EXERCISE 1

Recall that in this course all rings are commutative and unital.

(1) Let A be a ring, let $I \subseteq A$ be an ideal, and let

$$\sqrt{I} := \{f \in A \mid f^n \in I \text{ for some } n\}$$

Show that $\sqrt{I}$ is an ideal in A. This ideal is called the radical of I, and I is called a radical ideal if $I = \sqrt{I}$. Show that prime ideals are radical.

(2) Given an algebraic set X, show that $I(X)$ is a radical ideal.

(3) Let A be a noetherian ring, and let $I \subseteq A$ be an ideal. Show that A/I is a noetherian ring. If B is a subring of A, must B also be noetherian?

(4) Given a field k, and given $n > 0$, show that for any $(a_1, \ldots, a_n) \in \mathbb{A}^n_k$, the ideal $I = (x_1 - a_1, \ldots, x_n - a_n)$ is a maximal ideal of $k[x_1, \ldots, x_n]$.

(5) Given a field k, and given $n > 0$, show that if I is a maximal ideal of $k[x_1, \ldots, x_n]$ then $V(I)$ is either a point or is empty.