# CATEGORIES OF MODULES AND HOMOLOGICAL ALGEBRA

EXERCISE 3

(1) Prove the five lemma: Let $R$ be a ring. Given a commutative diagram with exact rows of left $R$-modules

$$\begin{array}{ccccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5 \end{array}$$

the following hold:
- if $f_2, f_4$ are surjective and $f_5$ is injective then $f_3$ is surjective.
- if $f_2, f_4$ are injective and $f_1$ is surjective then $f_3$ is injective.
- Deduce that if $f_1, f_2, f_4, f_5$ are isomorphisms then $f_3$ is an isomorphism.

(2) Let $K$ be a field. For all $n \geq 0$, compute $\operatorname{Ext}^n_{K[x]}(K, K[x])$ and $\operatorname{Tor}^{K[x]}_n(K, K)$.

(3) Let $R$ be a commutative ring, and let $I \subseteq R$ be an ideal.
- Show that for any $R$-module $M$ there is an isomorphism
$$R/I \otimes_R M \cong M/IM.$$
- Show that for all $n \in \mathbb{N}$
$$\operatorname{Tor}^R_n(R/I, M) \cong \operatorname{Tor}^R_{n+1}(I, M).$$

(4) Let $R$ be a ring, let $M$ be a right $R$-module, and let $\{N_i\}_{i \in I}$ be a family of left $R$-modules. Show that for all $i \geq 0$ there is a natural isomorphism
$$\operatorname{Tor}^R_i(M, \bigoplus_{i \in I} N_i) \cong \bigoplus_{i \in I} \operatorname{Tor}^R_i(M, N_i).$$

(5) Let $R$ be a commutative ring, let $I \subseteq R$ be an ideal. For any $R$-module $M$, let
$$\Gamma_I(M) = \{m \in M \mid \exists n, I^n \cdot m = 0\}$$
This is called the $I$-torsion submodule of $M$.
- Show that $\Gamma_I$ defines a functor $\operatorname{Mod}(R) \to \operatorname{Mod}(R)$.
- Show that this functor is left exact.
- Given a short exact sequence $0 \to M \to N \to K \to 0$ of $R$-modules, write down the long exact sequence of $R$-modules obtained from the right derived functor of $\Gamma_I$.