Definice 2.27. Mějme hladkou parametrickou křivku $\mathbf{c}(t)$, $t \in (\alpha, \beta)$ v $\mathbb{R}^n$ a reálnou funkci f definovanou na $\langle \mathbf{c} \rangle$. Pak definujeme *Křivkový integrál 1. druhu*

$$\int_{\mathbf{c}} f \mathrm{d}s := \int_{\alpha}^{\beta} f(\mathbf{c}(t)) ||\mathbf{c}'(t)|| \mathrm{d}t,$$

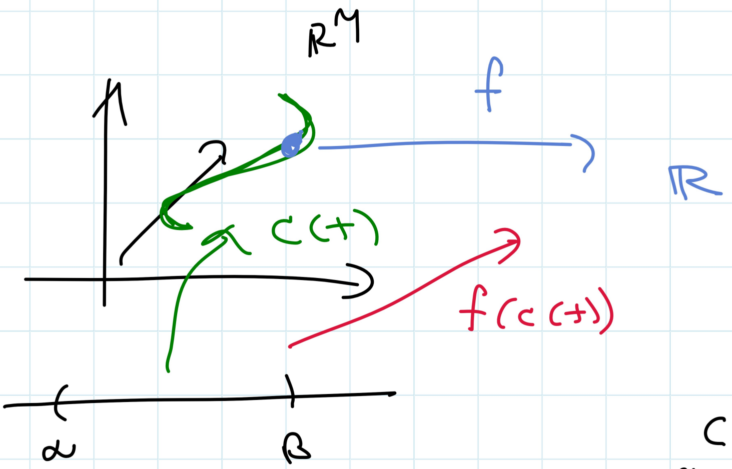
pokud integrál napravo existuje jako Lebesqueův integrál.

Věta 2.28. Křivkový integrál prvního druhu nezávisí na (re)parametrizaci.

Definice 2.29. Délku křivky definujeme jako integrál prvního druhu z konstantní jednotkové funkce

$$\ell(\mathbf{c}) := \int_{\mathbf{c}} 1 \mathrm{d}s.$$

$f(x,y,z) = x^2+y^2+z^2$



Dk 2.28

$c(s) = c(\phi(s))$ $\quad s \in (\tilde{\alpha}, \tilde{\beta})$

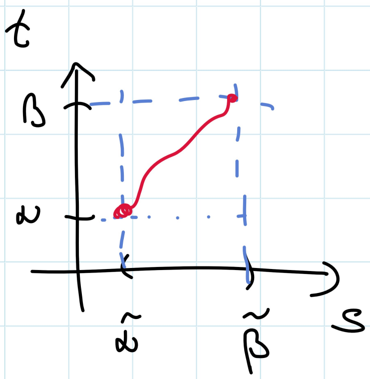
1) $\phi' > 0$

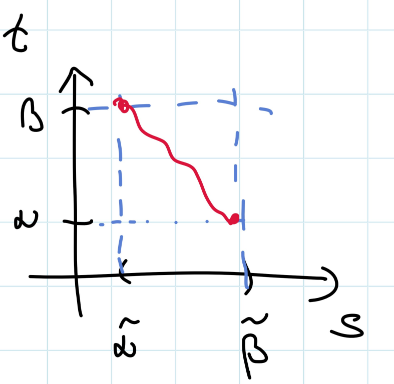
$$\int_{\tilde{\alpha}}^{\tilde{\beta}} f(c(s)) \, ||\dot{c}(s)|| \, ds =$$

$$= \int_{\tilde{\alpha}}^{\tilde{\beta}} f(c(\phi(s))) \, ||\dot{\phi}(s) \cdot c'(\phi(s))|| \, ds =$$

SUBTITUCE $t = \phi(s)$, $dt = \dot{\phi} \, ds$

$$= \int_{\alpha}^{\beta} f(c(t)) \, ||c'(t)|| \, \underbrace{\dot{\phi}(s) \, ds}_{dt}$$





2) $\dot{\phi} < 0$

$$\int_{\tilde{\alpha}}^{\tilde{\beta}} f(c(s)) \, ||\dot{c}(s)|| \, ds =$$

$$= \int_{\tilde{\alpha}}^{\tilde{\beta}} f(c(\phi(s))) \, ||\dot{\phi}(s) \cdot c'(\phi(s))|| \, ds =$$

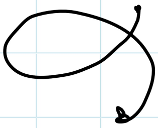
SUBS $dt = \dot{\phi} \cdot ds$

$$= - \int_{\beta}^{\alpha} f(c(t)) \cdot ||c'(t)|| \, \overbrace{\underbrace{|\dot{\phi}(s)|}_{-\dot{\phi}(s)} \, ds}^{-dt}$$

$\blacksquare$

PŘ

$$\|c'\| = \sqrt{2}$$



$$c(t) = (\cos t, \sin t, t) \qquad t \in (0, 2\pi)$$

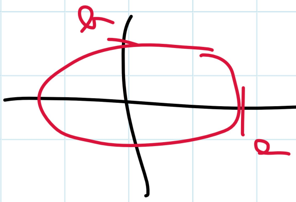
$$f(x,y,z) = x^2+y^2+z^2$$

$$\int_c f\, ds = \int_0^{2\pi} (\cos^2 t + \sin^2 t + t^2)\, \|c'(t)\|\, dt =$$

$$= \int_0^{2\pi} (t^2+1)\cdot\sqrt{2}\, dt = \sqrt{2}\left[\frac{t^3}{3}+t\right]_0^{2\pi} =$$

$$= \sqrt{2}\left(\frac{8\pi^3}{3}+2\pi\right)$$

PŘ délka elipsy



$$c(t) = (a\cdot\cos t, b\cdot\sin t) \qquad t\in[0,2\pi]$$

$$c' = (-a\sin t, b\cos t) \qquad \|c'\| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$$

$$\int_c 1\, ds = \int_0^{2\pi} 1\cdot\sqrt{a^2\sin^2 t+b^2\cos^2 t}\; dt$$

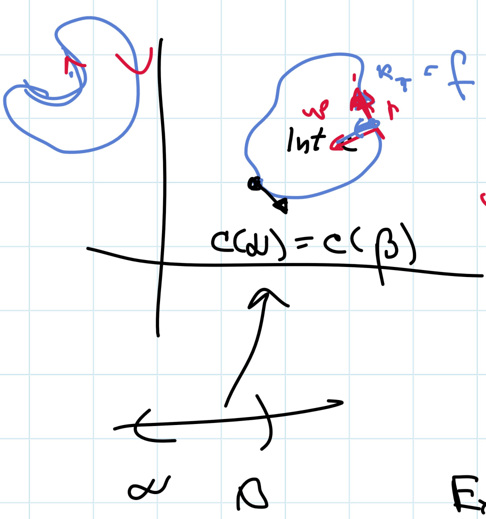
$$\boxed{\begin{array}{l} a = 2 \\ b = 1 \end{array}}$$

Definice 2.30. Parametrizovaná křivka $\mathbf{c} : [\alpha, \beta] \to \mathbb{R}^2$ se nazývá *uzavřená*, jestliže $\mathbf{c}(\alpha) = \mathbf{c}(\beta)$. Tuto křivku navíc nazveme *jednoduchou*, je-li $\mathbf{c}$ prosté na $[\alpha, \beta)$. Jednoduchá uzavřená rovinná křivka se rovněž nazývá *Jordanova*.

Věta 2.31 (Umlaufsatz)**.** Je-li $\mathbf{c}(t)$, $t \in [\alpha, \beta]$ hladká uzavřená křivka, pro kterou navíc $\mathbf{t}(\alpha) = \mathbf{t}(\beta)$, pak existuje $k \in \mathbf{Z}$ (nazývané index křivky) takové, že

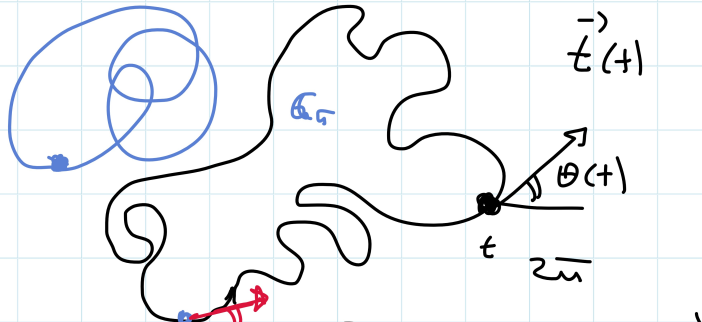
$$\int_c \kappa_z \, \mathrm{d}s = 2k\pi.$$

Je-li navíc $\mathbf{c}$ jednoduchá a kladně orientovaná (proti směru hodinových ručiček), pak $k = 1$.



→ JORDANOVA VĚTA

$\int \kappa_z = 2\pi$



c(α)=c(β)

Existuje fce θ

$$\vec t(t) = (\cos\theta(t), \sin\theta(t))$$

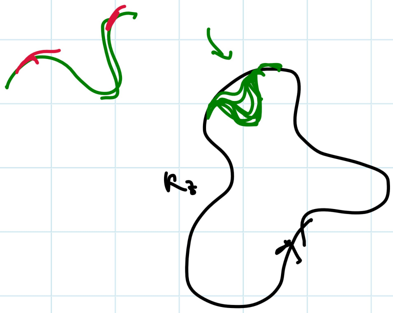
$$\vec t(\alpha) = (\cos\theta(\alpha), \sin\theta(\alpha))$$
$$\vec t(\beta) = (\cos\theta(\beta), \sin\theta(\beta))$$

$$\Rightarrow \theta(\beta) = \theta(\alpha) + 2k\pi$$

$$\frac{\theta'}{\|c'\|} = \kappa_z$$

$$\int_c \kappa_z \, ds = \int_\alpha^\beta \frac{\theta'(t)}{\|\cancel{c'(t)}\|} \cdot \|\cancel{c'}(t)\| \cdot dt = \int_\alpha^\beta \theta'(t)\, dt = [\theta(t)]_\alpha^\beta$$

$$= \theta(\beta) - \theta(\alpha) = 2k\pi$$



$$\int_c \kappa_z \, ds$$

$$t \in (0, 2\pi)$$

$$c(t) = [a \cdot \cos t, b \cdot \sin t] \qquad \| c' \| = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} = \sqrt{\ }$$

$$k_z = \frac{| \; |}{\sqrt{\ }^3} = \frac{a \cdot b}{\sqrt{\ }^3}$$

$$\int_c k_z \, ds = \int_0^{2\pi} \frac{a \cdot b}{\sqrt{\ }^3} \cdot \sqrt{\ } \, dt =$$

$$= \int_0^{2\pi} \frac{a b}{a^2 \sin^2 t + b^2 \cdot \cos^2 t} dt = \int_{-\pi}^{\pi} \frac{a b}{a^2 \sin^2 t + b^2 \cos^2 t} dt$$

$u = tg\, t$

$du = \frac{1}{\cos^2 t} dt$

$$= \int_{-\pi}^{\pi} \frac{a b}{a^2 \frac{\sin^2 t}{\cos^2 t} + b^2} \underbrace{\frac{1}{\cos^2 t} dt}_{du} = \int_{-\infty}^{+\infty} \frac{a b}{a^2 u^2 + b^2} du$$

$$= \int_{-\infty}^{\infty} \frac{\frac{a}{b}}{\left(\frac{a}{b}\right)^2 u^2 + 1} du = \int_{-\infty}^{+\infty} \frac{dv}{v^2 + 1} = [\operatorname{arctan} v]_{-\infty}^{+\infty}$$

$v = \frac{a}{b} u$

$dv = \frac{a}{b} du$

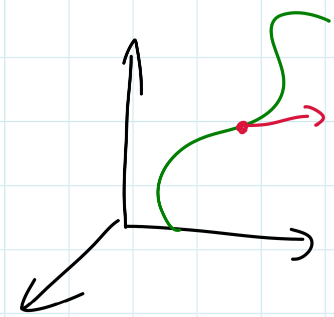
$$= \pi - (-\pi) = \underline{\underline{2\pi}}$$

$$\underline{\underline{K = 1}}$$

Definice 2.32. Mějme hladkou parametrickou křivku $\mathbf{c}(t)$, $t \in (\alpha, \beta)$ v $\mathbb{R}^n$ a zobrazení (vektorové pole) $\mathbf{F} : \langle \mathbf{c} \rangle \to \mathbb{R}^n$. Pak definujeme *Křivkový integrál 2. druhu*

$$\int_{\mathbf{c}} \mathbf{F} \mathrm{d}\mathbf{X} := \int_{\alpha}^{\beta} \mathbf{F}(\mathbf{c}(t)) \cdot \underline{\mathbf{c}'(t)\mathrm{d}t},$$

pokud integrál napravo existuje jako Lebesgueův integrál.

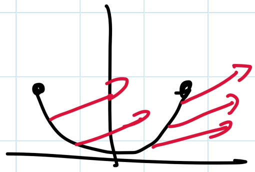


$F = (F_1, \dots, F_m)$

$F(x,y) = (2,1)$

$c(t) = (t, t^2)$ $\quad t \in (-1,1)$

$c' = (1, 2t)$

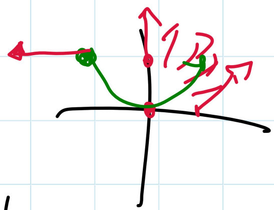


a) $\displaystyle\int_c F\,dX = \int_{-1}^{1} (1,2t) \cdot (2,1)\,dt =$

$\displaystyle = \int_{-1}^{1} (2+2t)\,dt \; \left[2t + t^2\right]_{-1}^{1}$

$= \underline{\underline{2}}$

b) $F(x,y) = \underline{(x, x+y)}$



$\displaystyle\int_c F\,\underline{dX} =$

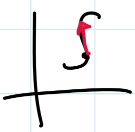
$\displaystyle\int_{-1}^{1} (t, t+t^2) \cdot (1, 2t)\,dt$

$\displaystyle = \int_c x\,dx + (x+y)\,dy$

$\displaystyle = \int_{-1}^{1} (t + 2t^2 + 2t^4)\,dt = \left[\frac{t^2}{2} + \frac{2t^3}{3} + \frac{2t^5}{5}\right]_{-1}^{1}$

$= \underline{\underline{1}}$

$\displaystyle \vec{t}(t) = \frac{c'(t)}{\|c'(t)\|}$ $\qquad \underline{c'} = \vec{t}(t)\,\|c'(t)\|$



F, c ... overload $\qquad \boxed{f = F \cdot \vec{t}}$

$\displaystyle \int_c f\,ds = \int_\alpha^\beta F_{(c(t))} \cdot \vec{t}(t) \cdot \|c'(t)\|\,dt = \int_\alpha^\beta F \cdot c'\,dt = \int_c F\,dX$

Věta 2.33 (Greenova věta)**.** Nechť $\mathbf{c}$ je jednoduchá, hladká, uzavřená, kladně orientovaná (proti směru hodinových ručiček) křivka v $\mathbf{R}^2$. Nechť $\mathbf{F}(x,y) = (F_1(x,y), F_2(x,y))$ je hladké vektorové pole definované na nějakém okolí $\bar{\Omega}$. Pak

$$\int_{\mathbf{c}} \mathbf{F}\mathrm{d}\mathbf{X} = \int_{\mathrm{Int}\,\mathbf{c}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}x\mathrm{d}y.$$

$(0,x)$ $\left(-\frac{1}{2}y, \frac{1}{2}x\right)$

$(-y,0)$

Lemma 2.34. Buď $\mathbf{c}(t) = (c_x(t), y_y(t))^T$, $t \in [\alpha, \beta]$ kladně orientovaná, jednoduchá, uzavřená křivka. Pak plošný obsah oblasti Int $\mathbf{c}$ je roven

$$A = \int_\alpha^\beta c_x(t)c_y'(t)dt = -\int_\alpha^\beta c_y(t)c_x'(t)dt = \frac{1}{2}\int_\alpha^\beta (c_x c_y' - c_x' c_y)dt.$$

$F(x,y) = (x, x+y)$

$c(\alpha) = c(\beta)$

$$\int_c x\,dx + (x+y)\,dy = \int_{\mathrm{Int}\,c} (1-1)\,dx\,dy = 0$$

$F(x,y) = (xy^2, y\cdot\sin x)$

$$\int_c xy^2\,dx + y\cdot\sin x\,dy = \int_{\mathrm{Int}\,c} (-y\cos x - 2xy)\,dx\,dy$$

$c(t) = (c_x(t), c_y(t))$

$$\int_c 0\,dx + x\,dy = \int_{\mathrm{Int}} 1\,dx\,dy$$

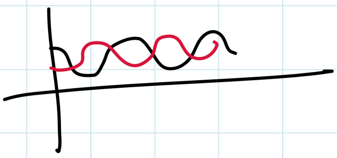
$$\int_\alpha^\beta (0, c_x(t))\cdot(c_x'(t), c_y'(t))\,dt = \int_\alpha^\beta c_x(t)\cdot c_y'(t)\,dt$$

D)

$$c(t) = \left(a\cdot\cos t,\ b\cdot\sin t\right) \qquad t\in(0,2\pi)$$

$$\int_0^{2\pi} (a\cdot\cos t)\cdot(b\cdot\cos t)\,dt = a\cdot b\int_0^{2\pi}\cos^2 t\,dt =$$

$$= \underline{a\cdot b\cdot\pi}$$



$$\overset{\pi}{\int_0^{2\pi}\cos^2 t\,dt} = \overset{\pi}{\int_0^{2\pi}\sin^2 t\,dt}$$

$$\int_0^{2\pi}(\cos^2 t+\sin^2 t)\,dt = 2\pi$$

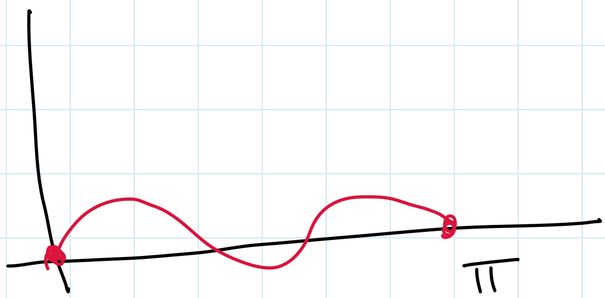
VARIAČNÍ POČET

∞ - dimenz

Lemma 2.35 (Wirtinger)**.** Nechť $f(t) : [0,\pi] \to \mathbb{R}$ je hladká funkce, pro kterou platí $f(0) = f(\pi) = 0$. Pak

$$\int_0^\pi f'^2(t)dt \geq \int_0^\pi f^2(t)dt$$

a rovnost nastane právě tehdy, když $f(t) = D\sin(t)$, kde D je konstanta.



$g(t) = \dfrac{f(t)}{\sin t}$ ~ na $(0,\pi)$ ok, ale lze dodefinovat $0,\pi$

$\lim\limits_{t\to 0+} \dfrac{f(t)}{\sin(t)} = f'(0)$

$f(t) = g(t)\cdot \sin t$

$$\int_0^\pi f'(t)^2 dt = \int_0^\pi (g'(t)\sin t + g(t)\cdot\cos t)^2 =$$

$$= \int_0^\pi g'(t)^2 \sin^2 t\, dt + \int_0^\pi g^2(t)\cancel{\cos^2 t}\, dt + \int_0^\pi 2\underbrace{g'(t)g(t)}_{[g(t)^2]'} \sin t \cos t\, dt =$$

$$\left[g(t)^2\cdot \sin t\cdot\cos t\right]_0^\pi - \int_0^\pi g(t)^2(\cancel{\cos^2 t} - \sin^2 t)dt$$

$$= \int_0^\pi g'(t)^2 \sin^2 t\, dt + \int_0^\pi g^2(t)\cdot\sin^2 t = \underbrace{\int_0^\pi g'(t)^2\cdot\sin^2 t}_{\geq 0} + \int_0^\pi f^2(t)dt$$

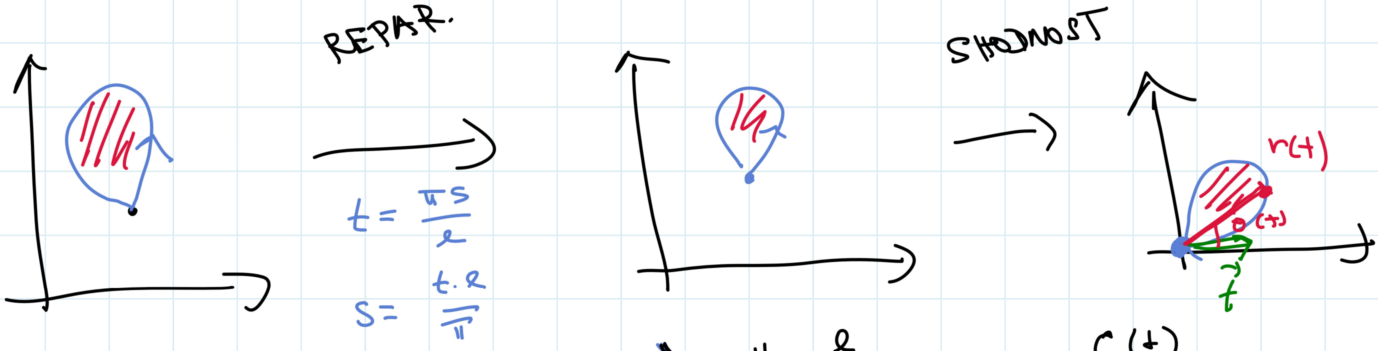
$=0 \Leftrightarrow g'(t) = 0 \Leftrightarrow g(t) = D$

$\boxed{\text{QED}}$

Věta 2.36 (Isoperimetrická nerovnost)**.** Buď $\mathbf{c} : [a,b] \to \mathbb{R}^2$ hladká jednoduchá uzavřená křivka délky l a buď A plošný obsah Int $\mathbf{c}$. Pak

$$\frac{l^2}{4\pi} \geq A,$$

přitom rovnost nastane, právě když $\mathbf{c}$ je kružnice.



$c(s) \ldots$ obloukem

$s \in (0, l)$

$\| c' \| = 1$

$\| c'(t) \| = \frac{l}{\pi}$

$t \in (0, \pi)$

$c(t)$

$c(0) = c(\pi) = (0,0)$

$c'(0) = (\underset{>0}{c_x'}, 0)$

$$c(t) = (c_x(t), c_y(t)) =$$

$$= r(t)\,(\cos\theta(t), \sin\theta(t))$$

$$\|c'(t)\|^2 = c_x'^2 + c_y'^2 = (r' \cdot \cos\theta - r\theta'\sin\theta)^2 + (r' \cdot \sin\theta + r\theta'\cos\theta)^2 =$$

$$= r'^2 + r^2\theta'^2 = \left(\frac{l}{\pi}\right)^2$$

$$\int_0^\pi (r'^2 + r^2\theta'^2)\,dt = \int_0^\pi \frac{l^2}{\pi^2}\,dt = \frac{l^2}{\pi}$$

$$A = \frac{1}{2}\int_0^\pi (c_x c_y' - c_y c_x')\,dt = \frac{1}{2}\int_0^\pi r^2\theta'\,dt$$

$$\frac{l^2}{4\pi} - A = \frac{1}{4}\int_0^\pi (r'^2 + r^2\theta'^2 - 2r^2\theta')\,dt =$$

$$= \frac{1}{4}\int_0^\pi \underbrace{r^2(\theta'-1)^2}_{\geq 0}\,dt + \frac{1}{4}\int_0^\pi \underbrace{(r'^2 - r^2)}_{\geq 0}\,dt$$

Wirt

ROVNOST

$\theta = t$

$r = D \sin t$

$\theta(t) = t$

$r(t) = D \sin t$

$c(t) = D \cdot \sin t \,(\cos t, \sin t)$