Barrier Options

Lukas Ernst

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Introduction to Barrier Options

Definition

A barrier option is a **path-dependent** derivative where the payoff depends on whether the underlying asset price S_t reaches a specific barrier level H during the time interval $[0, T]$.

Mathematical Distinction:

- **Vanilla Option:** Payoff depends only on terminal value S_T.
- **Barrier Option:** Payoff depends on S_T **AND** the path extrema:

$$M_T = \max_{0 \leq t \leq T} S_t \quad \text{or} \quad m_T = \min_{0 \leq t \leq T} S_t$$

Key Property

Strictly cheaper than vanilla options because the probability of a positive payoff is strictly lower (subset of the sample space).

Classification: The Fundamental Parity

Barrier options are classified by the **Barrier Event**:

1. **Knock-Out:** Option is extinguished (ceases to exist) if S_t touches H.
2. **Knock-In:** Option activates (comes into existence) only if S_t touches H.

The Static Replication (Arbitrage Relationship)

For a given Strike K and Barrier H, a standard vanilla option is the sum of the Knock-In and Knock-Out versions:

$$V_{\text{Vanilla}} = V_{\text{Knock-In}} + V_{\text{Knock-Out}}$$

The 8 Combinations: Mathematical Payoffs

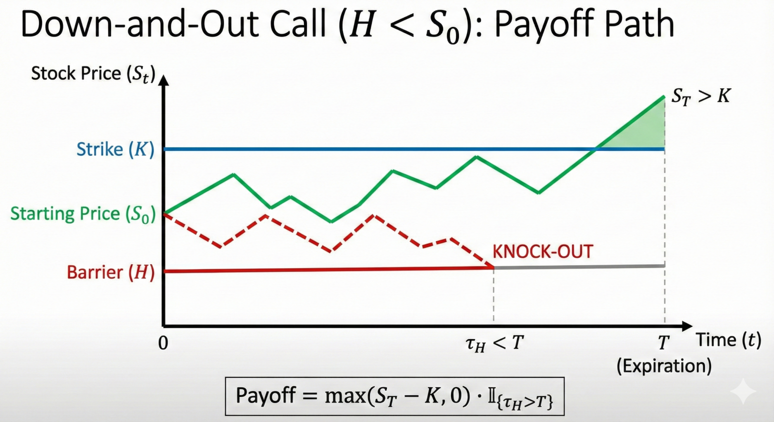
Let $\tau_H = \inf\{t : S_t = H\}$ be the first hitting time. The payoff depends on the final price S_T and whether the barrier was hit ($\tau_H \leq T$) or not ($\tau_H > T$).

Barrier Type	Option	Barrier Condition	Payoff Formula at T
Up-and-Out	Call	$H > S_0$	$(S_T - K)^+ \cdot \mathbb{I}_{\{\tau_H > T\}}$
	Put	$H > S_0$	$(K - S_T)^+ \cdot \mathbb{I}_{\{\tau_H > T\}}$
Up-and-In	Call	$H > S_0$	$(S_T - K)^+ \cdot \mathbb{I}_{\{\tau_H \leq T\}}$
	Put	$H > S_0$	$(K - S_T)^+ \cdot \mathbb{I}_{\{\tau_H \leq T\}}$
Down-and-Out	Call	$H < S_0$	$(S_T - K)^+ \cdot \mathbb{I}_{\{\tau_H > T\}}$
	Put	$H < S_0$	$(K - S_T)^+ \cdot \mathbb{I}_{\{\tau_H > T\}}$
Down-and-In	Call	$H < S_0$	$(S_T - K)^+ \cdot \mathbb{I}_{\{\tau_H \leq T\}}$
	Put	$H < S_0$	$(K - S_T)^+ \cdot \mathbb{I}_{\{\tau_H \leq T\}}$

*Note: $(x)^+ = \max(x, 0)$ and $\mathbb{I}_A$ is equal to 1 if event A occurs, 0 otherwise.

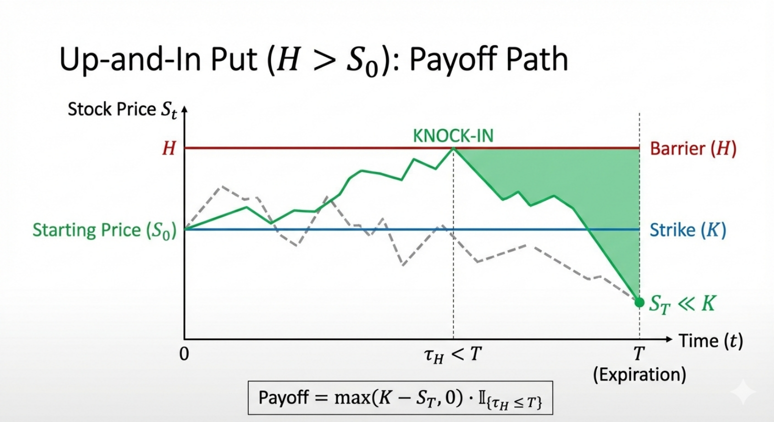
Example 1: Down-and-Out Call ($H < S_0$)

This option ceases to exist ("knocks out") if the asset price S_t touches or falls below the barrier level H at any time before expiration.



Example 2: Up-and-In Put ($H > S_0$)

This option is inactive and worthless until the asset price S_t touches or rises above the barrier level H, at which point it "knocks in" and becomes a standard put option.



Valuation: The Analytical Formula

We cannot use the standard Black-Scholes formula alone because it ignores the risk of the option being "Knocked Out."

The Logic:

$$\text{Barrier Value} = \text{Vanilla Value} - \text{Value of "Knock-Out" Paths}$$

The Formula (Merton, 1973): For a Down-and-Out Call ($H < S_0$), the price is:

$$V_{BO} = C_{BS}(S_0) - \left(\frac{S_0}{H}\right)^{1-\frac{2r}{\sigma^2}} C_{BS}\left(\frac{H^2}{S_0}\right)$$

- We subtract the value of a hypothetical option starting at the **Reflected Spot Price** $S' = H^2/S_0$.
- The term $\left(\frac{S_0}{H}\right)^{\cdots}$ represents the probability of hitting the barrier based on drift (r) and volatility (σ).

Valuation: Numerical Methods

Analytical formulas only work for constant barriers and constant volatility. For complex real-world contracts, we use:

1. **Monte Carlo Simulation:**
   - Simulate N paths (e.g., 10,000) using Geometric Brownian Motion.
   - Discard paths that touch H.
   - Average the payoff of surviving paths.

$$V \approx e^{-rT} \frac{1}{N} \sum_{i=1}^{N} \text{Payoff}_i \cdot_{\{\min S_t > H\}}$$

2. **Finite Difference Method (The "Grid" Approach):**
   - We discretize the world into a grid of Prices vs. Time.
   - **Step 1:** Start at Maturity (where values are known).
   - **Step 2:** Work backwards day-by-day to calculate today's price.
   - **Barrier handling:** We manually set the grid nodes at the barrier level H to Zero.

Advanced Barrier Conditions

In practice, barriers are often more complex than a simple continuous line.

1. **Monitoring Frequency (Discrete vs. Continuous):**
   - **Continuous:** Triggered if S_t hits H at *any* moment. (Risky, cheaper).
   - **Discrete:** Triggered only if Closing Price hits H. (Safer, expensive).
   - *Note: Discrete barriers reduce "intraday noise" risk.*

2. **Double Barriers:**
   - Two barriers: Upper (H_{up}) and Lower (H_{low}).
   - Option is knocked out if price exits the "Tunnel."
   - Used for betting on low volatility (Range Trading).

3. **Parisian Options ("Soft" Barriers):**
   - The barrier only triggers if the price *stays* beyond H for a specific duration (e.g., 24 hours).
   - Protects against temporary price spikes or manipulation.

Thank You

Questions?