EXERCISE 2

(1) Given a ring A, let M, N be A-modules, and let $f : M \to N$ be an A-linear map. Given $p \in \mathrm{Spec}(A)$, show that $\ker(f)_p \cong \ker(f_p)$, and that $\mathrm{Im}(f)_p \cong \mathrm{Im}(f_p)$.

(2) Show that an A-linear map $f : M \to N$ is injective if and only if for all $p \in \mathrm{Spec}(A)$, the map $f_p : M_p \to N_p$ is injective. Show that an A-linear map $f : M \to N$ is surjective if and only if for all $p \in \mathrm{Spec}(A)$, the map $f_p : M_p \to N_p$ is surjective. Deduce that an A-linear map f is an isomorphism if and only if for all $p \in \mathrm{Spec}(A)$, the map f_p is an isomorphism.

(3) Let $A = \mathbb{Z}$ be the ring of integers. Given a prime number p, show that the ideal $q = (p) \subseteq A$ is a prime ideal, and explicitly compute the localization map $A \to A_q$. What is its kernel?

(4) Given a multiplicatively closed subset $S \subseteq A$, such that $0 \notin S$, show that there exist an ideal $I \subseteq A$, such that $I \cap S = \emptyset$, and such that for any ideal J, if $I \subsetneq J$, then $J \cap S \neq \emptyset$. Show that any such ideal I must be a prime ideal, and deduce that the intersection of all prime ideals in a commutative ring is equal to the set of nilpotent elements in the ring.

(5) Given a ring A, show that the space $\mathrm{Spec}(A)$ is quasi-compact; that is, show that if $X = \mathrm{Spec}(A)$, and if

$$X = \bigcup_{i \in I} X_{f_i},$$

for some (possibly infinite) set I, and such that for each $i \in I$, we have that $f_i \in A$, then there exists finitely many indices $i_1, \ldots, i_n \in I$, such that

$$X = \bigcup_{j=1}^{n} X_{f_{i_j}}.$$