MASTER THESIS

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Maximum principles for elliptic systems of partial differential equations

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Title: Maximum principles for elliptic systems of partial differential equations

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Abstract: We consider nonlinear elliptic Bellman systems which arise in the theory of stochastic differential games. The right hand sides of the equations (which are called Hamiltonians) may have quadratic growth with respect to the gradient of the unknowns. Under certain assumptions on Lagrangians (from which the Hamiltonians are derived), that are satisfied for many types of stochastic games, we establish the existence and uniqueness of a Nash point and develop structural conditions on the Hamiltonians. From these conditions we establish a certain version of maximum and minimum principle. This result is then used to establish the existence of a bound solution.

Keywords: Elliptic systems Bellman systems Maximum principle Iterating exponentials method
I dedicate this thesis to my supervisor Miroslav Bulíček, Ph.D. for the interesting thoughts and ideas, to my family for unending support, to my friends for necessary moments of distraction and finally to my mother for the great patients she had with me. She supported me through the whole process and listened to my complaints. I am always so much grateful for you.
0.1 Notation

An open ball around $x$ with a radius $r$ is denoted by $B_r(x)$. The Lebesgue measure of $\Omega \subset \mathbb{R}^d$ will be denoted by $|\Omega|$ and the positive and negative parts are defined as

$$(x)_+ := \max(x, 0), \quad (x)_- = \min(0, x).$$

We use the following notation for partial derivatives of vector-valued function $u : \Omega \to \mathbb{R}^N$ with $\Omega \subset \mathbb{R}^d$, $d \geq 1$ and $N \geq 1$,

$$u := (u_1, \ldots, u_N), \quad (\nabla u)_{ij} := D_i u_j \quad \text{for } i = 1, \ldots, d; \ j = 1, \ldots, N.$$

The Lebesgue space with standard Lebesgue measure is denoted by $L^p(\Omega)$ with some $p \in [1, \infty]$. The Sobolev space is denoted by $W^{k,p}(\Omega)$ with $k \in \mathbb{N}$ and the space of Hölder continuous function is denoted by $C^{k,\alpha}(\Omega)$ with $\alpha \in (0, 1]$ (see definitions in Chapter 7). The corresponding norms are respectively

$$\| \cdot \|_p, \quad \| \cdot \|_{k,p} \quad \text{and} \quad \| \cdot \|_{k,\alpha}.$$

Note that for $k = 0$ we write $C^{k,\alpha} \equiv C^\alpha$. We denote by $W^{k,p}_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $W^{k,p}(\Omega)$. We will write $\Omega \in C^{k,\alpha}$ when $\partial \Omega$ is of the class $C^{k,\alpha}$.

The mapping $H : \Omega \times \mathbb{R}^M \to \mathbb{R}$ is called Carathéodory if

1. $x \mapsto H(x, p)$ is measurable for all $p \in \mathbb{R}^M$,
2. $p \mapsto H(x, p)$ is continuous for a.a. $x \in \Omega$.

The differential operator div is defined for a vector valued function $v \in L^1(\Omega, \mathbb{R}^d)$ as

$$\langle \text{div}(v), u \rangle = -\int_\Omega v \cdot \nabla u \quad \text{for all } u \in \mathcal{D}(\Omega).$$

The Laplace operator $\Delta$ is then defined for a scalar valued function $v \in W^{1,1}(\Omega)$ as $\text{div}(\nabla v)$ or

$$\langle \Delta v, u \rangle = -\int_\Omega \nabla v \cdot \nabla u \quad \text{for all } u \in \mathcal{D}(\Omega).$$

For $v \in L^p(\Omega, \mathbb{R}^d)$ and $v \in W^{1,p}(\Omega)$ respectively, these operators can be extended to operators to $(W^{1,q}(\Omega))^*$, where $q$ is the Hölder conjugate of $p$. For smooth $v$ and $u$ we get representation

$$\text{div}(v) = \sum_{n=1}^d D_nv_n \quad \text{and} \quad \Delta v = \sum_{n=1}^d (D_n)^2 v.$$

Finally, the Frobenius product of two rectangular matrices $A, B$ of the same size is defined as

$$A : B = \text{Tr}(AB^T).$$
1. Introduction

1.1 Problem setting

This thesis considers mainly the following diagonal system of elliptic PDEs

\[ Lu_n + \lambda_n u_n = H_n(x, \nabla u), \quad n = 1, \ldots, N \]  

(1.1)

in a domain \( \Omega \subset \mathbb{R}^d \), \( \Omega \in C^{1,1} \) with some \( d \geq 2 \). The coefficients \( \lambda_n \) are positive real numbers and the symbol \( L \) defines an elliptic operator, i.e.,

\[ L := -\text{div}(a \cdot \nabla) = -\sum_{i,j=1}^d D_i(a_{ij}(x)D_j) \]  

(1.2)

with \( a(x) := \{a_{ij}(x)\}_{i,j=1}^d \) uniformly positive definite in \( \Omega \), i.e., there exists \( \alpha_0 > 0 \) such that

\[ a(x)\xi \cdot \xi = \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq \alpha_0 |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d, \]  

(1.3)

\[ a_{ij}(x) \in L^\infty(\Omega). \]  

(1.4)

The main feature of diagonal systems is that the elliptic operator on the left-hand side is the same for all equations and the components react only through the non-linearity on the right-hand side. The functions \( H_n \) on the right-hand side of (1.1) are called Hamiltonians and satisfy for all \( n = 1, \ldots, N \) the following conditions:

\[ H_n(x, \mu, p) \text{ satisfies the Carathéodory conditions for } (x, \mu, p) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times N. \]  

(1.5)

There exists a constant \( K^* \) such that

\[ |H_n(x, \mu, p)| \leq K^*|p|^2 + K^* \quad \text{for all } (x, \mu, p) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times N. \]  

(1.6)

Here, we would like to empathize that the condition (1.6) is not sufficient for proving regularity nor existence of a weak solution, see Giaquinta [2016], Hildebrandt [1982], and Ladyzhenskaya and Ural’tseva [1968] for detailed description of the problem. The main reason is that, in general, solutions to (1.1) fail to be Hölder continuous and fail to satisfy corresponding a priori estimates.

Even in the scalar case \( N = 1 \) the solution may be unbounded, for example the equation

\[ -\Delta u = |\nabla u|^2 \quad \text{in } B_{e^{-1}}(0) \subset \mathbb{R}^2 \]  

(1.7)

has a weak solution

\[ u(x) = \ln |\ln |x|| \in W^{1,2}_0(B_{e^{-1}}(0)), \]  

(1.8)

Our goal in this thesis is to establish the existence of bounded solution to (1.1) provided some stronger conditions on \( H_n \) hold. However, even for \( N > 1 \) the bounded solutions can be discontinuous, as it is shown in Frehse [1973] that

\[ u_1 = \sin \ln |\ln |x||, \quad u_2 = \cos \ln |\ln |x|| \]  

(1.9)
solve
\[-\Delta u_1 = b_1(x)|\nabla u|^2, \quad -\Delta u_2 = b_2(x)|\nabla u|^2\] (1.10)
for certain $L^\infty$ functions $b_1$ and $b_2$.

The following assumptions will be of main interest of this thesis:
There exist constants $K_1^*$, $K_1$, $K_2^*$ and $K_2$ such that
\[H_n(x, \mu, p) \leq K_1^*|p||p_n| + K_1 \quad \text{for all } (x, \mu, p) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d \times N},\] (1.11)
\[\sum_{n=1}^{N} H_n(x, \mu, p) \geq -K_2^* \left| \sum_{n=1}^{N} p_n \right|^2 - K_2 \quad \text{for all } (x, \mu, p) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d \times N}.\] (1.12)

These conditions play an important role in the stochastic game theory as they often follow from the structure of underlying Lagrangians (from which the Hamiltonians are derived, see Section 1.4). It is shown in Beck et al. [2015] that these conditions guarantee the existence of regular $C^\alpha$ solution to (1.1). We follow these results in Chapter 5 to find a bounded weak solution to the system (1.1).

There are also other conditions often arising from stochastic games for Bellman systems such as
\[|H_n| \leq K|p_n|^2 + K\] (1.13)
or even more general
\[|H_n| \leq K|p_n|^2 + K \sum_{m=1}^{n} |p_m|^2 + K,\] (1.14)
for which it is possible to find uniform $C^\alpha$ bounds (see Bensoussan et al. [2010]).

Frequently, there are also weaker assumptions used in the stochastic game theory, e.g.,
\[H_n(x, \mu, p) \leq K|p_n|^2 + p_n \cdot g_0(x, p) + K,\] (1.15)
with $g_0$ being Carathéodory and satisfying
\[|g_0(x, p)| \leq K|p|,\] (1.16)
which is considered in Bulíček and Frehse [2011]. Another example is
\[H_n(x, \mu, p) = H_n^0(x, \mu, p) + p_n \cdot G(x, \mu, B(x, p)) + f_n(x)\] (1.17)
with
\[|B(x, p)| \leq K|p| + K,\]
\[|G(x, \mu, y)| \leq K|y| + K,\]
\[|f_n(x)| \leq K,\] (1.18)
\[\sum_{n=1}^{N} H_n^0(x, \mu, p) \geq c_0|B(x, p)|^2 - K,\]
\[H_n^0(x, \mu, p) \leq K|p_n|^2 + K||p_n||p| + K,\]
which is considered in Bensoussan et al. [2012].

Our main goal in this thesis is to derive the existence and uniqueness of Nash point for certain structure of underlying Lagrangians and establish the conditions (1.11) and (1.12). Then, we show that these structural conditions on Hamiltonians lead to certain version of maximum and minimum principle. Also, we establish the existence of the solution based on the inequality developed in Beck et al. [2015] using the iterating exponential method.

Outline. We present some applications of the systems of the type (1.1) in Section 1.2 and we define the notion of weak solution to (1.1) with the homogeneous Neumann and Dirichlet boundary conditions in Section 1.3. Next in Section 1.4, we establish the existence and uniqueness of a Nash point for certain structure of the Lagrangians and derive the structural assumptions (1.12) and (1.15) in Section 1.5.

We show that the assumptions (1.11)-(1.12) naturally lead to maximum principle for solutions of the class $C^2(\Omega)$ in Chapter 2. In Chapter 3 we proceed by applying the Galerkin method to find a solution to normalized system of equations

$$Lu^\delta_n + \lambda_n u^\delta_n = \frac{H_n(\cdot, u^\delta_n, \nabla u^\delta_n)}{1 + \delta|\nabla u^\delta_n|^2}. \quad (1.19)$$

In Chapter 4 we mollify the data of the problem to justify the procedure in Chapter 2. Finally, in Chapter 5 we set $\delta \to 0_+$ and show that $u^\delta$ converges to $u$, a solution to (1.1).

1.2 Motivation

Stochastic games. Such system of equations arises as a Bellman equation to a stochastic differential game

$$dy = g(y, v_1, \ldots, v_N)dt + \sigma(y)dw, \quad y(0) = x \quad (1.20)$$

with initial value $x \in \mathbb{R}^d$ and $v_n \in \mathbb{R}^{m_n}$. Here, $g$ is the drift term, $\sigma$ the diffusion term, $w(t)$ represents the standardized Wiener process and $y(t)$ is the state of the system at time $t$. Each player $n = 1, \ldots, N$ can affect the evolution of the corresponding dynamical system by a choice of admissible function $v_n(\cdot)$ to minimize his own cost functional

$$J_n(x, v) = E_n \left( \int_0^\tau e^{-\lambda_n t} l_n(y(t), v(t)) + e^{-\lambda_n \tau} f_n(y(\tau)) \right) \quad (1.21)$$

where $\tau$ is the exit time of $y(t)$ from the domain $\Omega$.

The dynamical programming principle for such stochastic control problems leads to an analytic problem

$$- \text{div}(a(x)\nabla u_n(x)) + \lambda_n u_n(x) = H_n(x, \nabla u(x)) \quad (1.22)$$

for all $n = 1, \ldots, N$ and where $a := 1/2\sigma\sigma^*$. The Hamiltonians $H_n(x, \nabla u(x))$ on the right-hand side are derived from Lagrangians of the form

$$L_n(x, p, v) = l_n(x, v) + p_n \cdot g(x, v_1, \ldots, v_N) + f_n(x), \quad n = 1, \ldots, N, \quad (1.23)$$

5
provided that the classical Nash game is considered: Let $v^* := v^*(x, p)$ be a Nash point of $L_n$, i.e., for given $(x, p)$ it satisfies
\[ L_n(x, p_n, v^*) \leq L_n(x, p_n, v^*_1, \ldots, v^*_{n-1}, v_n, v^*_n, \ldots, v^*_N) \] (1.24)
for all admissible $v_n \in \mathbb{R}^{m_n}$.

Then $H_n$ is defined by the formula
\[ H_n(x, p) := L_n(x, p_n, v^*(x, p)), \] (1.25)
$v^*(x, \nabla u(x))$ is a feedback control and the quantity $u_n(x)$ is the value function of the $n$-th player.

For more detailed study of stochastic games leading to Bellman equations see, for example, Fleming and Rishel [2012].

**Harmonic mappings.** The equations for harmonic mappings are another example of an application of system (1.1). In dimension $N = 3$ the equation
\[ -\Delta u = \frac{x}{|x|} |\nabla u|^2 \] (1.26)
is solved by $u = x/|x|$ which is another example of non-regular solution.

Another such system comes from Beck et al. [2015]: The equations presented have the following structure
\[ -\Delta u_n = u_n |\nabla u|^2. \] (1.27)
Assuming that all $u_n$ are positive we can rewrite the equation
\[ -\Delta \ln u_n = |\nabla u|^2 + |\nabla \ln u_n|^2. \] (1.28)
Defining new variables
\[ w_n := \ln u_n - \ln u_{n+1} \quad \text{for } n = 1, \ldots, N - 1, \]
\[ w_N := \ln u_N \] (1.29)
we obtain a system of equations
\[ -\Delta w_n = \nabla w_n \cdot \nabla \left( w_n + 2 \sum_{m=n+1}^{N} w_m \right) \quad \text{for } n = 1, \ldots, N - 1, \]
\[ -\Delta w_N = \sum_{m=1}^{N} e^2 \sum_{n=m}^{N} w_n \left| \nabla \sum_{n=m}^{N} w_n \right| + |\nabla w_N|^2. \] (1.30)
Provided that $u_n \in L^\infty$, this system of equations satisfy (1.14) and therefore, due to Bensoussan et al. [2010], we get uniform $C^\alpha$ bounds.

### 1.3 Notion of weak solution

In this thesis we consider the homogeneous Neumann boundary conditions, i.e., if $n = (n_1, \ldots, n_d)$ denotes the outer normal vector to $\partial \Omega$ then
\[ \sum_{i,j=1}^{d} a_{ij} D_j u_n n_i = 0 \quad \text{on } \partial \Omega \quad n = 1, \ldots, N, \] (1.31)
as well as the Dirichlet boundary conditions in the weak sense, i.e., the case
\[ u_n = 0 \text{ on } \partial \Omega \quad n = 1, \ldots, N. \] (1.32)
However, for simplicity in Chapter 5 we consider only the Neumann problem, for
the Dirichlet problem one could obtain very similar theory.

**Definition 1 (Neumann).** Let \( \Omega \) be a Lipschitz domain, \( a_{ij} \) satisfy (1.3)-(1.4) and \( H_n \) satisfy (1.5)-(1.6). We say that \( u \in W^{1,2}(\Omega, \mathbb{R}^N) \) is a weak solution to (1.1) and (1.31) if
\[
\int_{\Omega} a(x) \nabla u_n(x) \cdot \nabla \varphi(x) + \lambda_n u_n(x) \varphi(x) \, dx = \int_{\Omega} H_n(x, u, \nabla u) \varphi(x) \, dx
\]
for all \( \varphi \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \) and all \( n = 1, \ldots, N \). (1.33)

The case of Dirichlet data requires a slightly generalized notion of weak so-
olution that seems more suitable for problem (1.1) with the right-hand side \( H \)
satisfying the condition (1.6) of critical growth.

**Definition 2 (Dirichlet).** Let \( \Omega \) be a Lipschitz domain, \( a_{ij} \) satisfy (1.3)-(1.4) and \( H_n \) satisfy (1.5)-(1.6). We say that \( u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N) \) is a generalized weak solution to (1.1) and (1.32) if
\[
\int_{\Omega} a(x) \nabla u_n(x) \cdot \nabla \varphi(x) + \lambda_n u_n(x) \varphi(x) \, dx = \int_{\Omega} H_n(x, u, \nabla u) \varphi(x) \, dx
\]
for all \( \varphi \in D(\Omega) \) and all \( n = 1, \ldots, N \) (1.34)
and
\[
(\sum_{n=1}^N \sigma_n u_n)_+ \in W^{1,2}_0(\Omega), \quad (\sum_{n=1}^N u_n)_- \in W^{1,2}_0(\Omega)
\]
(1.35)
for all \( \sigma_n \gg 1 \).

This notion first appeared in Bulíček and Frehse [2011]. Let us point out that
the main difference is that we do not know a priori whether \( u_n \in W^{1,2}_0(\Omega) \) and we
are not able to identify a trace in the usual way. However, the condition (1.35)
gives us \( (u_n)_+ = 0 \) and \( (\sum_n u_n)_- = 0 \) on \( \partial \Omega \), so supposing that \( u_n \in W^{1,2}(\Omega) \)
we can conclude that \( u_n = 0 \) on \( \partial \Omega \) and therefore \( u_n \) is a weak solution in the
standard sense.
1.4 The Nash point

The main goal of this section is to establish existence and uniqueness of a Nash point for certain structure of Lagrangians. But first let us explain in details how the Hamiltonians are derived.

The Lagrangians considered here have a structure

$$L_n(x, p, v) := l_n(x, v) + p_n \cdot A v + f_n(x), \quad n = 1, \ldots, N,$$

(1.36)

where $A v := \sum_{n=1}^{N} A_n v_n$ with bounded measurable matrix functions $A_n(x) \in \mathbb{R}^{d \times m_n}$, $v$ denotes a vector $v = (v_1, \ldots, v_N)$ with $v_n \in \mathbb{R}^{m_n}$, $p$ stands for vectors $p = (p_1, \ldots, p_N)$ with $p_n \in \mathbb{R}^{d}$ and $f_n \in L^\infty(\Omega)$. The function $A v$ comes from the stochastic differential game (1.20) in case $g(x, v) = A v$.

We assume as in Bulíček and Frehse [2011] that for some constants $c_0 > 0$ and $K_3 \geq 0$ it holds

$$\sum_{n=1}^{N} l_n(x, v) \geq c_0 |v|^2 - K_3$$

for all $v \in \mathbb{R}^{\sum m_n}$. (1.37)

The Hamiltonians $H_n$ on the right-hand side of (1.1) are derived from the Lagrangians (1.36) in the following way:

Provided that the classical Nash game is considered, let $v^* := v^*(x, p)$ be a Nash point of $L_n$, i.e., for given $(x, p) \in \Omega \times \mathbb{R}^{dN}$ it satisfies

$$L_n(x, p_n, v^*) \leq L_n(x, p_n, v_1^*, \ldots, v_{n-1}^*, v_n, v_{n+1}^*, \ldots, v_N^*)$$

(1.38)

for all $v_n \in \mathbb{R}^{m_n}$. This problem need not always be uniquely solvable nor solvable but let us assume that the solution mapping exists and that

$$v^* : (x, p) \mapsto v^*(x, p)$$

is Carathéodory, (1.39)

then $H_n$ is defined by the formula

$$H_n(x, p) := L_n(x, p_n, v^*(x, p)).$$

(1.40)

To be able to say more about the existence of the Nash point, we need some further structure of $l_n$. In this thesis, we consider the case which is the most common in stochastic games

$$l_n(x, v) = \frac{1}{2} v_n \cdot Q_n(x) v_n + v_n \cdot G_n(x) \hat{v}^n,$$

(1.41)

where $\hat{v}^n$ is defined as $\hat{v}^n = (v_1, \ldots, v_{n-1}, v_{n+1}, \ldots, v_N)$. The $Q_n$ are bounded measurable symmetric $m_n \times m_n$ matrices and $G_n$ are bounded measurable $(\sum_{k \neq n} m_n) \times m_n$ matrices whose both coefficients may depend on $x$. The matrices $Q_n$ are uniformly positive definite in $\Omega$, i.e., there is $\kappa_0 > 0$ such that for $n = 1, \ldots, N$ we have

$$\frac{1}{2} \xi \cdot Q_n(x) \xi \geq \kappa_0 |\xi|^2$$

for all $\xi \in \mathbb{R}^{m_n}$ and a.a. $x \in \Omega$. (1.42)

The main goal of this chapter is the following result.
Theorem 1 (Existence and uniqueness of the Nash point). Let the Lagrangians have the structure \((1.36)\), \(l_n\) have structure \((1.41)\) and let \((1.37)\) and \((1.42)\) hold. Then there exists a unique Nash point \(v^*\) of \(L_n\) (defined in \((1.38)\)). Moreover, the function \(v^*(x,p)\) has the structure

\[
v^*(x,p) = B_0(x)p \quad \text{for some } B_0 \in L^\infty(\Omega, \mathbb{R}^{\sum n_m, \times \sum n_m}).
\]

Proof. We start by denoting the matrix

\[
M := \begin{pmatrix}
\frac{1}{2}Q_1 & G_1 \\
\frac{1}{2}Q_2 & G_2 \\
\vdots & \ddots & \ddots \\
G_N & \ddots & \ddots & \frac{1}{2}Q_N
\end{pmatrix},
\]

where \(\hat{G}_n\) denotes the first \(\sum_{k=1}^{n-1} m_n\) rows of \(G_n\) and \(\bar{G}_n\) last \(\sum_{k=n+1}^N m_n\) rows of \(G_n\).

Using the assumption \((1.41)\), the condition \((1.37)\) can be rewritten as

\[
c_0 |v|^2 - K_3 \leq \sum_{n=1}^N l_n(x,p) = \sum_{n=1}^N \left( \frac{1}{2} v_n \cdot Q_n v_n + v_n \cdot G_n \hat{v}^n \right) = v \cdot M v. \quad (1.45)
\]

Now, we denote \(\lambda_M\) the smallest eigenvalue of \((M + M^T)/2\) (all eigenvalues are real as it is symmetric matrix) and \(v_0\) its eigenvector. Then we set \(v := v_0\) in \((1.45)\)

\[
c_0 |v_0|^2 - K_3 \leq v_0 \cdot M v_0 = v_0 \cdot \frac{M + M^T}{2} v_0 = \lambda_M |v_0|^2. \quad (1.46)
\]

Subtracting \(\lambda_M |v_0|^2\) and \(K_3\) yields

\[
(c_0 - \lambda_M)|v_0|^2 \leq K_3. \quad (1.47)
\]

If \(c_0 > \lambda_M\) for some \(x \in \Omega\) then we let \(|v_0| \to \infty\) which leads to a contradiction so \(c_0 \leq \lambda_M\) for all \(x \in \Omega\) and we have

\[
v \cdot M v \geq \lambda_M |v|^2 \geq c_0 |v|^2. \quad (1.48)
\]

Next, we denote the matrix

\[
\hat{M} := \begin{pmatrix}
Q_1 & G_1 \\
G_2 & Q_2 \\
\vdots & \ddots & \ddots \\
G_N & \ddots & \ddots & Q_N
\end{pmatrix}
\]

and the matrix \(A_T p := (A_T^1 p_1, \ldots, A_T^N p_N)^T\). We use \((1.42)\) and \((1.48)\) to get

\[
v \cdot \hat{M} v = v \cdot M v + \sum_{n=1}^N \frac{1}{2} v_n \cdot Q_n v_n \geq (\kappa_0 + c_0) |v|^2. \quad (1.50)
\]

This means that \(\hat{M}\) is regular for every \(x \in \Omega\) and further \(\|\hat{M}^{-1}\|_2 \leq 1/(\kappa_0 + c_0)\). Since all norms on finitely dimensional vector spaces are equivalent and \(\max_{i,j} |a_{ij}|\) is a norm, we easily deduce that \(\hat{M}^{-1} \in L^\infty(\Omega, \mathbb{R}^{\sum n_m, \times \sum n_m})\).
A necessary condition for $v^*$ to be the Nash point of $L_n$ reads as
\[ Q_n v^*_n + G_n \hat{v}^*_n + A_n^T p_n = \frac{DL_n}{Dv_n} = 0 \quad \text{for } n = 1, \ldots, N. \tag{1.51} \]
or in matrix notation
\[ \hat{M} v^* + A^T p = 0. \tag{1.52} \]
Since $\hat{M}$ is regular we obtain a solution
\[ B_0(x)p := -\hat{M}^{-1} A^T p = v^*(x,p). \tag{1.53} \]
This solution is unique and $B_0 \in L^\infty(\Omega)$ as both $\hat{M}^{-1}$, $A^T \in L^\infty(\Omega)$. It is clearly continuous in $p$ for every $x \in \Omega$ and measurable in $x$ for all $p \in \mathbb{R}^d$, therefore it is Carathéodory.

Since we have that $D^2 L_n \frac{Dv^2_n}{Dv_n} = Q_n$ is positive definite from (1.42), the sufficient condition for minimum is satisfied and $v^*(x,p)$ is the Nash point of $L_n$. \hfill \Box

### 1.5 $H$ estimates

In this section we establish the structural conditions (1.11)-(1.12) of the Hamiltonians $H_n$ for certain assumptions on the structure of Lagrangians $L_n$. The results are summarized in the following two lemmas.

**Lemma 2.** Let $L_n$ have the structure (1.36) and let (1.37) hold, then there exist constants $K_2^*$ and $K_2$ such that
\[ \sum_{n=1}^N H_n(x,p) \geq -K_2^* \left| \sum_{n=1}^N p_n \right|^2 - K_2. \tag{1.55} \]

**Lemma 3.** Let $L_n$ have the structure (1.36), $l_n$ have the structure (1.41) and let (1.42) hold, then there exist constants $K_3$ and a Carathéodory function $g_0(x,p)$ such that
\[ |g_0(x,p)| \leq K_3 |p| \tag{1.56} \]
and
\[ H_n \leq K_3 |p|^2 + g_0(x,p) \cdot p + K_3. \tag{1.57} \]

**Convention:** From this point the positive constant $K$ is chosen sufficiently large such that all inequalities hold.

**Proof of Lemma 2.** Summing the Hamiltonians and using the definition (1.40) of $H_n$ and (1.36) of $L_n$, the assumption (1.37), $f_n \in L^\infty(\Omega)$ and Cauchy-
Schwartz inequality we obtain

\[
\sum_{n=1}^{N} H_n(x, p) = \sum_{n=1}^{N} L_n(x, p, B_0 p)
\]

\[
= \sum_{n=1}^{N} l_n(x, B_0 p) + p_n \cdot AB_0 p + f_n
\]

\[
\geq c_0 |B_0 p|^2 - |AB_0 p| \left| \sum_{n=1}^{N} p_n \right| - K
\]

\[
\geq c_0 |B_0 p|^2 - K |B_0 p| \left| \sum_{n=1}^{N} p_n \right| - K
\]

We employ Young’s inequality (6.21) to get

\[
K |B_0 p| \left| \sum_{n=1}^{N} p_n \right| \leq c_0 |B_0 p|^2 + \frac{K^2}{4c_0} \left| \sum_{n=1}^{N} p_n \right|^2.
\]

Using this in (1.58) we easily obtain (1.55).

**Proof of Lemma 3.** Since \( v^* \) is a Nash point the necessary condition (1.51) holds so we can substitute it in the definition (1.40) of \( H_n \) to get

\[
H_n(x, p) = - \frac{1}{2} v_n^* \cdot Q_n v_n^* - p_n \cdot A_n v_n^* + p_n \cdot Av^* + f_n(x).
\]

Using (1.42) the definiteness of \( Q_n \), \( f_n \in L^\infty \) and Cauchy-Schwartz inequality we obtain

\[
H_n(x, p) \leq -\kappa_0 |v_n^*|^2 + K |v_n^*||p_n| + p_n \cdot AB_0 p + K
\]

from which, using Young’s inequality (6.21)

\[
K |v_n^*||p_n| \leq \kappa_0 |v_n^*|^2 + \frac{K^2}{4\kappa_0} |p_n|^2.
\]

we get (1.57) with \( g_0(x, p) := AB_0 p \). It is easy to see that \( g_0 \) is Carathéodory and that \( |g_0| \leq K |p| \).
2. The maximum principle for continuous setting

From this point we assume that the structural conditions (1.11)-(1.12) on Hamiltonians $H_n$ hold. The goal of this chapter is to use these conditions to establish uniform bounds of a solution to (1.1) depending only on $\lambda_n, K_1, K_2$ and $N$, supposing the solution is of the class $C^2(\Omega)$.

**Assumptions of this chapter.** Suppose that $a \in C^1(\overline{\Omega}, \mathbb{R}^{d\times d}), u_n \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies the equation (1.1) everywhere in $\Omega$. For Neumann problem we further assume that $u_n \in C^2(\Omega)$.

Since $u_n$ is continuous it attains its maximum in $\overline{\Omega}$. Let us first assume that the maximum is attained at some point $x \in \Omega$. The first order condition for a maximum gives us

$$\nabla u_n(x) = 0. \quad \text{(2.1)}$$

Using (2.1), the fact that $\nabla u_n(x) = 0$ and substituting into the equation (1.1) we get

$$-a(x) : \nabla^2 u_n(x) + \lambda_n u_n(x) \quad \text{(2.2)}$$

The second order condition for a maximum is that $\nabla^2 u_n$ is negative semidefinite. The matrix $a(x)$ is positive definite from (1.3) so Lemma 22 yields

$$\lambda_n u_n(x) \leq -a(x) : \nabla^2 u_n(x) + \lambda_n u_n(x) \quad \text{(2.3)}$$

From which we conclude the upper bound

$$u_n \leq u_n(x) \leq \frac{K_1}{\lambda_n}. \quad \text{(2.4)}$$

For the lower bound we again assume that the minimum of $v := \sum_{n=1}^N u_n$ is attained at some $x \in \Omega$. Summing the equations (1.1) over $n$ and using the assumption (1.12) we obtain

$$-\text{div}(a(x) \nabla v(x)) + \sum_{n=1}^N \lambda_n u_n(x) = \sum_{n=1}^N H_n(x, u(x), \nabla u(x)) \geq -K_1 |\nabla v(x)|^2 - K_2. \quad \text{(2.5)}$$

Using the first order condition $\nabla v(x) = 0$ and the same computation as before we get

$$-a(x) : \nabla^2 v(x) + \sum_{n=1}^N \lambda_n u_n(x) \geq -K_2. \quad \text{(2.6)}$$
The second order condition with Lemma \ref{lemma22} yields
\[
\sum_{n=1}^{N} \lambda_n u_n(x) \geq -a(x) : \nabla^2 v(x) + \sum_{n=1}^{N} \lambda_n u_n(x) \geq -K_2.
\]  
(2.7)

We denote \( \tilde{\lambda} := \min_n (\lambda_n) \) and \( \Lambda := \sum_n (\lambda_n - \tilde{\lambda}) \) and use the upper bound (2.4) in (2.7) to obtain
\[
\tilde{\lambda} v \geq \tilde{\lambda} v(x) \geq -K_2 - \sum_{n=1}^{N} (\lambda_n - \tilde{\lambda}) u_n(x) \geq -K_2 - K_1 \Lambda.
\]  
(2.8)

Finally, using upper bound (2.3) and the lower bound (2.8) we get
\[
\frac{-K_2 - K_1 \Lambda}{\tilde{\lambda}} \leq v = u_n + \sum_{m \neq n} u_m \leq u_n + (N - 1) \frac{K_1}{\lambda_n}
\]  
(2.9)

so
\[
K_{3n} := \frac{-K_2 - K_1 \Lambda}{\tilde{\lambda}} - (N - 1) \frac{K_1}{\lambda_n} \leq u_n
\]  
(2.10)

and the lower bound is established.

Now, if the maximum of \( u_n \) lies on the boundary in the Dirichlet case it means that \( u_n \leq 0 \) so (2.4) holds and the same is true for the minimum of \( v \) and we are done. In the Neumann case we notice that the first order condition for maximum at some point \( x \) on the boundary is
\[
\frac{Du_n}{Dt_i}(x) = 0 \quad \text{for } i = 1, \ldots, d - 1,
\]  
(2.11)

with \( \{t_i\} \) being any base of a tangent space of \( \partial \Omega \) at \( x \). We know that
\[
\frac{Du_n}{Du}(x) = 0
\]  
(2.12)

where \( v = a^T(x) n \). Since \( a^T \) is positive definite \([1.3]\) we get that \( v \) is independent of the tangent space. Otherwise there would be scalars \( \alpha_n \) such that
\[
a^T n = \sum_{n=1}^{d-1} \alpha_n t_n
\]  
(2.13)

multiplying both side with \( n \) we get
\[
0 < a^T n \cdot n = \sum_{n=1}^{d-1} \alpha_n t_n \cdot n = 0,
\]  
(2.14)

contradiction. Together we get \( \nabla u_n(x) = 0 \) and we can follow the computations as if \( x \) was inside. The same holds for \( v \).
3. Galerkin method

In this section we establish the existence of solution to the normalized system of (1.1) with both Neumann (1.31) and Dirichlet (1.32) boundary conditions, i.e., solution $u \in W_N$ of

$$\int_{\Omega} a(x) \nabla u_n(x) \cdot \nabla \varphi(x) + \lambda_n u_n(x) \varphi(x) \, dx = \int_{\Omega} H_n^\delta(x, u, \nabla u) \varphi(x) \, dx$$

for all $n = 1, \ldots, N$ and $\varphi \in W$. (3.1)

Here, $W := W_1^1(\Omega)$ for the Neumann and $W := W_0^1(\Omega)$ for the Dirichlet boundary condition and

$$H_n^\delta(x, \mu, p) = \frac{H_n(x, \mu, p)}{1 + \delta|p|^2}$$

for $0 < \delta < 1$. Note that for fixed $\delta$ we have, from (1.6), that $H_n^\delta$ is bounded

$$|H_n^\delta(x, \mu, p)| \leq \frac{K^* + K^*|p|^2}{1 + \delta|p|^2} \leq \frac{K^*}{\delta}.$$  (3.2)

Convention. From now on in this chapter we omit the $\delta$ index which is to be understood over all $H_n$. Due to Theorem 10 the space $W$ is separable so we take its countable linearly independent subset $\{w_1, w_2, \ldots\}$ such that its linear hull is dense in $W$. We denote its subspaces $W_k := \text{Span}\{w_1, \ldots, w_k\}$.

We are looking for a solution $u_k \in W_N^k$ to

$$\int_{\Omega} a(x) \nabla u_n^k(x) \cdot \nabla w(x) + \lambda_n u_n(x) w(x) \, dx = \int_{\Omega} H_n(x, u^k, \nabla u^k) w(x) \, dx$$

for all $w \in W_k$ and all $n = 1, \ldots, N$. (3.4)

To find it we use Lemma 19. We denote $F : \mathbb{R}^{kN} \to \mathbb{R}^{kN}$ defined by

$$[F(\alpha)]_m := \int_{\Omega} a \nabla u_n \cdot \nabla w_i + \lambda_n u_n w_i - H_n(u, \nabla u) w_i$$

where $u_n(\alpha) := \sum_{i=1}^{k} \alpha_i \nabla w_i$. (3.5)

The mappings $\alpha \to u(\alpha)$ and $\alpha \to \nabla u(\alpha)$ are linear and therefore continuous so the first two terms are continuous. The continuity of the last term comes from continuity of $H_n$ in the last two coordinates (assumption (1.5)) and continuous dependence of an integral on a parameter.

To check the other hypothesis of Lemma 19 we compute

$$F(\alpha) \cdot \alpha = \sum_{n=1}^{N} \int_{\Omega} a \nabla u_n \cdot \nabla u_n + \lambda_n u_n u_n - H_n(u, \nabla u) u_n \, dx$$

$$\geq \alpha_0 \|\nabla u\|_2^2 + \lambda_n \|u\|_2^2 - \frac{K^*}{\delta} \|u\|_1$$

$$\geq \lambda_n \|u\|_2^2 - \frac{K^*C}{\delta} \|u\|_2$$

(3.6)
where we used the definiteness \((1.3)\) of the matrix \(a\) and boundedness \((3.3)\) of \(H_n\). The constant \(C\) comes from \(L^1 \hookrightarrow L^2\) embedding and finally \(\lambda_m\) denotes the minimum of \(\lambda_n\)’s.

Now, the mapping \(\alpha \to u(\alpha)\) is isomorphism of linear spaces of finite dimension and so there exists a constant \(d_0 > 0\) such that \(\|u\|_2 \geq d_0 |\alpha|\). Taking \(|\alpha| = \frac{K^*}{d_0 \lambda_{m_0}}\) we get \(\|u(\alpha)\|_2 \geq \frac{KC}{\lambda_{m_0}}\), which, considering the inequality \((3.6)\), implies that \(F(\alpha) \cdot \alpha \geq 0\). This concludes the hypotheses of Lemma \(19\) and therefore there exists \(\alpha_0\) such that \(F(\alpha_0) = 0\). The function \(u(\alpha_0)\) clearly solves the equation \((3.4)\). We denote this solution \(u^k := u(\alpha_0)\).

**Convention:** From now on, whenever we subtract a subsequence we renumber it so we use the same indices as before.

Due to boundedness \((3.3)\) of \(H^k_n(x) := H_n(x, u^k(x), \nabla u^k(x))\) and reflexivity of \(L^2\) we can use the Banach-Alaoglu theorem (Theorem \(4\)) to subtract weakly convergent subsequence in \(L^2(\Omega)\) and denote its limit \(\overline{H}_n\), i.e.,

\[
H^k_n \rightharpoonup \overline{H}_n \quad \text{in} \quad L^2(\Omega) \quad \text{for all} \quad n = 1, \ldots, N. \tag{3.7}
\]

Next, we have that \(u^k_n \in W_k\) so we can set it for \(w\) in \((3.4)\) to get an a priori estimate

\[
\alpha_0 \| \nabla u^k_n \|_2^2 + \lambda_n \| u^k_n \|_2^2 \leq \int_{\Omega} a(x) \nabla u^k_n(x) \cdot \nabla u^k_n(x) + \lambda_n u^k_n(x) u^k_n(x) dx
\]

\[
= \int_{\Omega} H^k_n(x) u^k_n(x) dx \leq \frac{K^*}{\delta} \int_{\Omega} |u^k_n(x)| dx
\]

\[
\leq \frac{1}{2} \lambda_n \| u^k_n \|_2^2 + \frac{K^* |\Omega|}{2\delta^2 \lambda_n}
\]

where we used the definiteness \((1.3)\) of matrix \(a\) and Young’s inequality \((6.21)\). Finally, subtracting \(1/2\lambda_n \| u^k_n \|_2^2\) from both sides we get a \(W^{1,2}(\Omega)\) bound independent of \(k\). Again the space \(W^{1,2}(\Omega)\) is reflexive so we use the Banach-Alaoglu theorem (Theorem \(4\)) to subtract weakly \(W^{1,2}(\Omega)\) convergent subsequence

\[
u^k_n \rightharpoonup u_n \quad \text{in} \quad W^{1,2}(\Omega) \quad \text{for all} \quad n = 1, \ldots, N. \tag{3.9}
\]

For any \(w \in W_\infty\) there is \(n_0\) such that for all \(k > n_0\) : \(w \in W_k\), so for fixed \(w\) we can pass the limit \(k \to \infty\) in \((3.4)\) and use \((3.7)\) and \((3.9)\) to conclude that

\[
\int_{\Omega} a(x) \nabla u_n(x) \cdot \nabla w(x) + \lambda_n u_n(x) w(x) dx = \int_{\Omega} \overline{H}_n(x) w(x) dx
\]

for all \(w \in W_\infty\) and all \(n = 1, \ldots, N\).

Since \(W_\infty\) is dense in \(W\), for any \(\varphi \in W\) we can take sequence \(w_l \in W_\infty\) such that \(w_l \to \varphi\) in \(W\), set them in \((3.10)\) and pass the limit \(l \to \infty\) to get

\[
\int_{\Omega} a \nabla u_n(x) \cdot \nabla \varphi(x) + \lambda_n u_n(x) \varphi(x) dx = \int_{\Omega} \overline{H}_n(x) \varphi(x) dx
\]

for all \(\varphi \in W\) and all \(n = 1, \ldots, N\).
Due to the weak convergence \( [3.9] \) of \( u_n^k \) and the compact embedding \( W^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \) (Theorem \[12\]) we have strong convergence in \( L^2(\Omega) \), i.e.,

\[
u_n^k \to u_n \quad \text{in } L^2(\Omega) \quad \text{for all } n = 1, \ldots, N.\]

(3.12)

Finally, we use Theorem \[6\] to subtract an almost everywhere convergent subsequence

\[
u_n^k(x) \to u_n(x) \quad \text{for a.a. } x \in \Omega \quad \text{and all } n = 1, \ldots, N.\]

(3.13)

Now we again set \( w = u_n^k \) in (3.4) and let \( k \to \infty \) to obtain

\[
\lim_{k \to \infty} \int_{\Omega} \alpha \nabla u_n^k \cdot \nabla u_n^k + \lambda_n u_n^k u_n^k = \lim_{k \to \infty} \int_{\Omega} H_n^k u_n^k = \int_{\Omega} \bar{H}_n u_n
\]

(3.14)

where the last equality follows from (3.7), (3.12) and Lemma \[13\].

The strong convergence of \( \nabla u_n^k \) in \( L^2 \) now follows from

\[
\alpha_0 \| \nabla (u_n^k - u_n) \|^2 + \lambda_n \| u_n^k - u_n \|^2 \leq \int_{\Omega} \alpha (\nabla u_n^k - \nabla u_n) \cdot (\nabla u_n^k - \nabla u_n) + \lambda_n (u_n^k - u_n)^2
\]

(3.15)

\[
= \int_{\Omega} (\alpha \nabla u_n^k \cdot \nabla u_n^k + \lambda_n u_n^k u_n^k) - \alpha \nabla u_n^k \cdot \nabla u_n + \lambda_n u_n^k u_n
\]

as we now prove that all the terms on the right-hand side tend to \( \int_{\Omega} \bar{H}_n u_n \).

The limit of the first term was established in (3.14). For the second term we use the weak convergence of \( u_n^k \) in \( W^{1,2}(\Omega) \) from (3.9) and the fact that \( a^T \nabla u_n, u_n \in L^2 \)

\[
\lim_{k \to \infty} \int_{\Omega} \alpha \nabla u_n^k \cdot \nabla u_n + \lambda_n u_n^k u_n^k = \lim_{k \to \infty} \int_{\Omega} \nabla u_n^k \cdot a^T \nabla u_n + \lambda_n u_n^k u_n^k
\]

(3.16)

\[
= \int_{\Omega} a \nabla u_n \cdot \nabla u_n + \lambda_n u_n u_n = \int_{\Omega} \bar{H}_n u_n.
\]

The last equality follows from (3.11) with \( w = u_n \in W \). The third term converges the same and the fourth is equal to \( \int_{\Omega} \bar{H}_n u_n \) from (3.11) with \( w = u_n \in W \).

Therefore, we conclude the strong convergence of \( \nabla u_n^k \) in \( L^2(\Omega) \) and again we subtract an almost everywhere convergent subsequence

\[
\nabla u_n^k(x) \to \nabla u_n(x) \quad \text{for a.a. } x \in \Omega \quad \text{and all } n = 1, \ldots, N.
\]

(3.17)

From the continuity (1.5) of \( H_n \) in \( u \) and \( \nabla u \) and a.e. convergence (3.13), (3.17) of \( u_n^k \) and \( \nabla u_n^k \), we conclude that

\[
H_n^k(x) \to H_n(x, u(x), \nabla u(x)) \quad \text{for a.a. } x \in \Omega \quad \text{and all } n = 1, \ldots, N.
\]

(3.18)

We use the Lebesgue dominant convergence theorem (Theorem \[7\]) with the bound \( [3.3] \) to obtain

\[
H_n^k \to H_n(u, \nabla u) \quad \text{in } L^2(\Omega) \quad \text{for all } n = 1, \ldots, N
\]

(3.19)

so finally \( \bar{H}_n(x) = H_n(x, u(x), \nabla u(x)) \) since the weak and strong limit must coincide. Substituting this into (3.11) we get (3.1) and the existence is established.
4. The maximum principle

In this chapter we are going to establish $L^\infty$ estimates of $u^\delta$ independent of the parameter $\delta$. To do this we use the result of Chapter 2 on mollified equation

$$L_\varepsilon u_n + \lambda_n u_n = H^{\delta}_{en}(\cdot, u, \nabla u)$$  \hspace{1cm} (4.1)

with $a_\varepsilon$ and $H_\varepsilon$ smooth and setting $\varepsilon, \epsilon \to 0_+$ to get the original solution.

We start by taking standard symmetric positive mollifier $\varphi$ and for any $\varepsilon > 0$ we denote $\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n} \varphi(x/\varepsilon)$. We extend $a(x)$ with $\alpha_0 I_d$ and $H_n$ with zero outside of $\Omega$, further we denote the mollifications of $a(x)$ and $H_n(x, \mu, p)$ as

$$a_{\varepsilon ij}(x) := (a_{ij} \ast \varphi_\varepsilon)(x)$$
$$H_{en}(x, \mu, p) := (H_n(\cdot, \mu, p) \ast \varphi_\varepsilon)(x)$$  \hspace{1cm} (4.2)

We denote $a_\varepsilon := \{a_{\varepsilon ij} \}_{i,j=1}^d$ and $L_\varepsilon$ the corresponding operator.

Now, due to $a_{\varepsilon ij}$ and $H_{en}(\cdot, \mu, p)$ are smooth and abide the same structural conditions mainly

$$|a_{\varepsilon ij}(x)| \leq \int_{\mathbb{R}^d} |a_{ij}(y)| \varphi_\varepsilon(x - y) dy \leq K \text{ so } a_{\varepsilon ij} \in L^\infty(\Omega),$$

$$a_\varepsilon(x) \xi \cdot \xi = \sum_{ij} \int_{\mathbb{R}^d} a_{\varepsilon ij}(y) \xi_i \xi_j \varphi_\varepsilon(x - y) dy \geq \alpha_0 |\xi|^2 \text{ for all } \xi \in \mathbb{R}^d$$  \hspace{1cm} (4.3)

and

$$|H_{en}(x, \mu, p)| \leq \int_{\mathbb{R}^d} |H_n(x - y, \mu, p)| \varphi_\varepsilon(y) dy \leq K^* |p|^2 + K^*$$
$$H_{en}(x, \mu, p) = \int_{\mathbb{R}^d} H_n(x - y, \mu, p) \varphi_\varepsilon(y) dy \leq K^*_1 |p||p_n| + K_1$$
$$\sum_{n=1}^N H_{en}(x, \mu, p) = \int_{\mathbb{R}^d} \sum_{n=1}^N H_n(x - y, \mu, p) \varphi_\varepsilon(y) dy \geq -K^*_2 \left| \sum_{n=1}^N p_n \right|^2 - K_2.$$  \hspace{1cm} (4.4)

Now, we could also mollify $H_n$ in the other two variables to get slightly modified conditions \[4.4\], which would in limit $\varepsilon \to 0_+$ give the same $L^\infty$ bounds. Instead for simplicity, we are going to assume that $(\mu, p) \mapsto H_n(x, \mu, p)$ is Hölder continuous for some $\alpha$ for almost every $x \in \Omega$.

Considering \[4.3\] and \[4.4\] we can use the Galerkin method developed in the previous chapter to obtain a weak solution $u^\delta_k \in W^N$ to the system of equations

$$\int_{\Omega} a_\varepsilon(x) \nabla u_n(x) \cdot \nabla \varphi(x) + \lambda_n u_n(x) \varphi(x) dx = H^{\delta}_{en}(x, u(x), \nabla u(x)) \varphi(x) dx$$  \hspace{1cm} (4.5)

for all $\varphi \in W$ and all $n = 1, \ldots, N$ with $\varepsilon = \frac{1}{k}$ and

$$H^{\delta}_{en}(x, \mu, p) := \frac{H_{en}(x, \mu, p)}{1 + \delta |p|^2}.$$  \hspace{1cm} (4.6)

\footnote{Here $I_d$ denotes $d$-dimensional identity matrix.}
We drop the $\delta$ superscript for now and we use the notation
\[ H_n^k(x) := H_n(x, u^k(x), \nabla u^k(x)). \]  
(4.7)

Using (4.4) we know that
\[ |H_n^k(x)| = \frac{|H_{1/kn}(x, u^k(x), \nabla u^k(x))|}{1 + \delta|\nabla u^k(x)|^2} \leq \frac{K^*|\nabla u^k(x)|^2 + K^*}{1 + \delta|\nabla u^k(x)|^2} \leq \frac{K^*}{\delta} \]  
(4.8)

so $H_n^k \in L^\infty(\Omega)$. Considering the system of equations
\[ \int_{\Omega} a(x)\nabla v_n(x) \cdot \nabla \varphi(x) + \lambda_n v_n(x) \varphi(x) dx = H_n^k(x) \varphi(x) dx \]
(4.9)

for all $\varphi \in W$ and all $n = 1, \ldots, N$ we can use Theorem 15 for Dirichlet and Theorem 16 for Neumann boundary condition to get that there exists a weak solution $v$ which must be equal to $u^k$ and is of the class $W^{2,p}(\Omega, \mathbb{R}^N)$ for all $p \geq 1$. So we get that $u^k \in W^{2,p}(\Omega, \mathbb{R}^N)$ for some fixed $p \geq \frac{d}{1-\alpha}$ and
\[ \|u^k\|_{2,p} \leq C(||u^k||_p + \|H_n^k\|_p) \leq CK + C\frac{K^*}{\delta} \leq \tilde{K} \]  
(4.10)

with $\tilde{K}$ independent of $k$. From the Sobolev embedding (Theorem 12) we get $u^k \in C^{1,\alpha}(\Omega, \mathbb{R}^N)$ and therefore $H_n^k \in C^\alpha(\Omega)$ as well.

For any point $x \in \Omega$ we choose $r$ such that $B_3r(x) \subset \Omega$. There exists a cut-off function $\tau \in C_c^\infty(\Omega)$ such that $\tau = 1$ in $B_r(x)$ and $\tau = 0$ outside of $B := B_{2r}(x)$. Now we can rewrite the formulation to strong form as
\[ -\text{div}(a\nabla u^k_n) + \lambda_n u^k_n = H_n^k. \]  
(4.11)

Multiplying it by $\tau$ we get
\[ H_n^k \tau = -\text{div}(a\nabla u^k_n)\tau + \lambda_n u^k_n \tau \]
\[ = -\text{div}(a\nabla u^k_n)\tau + a\nabla u^k_n \cdot \nabla \tau + \lambda_n u^k_n \tau \]
\[ = -\text{div}(a\nabla (u^k_n \tau)) + 2a\nabla u^k_n \cdot \nabla \tau + \nabla a\nabla u^k_n \cdot \nabla \tau + \lambda_n u^k_n \tau. \]  
(4.12)

We denote $z := u^k_n \tau$ and
\[ F := H_n^k \tau - \sum_{i,j=1}^d D_i a_{ij} u^k_n D_j \tau + 2a_{ij} D_i u^k_n D_j \tau \]
\[ = H_n^k \tau - 2a\nabla u^k_n \cdot \nabla \tau - \nabla a\nabla u^k_n \cdot \nabla \tau \]  
(4.13)

then $z$ solves the equation
\[ Lz + \lambda_n z = F \quad \text{in } B, \]
\[ z = 0 \quad \text{on } \partial B. \]  
(4.14)

Then since we have $H_n^k, a_{ij}, D_i a_{ij}, \tau, D_i \tau, D_i u^k_n, u^k_n \in C^\alpha(\overline{B})$, it is also true that $F \in C^\alpha(\overline{B})$ and we use Theorem 17 to deduce $z \in C^\alpha(\overline{B})$ and so $u^k_n \in C^\alpha(\overline{B},(x))$. Since $x$ was arbitrary we conclude $u^k_n \in C^2(\Omega)$.
For Neumann problem we use the mirroring technique to extend the equation locally outside of $\Omega$ as in the proof of Theorem 16 and use the previous computations to show that $u_n^k \in C^2(\overline{\Omega})$.

This justifies the assumptions in Chapter 2 and we get uniform $L^\infty$ bounds independent on $\delta$, $\varepsilon$ and $k$. It remains to set $k \to \infty$ and $\varepsilon \to 0_+$ and show that the limit satisfies (3.1).

We proceed as in Chapter 3, we have the $L^\infty$ bounds on $H_n^k$ so we can subtract $L^2$ weakly convergent subsequence

$$H_n^k \rightharpoonup \hat{H}_n \text{ in } L^2(\Omega). \quad (4.15)$$

We now have $W^{2,p}(\Omega)$ bounds from (4.10) for $u^k$ so we use the Banach-Alaoglu theorem (Theorem 4 to subtract weakly $W^{2,p}(\Omega)$ convergent sequence which converges strongly in $C^{1,\alpha}(\Omega)$, i.e.,

$$u_n^k \to u_n \text{ in } C^{1,\alpha}(\Omega). \quad (4.16)$$

due to the compact Sobolev embedding (Theorem 12). It only remains to show that $\hat{H}_n(x) = H_n(x, u(x), \nabla u(x))$. We are going to show that

$$H_n^k \to H_n(x, u(x), \nabla u(x)) \quad (4.17)$$

in the sense of distribution.

We fix $\eta \in C_0^\infty(\Omega)$ then

$$\left| \int_\Omega (H_n^k(x) - H_n(x, u(x), \nabla u(x))) \eta(x) dx \right| = \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (H_n(y, u^k(x), \nabla u^k(x)) - H_n(x, u(x), \nabla u(x))) \eta(x) \varphi(x,k(x-y)) dy dx \right| \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(H_n(y, u^k(x), \nabla u^k(x)) - H_n(x, u(x), \nabla u(x))) \eta(x) | \varphi(x,k(x-y)) dx dy. \quad (4.18)$$

The functions $F_k$ are defined by

$$y \mapsto \int_{\mathbb{R}^d} |(H_n(y, u^k(x), \nabla u^k(x)) - H_n(x, u(x), \nabla u(x))) \eta(x) | \varphi(x,k(x-y)) dx \quad (4.19)$$

and are bounded by

$$|F_k| \leq \frac{2K^*}{\delta} \max |\eta| \quad (4.20)$$

so to use the Lebesgue dominant convergence theorem (Theorem 7) we only need to know

$$F_k(y) \to 0 \quad \text{for almost every } y \in \Omega \quad (4.21)$$

and we are done.

We fix $\varepsilon_1 > 0$ and $y \in \Omega$ such that $y$ is a Lebesgue point of measurable function $H_n(\cdot, u(\cdot), \nabla u(\cdot))$ (which from Theorem 9 is almost everywhere) and such that the function $H_n(y, \cdot)$ is continuous (which from (1.5) is also almost everywhere).
For $B(y) \subset \subset \Omega$ we find compact $P \subset \mathbb{R}^N \times \mathbb{R}^{dN}$ such that $(u, \nabla u)(B(y)), (u, \nabla u)(B(y)) \subset P$. We find $\varepsilon$ such that
\[
\forall (\mu_1, p_1), (\mu_2, p_2) \in P, |\mu_1 - \mu_2| + |p_1 - p_2| < \varepsilon :
|H_n(y, \mu_1, p_1) - H_n(y, \mu_2, p_2)| < \varepsilon_1
\] (4.22)
since $H_n(y, \cdot)$ is continuous on compact set $P$. Then we find $k_1$ such that for all $k \geq k_1$
\[
|u^k - u| + |\nabla u^k - \nabla u| < \varepsilon \quad \text{in } \Omega
\] (4.23)
so
\[
|H_n(y, u^k, \nabla u^k) - H_n(y, u, \nabla u)| < \varepsilon_1 \quad \text{in } B(y).
\] (4.24)
We find $k_2$ such that for all $k \geq k_2$ we have
\[
|u - u(y)| + |\nabla u - \nabla u(y)| < \varepsilon \quad \text{in } B_{1/k}(y)
\] (4.25)
so
\[
|H_n(y, u(x), \nabla u(x)) - H_n(y, u(y), \nabla u(y))| < \varepsilon_1 \quad \text{in } B_{1/k}(y).
\] (4.26)
Next, we find $k_3$ such that for all $k \geq k_3$ we have
\[
\frac{1}{|B_{1/k}(y)|} \int_{B_{1/k}(y)} |H_n(y, u(y), \nabla u(y)) - H_n(x, u(x), \nabla u(x))| < \varepsilon_1
\] (4.27)
and we find $k_4$ such that $B_{1/k_4}(y) \subset B(y)$.

Finally, for $k \geq \max(k_1, k_2, k_3, k_4)$, we compute
\[
\int_{\mathbb{R}^d} |(H_n(y, u^k(x), \nabla u^k(x)) - H_n(x, u(x), \nabla u(x)))\eta(x)|\varphi_{1/k}(x - y)dx
\]
\[
\leq \int_{B_{1/k}(y)} |(H_n(y, u^k(x), \nabla u^k(x)) - H_n(y, u(x), \nabla u(x)))\eta(x)|\varphi_{1/k}(x - y)dx
\]
\[
+ \int_{B_{1/k}(y)} |(H_n(y, u(x), \nabla u(x)) - H_n(y, u(y), \nabla u(y)))\eta(x)|\varphi_{1/k}(x - y)dx
\]
\[
+ \int_{B_{1/k}(y)} |(H_n(y, u(y), \nabla u(y)) - H_n(x, u(x), \nabla u(x)))\eta(x)|\varphi_{1/k}(x - y)dx
\]
\[
\leq K\varepsilon + K\varepsilon_1 + K\varepsilon_1 = 3K\varepsilon_1
\] (4.28)
thus $F_k(y) \to 0$ for almost every $y \in \Omega$.

Therefore $H^k_n \to H_n(\cdot, u, \nabla u)$ in the sense of distribution and since the weak and distributional limit must coincide we deduce $H_n(x) = H_n(x, u(x), \nabla u(x))$.

Now, we have all convergences to set $k \to \infty$ in (4.5) to get that $u^\varepsilon$ solves
\[
\int_{\Omega} a_\varepsilon(x)\nabla u_n(x) \cdot \nabla \varphi + \lambda_n u_n(x)\varphi(x)dx = H^\varepsilon_n(x, u(x), \nabla u(x))\varphi(x)dx
\] (4.29)
for all $\varphi \in W$ and all $n = 1, \ldots, N$

with $L^\infty$ estimates of Chapter 2.
Last thing we need is to set $\varepsilon \to 0_+$: so we set $k := 1/\varepsilon$ and denote $a_k := a_\varepsilon$ and solution to (4.29) as $u^k$. We proceed the same way as in Chapter 3 to obtain $W^{1,2}(\Omega)$ bounds and subtract weakly $W^{1,2}(\Omega)$ convergent subsequence which converges almost everywhere. Then we subtract subsequence such that $\nabla u^k \to \nabla u$ almost everywhere and

$$H^k_n(x) := H_n(x, u^k(x), \nabla u^k(x)) \to \overline{H}_n(x) \quad \text{in } L^2. \quad (4.30)$$

Since $H^k_n(x) \to H_n(x, u(x), \nabla u(x))$ for almost every $x \in \Omega$ and it is bounded we get from the Lebesgue dominated convergence theorem (Theorem 7) that $\overline{H}_n = H_n(x, u(x), \nabla u(x))$ (for more details see Chapter 3).

To pass the limit on both sides of (4.29) it is now sufficient to show that

$$a^T_k \cdot \nabla \varphi \to a^T \cdot \nabla \varphi \quad \text{in } L^2. \quad (4.31)$$

For that we have bound on $a_k$ from (4.3) so we get an integrable dominant $K|\nabla \varphi| \in L^2(\Omega)$. An almost everywhere convergence follows from Theorem 23 so by the Lebesgue dominant convergence theorem (Theorem 7) and Lemma 13 we get

$$\lim_{k \to \infty} \int_\Omega a_k \nabla u_n^k \cdot \varphi = \lim_{k \to \infty} \int_\Omega \nabla u_n^k \cdot a_k^T \nabla \varphi = \int_\Omega \nabla u_n \cdot a^T \nabla \varphi = \int_\Omega a \nabla u_n \cdot \nabla \varphi. \quad (4.32)$$

Finally, we can set $k \to \infty$ in (4.29) to conclude that $u$ solves the equation (3.1) with $L^\infty$ estimates of Chapter 2.
5. Passing to a solution

In this section we want to set $\delta \to 0^+$ and prove that after a subtraction of subsequence we get a solution to the original problem \((1.1)\) with our $L^\infty$ estimates. From now on, we consider only the Neumann boundary condition.

We proceed in 5 steps:

1. Developing the point-wise inequality from Beck et al. [2015] and establishing uniform $W^{1,2}(\Omega)$ bounds.

2. Subtracting converging subsequence and proving point-wise convergence of $\nabla u^\delta$.

3. Showing that $u_n$ is a subsolution to \((1.1)\).

4. Showing that $\sum_n u_n$ is a supersolution to sum of equations \((1.1)\).

5. Concluding that $u$ is a solution.

5.1 $W^{1,2}(\Omega)$ estimates

First we have to find a uniform estimate (independent on $\delta$) of $\|\nabla u^\delta\|_2$. For this we use the method of iterating exponentials to get a point-wise inequality involving $|\nabla u|$:

$$
-K_1 \sum_{n=1}^N D_{u_n} \varphi_1(u) + \alpha_0 \sum_{n=1}^N Z_n(u) \gamma_n''(u_n) |\nabla u_n|^2 \prod_{m=1}^n \varphi_m(u) \\
\leq \sum_{n=1}^N a \nabla u_n \cdot \nabla D_{u_n} \varphi_1(u) - \sum_{n=1}^N D_{u_n} \varphi_1(u) H_n(u, \nabla u).
$$

(5.1)

This proof is the same as in Beck et al. [2015].

For $n = 1, \ldots, N$ we assume that $\gamma_n \in C^2(\mathbb{R})$ are given nonnegative, strictly convex nondecreasing functions and for any $u \in \mathbb{R}^N$ we define

$$
\varphi_N(u) := e^{\gamma_N(u_N)},
\varphi_n(u) := e^{\gamma_n(u_n) + \varphi_{n+1}(u)} \quad \text{for } n = 1, \ldots, N - 1
$$

(5.2)

and we reserve the symbol $D_{u_n} \varphi_m(u) := \partial \varphi_m(u)/\partial u_n$ for their partial derivative.

We start by observing

$$
D_{u_n} \varphi_n(u) = \begin{cases} 
0 & \text{if } m < n, \\
\gamma'_m(u_m) \prod_{\alpha=n}^m \varphi_\alpha(u) & \text{if } m \geq n,
\end{cases}
$$

(5.3)

which, from the properties of $\gamma$’s and $\varphi$’s, are positive functions. Now for any $u \in L^\infty(\Omega, \mathbb{R}^N) \cap W^{1,2}(\Omega, \mathbb{R}^N)$ we observe that

$$
\nabla \varphi_n(u) = \sum_{m=n}^N \nabla \gamma_m(u_m) \prod_{\alpha=n}^m \varphi_\alpha(u),
$$

(5.4)
\[ \nabla \gamma_n(u_n) = \nabla \ln(\varphi_n(u)) - \nabla \varphi_{n+1}(u). \quad (5.5) \]

First, we want to estimate the term

\[ \sum_{n=1}^{N} a \nabla u_n \cdot \nabla (D_n \varphi_1(u)) = \sum_{n=1}^{N} a \nabla u_n \cdot \nabla (\gamma_n(u_n)) \prod_{m=1}^{n} \varphi_m(u) \]
\[ = \sum_{n=1}^{N} \gamma_n(u_n) a \nabla u_n \cdot \nabla \prod_{m=1}^{n} \varphi_m(u) + a \nabla (\gamma_n(u_n)) \cdot \nabla \left( \prod_{m=1}^{n} \varphi_m(u) \right). \quad (5.6) \]

Using that \( \nabla \varphi_\alpha(u) = \varphi_\alpha(u) \nabla \ln(\varphi_\alpha(u)) \) and the identity (5.4) we rewrite the last term as

\[ \sum_{n=1}^{N} a \nabla (\gamma_n(u_n)) \cdot \nabla \left( \prod_{m=1}^{n} \varphi_m(u) \right) \]
\[ = \sum_{n=1}^{N} \sum_{\alpha=1}^{n} a \nabla (\gamma_n(u_n)) \nabla (\ln(\varphi_\alpha(u))) \prod_{m=1}^{n} \varphi_m(u) \]
\[ = \sum_{\alpha=1}^{N} a \nabla (\gamma_n(u_n)) \cdot \nabla (\ln(\varphi_\alpha(u))) \prod_{m=1}^{n} \varphi_m(u) \]
\[ = \sum_{\alpha=1}^{N} a \nabla \varphi_\alpha(u) \cdot \nabla (\ln(\varphi_\alpha(u))) \prod_{m=1}^{n} \varphi_m(u) \]
\[ = \sum_{\alpha=1}^{N} a \nabla (\ln(\varphi_\alpha(u))) \cdot \nabla (\ln(\varphi_\alpha(u))) \prod_{m=1}^{n} \varphi_m(u). \quad (5.7) \]

Substituting back and using the ellipticity (1.3), we get the first estimate

\[ \sum_{n=1}^{N} a \nabla u_n \cdot \nabla D_n \varphi_1(u) \]
\[ \geq \alpha_0 \sum_{n=1}^{N} (|\nabla \ln(\varphi_n(u))|^2 + \gamma_n(u_n)|\nabla u_n|^2) \prod_{m=1}^{n} \varphi_m(u). \quad (5.8) \]

Next, we use the assumption (1.11) on \( H_n \) to estimate

\[ \sum_{n=1}^{N} D_n \varphi_1(u)H_n(u, \nabla u) \]
\[ \leq K_1 \sum_{n=1}^{N} D_n \varphi_1(u) + K_1^* \sum_{n=1}^{N} D_n \varphi_1(u)|\nabla u||\nabla u_n|. \quad (5.9) \]

We use both (5.4) and (5.5) and Young’s inequality (6.21) to estimate the last
Using this inequality in (5.9) and summing it with (5.7) we conclude that

\[ \sum_{n=1}^{N} D_{u_n} \varphi_1(u) |\nabla u| |\nabla u_n| = \sum_{n=1}^{N} |\nabla u| |\nabla \gamma_n(u_n)| \prod_{m=1}^{n} \varphi_m(u) \]

\[ = |\nabla u| |\nabla \gamma_N(u_N)| \prod_{m=1}^{N} \varphi_m(u) + \sum_{n=1}^{N-1} |\nabla u| |\nabla \gamma_n(u_n)| \prod_{m=1}^{n} \varphi_m(u) \]

\[ \leq \sum_{n=1}^{N} |\nabla u| |\nabla (\ln \varphi_n(u))| \prod_{m=1}^{n} \varphi_m(u) + \sum_{n=1}^{N-1} |\nabla u| |\nabla \varphi_{n+1}(u)| \prod_{m=1}^{n} \varphi_m(u) \]

\[ = \sum_{n=1}^{N} |\nabla u| |\nabla (\ln \varphi_n(u))| \prod_{m=1}^{n} \varphi_m(u) + \sum_{n=2}^{N} |\nabla u| |\nabla \varphi_n(u)| \prod_{m=1}^{n} \varphi_m(u) \]

\[ \leq 2 \sum_{n=1}^{N} |\nabla u| |\nabla (\ln \varphi_n(u))| \prod_{m=1}^{n} \varphi_m(u) \]

\[ \leq \alpha_0 K_1^{-1} \sum_{n=1}^{N} |\nabla (\ln \varphi_n(u))|^2 \prod_{m=1}^{n} \varphi_m(u) + K_1^* \alpha_0^{-1} \sum_{n=1}^{N} |\nabla u|^2 \prod_{m=1}^{n} \varphi_m(u) \]

\[ \leq \alpha_0 K_1^{-1} \sum_{n=1}^{N} |\nabla (\ln \varphi_n(u))|^2 \prod_{m=1}^{n} \varphi_m(u) + K_1^* \alpha_0^{-1} \sum_{n=1}^{N} |\nabla u_n|^2 \prod_{m=1}^{N} \varphi_m(u). \]

Using this inequality in [5.9] and summing it with [5.7] we conclude that

\[ - K_1 \sum_{n=1}^{N} D_{u_n} \varphi_1(u) + \alpha_0 \sum_{n=1}^{N} Z_n(u) \gamma''_n(u_n)|\nabla u_n|^2 \prod_{m=1}^{n} \varphi_m(u) \]

\[ \leq \sum_{n=1}^{N} a \nabla u_n \cdot \nabla D_{u_n} \varphi_1(u) - \sum_{n=1}^{N} D_{u_n} \varphi_1(u) H_n(u, \nabla u), \quad (5.10) \]

where we denoted \( \varphi_{N+1} := 1 \) and

\[ Z_n(u) := 1 - \frac{K_1^* N \prod_{m=n+1}^{N+1} \varphi_m(u)}{\alpha_0^2 \gamma''_n(u_n)}. \quad (5.11) \]

Now, we want to choose the functions \( \gamma_n \) such that we have a uniform positivity of the terms \( Z_n(u) \). We set

\[ \gamma_n(u_n) := e^{u_n + M + c_n} \quad (5.12) \]

where \( M \) is such that

\[ \|u_n\|_{\infty} \leq M \quad \text{for all } n = 1, \ldots, N \quad (5.13) \]

and \( c_n \) are positive constants. From the definition of \( M \) we get

\[ \gamma''_n(u_n) \geq e^{c_n} \quad (5.14) \]
and from the definition of functions $\gamma_n$ in (5.2) we obtain

$$\varphi_N(u) \leq e^{c_{2M+cN}} =: \varphi_N(u_{\max}),$$

$$\varphi_n(u) \leq e^{c_{2M+c_n+\varphi_{n+1}(u_{\max})}} =: \varphi_n(u_{\max}) \quad \text{for } n = 1, \ldots, N - 1. \quad (5.15)$$

Finally, using the definition (5.11) of $Z_n(u)$ with previous estimates, we get

$$Z_n(u) \geq 1 - \frac{K_1^2 N \prod_{m=n+1}^{N+1} \varphi_m(u_{\max})}{\alpha_0^2 e^{c_n}}. \quad (5.16)$$

We note that the nominator on the right-hand side of the last inequality is independent of $c_1, \ldots, c_n$ so we can choose the constant $c_n$ so that

$$Z_n(u) \geq \frac{1}{2} \quad \text{for all } n = 1, \ldots, N. \quad (5.17)$$

We choose the constants to be

$$c_N := \ln \left( \frac{2K_1^2 N}{\alpha_0^2} \right),$$

$$c_n := c_{n+1} + \ln \varphi_{n+1}(u_{\max}) \quad \text{for } n = 2, \ldots, N-1,$$

$$c_1 \geq c_2 + \ln \varphi_2(u_{\max}),$$

then (5.17) is easily derived.

Next, we set $k := 1/\delta$, denote the solution to (3.1) as $u^k$ (it exists due to Chapter 3 and denote $H^k_n(x) := H^k_n(x, u^k, \nabla u^k)$. Thanks to Chapter 4 we know that $u^k \in L^\infty$ and we can set $M := \max_{K_1/\lambda_n, K_3 n}$. We want to prove that $D_n \varphi_1(u^k) \in W^{1,2}(\Omega)$ for all $n = 1, \ldots, N$.

First, we note that $D_n \varphi_1(u^k)$ is bounded. This follows from (5.3) and (5.15) as

$$D_n \varphi_1(u^k) = \gamma'_n(u^k) \prod_{m=1}^{n} \varphi_m(u^k) \leq \gamma'_n(M) \prod_{m=1}^{n} \varphi_m(u_{\max}). \quad (5.19)$$

It is also bound away from 0 as we can use the definitions (5.2) and (5.12) of $\varphi_n$ and $\gamma_n$ and the property (5.13) of $M$ to get

$$\varphi_N(u) \geq e^{c_N} =: \varphi_N(u_{\min}) > 0,$$

$$\varphi_n(u) \geq e^{c_n+\varphi_{n+1}(u_{min})} =: \varphi_n(u_{\min}) > 0 \quad \text{for } n = 1, \ldots, N - 1. \quad (5.20)$$

Thus, we have

$$D_n \varphi_1(u^k) = \gamma'_n(u^k) \prod_{m=1}^{n} \varphi_m(u^k) \geq \gamma'_n(-M) \prod_{m=1}^{n} \varphi_m(u_{\min}) > 0. \quad (5.21)$$

The gradient of $D_n \varphi_1(u^k)$ can be computed, using (5.3) and the fact that $\gamma_n = \gamma'_n$, as

$$\nabla(D_n \varphi_1(u^k)) = \nabla(\gamma'_n(u^k) \prod_{m=1}^{n} \varphi_m(u^k))$$

$$= \gamma''_n(u^k) \nabla u^k_n \prod_{m=1}^{n} \varphi_m(u^k) + \gamma'_n(u^k) \nabla \left( \prod_{m=1}^{n} \varphi_m(u^k) \right) \quad (5.22)$$

$$= \gamma_n(u^k) \nabla u^k_n \prod_{m=1}^{n} \varphi_m(u^k) + \gamma_n(u^k) \nabla \left( \prod_{m=1}^{n} \varphi_m(u^k) \right).$$
From (5.19) we get that \( \gamma_n(u_n^k) \prod_{m=1}^{n} \varphi_m(u^k) \in L^\infty(\Omega) \) and since \( \nabla u_n^k \in L^2(\Omega) \) we get that
\[
\gamma_n(u_n^k) \nabla u_n \prod_{m=1}^{n} \varphi_m(u^k) \in L^2(\Omega).
\] (5.23)

It remains to show that the second term on the right-hand side of (5.22) is in \( L^2 \) as well. We have
\[
\nabla \left( \prod_{m=1}^{n} \varphi_m(u^k) \right) = \sum_{\alpha=1}^{n} \nabla (\ln \varphi_\alpha(u^k)) \prod_{m=1}^{n} \varphi_m(u^k)
\] (5.24)

and again we have shown that \( \prod_{m} \varphi_m(u^k) \in L^\infty(\Omega) \), thus it remains to prove \( \nabla (\ln \varphi_\alpha(u^k)) \in L^2(\Omega) \). We proceed by induction: For \( \alpha = N \) by the definition (5.2) and (5.12) of \( \varphi_n \) and \( \gamma_n \), we have
\[
\nabla \ln \varphi_N(u^k)) = \nabla e^{u_N^k + M + c_N} = e^{u_N^k + M + c_N} \nabla u_N^k \in L^2(\Omega)
\] (5.25)
as \( u_N^k \) is bounded and so is \( e^{u_N^k + M + c_N} \).

Now suppose that \( \nabla (\ln \varphi_{\alpha+1}(u^k)) \in L^2 \). Then
\[
\nabla (\ln \varphi_{\alpha+1}) \varphi_{\alpha+1}(u^k) = \nabla \varphi_{\alpha+1}(u^k)
\] (5.26)
and \( \varphi_{\alpha+1}(u^k) \) is bound away from 0 by \( \varphi_{\alpha+1}(u_{\min}) \) so we get \( \nabla \varphi_{\alpha+1}(u^k) \in L^2 \) as well. Moreover, we have
\[
\nabla \ln \varphi_{\alpha}(u^k) = \nabla (e^{u_{\alpha}^k + M + c_N} + \varphi_{\alpha+1}(u^k))
\]
(5.27)

and the induction step is finished. This together with (5.23), (5.24) and boundedness of \( \gamma_n(u_n^k) \) proves that \( \nabla (D_{u_n} \varphi_1(u^k)) \in L^2(\Omega) \) and so \( D_{u_n} \varphi_1(u^k) \in W^{1,2}(\Omega) \).

Thus, we can set \( \varphi := D_{u_n} \varphi_1(u^k) \) in (3.1) and sum the equations to get
\[
\sum_{n=1}^{N} \int_{\Omega} a \nabla u_n \cdot \nabla D_{u_n} \varphi_1(u^k) - D_{u_n} \varphi_1(u^k) H_n^k = - \sum_{n=1}^{N} \int_{\Omega} \lambda_n u_n^k D_{u_n} \varphi_1(u^k).
\] (5.28)

We know that \( H_n^k \) satisfy the same structural condition as \( H_n \) since we have
\[
H_n^k(x, u, \nabla u) = \frac{H_n(x, u, \nabla u)}{1 + \delta |\nabla u_n|^2} \leq H_n(x, u, \nabla u) \leq K_1^* |\nabla u||\nabla u_n| + K_1
\] (5.29)

Therefore, we can use the inequality (5.10) together with (5.17) on left-hand side of (5.28) to get
\[
\frac{1}{2} \alpha_0 \int_{\Omega} \sum_{n=1}^{N} \gamma''_n(u_n^k) |\nabla u_n^k|^2 \prod_{m=1}^{n} \varphi_m(u^k)
\]
(5.30)

Since the constant \( M \) is independent of \( k \), and therefore \( c_n \)'s as well, all the terms on right-hand side are uniformly bound (from (5.13) and (5.19)) and all the terms on left-hand side apart from \( |\nabla u_n| \) are uniformly bound away from 0 (from (5.21) and \( \gamma''_n(u_n^k) \geq \gamma''_n(-M) > 0 \)). We conclude that \( \|\nabla u^k\|_2 \) is uniformly bound as well.
5.2 The almost everywhere convergence of $\nabla u^k$

Having the uniform $W^{1,2}(\Omega)$ bound for $u^k_n$ we use the Banach-Alaoglu theorem (Theorem 4) to subtract a weakly convergent subsequence, i.e.,

$$u^k_n \rightharpoonup u_n \quad \text{in} \ W^{1,2}(\Omega) \quad \text{for all} \ n = 1, \ldots, N.$$  \hfill (5.31)

Due to the compact embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ (Theorem 12) we have the strong convergence in $L^2(\Omega)$, i.e.,

$$u^k_n \rightarrow u_n \quad \text{in} \ L^2(\Omega) \quad \text{for all} \ n = 1, \ldots, N.$$  \hfill (5.32)

Using Theorem 6 we can subtract an almost everywhere convergent subsequence

$$u^k_n(x) \rightarrow u_n(x) \quad \text{for a.a.} \ x \in \Omega \quad \text{and all} \ n = 1, \ldots, N.$$  \hfill (5.33)

Note that now from (1.6) we also have uniform $L^1$ bound for $H^k_n$ since

$$\int_{\Omega} |H^k_n| \leq \int_{\Omega} K^* + K^*|\nabla u^k|^2 \leq K^*|\Omega| + K^*\|\nabla u^k\|_2^2 \leq K.$$  \hfill (5.34)

Now, we set $\varphi := T_\varepsilon(u^k_n - u_n)$, with $T_\varepsilon(s) := \text{sgn}(s) \max(|s|, \varepsilon)$. We have $|\varphi| \leq \varepsilon$ and therefore $\varphi \in L^\infty$. The function $T_\varepsilon$ is lipschitz so we can use Theorem 14 to get

$$\nabla \varphi = \begin{cases} \nabla(u^k_n - u_n) & \text{on} \ \Omega' := \{|u^k_n - u_n| < \varepsilon\}, \\ 0 & \text{otherwise}. \end{cases} \hfill (5.35)$$

Thus, we have $\nabla \varphi \in L^2$ and so $\varphi \in W^{1,2}(\Omega)$. This means we can set $\varphi$ in the equation (4.5) and use the positive definiteness (1.3) of matrix $a$ and (5.34) to obtain

$$\begin{align*}
\alpha_0\|\nabla T_\varepsilon(u^k_n - u_n)\|^2_2 &+ \int_{\Omega} a\nabla u_n \cdot \nabla T_\varepsilon(u^k_n - u_n) + \lambda_n u^k_n T_\varepsilon(u^k_n - u_n) \\
&\leq \int_{\Omega} a\nabla T_\varepsilon(u^k_n - u_n) \cdot \nabla T_\varepsilon(u^k_n - u_n) + a\nabla u_n \cdot \nabla T_\varepsilon(u^k_n - u_n) \\
&+ \int_{\Omega} \lambda_n u^k_n T_\varepsilon(u^k_n - u_n) \\
&= \int_{\Omega} a\nabla(u^k_n - u_n) \cdot \nabla(u^k_n - u_n) + a\nabla u_n \cdot \nabla(u^k_n - u_n) \\
&+ \int_{\Omega} \lambda_n u^k_n T_\varepsilon(u^k_n - u_n) \\
&= \int_{\Omega} a\nabla(u^k_n - u_n) \cdot \nabla(u^k_n - u_n) + \int_{\Omega} \lambda_n u^k_n T_\varepsilon(u^k_n - u_n) \\
&= \int_{\Omega} a\nabla(u^k_n - u_n) \cdot \nabla(u^k_n - u_n) + \lambda_n u^k_n T_\varepsilon(u^k_n - u_n) \\
&= \int_{\Omega} H^k_n T_\varepsilon(u^k_n - u_n) \leq \varepsilon K. \hfill (5.36)
\end{align*}$$

We want to prove that for a subsequence the last two terms on the left-hand side tend to 0 as $k \rightarrow \infty$. 

27
For the second term, we have \( a \nabla u_n \in L^2 \) so it would suffice that
\[
T_\varepsilon(u_n^k - u_n) \rightharpoonup 0 \quad \text{in} \quad L^2(\Omega). \tag{5.37}
\]
Since \( T_\varepsilon(u_n^k - u_n) = u_n^k - u_n \) on \( \Omega' \) we have uniform \( W^{1,2}(\Omega) \) bound, so we can use the Banach-Alaoglu theorem (Theorem 4) to subtract a weakly \( W^{1,2}(\Omega) \) convergent subsequence
\[
T_\varepsilon(u_n^k - u_n) \rightharpoonup v \quad \text{in} \quad W^{1,2}(\Omega). \tag{5.38}
\]
From the compact embedding \( W^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \) (see Theorem 12) we get
\[
T_\varepsilon(u_n^k - u_n) \rightharpoonup v \quad \text{in} \quad L^2(\Omega), \tag{5.39}
\]
but from (5.32) we know that \( T_\varepsilon(u_n^k - u_n) \to 0 \) in \( L^2(\Omega) \) so \( v = 0 \) and we are done. From this we have
\[
\int_\Omega a \nabla u_n \cdot \nabla T_\varepsilon(u_n^k - u_n) \to 0. \tag{5.40}
\]

The third term tends to 0 thanks to (5.32) and the fact that \( T_\varepsilon(u_n^k - u_n) \to 0 \) in \( L^2(\Omega) \), i.e.,
\[
\int_\Omega \lambda_n u_n^k T_\varepsilon(u_n^k - u_n) \to 0. \tag{5.41}
\]
Taking \( \lim \sup \) on both sides of (5.36) and using (5.40) and (5.41) we get
\[
\limsup_{k \to \infty} \| \nabla T_\varepsilon(u_n^k - u_n) \|_2 \leq \varepsilon K. \tag{5.42}
\]

Next, we can use Hölder’s inequality (6.22) to estimate
\[
\| \nabla (u_n^k - u_n) \|_1
= \int_{\{|u_n^k - u_n| < \varepsilon\}} |\nabla T_\varepsilon(u_n^k - u_n)| + \int_{\{|u_n^k - u_n| \geq \varepsilon\}} |\nabla (u_n^k - u_n)|
\leq \| \nabla T_\varepsilon(u_n^k - u_n) \|_2 |\Omega|^{1/2} + \| \nabla (u_n^k - u_n) \|_2 \{ |u_n^k - u_n| \geq \varepsilon \}^{1/2}.
\tag{5.43}
\]
Due to the Banach-Steinhaus theorem (Theorem 5) and the convergence (5.31), we know that \( \| \nabla (u_n^k - u_n) \|_2 \) is bounded and we know that \( u_n^k(x) \to u_n(x) \) almost everywhere from (5.33), so the term \( \{ |u_n^k - u_n| \geq \varepsilon \}^{1/2} \) tends to 0. Together, the last term on the right-hand side tends to 0 as well, therefore, taking lim sup on both sides of (5.43) and using (5.42) we deduce
\[
0 \leq \limsup_{k \to \infty} \| \nabla (u_n^k - u_n) \|_1 \leq \varepsilon K.
\tag{5.44}
\]
Setting \( \varepsilon \to 0^+ \) we obtain
\[
\limsup_{k \to \infty} \| \nabla (u_n^k - u_n) \|_1 = 0
\tag{5.45}
\]
and since \( \| \nabla (u_n^k - u_n) \|_1 \geq 0 \) we conclude that
\[
\nabla u_n^k \to \nabla u_n \quad \text{in} \quad L^1(\Omega, \mathbb{R}^d). \tag{5.46}
\]
Finally, using Theorem 6 we again subtract an almost everywhere convergence subsequence, i.e.,
\[
\nabla u_n^k(x) \to \nabla u_n(x) \quad \text{for a.a.} \; x \in \Omega \; \text{and} \; n = 1, \ldots, N. \tag{5.47}
\]
5.3 Weak lower-stability

In this section we show the weak lower-stability of our equation (1.1), i.e., that

\[
\int_{\Omega} a \nabla u_n \cdot \nabla \varphi + \lambda_n u_n \varphi \leq \int_{\Omega} H_n(u, \nabla u) \varphi
\]

(5.48)

for all \( n = 1, \ldots, N \) and nonnegative \( \varphi \in L^\infty(\Omega) \cap W^{1,2}(\Omega) \).

For any nonnegative \( \varphi \in L^\infty(\Omega) \cap W^{1,2}(\Omega) \) we set

\[
\psi := \varphi D_{u_n} \varphi_1(u^k)
\]

(5.49)

(where \( \varphi_1 \) is defined in (5.2)) and note that from (5.19) and (5.21) we know that \( \psi \in L^\infty(\Omega) \). Further we get

\[
\nabla \psi = D_{u_n} \varphi_1(u^k) \nabla \varphi + \varphi \nabla D_{u_n} \varphi_1(u^k)
\]

(5.50)

and as \( D_{u_n} \varphi_1(u^k) \in W^{1,2}(\Omega) \) (which we proved in Section 5.1) and \( \varphi \in W^{1,2}(\Omega) \) we conclude \( \psi \in W^{1,2}(\Omega) \) as well. Now, we can set \( \psi \) in (4.5) and sum the equations to get

\[
0 = \sum_{n=1}^N \int_{\Omega} a \nabla u_n^k \cdot \nabla \varphi D_{u_n} \varphi_1(u^k) dx + \int_{\Omega} \lambda_n u_n^k \varphi D_{u_n} \varphi_1(u^k) dx + \int_{\Omega} \varphi a_k \nabla u_n^k \cdot \nabla D_{u_n} \varphi_1(u^k) dx - \int_{\Omega} \varphi H_n^k D_{u_n} \varphi_1(u^k) dx =: \sum_{l=1}^4 \int_{\Omega} T_l^k(u^k) dx.
\]

(5.51)

First, we want to prove that

\[
\lim_{k \to \infty} \int_{\Omega} T_1^k(u^k) + T_2^k(u^k) = \int_{\Omega} T_1(u) + T_2(u).
\]

(5.52)

For the first term we have the weak convergence \([5.31]\) of \( \nabla u_n^k \) in \( L^2 \) so due to Lemma \([13]\) it is sufficient to prove that

\[
a^T \nabla \varphi D_{u_n} \varphi_1(u^k) \to a^T \nabla \varphi D_{u_n} \varphi_1(u) \quad \text{in} \quad L^2.
\]

(5.53)

We use the Lebesgue dominant convergence theorem (Theorem \([7]\)): We have point-wise convergence from \([5.33]\) and from the boundedness \([1.4]\) of \( a \), \([5.19]\) of \( D_{u_n} \varphi_1(u^k) \) and Cauchy-Schwarz inequality we obtain

\[
|a^T \nabla \varphi D_{u_n} \varphi_1(u^k)| \leq K|\nabla \varphi|,
\]

(5.54)

which is a \( L^2 \) dominant. Together we have

\[
\lim_{k \to \infty} \int_{\Omega} T_1^k(u^k) = \int_{\Omega} T_1(u).
\]

(5.55)
To prove the convergence of $\int T^k_2(u^k)$ we proceed as before except this time the dominant is a constant, since $\varphi \in L^\infty$, $u^k_n$ is uniformly bounded from Section 4 and $D_n\varphi_1(u^k)$ is bounded from (5.19). The conclusion (5.52) follows.

Now, we prove that

$$H_n^k(x) \to H_n(x, u, \nabla u) \quad \text{for a.e. } x \in \Omega. \quad (5.56)$$

We fix $x$ such that

$$(\mu, p) \mapsto H_n(x, \mu, p) \quad \text{is continuous,}$$

$$u^k(x) \to u(x) \quad \text{and}$$

$$\nabla u^k(x) \to \nabla u(x). \quad (5.57)$$

There are almost all such $x$. Let $\epsilon > 0$ be arbitrary then we find $k_1$ such that for all $k \geq k_1$ we have

$$\frac{K^*|\nabla u^k(x)|^2 + K^*}{k + |\nabla u^k(x)|^2} \leq \frac{\epsilon}{2}, \quad (5.58)$$

where $K^*$ is from (1.6), and which is possible since $\nabla u^k(x) \to \nabla u(x)$.

Next, we can find $\epsilon > 0$ such that

$$\forall |\mu - u(x)| + |p - \nabla u(x)| < \epsilon, \quad \text{where } (\mu, p) \in \mathbb{R}^N \times \mathbb{R}^{N,d}, \quad (5.59)$$

we have

$$|H_n(x, \mu, p) - H_n(x, u(x), \nabla u(x))| < \frac{\epsilon}{2}. \quad (5.60)$$

We find $k_2$ such that for all $k \geq k_2$ we have

$$|u^k(x) - u(x)| < \frac{\epsilon}{2} \quad \text{and} \quad |\nabla(u^k(x) - u(x))| < \frac{\epsilon}{2}, \quad (5.61)$$

so we have

$$|H_n(x, u^k(x), \nabla u^k(x)) - H_n(x, u(x), \nabla u(x))| < \frac{\epsilon}{2}. \quad (5.62)$$

Now we take $k \geq \max(k_1, k_2)$ and proceed by estimating

$$|H_n^k(x) - H_n(x, u(x), \nabla u(x))|$$

$$= |H_n^{1/k}(x, u^k(x), \nabla u^k(x)) - H_n(x, u(x), \nabla u(x))|$$

$$\leq |H_n^{1/k}(x, u^k(x), \nabla u^k(x)) - H_n(x, u^k(x), \nabla u^k(x))|$$

$$+ |H_n(x, u^k(x), \nabla u^k(x)) - H_n(x, u(x), \nabla u(x))|. \quad (5.63)$$

The first term can be estimated, using the definition (3.2) of $H_n^{1/k}$, as

$$|H_n^{1/k}(x, u^k(x), \nabla u^k(x)) - H_n(x, u^k(x), \nabla u^k(x))|$$

$$= \frac{|H_n(x, u^k(x), \nabla u^k(x))|}{k + |\nabla u^k(x)|^2}$$

$$\leq \frac{K^*|\nabla u^k(x)|^2 + K^*}{k + |\nabla u^k(x)|^2} < \frac{\epsilon}{2}, \quad (5.64)$$

where we used the assumption (1.5) and (5.58).
For the second term we use the inequality (5.62) and we get

\[ |H_n^k(x) - H_n(x, u(x), \nabla u(x))| < \varepsilon \]  

(5.65)

and thus (5.56) is established.

Next, from (5.33) and continuity of \( D_{u_n} \varphi_1 \) (see definition (5.2) and identity (5.3)) we get

\[ D_{u_n} \varphi_1(u^k(x)) \to D_{u_n} \varphi_1(u(x)) \quad \text{for a.a. } x \in \Omega \]  

(5.66)

and together with (5.56) we obtain

\[ T^k_3(u^k(x)) \to T_3(u(x)) \quad \text{for a.a. } x \in \Omega. \]  

(5.67)

We use (5.47) and the fact that

\[ \nabla D_{u_n} \varphi_1(u^k(x)) = \nabla u D_{u_n} \varphi_1(u^k(x)) \cdot \nabla u^k(x) \]  

\[ \to \nabla u D_{u_n} \varphi_1(u(x)) \cdot \nabla u(x) = \nabla D_{u_n} \varphi_1(u(x)), \]  

(5.68)

to get

\[ T^k_3(u^k(x)) \to T_3(u(x)) \quad \text{for a.a. } x \in \Omega. \]  

(5.69)

Now, we use the inequality (5.10) together with (5.17) and nonnegativity of \( \varphi \) to get

\[ -K \sum_{n=1}^{N} D_{u_n} \varphi_1(u^k(x)) \varphi \leq T^k_3(u^k(x)) + T^k_4(u^k(x)) \]  

(5.70)

and since we have uniform bound on \( D_{u_n} \varphi_1(u^k(x)) \) from (5.19) and \( \varphi \in L^\infty(\Omega) \) we obtain

\[ -\tilde{K} \leq T^k_3(u^k(x)) + T^k_4(u^k(x)). \]  

(5.71)

Thus, we can use Fatou’s lemma 8 to deduce that

\[ \liminf_{k \to \infty} \int_{\Omega} T^k_3(u^k(x)) + T^k_4(u^k(x)) \geq \int_{\Omega} T_3(u(x)) + T_4(u(x)). \]  

(5.72)

Using this in (5.51) with (5.52) we get

\[ 0 = \liminf_{k \to \infty} \sum_{l=1}^{4} \int_{\Omega} T^k_l(u^k(x)) \geq \sum_{l=1}^{4} \int_{\Omega} T_l(u(x)). \]  

(5.73)

Rewriting this back we obtain

\[ \sum_{n=1}^{N} \int_{\Omega} a \nabla u_n \cdot \nabla (\varphi D_{u_n} \varphi_1(u)) + \lambda_n u_n \varphi D_{u_n} \varphi_1(u) \leq \sum_{n=1}^{N} \int_{\Omega} H_n(u, \nabla u) \varphi D_{u_n} \varphi_1(u). \]  

(5.74)
Finally, we use the fact that $D_u \psi_1(u) = \gamma_1'(u) \psi_1(u) = e^{u_1+M+c_1} \psi_1(u)$ (see definitions (5.3) and (5.12)) and that the other derivatives are independent from $c_1$: we divide the inequality by $e^{c_1}$ and let $c_1 \to \infty$ to obtain
\[
\int_\Omega a \nabla u_1 \cdot \nabla (\varphi e^{u_1+M} \psi_1(u)) + \lambda_1 u_1 \varphi e^{u_1+M} \psi_1(u) \leq \int_\Omega H_1(u, \nabla u) \varphi e^{u_1+M} \psi_1(u). \tag{5.75}
\]

For $\eta \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$ we set $\varphi := (e^{u_1+M} \psi_1(u))^{-1} \eta$. We have bounds away from zero for $e^{u_1+M} \psi_1(u)$ from (5.21) so $\varphi \in L^\infty$. We compute its gradient
\[
\nabla \varphi = (e^{u_1+M} \psi_1(u))^{-1} \nabla \eta - (e^{u_1+M} \psi_1(u))^{-1} \eta \nabla u_1 - e^{-u_1-M}(\psi_1(u))^{-2} \nabla \psi_1(u) \cdot \nabla u \tag{5.76}
\]
which is a sum of $L^2$ functions, thus $\varphi \in L^\infty \cap W^{1,2}(\Omega)$ and it is positive. Therefore, we can substitute it to (5.75) to get
\[
\int_\Omega a \nabla u_1 \cdot \nabla \eta + \lambda_1 u_1 \eta \leq \int_\Omega H_1(u, \nabla u) \eta. \tag{5.77}
\]

Since the variable $u_1$ was chosen arbitrarily we can analogously proceed for the other variables and obtain (replacing $\eta$ by $\varphi$)
\[
\int_\Omega a \nabla u_n \cdot \nabla \varphi + \lambda_n u_n \varphi \leq \int_\Omega H_n(u, \nabla u) \varphi \quad \text{for all } n = 1, \ldots, N. \tag{5.78}
\]

### 5.4 Sum upper-stability

In this section we provide the reverse inequality for the sum of the equations. Again we provide the proof only for the Neumann boundary condition.

We proceed in the similarly as with the lower-stability. We denote
\[
v^k := \sum_{n=1}^N u_n^k \tag{5.79}
\]
and set $\psi := \varphi e^{-\lambda v^k} \in W^{1,2}(\Omega)$. Clearly $\psi \in L^\infty(\Omega)$ and the gradient can be easily computed as
\[
\nabla \psi = e^{-\lambda v^k} \nabla \varphi - \lambda \varphi e^{-\lambda v^k} \nabla v^k. \tag{5.80}
\]
Since $\nabla \varphi$, $\nabla v^k \in L^2$ and the other terms are in $L^\infty$ we get $\psi \in W^{1,2}(\Omega)$.

We set $\psi$ in (3.1) and sum the equations to get
\[
0 = -\int_\Omega a_k \nabla v^k \cdot \nabla \varphi e^{-\lambda v^k}
- \sum_{n=1}^N \int_\Omega \lambda_n u_n^k \varphi e^{-\lambda v^k}
- \int_\Omega \varphi a \nabla v^k \cdot \nabla e^{-\lambda v^k}
+ \sum_{n=1}^N \int_\Omega \varphi H_n^k e^{-\lambda v^k} =: \sum_{l=1}^4 \int_\Omega E_l^k(v^k). \tag{5.81}
\]
Using the same procedure as in the previous section we can check the convergence of the first two integrals and the point-wise convergence of \( E_3^k \) and \( E_4^k \). It remains to show the lower bound of \( E_3^k(v^k) + E_4^k(v^k) \).

We estimate the term \( E_3^k \) using the ellipticity (1.3) to get
\[
E_3^k(v^k) = \lambda \varphi a \nabla v^k \cdot \nabla v^k e^{-\lambda v^k} \\
\geq \alpha_0 \lambda \varphi |\nabla v^k|^2 e^{-\lambda v^k} \quad (5.82)
\]
and the term \( E_4^k \) using the assumption (1.11) to obtain
\[
E_4^k(v^k) \geq -K_2 \varphi e^{-\lambda v^k} - K_2^* |\nabla v^k|^2 e^{-\lambda v^k}. \quad (5.83)
\]
Setting \( \lambda = \alpha_0^{-1} K_2^* \) and summing the inequalities (5.82) and (5.83) (using the uniform bound of \( v^k \)) we obtain
\[
E_3^k(v^k) + E_4^k(v^k) \geq -K_2 \varphi e^{-\lambda v^k} \geq -\tilde{K}. \quad (5.84)
\]
Thus, again using Fatou’s lemma and setting \( \varphi = \eta e^{\lambda v} \) for \( \eta \in L^\infty \cap W^{1,2}(\Omega) \) (again now \( \varphi \in L^\infty \cap W^{1,2}(\Omega) \) and it is positive). We get
\[
\sum_{n=1}^{N} \int_{\Omega} a \nabla u_n \cdot \nabla \eta + \lambda_n u_n \eta \geq \int_{\Omega} \sum_{n=1}^{N} H_n(u, \nabla u) \eta \quad (5.85)
\]
and the upper-stability is proven.

### 5.5 Conclusion

For any \( \varphi \in L^\infty \cap W^{1,2}(\Omega) \) we denote
\[
I_n(\varphi) := \int_{\Omega} a \nabla u_n \cdot \nabla \varphi + \lambda_n u_n \varphi - \int_{\Omega} H_n(u, \nabla u) \varphi. \quad (5.86)
\]
From (5.78) and (5.85) we know that for \( \varphi \) positive we have
\[
I_n \leq 0 \quad \text{for all } n = 1, \ldots, N,
\]
\[
\sum_{n=1}^{N} I_n \geq 0 \quad (5.87)
\]
from which we easily deduce that
\[
\sum_{n=1}^{N} I_n = 0. \quad (5.88)
\]
Since all the summands are negative we conclude that
\[
I_n(\varphi) = 0 \quad \text{for all } n = 1, \ldots, N \quad (5.89)
\]
and for all nonnegative \( \varphi \in L^\infty(\Omega) \cap W^{1,2}(\Omega) \). For general \( \varphi \in L^\infty(\Omega) \cap W^{1,2}(\Omega) \) we have
\[
I_n(\varphi) = I_n((\varphi)_+) - I_n(-(\varphi)_-) = 0 \quad (5.90)
\]
and so \( u \) is a solution to (1.1) in the sense of Definition 1 with lower and upper bounds depending only on the constraints of the data.
6. Appendix

The set $\Omega$ is an open subset of $\mathbb{R}^d$.

**Theorem 4** (Banach-Alaoglu). Let $X$ be a reflexive Banach space, then every bounded sequence has a weakly convergent subsequence.

**Theorem 5** (Banach-Steinhaus). Let $X$ be a Banach space and $Y$ be a normed vector space. Suppose that $F \subset \mathcal{L}(X,Y)$. If for all $x \in X$ it holds that
\[
\sup_{T \in F} \|T(x)\|_Y < \infty,
\]
then
\[
\sup_{T \in F} \|T\| < \infty.
\]

**Note:** For $f_n \in X^*$ such that $f_n \rightharpoonup f$ we have $|f_n(x)|$ is bounded and therefore $\|f_n\|$ and $\|f_n - f\|$ are bounded as well.

**Theorem 6** (Almost everywhere convergence in $L^1$). For any convergent sequence in $L^1(\Omega)$ there is an almost everywhere point-wise convergent subsequence.

**Theorem 7** (Lebesgue dominant convergence theorem). Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that there exists a function $g \in L^p(\Omega)$ for some $1 \leq p$, with
\[
|f_n(x)| \leq g(x) \quad \text{a.e. in } \Omega
\]
and
\[
f_n(x) \to f(x) \quad \text{a.e. in } \Omega,
\]
then $f \in L^p(\Omega)$ and $f_n \to f$ in $L^p(\Omega)$.

**Theorem 8** (Fatou’s lemma). Let $\Omega \subset \mathbb{R}^d$ be measurable set and $\{f_n\}_{n=1}^{\infty}$ sequence of nonnegative measurable functions. Then
\[
\int_{\Omega} \liminf_{n \to \infty} f_n(x) dx \leq \liminf_{n \to \infty} \int_{\Omega} f_n(x) dx.
\]

**Note:** For $|\Omega| < \infty$ it is sufficient that there exists $K$ such that $f_n \geq -K$.

**Theorem 9** (Lebesgue differentiation theorem). Let $f \in L^1_{loc}(\Omega)$ then almost every point in $\Omega$ is a Lebesgue point of $f$.

**Theorem 10** (Sobolev spaces). For every $k \in \mathbb{N}_0$ and $p \in (1, \infty)$ the spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ are reflexive and separable.

**Theorem 11.** Let $f \in L^p(\Omega)$, $f(x) = 0$ for $x \notin \Omega$ and $p \in [1, \infty)$, then
\[
\forall \epsilon > 0 \exists \delta > 0 \forall h \in \mathbb{R}^d : |h| < \delta \implies \int_{\Omega} |f(x + h) - f(x)|^p dx < \epsilon^p.
\]

**Theorem 12** (Sobolev embedding). Let $d \geq 1$ be an integer, $p \in [1, \infty]$ and $\Omega$ be a Lipschitz domain, then
\[
W^{1,p}(\Omega) \hookrightarrow \begin{cases} 
L^q(\Omega) & \text{for } q \in [1, \frac{d}{d-p}) \text{ if } d > p \\
L^q(\Omega) & \text{for } q \in [1, \infty) \text{ if } d = p \\
C^\alpha(\Omega) & \text{for } \alpha \in (0, 1 - \frac{d}{p}) \text{ if } d < p 
\end{cases}
\]
Lemma 13. Let \( u_n \to u \) and \( v_n \to v \) in \( L^2(\Omega) \), then

\[
\int_\Omega u_n v_n \to \int_\Omega uv.
\]

Proof. Due to the Banach-Steinhaus theorem any weakly convergent sequence is bounded so there exists a constant \( K \) such that \( \|u_n\|_2 \leq K \) then

\[
\int_\Omega u_n v_n - uv = \int_\Omega u_n v_n - u_n v + u_n v - uv = \int_\Omega u_n (v_n - v) + \int_\Omega v (u_n - u).
\]

The first term goes to 0 because

\[
|\int_\Omega u_n (v_n - v)| \leq \|u_n\|_2 \|v_n - v\|_2 \leq K \|v_n - v\|_2 \to 0.
\]

The second term tends to 0 from the definition of weak convergence and the proof is finished.

\[
\square
\]

Theorem 14 (Sobolev chain rule). Let \( f : \mathbb{R} \to \mathbb{R} \) be a Lipschitz function and \( u \in W^{1,p}(\Omega), p \geq 1 \). If \( f(u) \in L^p \) or \( f \) is bounded, then \( f(u) \in W^{1,p}(\Omega) \) and for almost all \( x \in \Omega \),

\[
\nabla(f(u))(x) = f'(u(x))\nabla u(x).
\] (6.5)

Theorem 15 (Elliptic regularity - Dirichlet). Let \( L \) be an elliptic operator as considered in this thesis

\[
L = -\text{div}(a \cdot \nabla),
\]

\[
a\xi \cdot \xi \geq \alpha_0 |\xi|^2,
\]

\[
a \in L^\infty(\Omega, \mathbb{R}^{d \times d})
\]

for some \( \alpha_0 > 0 \) and all \( \xi \in \mathbb{R}^d \). Further, let \( a_{ij} \in C^{0,1}(\Omega) \) and \( f \in L^p(\Omega) \) for some \( p > 1 \). Then there exists a unique weak solution \( u \in W_0^{1,p}(\Omega) \) of

\[
Lu + \lambda u = f(x) \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{on } \partial \Omega.
\] (6.7)

and it belongs to \( W^{2,p}_{\text{loc}}(\Omega) \). Moreover, if \( \Omega \in C^{1,1} \) and \( a_{ij} \in C^{0,1}(\overline{\Omega}) \), then \( u \in W^{2,p}(\Omega) \) and there exist \( C \) depending on \( \alpha_0, \|a\|_\infty, n \) and \( p \) such that

\[
\|u\|_{2,p} \leq C(\|u\|_p + \|f\|_p)
\] (6.8)

For proof see [Gilbarg and Trudinger 1977].

Theorem 16 (Elliptic regularity - Neumann). Let \( L \) be an elliptic operator as considered in this thesis

\[
L = -\text{div}(a \cdot \nabla),
\]

\[
a\xi \cdot \xi \geq \alpha_0 |\xi|^2,
\]

\[
a \in L^\infty(\Omega, \mathbb{R}^{d \times d})
\]

35
for some $\alpha_0 > 0$ and all $\xi \in \mathbb{R}^d$. Further, let $a_{ij} \in C^{0,1}(\Omega)$ and $f \in L^p(\Omega)$ for some $p > 1$. Then there exists a unique weak solution $u \in W^{1,p}(\Omega)$ of
\[
Lu + \lambda u = f(x) \quad \text{in } \Omega,
\]
\[
a \nabla u \cdot n = 0 \quad \text{on } \partial \Omega.
\]
and it belongs to $W^{2,p}_{\text{loc}}(\Omega)$. Moreover, if $\Omega \in C^{1,1}$ and $a_{ij} \in C^{0,1}(\overline{\Omega})$ then $u \in W^{2,p}(\Omega)$ and there exist $C$ depending on $\alpha_0$, $\|a\|_\infty$, $n$ and $p$ such that
\[
\|u\|_{2,p} \leq C(\|u\|_p + \|f\|_p)
\]

Proof. We only sketch the proof here. The proof for regularity inside the domain is the same as the previous theorem. For the result near the boundary we extend the problem locally to a bigger domain. We do this in two steps:

1. Let $\Omega = (-1,1)^{d-1} \times (0,1)$ and the homogeneous Neumann condition be prescribed on the part of boundary $\{x_d = 0\} \cap \Omega$. We can mirror the solution $u$ and the data $a$ and $f$ to $\Omega' = (-1,1)^{d}$ by

\[
u(x',x_d) := u(x', -x_d), \quad a(x', x_d) := a(x', -x_d) \quad \text{and} \quad f(x', x_d) := f(x', -x_d) \quad \text{for } x_d < 0.
\]

This new $u$ solves the problem $Lu + \lambda u = f$ in $\Omega'$ and $f \in L^p(\Omega')$, $a \in C^{0,1}(\Omega')$.

For any $U$ open such that $U \subset \subset \Omega'$ there is a cut-off function $\tau \in C^\infty_0(\Omega')$ such that $\tau = 1$ in $U$. We multiply the equation by $\tau$ and compute

\[
f \tau = -\nabla (a \nabla u) \tau + \lambda u \tau
\]
\[
= -\nabla (a \nabla u \tau) + a \nabla u \cdot \nabla \tau + \lambda u \tau
\]
\[
= -\nabla (a \nabla (u \tau)) + 2a \nabla u \cdot \nabla \tau + \nabla a \nabla u \tau + \lambda u \tau.
\]

We denote $z := u \tau$ and
\[
F := f \tau - \sum_{i,j=1}^d D_i a_{ij} u D_j \tau + 2a_{ij} D_i u D_j \tau
\]
then $z$ solves the equation
\[
Lz + \lambda z = F \quad \text{in } \Omega',
\]
\[
z = 0 \quad \text{on } \partial \Omega'.
\]

The right hand side is $L^p$ and therefore by Theorem 15 we have $z \in W^{2,p}_{\text{loc}}(\Omega')$ so $u \in W^{2,p}_{\text{loc}}(U)$ and since $U$ was arbitrary we get $u \in W^{2,p}_{\text{loc}}(\Omega')$.

2. For each point $x$ on the boundary $\partial \Omega$ there is a ball $B$ and a bi-$C^{1,1}$ mapping $T$ with $\det(\nabla T) = 1$ such that $T(B) \subset (-1,1)^d$, $T(B \cap \partial \Omega) \subset (-1,1)^{d-1} \times \{0\}$ and $T(B \cap \Omega) \subset (-1,1)^{d-1} \times (0,1)$. Since $\partial \Omega$ is compact there are finitely many balls $B_i$ and mappings $T_i$ such that the balls cover $\partial \Omega$ it is sufficient to prove that $u$ can be extended to $B_i$ of the class $W^{2,p}_{\text{loc}}(B_i)$ for each $i$. 

36
We fix some $B := B_i$ and denote $\Omega_+ := \Omega \cap B$, $\Omega' := T(B)$, $y \in \Omega_+ := \Omega' \cap \{x_d > 0\} = T(\Omega_+)$, $R := T^{-1}$, $v(y) := u(Ry)$ and $\tilde{f}(y) := f(Ry)$ and
\[
\tilde{a}(y) := (\nabla T(Ry))^T a(Ry) \nabla T(Ry), \quad \text{for } y \in \Omega'_+ \tag{6.16}
\]
so we have
\[
a(x) = (\nabla R(Tx))^T \tilde{a}(Tx) \nabla R(Tx). \tag{6.17}
\]
Finally, for $\tilde{\varphi} \in W^{1,q}(\Omega'_+) \text{ we denote } \varphi(x) := \varphi(\tilde{T}x)$, clearly $\varphi \in W^{1,q}(\Omega \cap B)$. We compute
\[
\int_{\Omega'_+} \tilde{a}(y) \nabla v(y) \cdot \nabla \tilde{\varphi}(y) dy 
= \int_{\Omega_+} \tilde{a}(Tx) \nabla R(Tx) \nabla u(x) \cdot \nabla R(Tx) \nabla \varphi(x) dx 
= \int_{\Omega_+} a(x) \nabla u(x) \cdot \nabla \varphi(x) dx 
= \int_{\Omega_+} f(x) \varphi(x) = \int_{\Omega'_+} \tilde{f}(y) \tilde{\varphi}(y) dy.
\tag{6.18}
\]
Therefore, $v$ solves the equation in $\Omega'_+$ with $\tilde{a}$ and $\tilde{f}$. The regularity of data remains: $\tilde{a} \in C^{0,1}(T(\Omega \cap B))$, $\tilde{f} \in L^p(T(B) \cap \Omega'_+)$ and from bounds of $\nabla R$ we get ellipticity of $\tilde{a}$, i.e.,
\[
\xi \cdot \tilde{a} \xi = (\nabla T \xi) \cdot a(\nabla T \xi) \geq a_0 |\nabla T \xi|^2 \geq a_1 |\xi|^2. \tag{6.19}
\]
So we can use step 1 to get $v \in W^{2,p}_{loc}(\Omega')$ pulling back to $B$ we get $u(x) := v(Tx) \in W^{2,p}_{loc}(B)$. Since we have local estimates over the whole boundary and inside $\Omega$ we conclude that $u \in W^{2,p}(\Omega)$.

\[\square\]

**Theorem 17** (Classical solution). Let $L$ be an elliptic operator as in the previous theorem, $\Omega$ be a open ball $B \subset \mathbb{R}^d$, $a_{ij} \in C^{1,1}(\Omega)$ and $f \in C^0(\Omega)$. Then the problem
\[
Lu + \lambda u = f(x) \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega 
\tag{6.20}
\]
has a unique solution $u \in W^{1,2}_0(\Omega)$ and this solution is of the class $C^{2,\alpha}(\overline{\Omega})$.

For proof see [Gilbarg and Trudinger 1977].

**Theorem 18** (Brouwer fixed point). Let $K \subset \mathbb{R}^n$ be a convex compact set and $f : K \to K$ be continuous, then there is $x \in K$ such that $f(x) = x$.

**Lemma 19.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and $R > 0$. Suppose that $f(x) \cdot x \geq 0$ for all $|x| = R$, then there exists $x_0 \in B_R(0)$ such that $f(x_0) = 0$.

**Proof.** Suppose that $f(x) \neq 0$ in $\overline{B_R(0)}$ then the function $-Rf(x)/|f(x)|$ is defined continuous and into $\overline{B_R(0)}$. Since $\overline{B_R(0)}$ is convex and compact we can use the Brouwer fixed point theorem to get $y = -Rf(y)/|f(y)|$. Now we have that $|y| = R|f(y)|/|f(y)| = R$ but $f(y) \cdot y = -R|f(y)|^2/|f(y)| < 0$ which is a contradiction. \[\square\]
Theorem 20 (Young’s inequality). Let $a, b \geq 0$ and $p, q \geq 1$ such that $1/p + 1/q = 1$, then the following inequality holds

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \quad (6.21)$$

Theorem 21 (Hölder’s inequality). Let $(\Omega, \Sigma, \mu)$ be a measurable space and $f, g : \Omega \to \mathbb{R}$ measurable functions, then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \quad (6.22)$$

where $p, q \geq 1$ are the so called Hölder conjugates, i.e., $1/p + 1/q = 1$.

Lemma 22. Let $A, B \in \mathbb{R}^{d \times d}$ be positive semidefinite matrices and let $B$ be symmetric, then $A : B \geq 0$.

Proof. WLOG $A$ is symmetric too, otherwise we can replace $A$ with $(A + A^T)/2$ since

$$A : B = \text{Tr}(AB^T) = \frac{1}{2}(\text{Tr}(AB^T) + \text{Tr}(A^T B^T)) = \frac{A + A^T}{2} : B. \quad (6.23)$$

We denote symmetric matrices $X, Y$ such that $X^2 = A$ and $Y^2 = B$ then

$$A : B = \text{Tr}(XXYY) = \text{Tr}((XY)^T(XY)) = XY : XY \geq 0. \quad (6.24)$$

Note: If one of the matrices in negative semidefinite we can replace it with $-A$ and get the opposite inequality.

Theorem 23 (Properties of mollification). Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $u \in L^1(\Omega)$, $u(x) = K$ for $x \notin \Omega$ and $u_\varepsilon$ its mollification with standard symmetric positive mollifier $\varphi_\varepsilon$, i.e.,

$$u_\varepsilon(x) = (u * \varphi_\varepsilon)(x) \quad \text{for } x \in \mathbb{R}^d. \quad (6.25)$$

Then

1. $u_\varepsilon \in C^\infty(\mathbb{R}^d)$,

2. $u_\varepsilon \to u$ almost everywhere in $\Omega$.

3. If $u \in L^p(\Omega)$, then $u_\varepsilon \to u$ in $L^p(\Omega)$.

Proof. .

1. Follows easily from $\varphi_\varepsilon \in C^\infty(\mathbb{R}^d)$ and the Leibniz rule for differentiation under the integral sign.
2. It is easy to compute that for $x \in \Omega$ and $\varepsilon \leq \text{dist}(x, \partial \Omega)$

$$\frac{1}{\varepsilon^d} \left| \int_{B_\varepsilon(x)} \varphi \left( \frac{x-y}{\varepsilon} \right) (u(x) - u(y)) dy \right| \leq \frac{C}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} |u(x) - u(y)| dy.$$  \hfill (6.26)

The conclusion now follows from the Lebesgue differentiation theorem $\null^9$

3. For $\varepsilon < 1$ we have $u - u_\varepsilon = 0$ outside of an open ball $B := B_{R+1}(0)$ such that $B_R(0)$ contains $\Omega$. We have

$$\|u_\varepsilon - u\|_{L^p(B)}^p = \int_B |u_\varepsilon(x) - u(x)|^p dx$$

$$= \int_B \left( \int_{B_\varepsilon(x)} \varphi_\varepsilon(x-y) |u(y) - u(x)| dy \right)^p dx$$

$$= \int_B \left( \int_{B_1(0)} \varphi(z) |u(x - \varepsilon z) - u(x)| dz \right)^p dx$$  \hfill (6.27)

$$\leq C \int_B \int_{B_1(0)} |u(x - \varepsilon z) - u(x)|^p dz dx$$

$$= C \int_{B_1(0)} \int_B |u(x - \varepsilon z) - u(x)|^p dz dx.$$  

The inner integral goes to $0$ as $\varepsilon \to 0^+$ due to Theorem $\null^{11}$ and so $u_\varepsilon \to u$ in $L^p(B)$ and therefore in $L^p(\Omega)$ as well.
7. Some definitions

**Definition** (Lebesgue spaces). Let \((\Omega, \Sigma, \mu)\) be a measurable space. For \(u : \Omega \to \mathbb{R}\) measurable we denote its \(p \geq 1\) norm \(\|u\|_p\). It is defined by

\[
\|u\|_p := \left( \int_{\Omega} |u|^p \, d\mu \right)^{1/p}.
\]

The space

\[
L^p(\Omega, \mu) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable}; \int_{\Omega} |u|^p \, d\mu < +\infty \right\}
\]

with identification

\[ u = v \quad \text{iff} \quad u(x) = v(x) \text{ for } \mu\text{-almost all } x \in \Omega \]

and equipped with the norm \(\|\cdot\|_p\) is a Banach space.

**Note.** If \(\mu\) is omitted we mean standard \(\mathbb{R}^n\) Lebesgue measure.

**Definition** (Set of test functions). Let \(\Omega \subset \mathbb{R}^n\) be an open set then we denote

\[
C^\infty_0(\Omega) := \left\{ u \in C^\infty(\Omega); \text{supp}(u) \text{ is compact} \right\}
\]

where \(\text{supp}(u) = \{x \in \Omega; u(x) \neq 0\}\) is support of \(u\).

**Note:** We will sometimes use notation \(D(\Omega) := C^\infty_0(\Omega)\).

**Definition** (Multi-index). An \(n\)-dimensional **multi-index** is an \(n\)-tuple \(\alpha = (\alpha_1, \ldots, \alpha_n)\) with \(\alpha_i \in \mathbb{N}_0\). The norm of multi-index is defined by

\[ |\alpha| = \alpha_1 + \ldots + \alpha_n. \]

The \(\alpha\)-power of a derivative operator is given by

\[
D_\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.
\]

For \(i \in \mathbb{N}\) we can identify \(i = (i)\) and simply write \(D_i\) for derivative in the \(i\)-th direction.

**Definition** (Weak derivative). Let \(\Omega \subset \mathbb{R}^n\) be an open set, \(\alpha = (\alpha_1, \ldots, \alpha_n)\) an multi-index and \(u, v_\alpha \in L^1_{\text{loc}}(\Omega)\). We say that \(v_\alpha\) is a **weak derivative** of \(u\) with respect to \(x^\alpha\) iff for all \(\varphi \in C^\infty_0(\Omega)\) it holds

\[
\int_{\Omega} u(x) D_\alpha \varphi(x) \, dx = (-1)^{|\alpha|} \int_{\Omega} v_\alpha(x) \varphi(x) \, dx.
\]

**Note:** We denote \(D_\alpha u := v_\alpha\).

**Definition** (Sobolev spaces). Let \(\Omega \subset \mathbb{R}^n\) be an open set, \(k \in \mathbb{N}_0\) and \(p \in [1, \infty]\). The **Sobolev space** \(W^{k,p}(\Omega)\) is defined as

\[
W^{k,p}(\Omega) := \{ u \in L^p(\Omega); \forall |\alpha| \leq k : D_\alpha u \in L^p(\Omega) \}.
\]
This space equipped with norm
\[ \|u\|_{k,p} = \|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left( \sum_{|\alpha| \leq k}\|D^\alpha u\|_p^p \right)^{1/p} & \text{for } p \in [1, \infty) \\ \max_{|\alpha| \leq k}\|D^\alpha u\|_\infty & \text{for } p = \infty \end{cases} \]
is a Banach space.

**Definition (Hölder spaces).** Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( k \in \mathbb{N}_0 \) and \( \lambda \in (0,1] \). For some function \( u : \Omega \to \mathbb{R} \) we denote the Hölder coefficient
\[ H_{\Omega,\lambda}(u) := \sup_{x,y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\lambda} \]
and the Hölder space
\[ C^{k,\lambda}(\Omega) := \{ u \in C^k(\Omega); \; H_{\Omega,\lambda}(D^\alpha u) < \infty \; \text{for all } |\alpha| = k \}. \]
Similarly we define \( H_{\overline{\Omega},\lambda} \) and \( C^{k,\lambda}(\overline{\Omega}) \). The space \( C^{k,\lambda}(\Omega) \) with the norm
\[ \|u\|_{k,\alpha} = \|u\|_{C^{k,\alpha}(\Omega)} := \|u\|_{C^k} + \sum_{|\alpha| = k} H_{\Omega,\lambda}(D^\alpha u) \]
is a Banach space.

**Notes.** For \( \lambda = 0 \) we simply mean the continuous functions on the corresponding set. For \( k = 0 \) we simply write \( C^\lambda(\Omega) \). For \( \lambda = 1 \) the function from \( C^{k,\lambda} \) have Lipschitz continuous derivatives of order \( k \) with the Lipschitz constant being \( H_{\Omega,\lambda} \).

**Definition.** Let \( f \in L^1_{\text{loc}}(\Omega) \) then \( x \in \Omega \) is called a Lebesgue point of \( f \) iff
\[ \lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| \, dy = 0. \]
Bibliography


