Quasigroups and their use in cryptography

Andrea Frisová

March 24, 2010

Latin square

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1	2	3	4	5
2	3	5	1	4
3	5	4	2	1
4	1	2	5	3
5	4	1	3	2

3	1	2
2	3	1
1	2	3

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2	1	0	3
3	2	1	0

$\cdot$	0	1	2	3
0	1	0	3	2
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2	2	1	0	3
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The number of Latin square of order n = The number of quasigroups of order n.

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A *quasigroup* is a set Q with a binary operation $*\colon Q \times Q \to Q$ such that for each $u, v \in Q$ there exist unique $x, y \in Q$ which satisfy $u * x = v$ and $y * u = v$.

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Equivalently:

A *quasigroup* is a set Q with binary operations $*, \backslash, /$ such that for all $u, v \in Q$ the following conditions are satisfied:

$$u\backslash(u \cdot v) = v \qquad u \cdot (u\backslash v) = v$$

$$(v \cdot u)/u = v \qquad (v/u) \cdot u = v$$

The number of Latin squares/quasigroups of order n

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There is no known easily-computable formula for the number of quasigroups of order n.
There exist some lower and upper bounds for the number of the Latin squares/quasigroups, $L(n)$, where n is large. For example

$$\prod_{k=1}^{n} (k!)^{n/k} \geq L(n) \geq \frac{(n!)^{2n}}{n^{n^2}}$$

(given by van Lint and Wilson).

The number of Latin squares of order n

n	Number of Latin squares of order n	Reference
1	1	
2	2	
3	12	
4	576	
5	161280	Euler (1782), Cayley (1890), MacMahon (1915; incorrect value)
6	812851200	Frolov (1890) and Tarry (1900)
7	61479419904000	Frolov (1890, incorrect), Norton (1939, incomplete), Sade (1948), Saxena (1951)
8	108776032459082956800	Wells (1967)
9	5524751496156892842531225600	Bammel and Rothstein (1975)
10	9982437658213039871725064756920320000	McKay and Rogoyski (1995)
11	776966836171770144107444346734230682311065600000	McKay and Wanless (2005)
12	$12! \cdot 11! \cdot 1.62 \cdot 10^{44}$	McKay and Rogoyski (1995)
13	$13! \cdot 12! \cdot 2.51 \cdot 10^{56}$	McKay and Rogoyski (1995)
14	$14! \cdot 13! \cdot 2.33 \cdot 10^{70}$	McKay and Rogoyski (1995)
15	$15! \cdot 14! \cdot 1.5 \cdot 10^{86}$	McKay and Rogoyski (1995)

Isotopy

An *isotopy* between quasigroups $(Q, *)$ and $(P, \cdot)$ is a triple (α, β, γ) of bijections from Q onto P such that

$$\alpha(x) \cdot \beta(y) = \gamma(x * y) \quad \text{for all } x, y \text{ in } Q.$$

We then say that quasigroups $(Q, *)$ and $(P, \cdot)$ are *isotopic*.

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2	1	0	3	0	1	2	3	3	0	1	2
3	2	1	0	1	2	3	0	1	2	3	0

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0	3	2	1		2	3	0	1		2	3	0	1		2	3	1	0
2	1	0	3		0	1	2	3		3	0	1	2		3	1	0	2
3	2	1	0		1	2	3	0		1	2	3	0		0	2	3	1

Quasigroups

Quasigroups

Isotopic to a group

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Proposition 1

Each quasigroup is isotopic to a loop.

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Proposition 2

Each loop isotopic to a group is a group.

Quasigroups

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Loops

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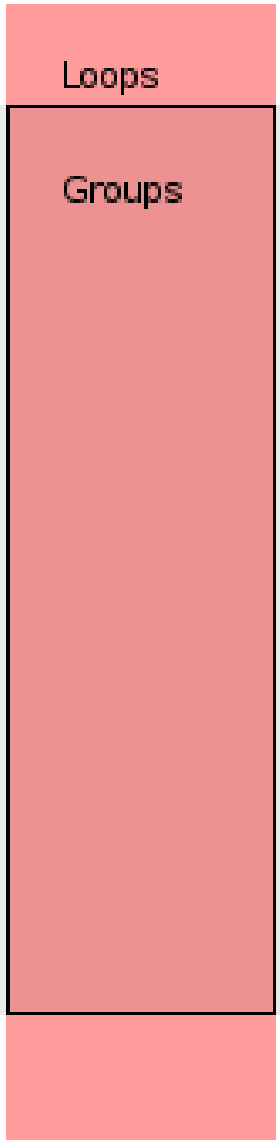
Quasigroups

Isotopic to a group

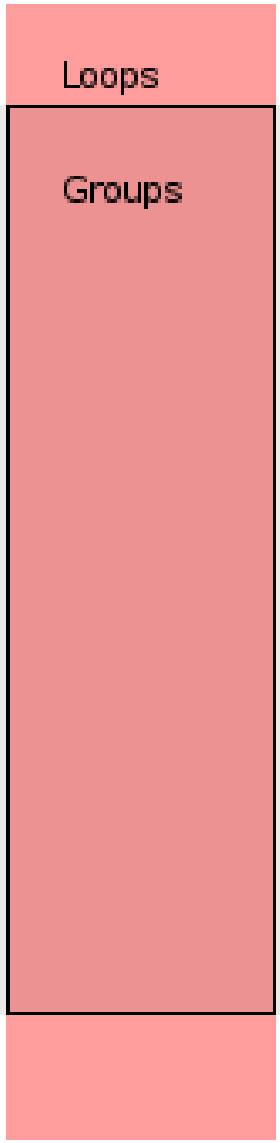


Proposition 3

Isotopic groups are isomorphic.

Quasigroups

Isotopic to a group



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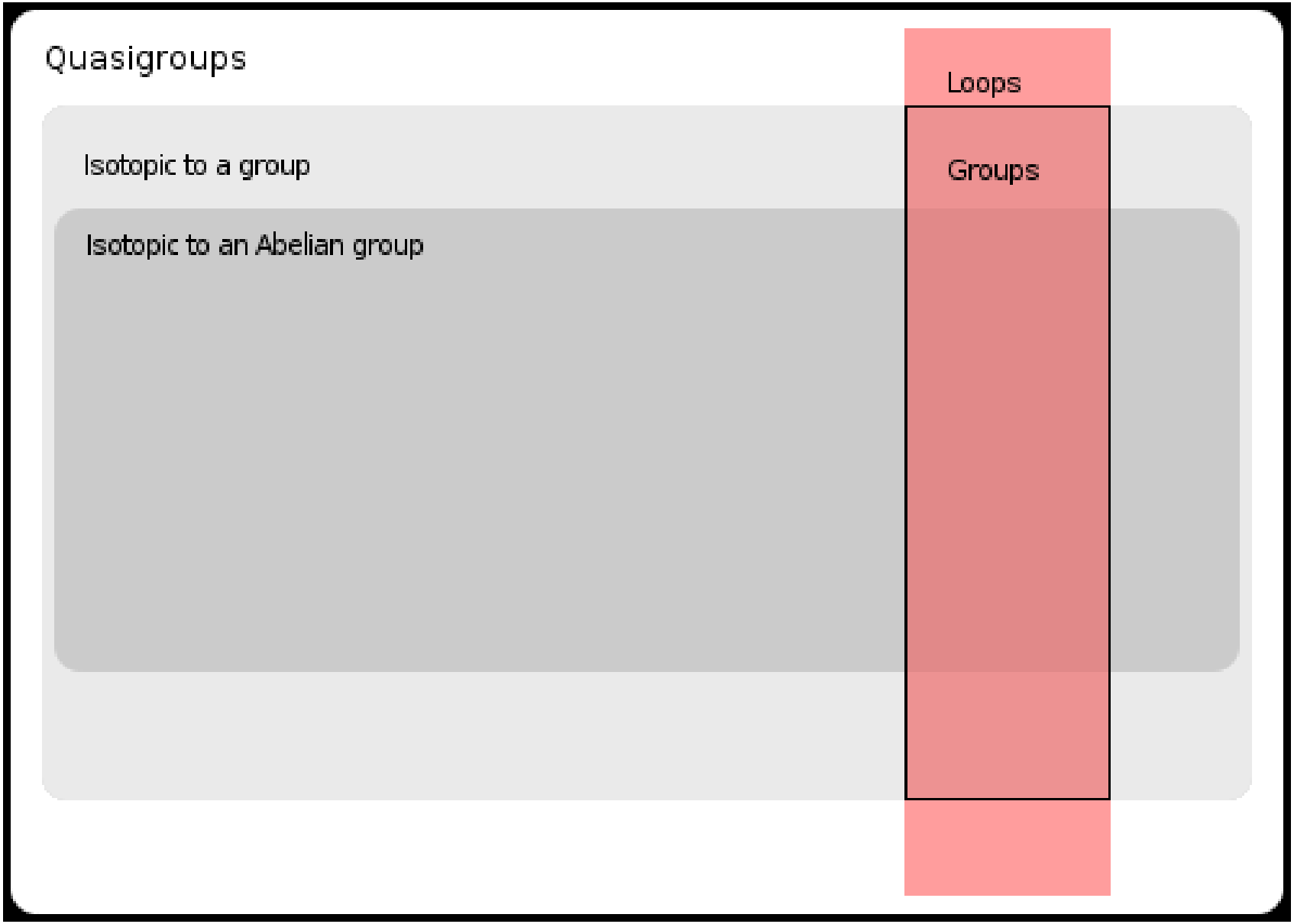

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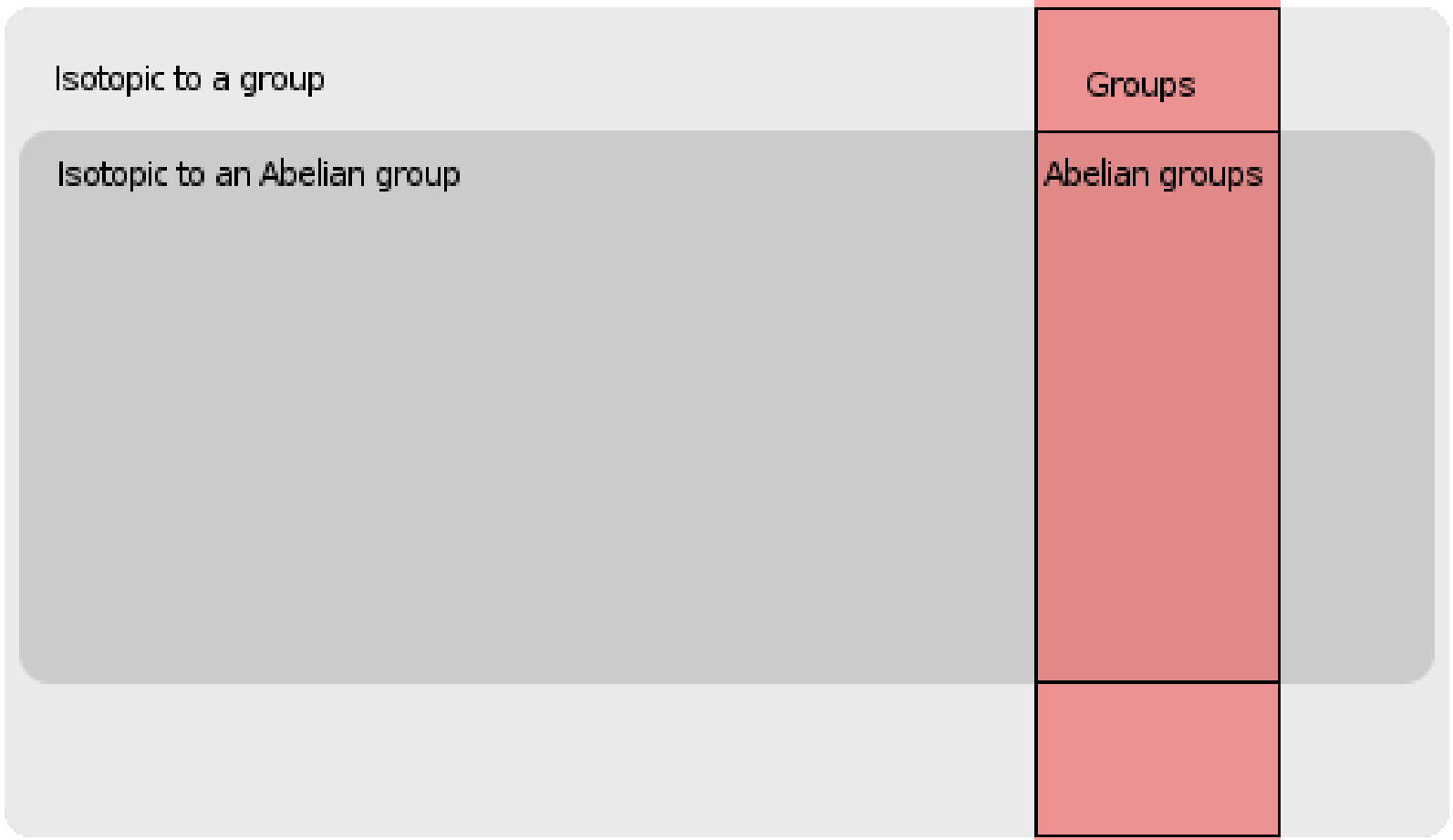

Central and medial quasigroups

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A *central quasigroup* is a quasigroup $(Q, *)$ such that there exists an Abelian group $(Q, +)$, $\alpha, \beta \in \mathrm{Aut}((Q, +))$, and $c \in Q$ such that

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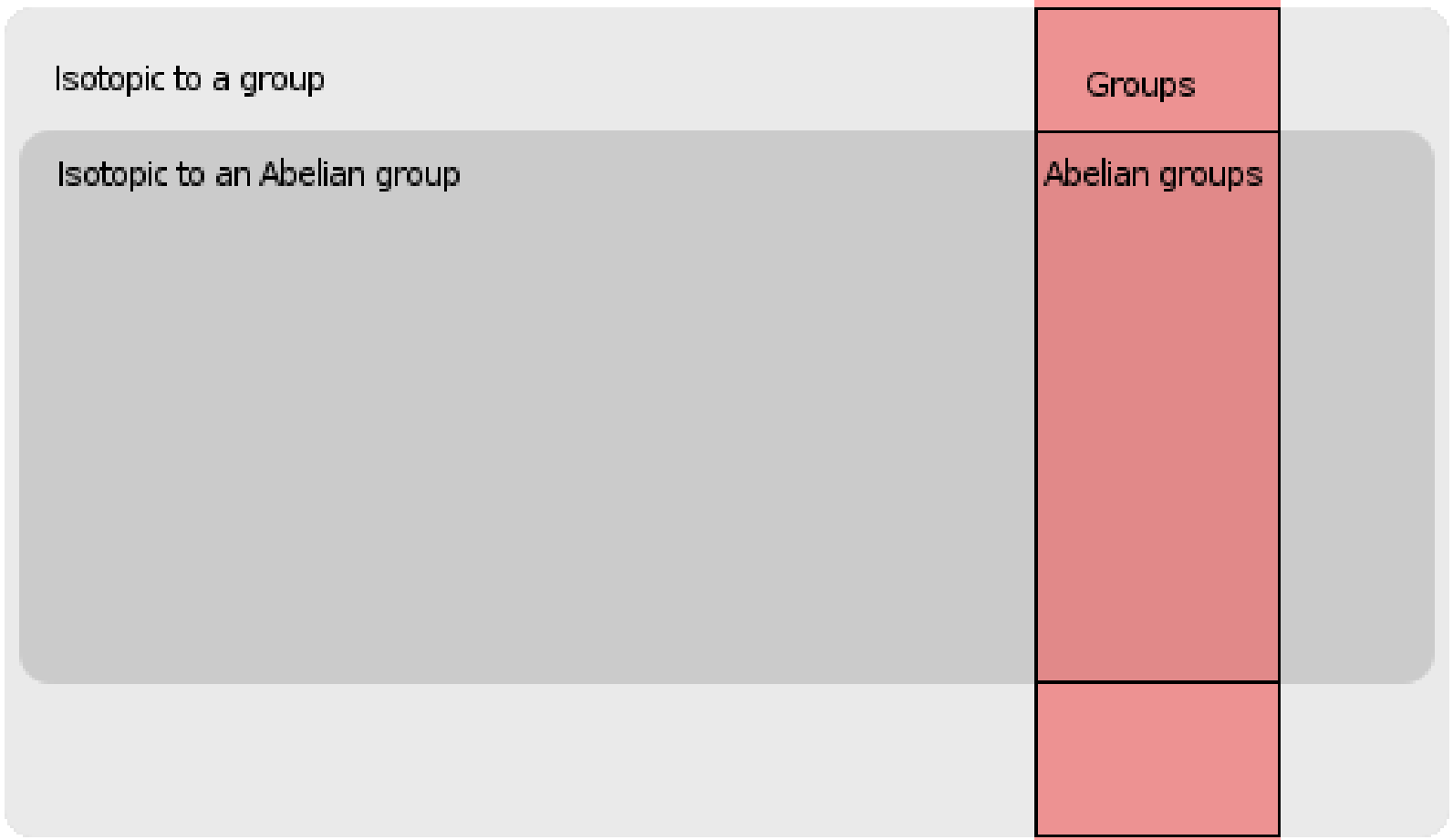
Proposition 4

Each central quasigroup is isotopic to an Abelian group.

Quasigroups



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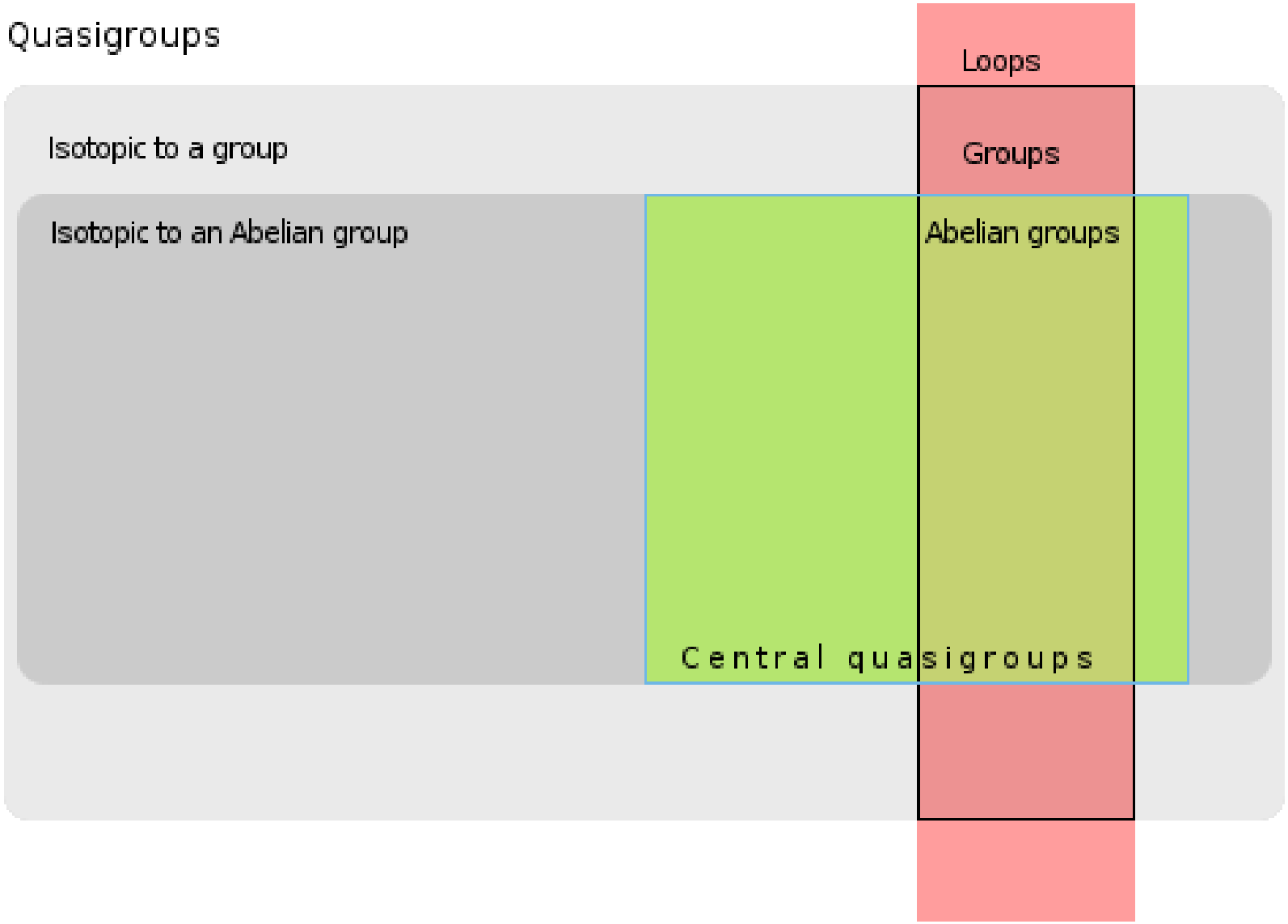

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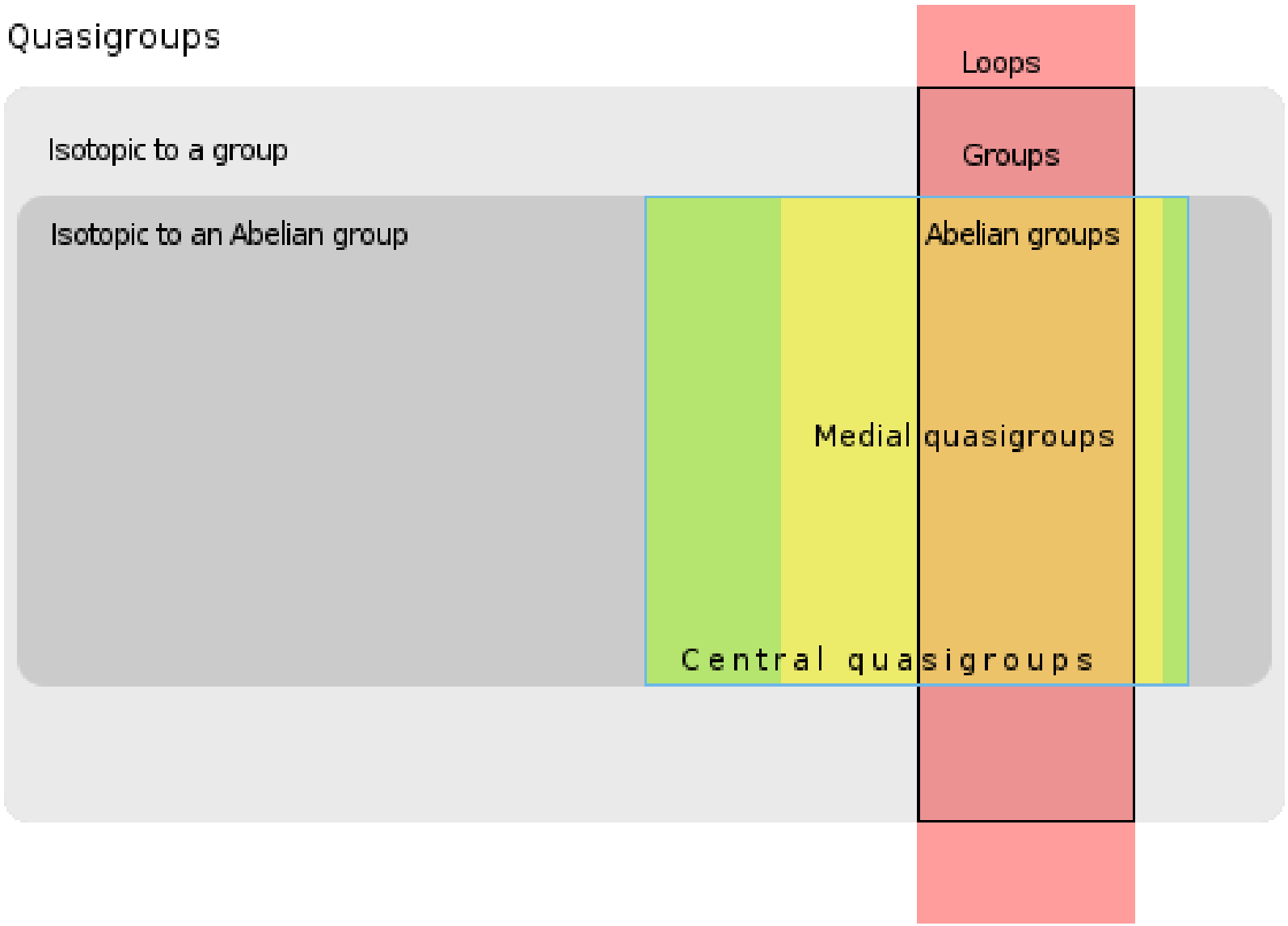

Quasigroups in cryptography

Quasigroups can be used in

- symmetric-key cryptography – stream cipher Edon-80

All designed by D. Gligoroski and collaborators.

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- public-key cryptography – MQQ (next talk by Adam Christov)

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Quasigroups can be used in

- symmetric-key cryptography – stream cipher Edon-80
- public-key cryptography – MQQ (next talk by Adam Christov)
- hash functions – Edon-R.

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Edon-R

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Bijections α and β are generated by using certain matrices.

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- isotopic to an Abelian group $(\mathbb{Z}_4, +)$
- non-medial
- non-central

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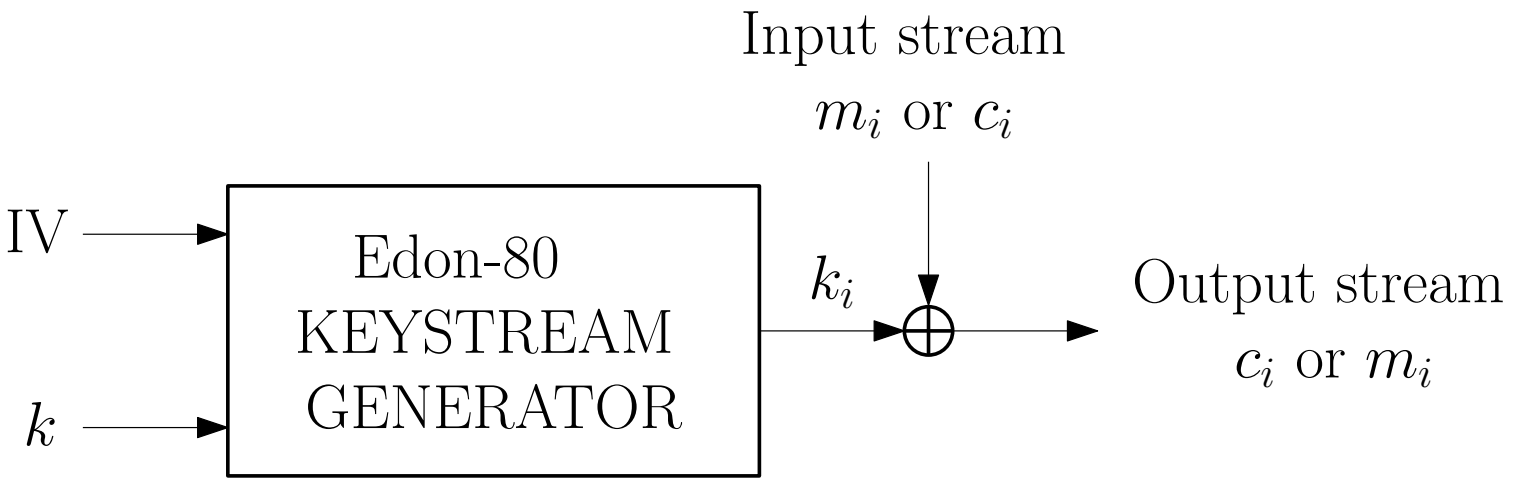

$k = K_0 \ldots K_{39}$ is a key
$IV = v_0 \ldots v_{39}$ is an initialisation value

Edon-80 – Description

A keystream is generated using the following map:

$$\tau(*, y)\colon (a_i) \to (b_i).$$

		a_0	a_1	a_2	a_3	$\ldots$
$*$	y					

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$$b_0 = y * a_0,$$
$$b_i = b_{i-1} * a_i \quad \text{for } i > 0.$$

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$*_i$		0	1	2	3	0	1	2	$\cdots$
$*_0$	y_0								
$*_1$	y_1								
$\vdots$	$\vdots$								
$*_{79}$	y_{79}								

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$*_i$		0	1	2	3	0	1	2	$\cdots$
$*_0$	y_0	$t_{0,0}$	$t_{0,1}$	$t_{0,2}$	$t_{0,3}$	$t_{0,4}$	$t_{0,5}$	$t_{0,6}$	$\cdots$
$*_1$	y_1								
$\vdots$	$\vdots$								
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$*_1$	y_1	$t_{1,0}$	$t_{1,1}$	$t_{1,2}$	$t_{1,3}$	$t_{1,4}$	$t_{1,5}$	$t_{1,6}$	$\cdots$
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$*_1$	y_1	$t_{1,0}$	$t_{1,1}$	$t_{1,2}$	$t_{1,3}$	$t_{1,4}$	$t_{1,5}$	$t_{1,6}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
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$*_0$	y_0	$t_{0,0}$	$t_{0,1}$	$t_{0,2}$	$t_{0,3}$	$t_{0,4}$	$t_{0,5}$	$t_{0,6}$	$\cdots$
$*_1$	y_1	$t_{1,0}$	$t_{1,1}$	$t_{1,2}$	$t_{1,3}$	$t_{1,4}$	$t_{1,5}$	$t_{1,6}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$*_{79}$	y_{79}	$t_{79,0}$	$\mathbf{t_{79,1}}$	$t_{79,2}$	$\mathbf{t_{79,3}}$	$t_{79,4}$	$\mathbf{t_{79,5}}$	$t_{79,6}$	$\cdots$

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- $* = *_i$ for all $i = 1, 2, \ldots, 79$ (we use only one quasigroup),

$*$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	$\cdots$
y_0								
y_1								
y_2								
y_3								
$\vdots$								

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We suppose that

- $* = *_i$ for all $i = 1, 2, \ldots, 79$ (we use only one quasigroup),
- $X = (x_i)$ a periodic sequence with a period P_X instead of sequence $012301230123\ldots$, and

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y_0								
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- $* = *_i$ for all $i = 1, 2, \ldots, 79$ (we use only one quasigroup),
- $X = (x_i)$ a periodic sequence with a period P_X instead of sequence $012301230123\ldots$, and
- $Y = (y_i)$ a sequence with no special property (we have arbitrary number of rows).

$*$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	$\cdots$
y_0								
y_1								
y_2								
y_3								
$\vdots$								

Periods

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- There are some indications that Edon-80 may generate a keystream with short periods (for some input values). This is a main reason for its failure in eSTREAM, ECRYPT stream cipher project.

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- There are some indications that Edon-80 may generate a keystream with short periods (for some input values). This is a main reason for its failure in eSTREAM, ECRYPT stream cipher project.
- Which quasigroups are the most suitable for Edon-80?

Periods for central quasigroups

For central quasigroups, this problem leads to the problem in the group ring $\mathbb{Z}_{e_G}[Aut(G)]$.

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Quasigroup type	Underlying group	P_i	$e_G \operatorname{lcm}(P_X, P_i)$
Medial	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$2^{\lceil \log_2 i \rceil}$	$2 \operatorname{lcm}(P_X, 2^{\lceil \log_2 i \rceil})$
Central, non-medial	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\frac{3}{2} \cdot 2^{\lceil \log_2 i \rceil}$	$2 \operatorname{lcm}(P_X, \frac{3}{2} \cdot 2^{\lceil \log_2 i \rceil})$
Central, non-medial	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$2 \cdot 2^{\lceil \log_2 i \rceil}$	$2 \operatorname{lcm}(P_X, 2 \cdot 2^{\lceil \log_2 i \rceil})$
Medial	$\mathbb{Z}_4$	$2 \cdot 2^{\lceil \log_2 i \rceil}$	$4 \operatorname{lcm}(P_X, 2 \cdot 2^{\lceil \log_2 i \rceil})$
Medial	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$3 \cdot 2^{\lceil \log_2 i \rceil}$	$2 \operatorname{lcm}(P_X, 3 \cdot 2^{\lceil \log_2 i \rceil})$

Thank you for your attention!