CATEGORIES OF MODULES AND HOMOLOGICAL ALGEBRA

EXERCISE 1

(1) Let R be a ring, and let $\{E_k\}_{k\in K}$ be a family of injective R-modules. Show that
$$\prod_{k\in K} E_k$$
is an injective R module. Deduce that if $E_1, \ldots, E_n$ are finitely many injective R-modules, then $\oplus_{k=1}^n E_k$ is an injective R-module.

(2) Let K be a field, and let $R = K[x]$ be the polynomial ring over K. Let $Q = K(X)$ be the quotient field of R, which is the same as the field of rational functions over K. Show that Q and Q/R are injective R-modules.

(3) Let R be a ring. Recall that a short exact sequence of left R-modules
$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$
is called split if there exists an R-linear map $h : C \to B$ such that $p \circ h = 1_C$. Prove that a short exact sequence is split if and only if there exists an R-linear map $j : B \to A$, such that $j \circ i = 1_A$.

(4) Use (3) to show that an R-module E is injective if and only if every short exact sequence of the form
$$0 \to E \xrightarrow{i} B \xrightarrow{p} C \to 0$$
is split.

(5) Let R be an integral domain, and let R be an R-module. Show that if E is injective then E is divisible. Deduce that over a principal ideal domain, the injective modules are exactly the divisible modules.