Finite model theory

Pavel Paták

Charles University in Prague

Faculty of Mathematics and Physics

Spring School of Algebra, April 7, 2010

Definitions

We will use first order logic (FO) and assume that the language contains at least one binary symbol.

Let φ be a formula, T a theory and Γ a set of formulas:

Definition

- φ is valid (in the finite case) in T if all (finite) models of T satisfy φ
- φ is valid (in the finite case) if all (finite) models satisfy φ
- φ Γ implies φ (in the finite case) iff φ is valid in all (finite) models of Γ

Classical infinite logic

Let us assume that we have some kind of complete logical calculus.

Theorem

- *Sound* *We cannot prove contradiction.*
- *Complete* *If φ is true, there exists a proof of φ.*
- *Compact* *If φ is a consequence of some set Γ, it is a consequence of some finite subset of Γ*

Example

If Γ implies φ in the finite, is there necessarily some finite subset $\Gamma' \subseteq \Gamma$?

Completeness in finite models

Theorem (Church)

The set of all valid FO sentences is recursively enumerative (r.e.) but not co-r.e.

Theorem (Trakhtenbrot)

The set of all FO sentences valid in the finite case is not r.e. but is co-r.e.

Corollary

There is no logic calculus which can decide whether a sentence is valid in the finite case.

Craig's interpolation theorem, Beth's definability theorems

Theorem (Craig's interpolation formula)

If T implies $\varphi \to \psi$, then there is a formula χ containing only logical connectives and non-logical symbols which occurs both in φ and ψ, such that T implies $\varphi \to \chi$ and $\chi \to \psi$.

Theorem (Beth definability theorem)

A relation is defined implicitly if and only if it can be defined explicitly.

Theorem

There is no recursive function which bounds the size of the interpolant in the terms of the length of input formulas.

Other examples

Example

- Substructure preserving theorem
- Elementary equivalence
- Löwenheim-Skolem theorem

Theorems and conjectures in the finite case

Theorem (Robinson's joint consistency theorem)

If $T_1 \supseteq T$ and $T_2 \supseteq T$ are theories such that T is a complete theory in the intersection of languages of T_1 and T_2 then $T_1 \cup T_2$ is consistent (in the finite case).

Example (Generalized Asser's conjecture)

Is the class of generalized spectra closed under complements?

1-0 law

Definition

We say that a sentence φ is almost surely true, iff

$$\lim_{n \to \infty} p_n = 1,$$

where p_n denotes the probability that a random $\mathcal{L}$-structure on $\{1, 2, 3, \ldots, n\}$ will satisfy φ.

Theorem (1-0 law)

Let σ be any sentence. Then σ or its negation is almost surely true.

Stronger logic

Fixed point logic

- Better describes the properties of finite structures (can e.g. capture if the graph is connected)
- In the presence of linear order describes PTIME
- Can not express evenness

Second order logic

- Better describes the properties of finite structures (can even capture evenness)
- Deciding whether $A \models \varphi$ is not in PTIME (unless P=NP)