Fully Homomorphic Encryption

based on Craig Gentry's PhD thesis

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RSA homomorphic encryption

- plaintexts π_i
- ciphertexts $\psi_i = (\pi_i)^e \mod N$
- RSA is multiplicatively homomorphic

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- Indeed NAND is a universal gate

Fully Homomorphic Encryption

We are looking for encryption scheme $\mathcal{E}$ that consists of

- $\text{KeyGen}_{\mathcal{E}}$ key-generation algorithm
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- $\text{Evaluate}_{\mathcal{E}}$ evaluation algorithm

Fully Homomorphic Encryption

Algorithm Evaluate$_{\mathcal{E}}$ is efficient and

- for any public key pk, and
- for any circuit C, and
- for any ciphertexts $\psi_i = \text{Encrypt}_{\mathcal{E}}(pk, \pi_i)$

outputs

$$\psi \leftarrow \text{Evaluate}_{\mathcal{E}}(pk, C, \psi_1, \ldots, \psi_t)$$

a valid ciphertext of $C(\pi_1, \ldots, \pi_t)$
(i.e. $C(\pi_1, \ldots, \pi_t) = \text{Decrypt}_{\mathcal{E}}(sk, \psi)$)

Of course, we want to avoid trivial "do-it-yourself" solutions such as

$$\text{Evaluate}_{\mathcal{E}}\left(pk, C, \psi_1, \ldots, \psi_t\right) = \left(C, \psi_1, \ldots, \psi_t\right)$$

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thus, we require the output of $\text{Evaluate}_{\mathcal{E}}$ to be bounded or even to look like a "normal" ciphertext

Definition of $\mathcal{C}_{\mathcal{E}}$

Definition (Homomorphic encryption)

Scheme $\mathcal{E}$ is homomorphic for circuits in $\mathcal{C}_{\mathcal{E}}$ if $\mathcal{E}$ is correct for $\mathcal{C}_{\mathcal{E}}$ and $\text{Decrypt}_{\mathcal{E}}$ can be expressed as a circuit $D_{\mathcal{E}}$ of size $\text{poly}(\lambda)$ where λ is the security parameter.

Definition (Fully Homomorphic encryption)

Scheme $\mathcal{E}$ is fully homomorphic if it is homomorphic for all circuits.

Definition (Leveled Fully Homomorphic encryption)

A family of schemes $\mathcal{E}^{(d)}$ is leveled fully homomorphic if they use same decryption circuit and $\mathcal{E}^{(d)}$ is fully homomorphic for all circuits up to depth d.

3-step Construction

- "Bootstraping" encryption scheme

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- Ideal Lattices

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- "Bootstraping" encryption scheme
- Ideal Lattices
- Shortening decryption circuit

Bootstrapable Scheme is the one that can evaluate its own decryption circuit with a "little" add-on.
Evaluating the decryption circuit means finding different ciphertexts for the same plaintext without knowing the plaintext and hoping the new plaintext is "better".

If $\mathcal{E}$ is bootstrapable with plaintext space $\mathcal{P} = \{0, 1\}$ and

- $\psi_1 = \text{Encrypt}_{\mathcal{E}}(pk_1, \pi)$ is old noisy ciphertext
- $\overline{sk_{1j}} = \text{Encrypt}_{\mathcal{E}}(pk_2, sk_1)$
  - sk_1 is the secrect key corresponding to public key pk_1
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then we can consider algorithm

$\mathsf{Recrypt}_{\mathcal{E}}(pk, D_{\mathcal{E}}, \langle \overline{sk_{1j}} \rangle, \psi_1)$

1. Set $\overline{\psi_{1j}} \xleftarrow{R} \mathsf{Encrypt}_{\mathcal{E}}(pk_2, \psi_{1j})$
2. Output $\psi_2 \leftarrow \mathsf{Evaluate}_{\mathcal{E}}(pk_2, D_{\mathcal{E}}, \langle \langle \overline{sk_{1j}} \rangle, \langle \overline{\psi_{1j}} \rangle \rangle)$

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We "refreshed" ciphertext old ψ_1 to (hopefully) fresh $\psi_2 = \text{Encrypt}_{\mathcal{E}}(sk_1, \pi)$

If $\mathcal{E}$ is bootstrapable, we can evaluate circuits of arbitrary length since we can

- do a little work - NAND gate
- when necessary, refresh ciphertext, i.e. lower error

Definition

Augmented Decryption Circuit Let Γ be a set of gates. For a gate $g \in \Gamma$, the g-augmented decryption circuit consist of g-gate connecting multiple copies of $D_{\mathcal{E}}$ on input. We denote the set of g-augmented decryption circuits with $g \in \Gamma$ by $D_{\mathcal{E}}(\Gamma)$.

Definition

Bootstrapable Encryption Let $\mathcal{C}_{\mathcal{E}}$ be a set of circuits with respect to which $\mathcal{E}$ is homomorphic. We say that $\mathcal{E}$ is bootstrapable with respect to Γ if

$$D_{\mathcal{E}}(\Gamma) \subseteq \mathcal{C}_{\mathcal{E}}$$

Circuicts of arbitrary length

- Suppose scheme $\mathcal{E}$ is bootstrapable, i.e. evaluates its own decryption circuit augmented by gates in Γ
- Then there exists $\mathcal{E}_\delta$ that evaluates arbitrary circuits with gates in Γ with depth at most δ
- Ciphertexts same size in $\mathcal{E}_\delta$ as in $\mathcal{E}$
- Public key:
  - $\delta + 1$ $\mathcal{E}$ public keys $pk_0, \ldots, pk_\delta$
  - δ encrypted secret keys $\text{Encrypt}(pk_i, sk_{i-1})$
  - length linear in δ
  - if $\mathcal{E}$ KDM[1]-secure, constant length public key is $pk, \text{Encrypt}(pk, sk)$

[1] Key Dependent Message, specifically circular-secure, we can encrypt private key with its public key

So what's the lesson learned from bootstrapability?

If we want to create a fully homomorphic encryption, we have to find one that

- can evaluate its (augmented) decryption circuit
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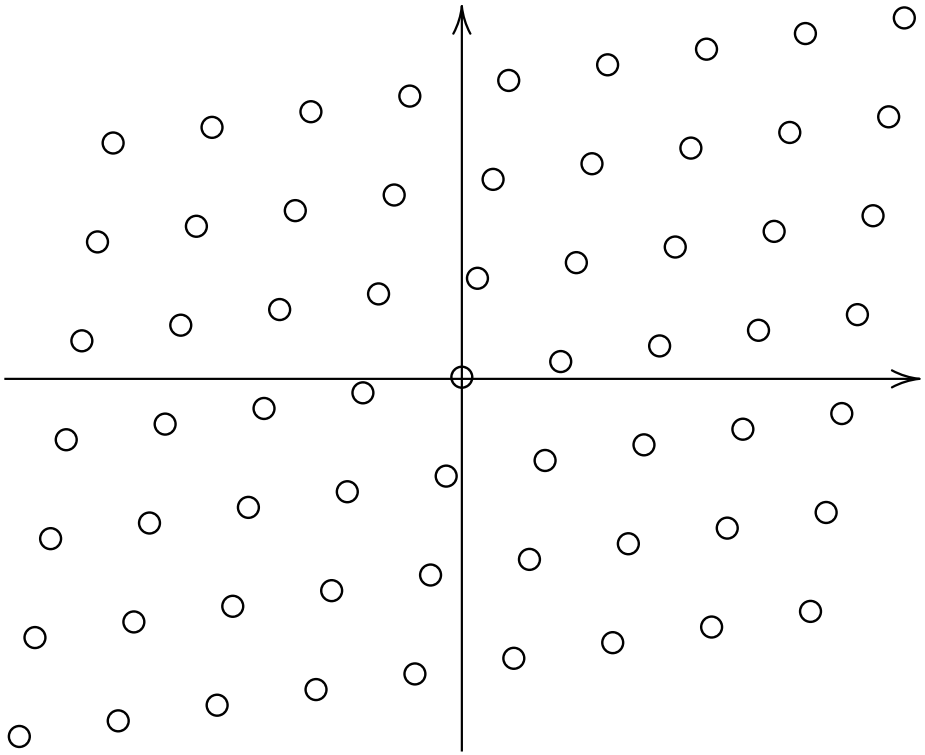
If we want to create a fully homomorphic encryption, we have to find one that

- can evaluate its (augmented) decryption circuit
- thus, it has "shallow" (short) decryption circuit
- thus RSA is out of the question (exponentiation is expensive and long, "unparallelizable")
- lattice based cryptosystems' decryption is often a simple inner product (shallow circuit)

What is a lattice?

2-dimensional lattice $\mathcal{L}$ is a set of $\{c_1\mathbf{u_1} + c_2\mathbf{u_2}\}$ where $c_1, c_2 \in \mathbb{Z}$ and $\mathbf{u_1}, \mathbf{u_1}$ are linearly independent



(Watch the blackboard for the example of a simple lattice encryption scheme :-))

What is an ideal lattice?

- Lattice: $\mathcal{L} = \left\{ \sum_{1 \leq i \leq n} \alpha_i \mathbf{u}_i, \alpha_i \in \mathbb{Z} \right\}$, $\mathbf{u}_i$ linearly independent
- Ideal I in ring R satisfies $I \subseteq R$ and $ri \in I$ for any $i \in I, r \in R$

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- Ideal I in ring R satisfies $I \subseteq R$ and $ri \in I$ for any $i \in I, r \in R$
- for $f(x)$ monic and $R = \mathbb{Z}[x]/f(x)$ ideal lattice is two objects at once
  - ideal $(a(x))$ in R, $\deg a = n$, (i.e. multiples of $a(x)$ modulo $f(x)$), multiplication structure
  - lattice in $\mathbb{Z}$ generated by coefficient vectors of $(x^i a(x) \mod f(x))$, for $1 \leq i \leq n-1$, addition structure

Initial construction

- $\mathrm{KeyGen}_{\mathcal{E}}(R, B_I)$
  - $(B_J^{sk}, B_J^{pk}) \leftarrow \mathrm{IdealGen}(R, B_I)$
  - public key pk includes R, B_I, B_J^{pk}, sampling algorithm Samp from $R \mod I$ to R
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- $\text{Encrypt}_{\mathcal{E}}(pk, \pi)$
  - $\psi' \xleftarrow{R} \pi + I$ (using $\text{Samp}(\pi, B_I, R, B_J^{pk})$ algorithm)
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  - $\psi \leftarrow \psi' \mod B_J^{pk}$
- $\text{Decrypt}_{\mathcal{E}}(sk, \psi)$
  - $\pi \leftarrow (\psi \mod B_J^{sk}) \mod B_I$
- $\text{Evaluate}_{\mathcal{E}}(pk, C, \Psi)$
  - circuit C on input is "$\mod B_I$" circuit on plaintexts
  - to create generalized circuit $g(C)$ replace Add_{B_I} and Mul_{B_I} by standard ring operations, i.e. drop "$\mod B_I$"
  - output the result of $g(C)$ with Ψ on input

Necessary notation

- X_{ENC} - image of Samp
- X_{DEC} - representatives of mod B_J^{sk} classes, i.e. R mod B_J^{sk}
- permitted circuit - any circuit satisfying $g(C)(X_{ENC}) \subseteq X_{DEC}$
- $C_{\mathcal{E}}$ - set of all permitted circuits
- valid ciphertext - output of Evaluate$_{\mathcal{E}}$ with any valid input

Theorem (Correctness of $\mathcal{E}$)

For any permitted circuit $\mathcal{E}$ is correct, i.e. Decrypt *correctly decrypts all valid ciphertexts.*

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Proof.

$\Psi = \{\psi_1, \ldots, \psi_t\}$, *where* $\psi_k = \pi_k + i_k + j_k$ *and* $\pi_k + i_k \in X_{ENC}$

$$\text{Evaluate}_{\mathcal{E}}(pk, C, \Psi) = g(C)(\Psi) \mod B_J^{pk} \in g(C)(\pi_1 + i_1, \ldots, \pi_t + i_t) + J$$

Since C is permitted circuit $g(C)(\Psi) \in X_{DEC}$.

$$\text{Decrypt}(sk, \text{Evaluate}_{\mathcal{E}}(pk, C, \Psi)) = g(C)(\pi_1 + i_1, \ldots, \pi_t + i_t) \mod B_I$$
$$= g(C)(\pi_1, \ldots, \pi_t) \mod B_I = C(\pi_1, \ldots, \pi_t)$$ □

Addition and Multiplication

Geometrically, we see circuit is permitted if it "doesn't expand too much". Specifically

- addition is cheap: $\|u_1 + u_2\| \leq \|u_1\| + \|u_2\|$
- multiplication is worse. We can show for each ring $R = \mathbb{Z}[x]/f(x)$ there exists $\gamma_{\mathrm{MULT}}(R)$ such that for any two vectors u_1, u_2

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  in lattice associated with R
- Moreover, if $f(x) = x^n - g(x)$ with $\deg g \leq n - \frac{n-1}{k}$, $k \geq 2$ it holds $\gamma_{\text{MULT}}(R) \leq \sqrt{2n}(1 + 2n + (\sqrt{(k-1)n}\|f\|)^k)$.
- Note: There's "some" evidence $f(x) = x^n - 1$ is not be best choice so we'll avoid it.

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- Compute $\mathbf{v}_J^{sk} \in J^{-1} \subseteq \mathbb{Q}[x]/f(x)$ such that decryption simplifies to $\psi - \lfloor \mathbf{v}_J^{sk} \times \psi \rceil \mod B_I$ (after radius r_{DEC} adjustment)

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- Decreasing the complexity of decryption circuit ... next time.

Finally, the scheme is bootstrapable, i.e. fully homomorphic.

Thank you for your attention!
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