CATEGORIES OF MODULES AND HOMOLOGICAL ALGEBRA

EXERCISE 2

(1) Let R be a commutative ring, let F be a flat R-module, and let I be an injective R-module. Show that $\mathrm{Hom}_R(F, I)$ is an injective R-module.

(2) Suppose we are given a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & M' & \longrightarrow & K' & \longrightarrow & 0 \end{array}$$

of R-modules. Show that if all the vertical maps in this diagram are isomorphisms, then the top row is a short exact sequence if and only if the bottom row is a short exact sequence.

(3) An R-module M is called finitely presented if there exist $n, m \in \mathbb{N}$ and an exact sequence of the form

$$R^m \to R^n \to M \to 0.$$

Every finitely presented module is finitely generated, but the converse is false.

- Show that if P is a finitely generated projective R-module, then P is finitely presented.
- Give an example of a ring R, and a finitely generated R-module M, such that M is not finitely presented. When proving M is not finitely presented, you may use without proof the following fact:
  (*) If M is a finitely presented module, then for any surjective R-linear map $\varphi : R^n \to M$, the R-module $\ker(\varphi)$ is a finitely generated R-module.

(4) Let R be a ring, and M an R-module. Prove that there exist injective R-modules I, J and an exact sequence of the form

$$0 \to M \to I \to J.$$

(5) The following problem is more challenging. Feel free to try it. Let R be an integral domain that is not a field. Let M be an R-module which is both injective and projective. Show that $M = 0$.