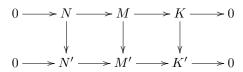
## CATEGORIES OF MODULES AND HOMOLOGICAL ALGEBRA

## EXERCISE 2

- (1) Let R be a commutative ring, let F be a flat R-module, and let I be an injective R-module. Show that  $\operatorname{Hom}_R(F,I)$  is an injective R-module.
- (2) Suppose we are given a commutative diagram



of R-modules. Show that if all the vertical maps in this diagram are isomorphisms, then the top row is a short exact sequence if and only if the bottom row is a short exact sequence.

(3) An R-module M is called finitely presented if there exist  $n, m \in \mathbb{N}$  and an exact sequence of the form

$$R^m \to R^n \to M \to 0.$$

Every finitely presented module is finitely generated, but the converse is false.

- ullet Show that if P is a finitely generated projective R-module, then P is finitely presented.
- Give an example of a ring R, and a finitely generated R-module M, such that M is not finitely presented. When proving M is not finitely presented, you may use without proof the following fact:
  - (\*) If M is a finitely presented module, then for any surjective R-linear map  $\varphi: R^n \to M$ , the R-module  $\ker(\varphi)$  is a finitely generated R-module.
- (4) Let R be a ring, and M an R-module. Prove that there exist injective R-modules I,J and an exact sequence of the form

$$0 \to M \to I \to J$$
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(5) The following problem is more challenging. Feel free to try it. Let R be an integral domain that is not a field. Let M be an R-module which is both injective and projective. Show that M=0.