TUTORIAL FOR THE SUBJECT NMAG336
INTRODUCTION TO THE CATEGORY THEORY

TUTORIAL 2 / MARCH 10 2023

Problem 2.1. *Let* $\mathbf{Fld}$ *denote the category of fields and* $\mathbf{Grp}$ *the category of groups. The functors* $\mathbf{Gl}_n, (-)^*\colon \mathbf{Fld} \to \mathbf{Grp}$ *are defined as follows:*

- *the functor* $\mathbf{Gl}_n$ *assigns to field* F *the group* $\mathbf{Gl}_n(F)$ *of all regular* $n \times n$ *matrices over* F*;*
- *the functor* $(-)^*$ *assigns to a field* F *its multiplicative group* F^**.*

Verify that $\det = \{\det_F\colon \mathbf{Gl}_n(F) \to F^* \mid F \in \mathbf{ob}(\mathbf{Fld})\}$ *is a natural transformation* $\mathbf{Gl}_n \to (-)^*$*.*

Problem 2.2. *Given a set* A*, let* $A \times -\colon \mathbf{Set} \to \mathbf{Set}$ *be the functor that assigns to a set* B *the set* $A \times B$ *and to a morphism* $g\colon B \to C$ *the morphism* $A \times g\colon A \times B \to A \times B$ *which sends* $\langle a, b\rangle$ *to* $\langle a, f(b)\rangle$*. Show that every mapping* $A \to B$ *corresponds to a natural transformation* $A \times - \to B \times -$*.*

Problem 2.3. *Let* $\mathbf{A}$ *be a category and* $\mathbf{P}$ *an ordered set (viewed as a category). Let* $F, G\colon \mathbf{A} \to \mathbf{P}$ *be a pair of functors.*

(1) *Decide when there is a natural transformation* $F \to G$*.*
(2) *Show that there is at most one natural transformation* $F \to G$*.*

Problem 2.4. *For a field* F*, let* $\mathbf{Mat}_F$ *denote the category whose*

- *objects are natural numbers,*
- *morphisms* $n \to m$ *are* $m \times n$ *matrices over* F *(composition of the morphisms corresponds to matrix multiplication).*

Prove that the category $\mathbf{Mat}_F$ *is equivalent to the category* $\mathbf{vec}_F$ *of all finitely dimensional vector spaces over the field* F *(with morphisms being linear maps).*

Problem 2.5. *For sets* A*,* B*, let* B^A *denote the set of all mappings* $f\colon A \to B$*. Consider the functor* $(-)^A\colon \mathbf{Set} \to \mathbf{Set}$*. For the set* X*, let us define a map* $\varepsilon_X\colon X^A \times A \to X$ *by* $\varepsilon_X(f, a) = f(a)$*. Show that* $\varepsilon = \{\varepsilon_X \mid X \in \mathbf{ob}(\mathbf{Set})\}$ *is a natural transformation from the functor* $(-)^A \times A$ *to the identity functor on* $\mathbf{Set}$*.*

Problem 2.6. *Let* $\mathbf{Vec}_F$ *denote the category of vector spaces (and* F*-linear maps) over a field* F*. Let* $(-)^*\colon \mathbf{Vec}_F \to \mathbf{Vec}_F$ *be the contravariant functor that assigns to a vector space* V *the vector prosotor* $V^* = \hom_T(V, F)$ *of all linear forms* $f\colon V \to F$*. For the vector space* V*, consider the mapping* $\varepsilon_V\colon V \to (V^*)^*$ *given by* $\varepsilon(v)(f) = f(v)$ *for each* $v \in V$ *and each* $f \in V^*$*.*

(1) *Show that* $\varepsilon = \langle \varepsilon_V \mid V \in \mathbf{ob}(\mathbf{Vec}_F)\rangle$ *is a natural transformation from the identity functor on* $\mathbf{Vec}_F$ *to the functor* $((-)^*)^*$*.*
(2) *Decide whether* ε *is a natural equivalence.*
(3) *Let* $\mathbf{vec}_F$ *denote the subcategory of the category* $\mathbf{Vec}_F$ *of all finite dimensional spaces. Decide whether the restriction* $\varepsilon \restriction \mathbf{vec}_F$ *is a natural equivalence.*

Problem 2.7. *Let* G, H *be groups. Each of them view as a category with one object (and morphisms corresponding to elements of the groups).*

(1) *Verify that the functors* $G \to H$ *correspond to group homomorphisms.*
(2) *Show that for a pair of functors* $S, T\colon G \to H$ *there exists a natural transformation* $S \to T$ *if and only if there exists* $h \in H$ *such that* $S(g) = hT(g)h^{-1}$ *for every* $g \in G$*. Describe the corresponding natural transformation and decide whether it is a natural equivalence.*

Problem 2.8. *Let* G *be a group and* F *a field. As abocve, view* G *as an one object category. Show that the category* $\mathbf{Vec}_F^G$ *of functors from* $G \to \mathbf{Vec}_F$ *can be identified with the category of all* F*-representations of the group* G*.*