EXERCISE 3

(1) Let $V = V(x+y, z-1) \subset A^3_k$, and let $W = V(x-z^2, y+z) \subset A^3_k$. Show that V and W are isomorphic varieties.

(2) Assume k is a field of characteristic 0. Let $V = V(x^2 + y^2 - 1)$. Show that V is a rational variety.

(3) Given a field k, the power series ring over k, denoted by $k[[x]]$ is defined to be the set of elements of the form $\sum_{n=0}^{\infty} a_n x^n$, where $a_n \in k$. There is no restriction on "convergence". Sum is defined by

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n,$$

and the product is defined by

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n,$$

where

$$c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k}.$$

Show that $k[[x]]$ is a commutative ring that contains $k[x]$, and that moreover, it is a discrete valuation ring.

(4) For each of the following polynomials, consider the plane algebraic curve $V(f)$. Find all points where this curve is smooth. At points where it is non-smooth, you can try to show that the local ring is not a discrete valuation ring.
- $f(x, y) = y - x^2$.
- $f(x, y) = x^2 - y^3$.
- $f(x, y) = x^2 + y^2 - 1$.
- $f(x, y) = x^4 + y^4 - x^2 y^2$.
- $f(x, y) = x^4 - 5y^2 x + y^2$.

(5) Let $k = \bar{k}$, and let $f \in k[x, y]$ be an irreducible polynomial. Let $V = V(f)$ be the corresponding plane curve. Show that the set of points where V is non-smooth is a finite set.