ are pairs of opposite sides of the hexagon. Now "if two of these pairs are parallel, so is the third" (PT).

Theorem 9 (Artin E., Geometric algebra, 1964)**.** *The field $\mathcal{K}$ is commutative iff in given geometry the Pappus' theorem holds.*

Lemma 10. *In a Desarguesian plane with only finite number of points the Pappus' theorem holds.*

2 Conclusion

In previous parts of talk about geometric algebra it was said that

- linear subspaces of projective space form a modular geometric lattice.
- linear subspaces define the geometry whose subspaces are exactly the linear spaces (projective geometry).

All proven theorems lead to result:

Theorem 11. *Let L be modular geometric lattice. Then L satisifies the arguesian identity iff Desargues' theorem holds in the associated projective geometry and there exists a field $\mathcal{K}$ whose elements are exactly the atoms in L.*