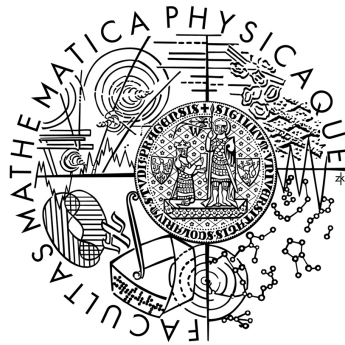Charles University in Prague

Faculty of Mathematics and Physics

MASTER THESIS

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Weighted rearrangement-invariant function spaces

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Název práce: Váhové prostory funkcí s normou invariantní vzhledem k nerostoucímu přerovnání

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Abstrakt: V této práci se zaměříme na studium zobecněných Gamma prostorů $G\Gamma(p, m, v)$ a určíme některé jejich důležité vlastnosti. V [3], se autoři pokusili charakterizovat jejich asociovanou normu, ale získali pouze několik jejích jednostranných odhadů, pomocí nichž pak ukázali reflexivitu prostorů pro $p \geq 2$ a $m > 1$, navíc vše na prostorech konečné míry. Avšak charakterizace asociované normy a otázka reflexivity pro $2 > p > 1$ zůstaly otevřenými problémy. V této práci zobecníme úlohu na σ-konečné prostory a tyto otevřené problémy vyřešíme.

Klíčová slova: Prostory s normou invariantní vůči nerostoucímu přerovnání, klasické Lorentzovy prostory, Gamma prostory

Title: Weighted rearrangement-invariant function spaces

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Abstract: In this thesis we focus on generalized Gamma spaces $G\Gamma(p, m, v)$ and classify some of their intrinsic properties. In [3], the authors attempted to characterize their associate norms but obtained only several one-sided estimates. Equipped with these, they further showed reflexivity of generalized Gamma spaces for $p \geq 2$ and $m > 1$ under an additional restriction that the underlying measure space is of finite measure. However, the full characterization of the associate norm and of the reflexivity of such spaces for $2 > p > 1$ remained an open problem. In this thesis we shall fill this gap. We extend the theory to a σ-finite measure space. We present a complete characterization of the associate norm, and we find necessary and sufficient conditions for the reflexivity of such spaces.

Keywords: rearrangement-invariant spaces, classical Lorentz spaces, Gamma spaces

Contents

WEIGHTED REARRANGEMENT-INVARIANT FUNCTION SPACES

FILIP SOUDSKÝ

1. INTRODUCTION

Weighted rearrangement invariant spaces play an important role in contemporary mathematics. They have many applications in various branches of analysis including the theory of function spaces, interpolation theory, mathematical physics and probability theory (see, for example, [6] and the references therein). A basic example of an important class of such spaces is that of the so-called *classical Lorentz spaces*. Their first appearance can be traced to 1951 when G. Lorentz introduced the spaces $\Lambda^p(v)$ in [5]. Later, in 1990, another type of related spaces, namely the $\Gamma^p(v)$ spaces, were introduced by Sawyer in [4], in order to characterize the associate space to the space $\Lambda^p(v)$ and to find conditions for boundedness of some classical operators from $\Lambda^p(v)$ to $\Lambda^q(w)$. A generalization of the $\Gamma^p(v)$ spaces was then made in [9], where the notion of the $G\Gamma(p,m,v)$ space was introduced. The generalized classical Lorentz spaces of type Gamma cover some other classes of important function spaces including the so-called Grand Lebesgue spaces. Two years later, the same authors classified some of the basic properties of such function spaces in [8]. Finally, in [3], the same authors focused on the dual norm of these spaces. However, their results are incomplete in the sense that they obtain upper and lower estimates for the associate norms which do not meet. In this thesis we obtain a full characterization of the dual norms. Further, using the James characterization, the authors of [3] also prove reflexivity of these spaces, however, restricted to $p \geq 2$ and $m > 1$ and to a finite measure underlying space. In this thesis we obtain analogous results in full generality.

The thesis is divided into six sections. The following section provides the reader with basic definitions. In the third section, two characterizations of certain useful inequalities are presented. The fourth section contains some known results about the classical Lambda spaces. Also, the normability problem is studied there. The fifth section introduces the Gamma spaces and provides some motivation that lead to the appearance of such spaces in [4]. The final section contains our main original results. It is dedicated to the study of the generalized Gamma spaces that have been introduced only recently in [9]. We present some motivation for their definition, and we also point out some of their important particular instances in which $G\Gamma$ coincide with some other important function spaces. Conditions on the parameters are given, for which the functional $G\Gamma_{(p,m,v)}$ is a Banach function space or at least a quasi-Banach function space. Also, the dual norm is characterized. Finally, certain equivalent conditions for reflexivity of such spaces are given.

2. Preliminaries

Definition 1. Let $(\mathcal{R}, \mu)$ be a measure space. We call $(\mathcal{R}, \mu)$ non-atomic if for every $x \in X$ we have
$$\mu(\{x\}) = 0.$$

Definition 2. Let $(\mathcal{R}, \mu)$ be a measure space. We call $(\mathcal{R}, \mu)$ σ-finite if there exists a sequence of sets $A_n \subset \mathcal{R}$, such that $\mu(A_n) < \infty$ and
$$\bigcup_{n=1}^{\infty} A_n = \mathcal{R}.$$

Definition 3. Let $(\mathcal{R}, \mu)$ be a σ-finite measure space. Denote $\mathcal{M}^+$ the space of all nonnegative real-valued μ-measurable functions and let $\rho : \mathcal{M}^+ \to [0, \infty]$ be a functional. We call ρ a Banach function norm if for all $f, g, f_n \in \mathcal{M}^+$ the following conditions hold:

(1) $\rho(af) = a\rho(f)$, $\rho(f + g) \leq \rho(f) + \rho(g)$ and
   $\rho(f) = 0$ if and only if $f = 0$ μ-a.e;
(2) if $f \leq g$ μ-a.e. then $\rho(f) \leq \rho(g)$;
(3) if $f_n \uparrow f$ then $\rho(f_n) \uparrow \rho(f)$;
(4) if $\mu(E) < \infty$ then $\rho(\chi_E) < \infty$;
(5) if $\mu(E) < \infty$ then $\int_E f \mathrm{d}\mu \leq c_E \rho(f)$, where $0 < c_E < \infty$ depends only on E.

Define a Banach function space (BFS in short) by
$$X := \{f \in \mathcal{M}(\mathcal{R}) : \|f\|_X := \rho(|f|) < \infty\}.$$

Definition 4. If we consider the same setting as in the previous definition without the axiom (5) and the first axiom requires only a quasinorm (the triangle inequality is satisfied with some constant), then we call ρ a Banach quasinorm and X defined by
$$X := \{f \in \mathcal{M}(\mathcal{R}) : \|f\|_X := \rho(|f|) < \infty\}$$
a quasi-Banach function space (QBFS in short).

The couple $(\mathcal{R}, \mu)$ will in the whole paper denote a σ-finite measure space. We shall also use the symbol $\mathcal{M}(\mathcal{R})$ for the set of all measurable real-valued functions on $\mathcal{R}$. Positive real constant C used in the proofs is not fixed and can be different on every line of the proof.

Let f be a real-valued function on $\mathcal{R}$, and let $t \in \mathbb{R}$. We will denote the set of all $x \in \mathcal{R}$ for which $f(x) > t$ by $\{f > t\}$. For f real-valued measurable function on $\mathcal{R}$ we shall use the following notation. We denote by
$$\lambda_f(t) := \mu(\{|f| > t\}), \quad t \in [0, \infty),$$
the distribution function of measurable function f, and by
$$f^*(t) := \inf\{s : \lambda_f(s) \leq t\}, \quad t \in [0, \mu(\mathcal{R})),$$
the nonincreasing rearrangement of f. We also define
$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\mathrm{d}s, \quad t \in (0, \mu(\mathcal{R})).$$

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Remark 1. Since f^* is a nonincreasing function. The mean value of f^* on $(0,t)$ is greater or equal to $f^*(t)$. Hence,

$$f^{**}(t) \geq f^*(t)$$

for all $t \in (0, \mu(\mathcal{R}))$.

Lemma 1. *Let $(\mathcal{R}, \mu)$ be a σ-finite measure space. Let $f, g \in \mathcal{M}(\mathcal{R})$ and denote $a := \mu(\mathcal{R})$. Then*

$$\int_{\mathcal{R}} fg\mathrm{d}\mu \leq \int_0^a f^*(s)g^*(s)\mathrm{d}s.$$

Proof. See [1, Theorem 2.2]. □

Lemma 2. *Let $(\mathcal{R}, \mu)$ be a σ-finite non-atomic measure space. Suppose f belongs to $\mathcal{M}(\mathcal{R})$ and let $0 < t < \mu(\mathcal{R})$. Then there exists measurable set E_t with $\mu(E_t) = t$, such that*

$$\int_{E_t} |f|\mathrm{d}\mu = \int_0^t f^*(s)\mathrm{d}s.$$

Proof. See [1, Lemma 2.5]. □

As an immediate consequence of this lemma, we obtain that for $(\mathcal{R}, \mu)$ be a σ-finite non-atomic and $t \in (0, \mu(\mathcal{R}))$, we can construct a set A with $\mu(A) = t$.

Definition 5. Measure space $(\mathcal{R}, \mu)$ is said to be resonant if for every pair $f, g \in \mathcal{M}(\mathcal{R})$ we have

$$\sup_{\tilde{g}^* = g^*} \int_{\mathcal{R}} f\tilde{g}\mathrm{d}\mu = \int_0^a f^*(s)g^*(s)\mathrm{d}s.$$

Lemma 3. *Every non-atomic σ-finite measure space is resonant.*

Proof. See [1, Theorem 2.7] □

In the whole paper the symbol p' shall be defined by

$$p' := \frac{p}{p-1} \quad \text{for} \quad p \in (1, \infty).$$

The term weight function (or briefly weight) will always mean a positive real-valued measurable function defined on $(0, a)$, where $a \in (0, \infty]$.

Let A be a set and let $F, G : A \to [0, \infty)$ be two functions. We say F and G are equivalent, using the notation

$$F \approx G,$$

if there exists a constant $C \in [1, \infty)$, such that

$$C^{-1}F(x) \leq G(x) \leq CF(x) \quad \text{for all} \quad x \in A.$$

Now we need to recall some propositions and terms from the theory of Banach function spaces which shall be useful in the sequel.

Definition 6. Let X be BFS. The associate space to X is the space of all measurable real-valued functions g satisfying

$$\|g\|_{X'} := \sup_{\|f\|_X \leq 1} \int_{\mathcal{R}} fg < \infty.$$

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As the associated norm is quite difficult to investigate from the expression in definition, so for a given BFS X, it is quite useful to find some expression that is at least equivalent to associate norm and can be expressed more explicitly.

Remark 2. Let $(\mathcal{R}, \mu)$ be a non-atomic σ-finite measure space. If we consider a rearrangement invariant Banach function space (that is a Banach function space X, for which $\|f\|_X = \|g\|_X$ whenever $f^* = g^*$), we may rewrite it's associate norm as follows:

$$\|g\|_{X'} = \frac{\int_{\mathcal{R}} fg \mathrm{d}\mu}{\|f\|_X} = \frac{\int_0^a f^*(s)g^*(s)\mathrm{d}s}{\|f\|_X},$$

where $a = \mu(\mathcal{R})$, since, by Lemma 3, $(\mathcal{R}, \mu)$ is resonant.

Definition 7. Let X, Y be normed linear spaces. We say that X is continuously embedded into Y (using the notation $X \hookrightarrow Y$) if $X \subset Y$ and there exists a constant $C \in (0, \infty)$, such that

$$\|z\|_Y \leq C \|z\|_X$$

for all $z \in X$, that is, whenever the identity operator is continuous from X to Y.

Definition 8. Let X, Y be two normed linear spaces. Assume that X is continuously embedded into Y. We call the constant

$$\mathrm{Opt}(X, Y) := \sup_{\|z\|_X = 1} \|z\|_Y < \infty$$

an optimal constant of the embedding $X \hookrightarrow Y$.

Definition 9. Let X be a Banach function space. We say that a function $f \in X$ has absolutely continuous norm if

$$\|f\chi_{E_n}\|_X \to 0$$

whenever $\chi_{E_n} \to 0$ μ-a.e. on $\mathcal{R}$.

Definition 10. A BFS X is said to have absolutely continuous norm whenever every $f \in X$ has absolutely continuous norm.

Theorem 1. *Let X be a Banach function space. Then X is reflexive if and only if both X and the associated space X' have absolutely continuous norm.*

Proof. See [1, Corollary 4.4]. $\square$

3. Some useful integral inequalities

Theorem 2. *Let $(\mathcal{R}, \mu)$ be a σ-finite non-atomic measure space. Denote $r := \mu(\mathcal{R})$. Let $f \in L^1_{loc}$ and $p \in (1, \infty)$. Then there exists $a \in (0, r)$ such that*

$$f^{**}(t) + \left(\frac{1}{t} \int_t^r f^{**}(s)^{\frac{1}{p-1}} f^*(s) \mathrm{d}s \right)^{\frac{1}{p'}} \approx \left(\frac{1}{t} \int_t^r f^{**}(s)^{p'} \mathrm{d}s \right)^{\frac{1}{p'}}$$

for $t \in (0, a)$.

Proof. Note that for $\alpha \in (1, \infty)$ and all $t \in (0, \frac{r}{\alpha})$ we have

$$f^{**}(\alpha t) \geq \frac{1}{\alpha} f^{**}(t).$$

So, applying this to $\alpha = \frac{s}{t}$, we get

$$\frac{1}{t} \int_t^r f^{**}(s)^{p'} \mathrm{d}s = \frac{1}{t} \int_t^r f^{**}(\frac{s}{t} t)^{p'} \mathrm{d}s \geq \frac{1}{t} \int_t^r \frac{t^{p'}}{s^{p'}} f^{**}(t)^{p'} \mathrm{d}s$$

$$= t^{p'-1} f^{**}(t)^{p'} \int_t^r \frac{1}{s^{p'}} \mathrm{d}s.$$

Now, if $r = \infty$, then

$$\int_t^r \frac{1}{s^{p'}} \mathrm{d}s = \frac{1}{p'-1} t^{-p'+1}.$$

For $r < \infty$ we have

$$\int_t^r \frac{1}{s^{p'}} \mathrm{d}s = \frac{1}{p'-1} (t^{-p'+1} - r^{p'-1}).$$

But we may assume that $2t < r$ (if we choose a small enough a). Hence,

$$\int_t^r \frac{1}{s^{p'}} \mathrm{d}s \geq \int_t^{2t} \frac{1}{s^{p'}} \mathrm{d}s = C t^{-p'+1}$$

for all $t \in (0, a)$. Applying this to the original inequality, we get

$$f^{**}(t)^{p'} \leq C \frac{1}{t} \int_t^r f^{**}(s)^{p'} \mathrm{d}s.$$

Obviously also

$$\frac{1}{t} \int_t^r f^{**}(s)^{\frac{1}{p-1}} f^*(s) \mathrm{d}s \leq \frac{1}{t} \int_t^r f^{**}(s)^{p'} \mathrm{d}s,$$

since $f^*(s)$ is nonincreasing, which implies $f^{**}(x) \geq f^*(x)$ for all $x > 0$. Therefore

$$f^{**}(t)^{p'} + \frac{1}{t} \int_t^r f^{**}(s)^{\frac{1}{p-1}} f^*(s) \mathrm{d}s \leq C \frac{1}{t} \int_t^r f^{**}(s)^{p'} \mathrm{d}s,$$

which implies

$$f^{**}(t) + \left(\frac{1}{t} \int_t^r f^{**}(s)^{\frac{1}{p-1}} f^*(s) \mathrm{d}s \right)^{\frac{1}{p'}} \leq C \left(\frac{1}{t} \int_t^r f^{**}(s)^{p'} \mathrm{d}s \right)^{\frac{1}{p'}}.$$

Let's prove the reverse inequality. It remains to show

$$f^{**}(t)^{p'} + \frac{1}{t} \int_t^r f^{**}(s)^{\frac{1}{p-1}} f^*(s) \mathrm{d}s \geq C \frac{1}{t} \int_t^r f^{**}(s)^{p'} \mathrm{d}s$$

for some $C > 0$. Denote $F(t) := \int_0^t f^*(s) \mathrm{d}s$ and note that

$$f^{**}(t) = \frac{F(t)}{t}, \quad t \in (0, r).$$

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Rewriting the integral on the left with F instead of f^{**} and integrating it by parts, we get

$$\frac{1}{t}\int_t^r f^{**}(s)^{\frac{1}{p-1}}f^*(s)\mathrm{d}s = \frac{1}{t}\int_t^r \frac{F(s)^{\frac{1}{p-1}}}{s^{\frac{1}{p-1}}}f^*(s)\mathrm{d}s$$

$$= \frac{1}{t}\left(\frac{1}{p'}\left[\frac{1}{s^{\frac{1}{p-1}}}F(s)^{p'}\right]_t^r + \frac{1}{p}\int_t^r f^{**}(s)^{p'}\mathrm{d}s\right)$$

$$= \frac{1}{t}\left(\frac{1}{p'}\left(\lim_{s\to r}\frac{1}{s^{\frac{1}{p-1}}}F(s)^{p'} - \frac{1}{t^{\frac{1}{p-1}}}F(t)^{p'}\right) + \frac{1}{p}\int_t^r f^{**}(s)^{p'}\mathrm{d}s\right)$$

$$= \frac{1}{t}\left(\frac{1}{p'}\left(\lim_{s\to r}\frac{F(s)^{p'}}{s^{\frac{1}{p-1}}} - tf^{**}(t)^{p'}\right) + \frac{1}{p}\int_t^r f^{**}(s)^{p'}\mathrm{d}s\right).$$

Neglecting the limit in the last expression and summing it with $f^{**}(t)^{p'}$ we see that it suffices to show

$$\frac{1}{p}f^{**}(t)^{p'} + \frac{1}{pt}\int_t^r f^{**}(s)^{p'}\mathrm{d}s \geq C\frac{1}{t}\int_t^r f^{**}(s)^{p'}\mathrm{d}s$$

for some $C > 0$, which, however, trivially holds. $\qquad\square$

For $p \in (1,2)$ we can get even a better result.

Theorem 3. *Let $(\mathcal{R}, \mu)$ be a σ-finite non-atomic measure space. Denote $r := \mu(\mathcal{R})$. Let $f \in L^1_{loc}$ and $p \in (1,2)$. Denote $p' := \frac{p}{p-1}$. Then there exists $a \in (0,r)$ such that*

$$\left(\frac{1}{t}\int_t^r f^{**}(s)^{\frac{1}{p-1}}f^*(s)\mathrm{d}s\right)^{\frac{1}{p'}} \approx \left(\frac{1}{t}\int_t^r f^{**}(s)^{p'}\mathrm{d}s\right)^{\frac{1}{p'}}$$

for $t \in (0,a)$.

Proof. Obviously, since $f^* \leq f^{**}$,

$$\left(\frac{1}{t}\int_t^r f^{**}(s)^{\frac{1}{p-1}}f^*(s)\mathrm{d}s\right)^{\frac{1}{p'}} \leq \left(\frac{1}{t}\int_t^r f^{**}(s)^{p'}\mathrm{d}s\right)^{\frac{1}{p'}}.$$

Similarly as in the proof of the previous theorem we obtain

$$\frac{1}{t}\int_t^r f^{**}(s)^{\frac{1}{p-1}}f^*(s)\mathrm{d}s \geq -\frac{1}{p'}f^{**}(t)^{p'} + \frac{1}{pt}\int_t^r f^{**}(s)^{p'}\mathrm{d}s.$$

So now we would like to prove

$$-\frac{1}{p'}f^{**}(t)^{p'} + \frac{1}{pt}\int_t^r f^{**}(s)^{p'}\mathrm{d}s \geq C\frac{1}{t}\int_t^r f^{**}(s)^{p'}\mathrm{d}s,$$

which is equivalent to

$$\left(1 - \frac{1}{p'} - C\right)\frac{1}{t}\int_t^r f^{**}(s)^{p'}\mathrm{d}s \geq \frac{1}{p'}f^{**}(t)^{p'}.$$

Using the fact mentioned at the beginning of the proof of the previous theorem, we see that

$$\frac{1}{t}\int_t^r f^{**}(s)^{p'}\mathrm{d}s = \int_t^r \frac{1}{t}f^{**}\left(t\frac{s}{t}\right)^{p'}\mathrm{d}s \geq t^{p'-1}f^{**}(t)\int_t^r \frac{\mathrm{d}s}{s^{p'}}$$

$$= (p'-1)(t^{1-p'} - r^{1-p'})t^{p'-1}f^{**}(t)$$

$$= (p'-1)\left(1 - \frac{t^{p'-1}}{r^{p'-1}}\right)f^{**}(t).$$

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If we use this calculation in the original equivalence, we obtain that it suffices to show

$$\left(p'(p'-1)\right)\left(1 - \frac{1}{p'} - C\right)\left(1 - \frac{t^{p'-1}}{r^{1-p}}\right) f^{**}(t) \geq f^{**}(t).$$

Now if we consider $t \in (0, a)$ and we realize that $a > 0$ and $C > 0$ can be chosen arbitrarily small, it suffices to show

$$\left(p'(p'-1)\right)\left(1 - \frac{1}{p'}\right) = (p'-1)^2 > 1,$$

which for $1 < p < 2$ trivially holds. This completes the proof. $\qquad\square$

Definition 11. Let $0 < a \leq \infty$ and let $(\mathcal{R}, \mu)$ be a measure space with $\mu(\mathcal{R}) = a$, where μ is non-atomic and σ-finite, and let v be a weight function defined on $(0, a)$. Let $p \in (0, \infty)$.

The set $\Lambda^p(v)$ is the set of all measurable real-valued functions, such that

$$\|f\|_{\Lambda^p(v)} := \left(\int_0^a f^*(x)^p v(x) \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

Remark 3. The functional $\|.\|_{\Lambda^p(v)}$ is not always a norm, because the triangle inequality doesn't have to hold. Even a sum of two functions belonging to the $\Lambda^p(v)$ space doesn't have to belong there, as the following example shows.

Example 1. Let $\mathcal{R} := \mathbb{R}$ and let μ be the Lebesgue measure. Set

$$v(t) := \chi_{(0,1)}(t) + \frac{1}{t-1} \chi_{(1,\infty)}(t).$$

Now taking

$$f(x) := \chi_{(0,1)}(x) \quad \text{and} \quad g(x) := \chi_{(1,2)}(x),$$

we have $f \in \Lambda^p(v)$, $g \in \Lambda^p(v)$ but $f + g \notin \Lambda^p(v)$.

G. Lorentz in [5] found a necessary and sufficient condition on the weight function v for which $\Lambda^p(v)$ forms a Banach space.

Theorem 4. *Let $(\mathcal{R}, \mu)$ be a σ-finite non-atomic measure space with $\mu(\mathcal{R}) = a$. Let $p \in [1, \infty)$, and let $v \in L^1_{loc}[0, \infty)$ be a weight defined on $(0, a)$ (where $a = \mu(\mathcal{R})$). Then the set $\Lambda^p(v)$ is a Banach function space if and only if v is a nonincreasing function.*

Proof. Let v be nonincreasing and let $f, g \in \Lambda^p(v)$. Without loss of generality suppose $f, g \geq 0$. Then we have

$$\|f + g\|_{\Lambda^p(v)} = \left(\int_0^a (f + g)^*(s)^p v(s) \mathrm{d}s \right)^{\frac{1}{p}}.$$

Choose $\varepsilon > 0$. Now, since $(\mathcal{R}, \mu)$ is non-atomic, we know it's also resonant (for proof see [1, Theorem 2.6]). Therefore we can find $h \in \mathcal{M}(\mathcal{R})$ with $h^* = v$, such that

$$\left(\int_0^a (f + g)^*(s)^p v(s) \mathrm{d}s \right)^{\frac{1}{p}} \geq \left(\int_{\mathcal{R}} (f + g)^p h \mathrm{d}\mu \right)^{\frac{1}{p}}$$

$$\geq \left(\int_0^a (f + g)^*(s)^p v(s) \mathrm{d}s \right)^{\frac{1}{p}} - \varepsilon.$$

Define the measure ν by

$$\nu(A) := \int_A h \mathrm{d}\mu.$$

Using the Minkovski inequality in the space $(\mathcal{R}, \nu)$, we obtain:

$$\|f + g\|_{\Lambda^p(v)} - \varepsilon \leq \left(\int_{\mathcal{R}} (f + g)^p \mathrm{d}\nu \right)^{\frac{1}{p}} \leq \left(\int_{\mathcal{R}} f^p \mathrm{d}\nu \right)^{\frac{1}{p}} + \left(\int_{\mathcal{R}} g^p \mathrm{d}\nu \right)^{\frac{1}{p}}$$

by the Hardy-Littlewood theorem (see [1, Theorem 2.2]). The latter expression is less than or equal to

$$\left(\int_0^a f^*(s)^p v(s) \mathrm{d}s \right)^{\frac{1}{p}} + \left(\int_0^a g^*(s)^p v(s) \mathrm{d}s \right)^{\frac{1}{p}} = \|f\|_{\Lambda^p(v)} + \|g\|_{\Lambda^p(v)}.$$

Letting $\varepsilon \to 0$, we get

$$\|f + g\|_{\Lambda^p(v)} \leq \|f\|_{\Lambda^p(v)} + \|g\|_{\Lambda^p(v)}.$$

Now let's suppose the triangle inequality holds and choose $a, h, \delta > 0$, such that $a + h < \mu(\mathcal{R})$. Again since $\mathcal{R}$ is non-atomic we can find sets A, B, such that $\mu(A) = \mu(B) = a + h$, $\mu(A \cap B) = a$. Set $f := \chi_{A \cup B} + \delta \chi_A$ and $g := \chi_{A \cup B} + \delta \chi_B$. Now the triangle inequality applied on f, g yields:

$$\left[(2 + 2\delta)^p \int_0^a v(s) \mathrm{d}s + (2 + \delta)^p \int_a^{a+2h} v(s) \mathrm{d}s \right]^{\frac{1}{p}}$$

$$\leq 2 \left[(1 + \delta)^p \int_0^a v(s) \mathrm{d}s + \int_{a+h}^{a+2h} v(s) \mathrm{d}s \right]^{\frac{1}{p}},$$

which is equivalent to

$$(2 + \delta)^p \int_a^{a+2h} v(s) \mathrm{d}s \leq (2 + 2\delta)^p \int_a^{a+h} v(s) \mathrm{d}s + 2^p \int_{a+h}^{a+2h} v(s) \mathrm{d}s$$

and thus

$$\frac{(2 + \delta)^p - 2^p}{(2 + 2\delta)^p - (2 + \delta)^p} \int_{a+h}^{a+2h} v(s) \mathrm{d}s \leq \int_a^{a+h} v(s) \mathrm{d}s.$$

If we denote $V(t) := \int_0^t v(s) \mathrm{d}s$ and let $\delta \to 0$ we obtain:

$$V(a + 2h) - V(a + h) \leq V(a + h) - V(a),$$

therefore V is concave, so v is nonincreasing. $\qquad \square$

Note that the requirements on the weight function are quite strong. We could therefore ask when the functional $\|.\|_{\Lambda^p(v)}$ is at least a quasinorm. The condition on the weight function is quite simple.

Definition 12. Let v be a positive real-valued function defined on $(0, a)$. We say that v satisfies the Δ_2-condition if there exists a constant c such that

$$V(2t) \leq cV(t) \quad \text{for all} \quad t \in (0, a),$$

where V is defined by $V(t) := \int_0^t v(s) \mathrm{d}s$.

Theorem 5. *Let* $p \in [1, \infty)$ *and* v *be a weight defined on* $(0, \infty)$. *Then the functional* $\|.\|_{\Lambda^p(v)}$ *is a quasinorm if and only if* v *satisfies the* Δ_2-*condition.*

Proof. Let the weight not satisfy the Δ_2-condition. Then we can find a sequence $t_n \in (0, a)$, such that

$$2^p n^p V(t_n) \leq V(2t_n).$$

Then again since $(\mathcal{R}, \mu)$ is σ-finite and non-atomic, for each $n \in \mathbb{N}$ we can find disjoint sets A_n, B_n, such that $\mu(A_n) = \mu(B_n) = t_n < \frac{\mu(\mathcal{R})}{2}$. Set $f_n := \chi_{A_n}$, $g_n := \chi_{B_n}$. We have

$$f_n^*(s)^p = \chi_{(0, t_n)} = g_n^*(s)^p$$

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and
$$(f_n + g_n)^*(s)^p = \chi_{(0,2t_n)}(s).$$

Hence
$$\|g_n + f_n\|_{\Lambda^p(v)} = \left(\int_0^a (f_n + g_n)^*(s)^p v(s) \mathrm{d}s \right)^{\frac{1}{p}} = (V(2t_n))^{\frac{1}{p}} \geq 2nV(t_n)$$

and
$$\|g_n\|_{\Lambda^p(v)} = \|f_n\|_{\Lambda^p(v)} = (V(t_n))^{\frac{1}{p}},$$

therefore
$$\|f_n\|_{\Lambda^p(v)} + \|g_n\|_{\Lambda^p(v)} \leq n \, \|f_n + g_n\|_{\Lambda^p(v)},$$

and so $\|.\|_{\Lambda^p(v)}$ is not a quasinorm.

Let the Δ_2-condition be satisfied. Define the measure μ_v on $(0, \infty)$ by
$$\mu_v(A) := \int_A v(s) \mathrm{d}s.$$

Then, using Fubini's theorem, we can express the norm in a different form
$$\|f\|_{\Lambda^p(v)}^p = \int_0^a f^*(s)^p v(s) \mathrm{d}s = \int_0^\infty \mu_v \left\{ f^*(s)^p > t \right\} \mathrm{d}t = \int_0^\infty \mu_v \left\{ f^* > t^{\frac{1}{p}} \right\} \mathrm{d}t.$$

Now after the change of variable $s = t^{\frac{1}{p}}$ we obtain
$$\|f\|_{\Lambda^p(v)}^p = \int_0^\infty p s^{p-1} \mu_v \left\{ f^* > s \right\} \mathrm{d}s = \int_0^\infty p s^{p-1} V(\lambda_f(s)) \mathrm{d}s.$$

Finally, using the last expression, we have
$$\|f + g\|_{\Lambda^p(v)}^p = \int_0^\infty p s^{p-1} V(\lambda_{f+g}(s)) \mathrm{d}s \leq \int_0^\infty p s^{p-1} V \left(\lambda_f \left(\frac{s}{2} \right) + \lambda_g \left(\frac{s}{2} \right) \right) \mathrm{d}s$$

$$\leq 2 \int_0^\infty p t^{p-1} V(\lambda_f(t) + \lambda_g(t)) \mathrm{d}t$$

$$\leq 2C \left(\int_0^\infty p t^{p-1} V(\lambda_f(t)) \mathrm{d}t + \int_0^\infty p t^{p-1} V(\lambda_g(t)) \mathrm{d}t \right) = 2C \left(\|f\|_{\Lambda^p(v)}^p + \|g\|_{\Lambda^p(v)}^p \right).$$

Therefore $\|.\|_{\Lambda^p(v)}$ is a quasinorm. Verifying the other axioms of QBFS is an easy job. $\qquad\square$

For more information about this topic see [6] or [12]. Seeing that $\Lambda^p(v)$ is not even a QBFS, we would like to know if everything still works. In order to do so we need to define some more general class then BFS.

Definition 13. Let ρ be a positive functional on $\mathcal{M}^+(\mathcal{R})$ satisfying the following conditions: For all $f, g \in \mathcal{M}^+$ and all $a > 0$

    (1) $\rho(af) = a\rho(f)$, $\rho(f) = 0 \Leftrightarrow f = 0$;
    (2) $f \leq g \Rightarrow \rho(f) \leq \rho(g)$;
    (3) $f^* = g^* \Rightarrow \rho(f) = \rho(g)$.

Set
$$X := \{f \in \mathcal{M} : \|f\|_X := \rho(|f|) < \infty\}.$$

Then we call X a rearrangement invariant lattice.

Remark 4. It is obvious that both $\Gamma^p(v)$ and $\Lambda^q(w)$ are r.i. lattices for arbitrary weight functions v, w and $p, q \in (0, \infty)$.

We can define an embedding between lattices in a similar way as in the case of normed spaces. It is essential to observe that every embedding

$$\Gamma^{\frac{m}{p}}(w) \hookrightarrow \Lambda^{\frac{1}{p}}(u),$$

mentioned in [2], was considered as an embedding between r.i. lattices.

5. Gamma spaces

Definition 14. Let $0 < a \leq \infty$ and let $(\mathcal{R}, \mu)$ be a measure space with $\mu(\mathcal{R}) = a$, where μ is non-atomic, and let v be a weight function defined on $(0, a)$. Let $p, m \in (0, \infty)$. The function space $\Gamma^p(v)$ is the space of all measurable real-valued functions, such that

$$\|f\|_{\Gamma^p(v)} := \left(\int_0^a f^{**}(x)^p v(x) \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

This space first appeared in [4] in connection with the duality problem for the space $\Lambda^p(v)$. In the following section we consider for simplicity a σ-finite, non-atomic measure space $(\mathcal{R}, \mu)$ with $\mu(\mathcal{R}) = \infty$. The importance of the space arises from the following theorem.

Theorem 6. *Let $1 < p < \infty$, let g, v be nonnegative measurable functions, and let $v \in L^1_{loc}[0, \infty)$. Denote $G(t) := \int_0^t g(s) \mathrm{d}s$ and $V(t) := \int_0^t v(s) \mathrm{d}s$. Then*

$$\sup_{f \geq 0, f \downarrow} \frac{\int_0^\infty f(x) g(x) \mathrm{d}x}{\left(\int_\infty^a f(x)^p v(x) \mathrm{d}x \right)^{\frac{1}{p}}} \approx \int_0^\infty \left(\int_x^\infty \frac{g(t)}{V(t)} \mathrm{d}t \right)^{p'} v(x) \mathrm{d}x$$

$$\approx \left(\int_0^\infty G(x)^{p'-1} V^{1-p'}(x) g(x) \mathrm{d}x \right)^{\frac{1}{p'}}$$

$$\approx \left(\int_0^\infty G(x)^{p'} \frac{v(x)}{V(x)^{p'}} \mathrm{d}x \right)^{\frac{1}{p'}} + \frac{\int_0^\infty g(s) \mathrm{d}s}{\left(\int_0^\infty v(s) \mathrm{d}s \right)^{\frac{1}{p}}},$$

where the constant from $\approx$ depends only on p.

Now, if we manage to prove this theorem we can easily get to it's important corollary, that is, if we add some condition on the weight v, characterizing the norm of the associated space to $\Lambda^p(v)$.

Corollary 1. *Let $v \in L^1_{loc}$, $v > 0$ be a nonincreasing function and $\int_0^\infty v(t) \mathrm{d}t = \infty$. Set $X := \Lambda^p(v)$, then*

$$\|g\|_{X'} \approx \|g\|_{\Gamma^{p'}(w)},$$

where $w(x) := x^{p'} \frac{v(x)}{V^{p'}(x)}$.

Proof. As we know from the previous section, if a weight function v is nonincreasing on $[0, \infty)$ and $v \in L^1_{loc}[0, \infty)$, then $\Lambda_p(v)$ is a Banach function space. Therefore we may consider it's associate space with the norm defined by:

$$\|g\|_{X'} := \sup_{\|f\|_{\Lambda^p(v)} \leq 1} \int_{\mathcal{R}} f g \mathrm{d}\mu = \sup_{f \geq 0} \frac{\int_{\mathcal{R}} f g \mathrm{d}\mu}{\left(\int_0^\infty f^*(s)^p v(s) \mathrm{d}s \right)^{\frac{1}{p}}},$$

where we use the convention $\frac{0}{0} = 0$. Since $(\mathcal{R}, \mu)$ is non-atomic and therefore resonant, we can replace $\int_{\mathcal{R}} f g \mathrm{d}\mu$ by $\int_0^\infty f^*(s) g^*(s) \mathrm{d}s$. Hence

$$\|g\|_{X'} = \sup_{f \geq 0} \frac{\int_0^\infty f^*(s) g^*(s) \mathrm{d}s}{\left(\int_0^\infty f^*(s)^p v(s) \mathrm{d}s \right)^{\frac{1}{p}}},$$

which is by the previous theorem equivalent to

$$\left(\int_0^\infty G(x)^{p'} \frac{v(x)}{V(x)^{p'}} \mathrm{d}x \right)^{\frac{1}{p'}} + \frac{\int_0^\infty g(t) \mathrm{d}t}{\left(\int_0^\infty v(t) \mathrm{d}t \right)^{\frac{1}{p}}}$$

and since $v \notin L^1$, the latter expression in the sum equals zero. We have

$$\left(\int_0^\infty G(x)^{p'} \frac{v(x)}{V(x)^{p'}} \mathrm{d}x\right)^{\frac{1}{p'}} = \left(\int_0^\infty g^{**}(x)^{p'} x^{p'} \frac{v(x)}{V(x)^{p'}} \mathrm{d}x\right)^{\frac{1}{p'}} = \|g\|_{\Gamma^{p'}(w)}.$$

$\square$

Lemma 4. *Let $p \in (1, \infty)$. Then there exists a constant C, such that for all $a_k \geq 0$,*

$$\sum_{j=-\infty}^\infty 2^j \left(\sum_{k=j}^\infty a_k\right)^p \leq C \sum_{j=-\infty}^\infty 2^k a_k^p$$

holds.

Proof. We have, by the Hölder inequality for sums,

$$\sum_{j=-\infty}^\infty 2^j \left(\sum_{k=j}^\infty a_k\right)^p = \sum_{j=-\infty}^\infty 2^j \left(\sum_{k=j}^\infty 2^{\frac{j-k}{pp'}} 2^{\frac{k-j}{pp'}} a_k\right)^p$$

$$\leq \sum_{j=-\infty}^\infty 2^j \left(\sum_{k=j}^\infty 2^{\frac{j-k}{p}}\right)^{p-1} \left(\sum_{k=j}^\infty 2^{\frac{k-j}{p'}} a_k^p\right)$$

$$\leq C \sum_{j=-\infty}^\infty 2^j \left(\sum_{k=j}^\infty 2^{\frac{k-j}{p'}} a_k^p\right)$$

$$= C \sum_{k=-\infty}^\infty 2^{\frac{k}{p'}} a_k^p \sum_{j=-\infty}^k 2^{\frac{j}{p}} \leq C \sum_{k=-\infty}^\infty 2^k a_k^p.$$

$\square$

Now we are prepared to proceed with the proof of theorem 6.

Proof. Due to the monotone convergence theorem we may assume that g has compact support (for g without compact support consider $g_n := \chi_{(0,n)} g$ and then if for all g_n the inequalities hold then by the monotone convergence theorem it holds for g as well). Also without loss of generality suppose $\int_0^\infty g(s) \mathrm{d}s = 1$ (otherwise we can take αg for an appropriate $\alpha \in (0, \infty)$). Define $V(t) := \int_0^t v(s) \mathrm{d}s$. Then:

$$0 < V(t) < \infty$$

for all $t > 0$. Set

$$\varphi(x) := \left(\int_x^\infty \frac{g(s)}{V(s)} \mathrm{d}s\right)^{p'-1} \quad \text{for} \quad x \in (0, \infty).$$

Then φ is bounded and nonincreasing on $(0, \infty)$. Integration by parts yields:

$$\int_0^\infty \varphi^p(x) v(x) \mathrm{d}x$$

$$= \left[V(x) \int_x^\infty \left(\frac{g(t)}{V(t)}\right)^{p'} \mathrm{d}t\right]_0^\infty + p' \int_0^\infty g(x) \left(\int_x^\infty \frac{g(t)}{V(t)} \mathrm{d}t\right)^{p'-1} \mathrm{d}x$$

$$= p' \int_0^\infty g(x) \varphi(x) \mathrm{d}x.$$

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The expression $\left[V(x)\int_x^\infty \frac{g(t)}{V(t)}\mathrm{d}t\right]_0^\infty$ equals zero since $\mathrm{supp}(g) \subset (0,\infty)$ is compact. Therefore taking $f(x) := \varphi(x)$ we get that the supremum on the left side is at least

$$\frac{\int_0^\infty \varphi(s)g(s)\mathrm{d}s}{\left(\int_0^\infty \varphi(s)^p v(s)\right)^{\frac{1}{p}}} = \frac{\int_0^\infty \varphi(s)^p v(s)\mathrm{d}s}{p'\left(\int_0^\infty \varphi(s)^p v(s)\mathrm{d}s\right)^{\frac{1}{p}}} = \frac{1}{p'}\left(\int_0^\infty \varphi^p(s)v(s)\mathrm{d}s\right)^{\frac{1}{p'}}$$

$$= \frac{1}{p'}\left(\left(\int_t^\infty \frac{g(s)}{V(s)}\mathrm{d}s\right)^{p'}v(t)\mathrm{d}t\right)^{\frac{1}{p}}.$$

On the other hand, consider f a nonnegative and nonincreasing function. We have

$$\int_0^\infty f(s)g(s)\mathrm{d}s = \int_0^\infty f(s)\frac{g(s)}{V(s)}\int_0^s v(t)\mathrm{d}t\mathrm{d}s = \int_0^\infty \int_t^\infty \frac{f(s)g(s)}{V(s)}\mathrm{d}s\, v(t)\mathrm{d}t$$

$$\leq \int_0^\infty f(t)\left(\int_t^\infty \frac{g(s)}{V(s)}\mathrm{d}s\right)v(t)\mathrm{d}t$$

$$\leq \left(\int_0^\infty f(t)^p v(t)\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^\infty \left(\int_t^a \frac{g(s)}{V(s)}\mathrm{d}s\right)^{p'}v(t)\mathrm{d}t\right)^{\frac{1}{p'}}.$$

The last inequality follows from the Hölder inequality, and the previous one from the fact that f is nonincreasing. So we are done with the equivalence of the left hand side and the first expression on the right.

Now let $\{x_j\}_{j=0}^\infty$ be a sequence satisfying $\int_0^{x_j} g(s)\mathrm{d}s = 2^{-j}$. Then

$$\int_0^\infty \left(\int_t^\infty \frac{g(s)}{V(s)}\mathrm{d}s\right)^{p'-1}g(t)\mathrm{d}t = \sum_{j=0}^\infty \int_{x_{j+1}}^{x_j}\left(\int_t^\infty \frac{g(s)}{V(s)}\mathrm{d}s\right)^{p'-1}g(t)\mathrm{d}t$$

$$\geq \sum_{j=0}^\infty \int_{x_{j+1}}^{x_j}\left(\int_t^\infty \frac{g(s)}{V(x_j)}\mathrm{d}s\right)^{p'-1}g(t)\mathrm{d}t$$

$$= \sum_{j=0}^\infty \int_{x_{j+1}}^{x_j} V(x_j)^{1-p'}\left(\int_{x_j}^\infty g(s)\mathrm{d}s\right)^{p'-1}g(t)\mathrm{d}t$$

$$= \sum_{j=1}^\infty V(x_j)^{1-p'}\left(\int_{x_j}^\infty g(s)\mathrm{d}s\right)^{p'-1}\int_{x_{j+1}}^{x_j}g(t)\mathrm{d}t$$

$$\geq C\sum_{j=1}^\infty V(x_j)^{1-p'}\left(\int_{x_j}^\infty g(s)\mathrm{d}s\right)^{p'-1}\int_{x_j}^{x_{j-1}}g(t)\mathrm{d}t$$

$$= C\sum_{j=0}^\infty V(x_{j+1})^{1-p'}\left(\int_{x_{j+1}}^\infty g(s)\mathrm{d}s\right)^{p'-1}\int_{x_{j+1}}^{x_j}g(t)\mathrm{d}t$$

$$\geq C\sum_{j=0}^\infty \int_{x_{j+1}}^{x_j} G(x_j)^{p'-1}V(x_{j+1})^{1-p'}g(t)\mathrm{d}t$$

$$\geq C\int_0^\infty G(t)^{p'-1}V(t)^{1-p'}g(t)\mathrm{d}t.$$

Conversely, if $\int_0^\infty v(t)\mathrm{d}t = \infty$, set $N := \infty$. In the case of $\int_0^\infty v(t) < \infty$, let N denote the largest integer, for which $2^{N-1} < \int_0^\infty v(t)\mathrm{d}t$. Then

$$\int_0^\infty \left(\int_t^\infty \frac{g(s)}{V(s)}\mathrm{d}s\right)^{p'} v(t)\mathrm{d}t$$

$$= \sum_{j=-\infty}^{N-1} \int_{x_j}^{x_{j+1}} \left(\int_t^\infty \frac{g(s)}{V(s)}\mathrm{d}s\right)^{p'} v(t)\mathrm{d}t$$

$$\leq \sum_{j=-\infty}^{N-1} \int_{x_j}^{x_{j+1}} v(t)\mathrm{d}t \left(\int_{x_j}^\infty \frac{g(s)}{V(s)}\mathrm{d}s\right)^{p'}$$

$$\leq \sum_{j=-\infty}^{N} \left(\int_{x_j}^{x_{j+1}} v(t)\mathrm{d}t\right) \left(\sum_{k=j}^{N-1} \frac{\int_{x_k}^{x_{k+1}} g(s)\mathrm{d}s}{V(x_k)}\right)^{p'}$$

$$\leq C \sum_{j=-\infty}^{N-1} 2^j \left(\sum_{k=j}^{N-1} 2^{-k} \int_{x_k}^{x_{k+1}} g(s)\mathrm{d}s\right)^{p'}$$

$$\leq C \sum_{j=-\infty}^{N-1} 2^j \left(2^{-j} \int_{x_j}^{x_{j+1}} g(s)\mathrm{d}s\right)^{p'}$$

$$= C \sum_{j=-\infty}^{N-1} \left(\frac{V(x_{j+1})}{2}\right)^{p'-1} \int_{x_j}^{x_{j+1}} p' \left(\int_{x_j}^t g(s)\mathrm{d}s\right)^{p'-1} g(t)\mathrm{d}t$$

$$\leq C \sum_{j=-\infty}^{N-1} \int_{x_j}^{x_{j+1}} G(t)^{p'-1} V(t)^{1-p'} g(t)\mathrm{d}t$$

$$= C \int_0^\infty G(t)^{p'-1} V(t)^{1-p'} g(t)\mathrm{d}t,$$

where the inequality between the fourth and fifth line follows from the previous lemma, applied on the sequence

$$a_k := 2^{-k} \int_{x_k}^{x_{k+1}} g(s)\mathrm{d}s \quad \text{for} \quad -\infty \leq k < N,$$
$$a_k := 0 \quad \text{for} \quad k \geq N.$$

To complete the proof, let's integrate the last expression by parts:

$$\int_0^\infty G(t)^{p'-1} V(t)^{1-p'} g(t)\mathrm{d}t = \frac{1}{p'}\left[G(t)^{p'} V(t)^{1-p'}\right]_0^\infty + \frac{1}{p}\int_0^\infty G(t)^{p'} V(t)^{-p'} v(t)\mathrm{d}t$$

$$= \frac{1}{p}\int_0^\infty G(t)^{p'} \frac{v(t)}{V(t)^{p'}}\mathrm{d}t + \frac{1}{p'}\left(\int_0^\infty g(s)\mathrm{d}s\right)^{p'} \left(\int_0^\infty v(s)\mathrm{d}s\right)^{1-p'},$$

which proves the equivalence between second and third expression on the right-hand side. The proof is complete. $\qquad\square$

Other applications of Γ spaces can be found in many branches of mathematics including probability theory, or the theory of the optimal function spaces in Sobolev embeddings, see, for example [10].

6. Generalized Gamma spaces

6.1. Normability and quasinormability.
Let's start with the definition.

Definition 15. Let $0 < a \leq \infty$ and let $(\mathcal{R}, \mu)$ be a measure space with $\mu(\mathcal{R}) = a$, where μ is non-atomic σ-finite, and let v be a weight defined on $(0, a)$. Let $p, m \in (0, \infty)$. The function space $G\Gamma(p, m, v)$ is the space of all measurable real-valued functions such that

$$\|f\|_{G\Gamma(p,m,v)} := \left(\int_0^a \left(\int_0^t f^*(s)^p \mathrm{d}s \right)^{\frac{m}{p}} v(t)\mathrm{d}t \right)^{\frac{1}{m}} < \infty.$$

Generalized Gamma spaces cover some well-known function spaces.

Example 2. Let $m \in (0, \infty)$ and let v be a positive weight function on $(0, \infty)$. If we set

$$w(s) := v(s)s^{-m}, \quad p := 1,$$

then we have $G\Gamma(p, m, v) = \Gamma^m(w)$ and their norms coincide.

The other type of function space covered by $G\Gamma$ space is the Grand Lebesgue space $L^{p)}$. Consider $\Omega \subset \mathbb{R}^n$ with $|\Omega| < \infty$. The norm in the Grand Lebesgue space is given by

$$\|f\|_{p)} := \sup_{0 < \varepsilon < p-1} \left(\frac{\varepsilon}{|\Omega|} \int_\Omega |f(x)|^{p-\varepsilon}\mathrm{d}x \right)^{\frac{1}{p-\varepsilon}}.$$

Lemma 5. *Let $1 < p < \infty$. Suppose $|\Omega| = 1$ and let f be a measurable function on Ω. For such setting denote*

$$\|f\|_{0,p)} := \int_0^1 \frac{(1 - \log t)^{-\frac{1}{p}}}{t} \left(\int_0^t f^*(s)^p \mathrm{d}s \right)^{\frac{1}{p}} \mathrm{d}t.$$

Then, for all $f \in L^{p)}$,

$$\|f\|_{p)} \approx \|f\|_{0,p)},$$

where the constants in $\approx$ depend only on p.

Proof. See [13]. $\qquad\qquad\square$

Example 3. If we set

$$v(t) := \frac{1}{t(1 - \log t)^{\frac{1}{p}}}, \quad m := 1,$$

we get

$$\|f\|_{G\Gamma(p,m,v)} = \|f\|_{p)}.$$

Since the definition of the functional $\|.\|_{G\Gamma(p,m,v)}$ is quite general, one cannot expect $G\Gamma$ to be a Banach function space unless we add some decent assumptions on p, m and v.

Theorem 7. *Suppose $1 \leq p, m < \infty$, and let v be a weight function defined on $(0, a)$. The space $G\Gamma(p, m, v)$ is a Banach function space if and only if at least one of the following conditions hold:*

(1) $\mu(\mathcal{R}) < \infty$ *and* $v \in L^1_{(t^{\frac{m}{p}})}(0, a)$;

(2) $\mu(\mathcal{R}) = \infty$, $v\chi_{(0,1)} \in L^1_{(t^{\frac{m}{p}})}(0, 1)$ *and* $v\chi_{(1,\infty)} \in L^1(1, \infty)$.

Proof. Set $\rho(f) := \|f\|_{G\Gamma}$ for $f \in \mathcal{M}^+(\mathcal{R})$. We claim that ρ is a Banach function norm. First note that (P2) follows immediately from the definition. Next, if we apply the monotone convergence theorem at first on the inner integral and then on the outer one, we obtain (P3).

Let us prove (P1). Obviously, $\rho(af) = a\rho(f)$ for every $a > 0$ and $\rho(f) = 0$ iff $f = 0$ μ-a.e. It remains to show the triangle inequality. Define $E_t := \{f + g > (f + g)^*(t)\}$. Now using the Minkowski inequality in the space $L^p(E_t)$, we obtain

$$
\left(\int_0^t (f + g)^*(s)^p ds \right)^{\frac{1}{p}} = \left(\int_{E_t} (f + g)(s)^p d\mu \right)^{\frac{1}{p}}
$$
$$
\leq \left(\int_{E_t} f(s)^p d\mu \right)^{\frac{1}{p}} + \left(\int_{E_t} g(s)^p d\mu \right)^{\frac{1}{p}}
$$
$$
\leq \left(\int_0^t f^*(s)^p ds \right)^{\frac{1}{p}} + \left(\int_0^t g^*(s)^p ds \right)^{\frac{1}{p}}.
$$

Therefore:

$$
\|f + g\|_{G\Gamma(p,m,v)} = \left(\int_0^a \left(\int_0^t (f + g)^*(s)^p ds \right)^{\frac{m}{p}} v(t)dt \right)^{\frac{1}{m}}
$$
$$
= \left(\int_0^a \|(f + g)^*\|^m_{L^p(0,t)} v(t)dt \right)^{\frac{1}{m}}
$$
$$
\leq \left(\int_0^a (\|f^*\|_{L^p(0,t)} + \|g^*\|_{L^p(0,t)})^m v(t)dt \right)^{\frac{1}{m}}
$$
$$
= \left\| \|g^*\|_{L^p(0,t)} + \|f^*\|_{L^p(0,t)} \right\|_{L^m_w(0,a)}
$$
$$
\leq \left\| \|g^*\|_{L^p(0,t)} \right\|_{L^m_w(0,a)} + \left\| \|f^*\|_{L^p(0,t)} \right\|_{L^m_w(0,a)}
$$
$$
= \|f\|_{G\Gamma(p,m,v)} + \|g\|_{G\Gamma(p,m,v)}.
$$

As for (P4), let $\mu(E) < \infty$. Then

$$
\|\chi_E\|_{G\Gamma(p,m,v)} = \left(\int_0^a \min(t, \mu(E))^{\frac{m}{p}} v(t)dt \right)^{\frac{1}{m}} < \infty
$$

whenever one of the two conditions on the weight in the theorem is satisfied.

Finally, in order to prove (P5), let $\mu(E) < \infty$. Then

$$
\left(\int_0^a \left(\int_0^t f^*(s)^p ds \right)^{\frac{m}{p}} v(t)dt \right)^{\frac{1}{m}} \geq \left(\int_{\mu(E)}^a \left(\int_0^{\mu(E)} f^*(s)^p ds \right)^{\frac{m}{p}} v(t)dt \right)^{\frac{1}{m}}
$$

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$$\geq \left(\int_{\mu(E)}^a \left(\int_E f(s)^p ds \right)^{\frac{m}{p}} v(t) dt \right)^{\frac{1}{m}} = \|f\|_{L^p(E)} \left(\int_{\mu(E)}^a v(t) dt \right)^{\frac{1}{m}}.$$

Denoting $\frac{1}{c_E} := \mathrm{Opt}(L^p(E), L^1(E)) \left(\int_{\mu(E)}^a v(t) dt \right)^{\frac{1}{m}}$, we have

$$\int_E f \leq c_E \|f\|_{G\Gamma(p,m,v)}$$

what was to be proved. $\qquad\square$

Although the case of $G\Gamma(p,m,v)$ with $p \in (0,1)$ or $m \in (0,1)$ was not discussed in [3] we can wonder what we could find out in that case. As $p < 1$ or $m < 1$, we cannot expect $\|.\|_{G\Gamma(p,m,v)}$ to be a norm as the following counterexample shows.

Example 4. Let $\mu(\mathcal{R}) = \infty$, $v(t) := \frac{2}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$, $p < 1$ and $m \in (0,\infty)$. Set

$$f_n := \chi_{A_n} n^{\frac{1}{p}} \quad \text{and} \quad g_n := \chi_{B_n} n^{\frac{2}{p}},$$

where $B_n \subset A_n$, $\mu(A_n) = \frac{1}{n}$ and $\mu(B_n) = \frac{1}{n^2}$ (Since $\mathcal{R}$ is non-atomic we may assume existence of such sets). We have

$$f_n^*(s)^p = n\chi_{(0,\frac{1}{n})} \quad \text{and} \quad g_n^*(s)^p = n^2 \chi_{(0,\frac{1}{n^2})}.$$

So we can easily compute

$$\|f_n\|_{G\Gamma(p,m,v)}^m = \int_0^\infty \left(\int_0^t n\chi_{(0,\frac{1}{n})} \right)^{\frac{m}{p}} v(t) dt$$

$$= \int_0^\infty \left(nt\chi_{(0,\frac{1}{n})} + \chi_{(\frac{1}{n},\infty)} \right)^{\frac{m}{p}} dv(t) \overset{n\to\infty}{\to} 1.$$

Similarly, we find out that $\lim_{n\to\infty} \|g_n\|_{G\Gamma(p,m,v)} = 1$. Now we calculate

$$\|f_n + g_n\|_{(p,m,v)}^m = \int_0^\infty \left(\int_0^t \left(n^{\frac{2}{p}} \chi_{(0,\frac{1}{n^2})} + n^{\frac{1}{p}} \chi_{(0,\frac{1}{n})} \right)^p ds \right)^{\frac{m}{p}} v(t) dt$$

$$\geq \int_0^\infty \left(\int_0^t \left(n^{\frac{2}{p}} \chi_{(0,\frac{1}{n^2})} + n^{\frac{1}{p}} \chi_{(\frac{1}{n^2},\frac{1}{n})} \right)^p ds \right)^{\frac{m}{p}} v(t) dt$$

$$\geq \int_{\frac{1}{n}}^\infty \left(2 - \frac{1}{n} \right)^{\frac{m}{p}} v(t) dt \overset{n\to\infty}{\to} 2^{\frac{m}{p}}.$$

Therefore, for every $\varepsilon > 0$ and $n \in \mathbb{N}$ we find f_n, g_n, such that

$$\|f_n\|_{(p,m,v)} < 1, \quad \|g\|_{(p,m,v)} < 1$$

and

$$\|f_n + g_n\|_{(p,m,v)}^m > 2^{\frac{m}{p}} - \varepsilon.$$

Now if we choose an $\varepsilon > 0$ small enough, we have f_n, g_n with

$$\|f_n\|_{(p,m,v)} < 1, \quad \|g_n\|_{(p,m,v)} < 1$$

and

$$\|f + g\|_{(p,m,v)} > 2^{\frac{1}{p}} - \delta$$

for an arbitrary $\delta > 0$. Thus $\|.\|_{(p,m,v)}$ is not a norm.

Observe that the method in the previous example can be easily adopted for every $v \in L^1[0, \infty)$.

Although we found out that for the general setting presented in the definition of $G\Gamma(p, m, v)$ is not BFS, we could still try to show it's a quasi-Banach function space, which also enjoys some nice properties.

Lemma 6. *Let $(\mathcal{R}, \mu)$ be a measure space and $p \in (0, 1)$. Then the functional $\|.\|_p$ defined by*

$$\|f\|_p := \left(\int_{\mathcal{R}} |f|^p \mathrm{d}\mu \right)^{\frac{1}{p}}$$

is a quasinorm.

Proof. We have

$$\|f + g\|_p^p = \int_{\mathcal{R}} |f + g|^p \mathrm{d}\mu = 2^p \int_{\mathcal{R}} \left(\frac{|f + g|}{2} \right)^p \mathrm{d}\mu \le 2^p \int_{\mathcal{R}} \max \left\{ |f|^p, |g|^p \right\} \mathrm{d}\mu.$$

The last expression equals

$$2^p \left(\int_{\{|f| \ge |g|\}} |f|^p \mathrm{d}\mu + \int_{\{|g| > |f|\}} |g|^p \mathrm{d}\mu \right) \le 2^p \int_{\mathcal{R}} \left(|f|^p + |g|^p \right) \mathrm{d}\mu$$
$$= 2^p (\|f\|_p^p + \|g\|_p^p).$$

Since

$$\|f\|_p^p + \|g\|_p^p \le 2 \max \left\{ \|f\|_p^p, \|g\|_p^p \right\},$$

we have $\|f + g\|_p \le 2^{1+\frac{1}{p}} \max \left\{ \|f\|_p, \|g\|_p \right\} \le 2^{1+\frac{1}{p}} (\|f\|_p + \|g\|_p)$. $\square$

Theorem 8. *Denote $a := \mu(\mathcal{R})$. Let $p, q > 0$, and let v be a weight function defined on $(0, a)$ satisfying:*

(1) $v \in L^1_{(t^{\frac{m}{p}})}(0, a)$ *for* $\mu(\mathcal{R}) < \infty$,

(2) $v\chi_{(0,1)} \in L^1_{(t^{\frac{m}{p}})}(0, 1)$ *and* $v\chi_{(1,\infty)} \in L^1(1, \infty)$ *for* $\mu(\mathcal{R}) = \infty$.

Then $G\Gamma(p, m, v)$ is a QBFS.

Proof. Choose $f, g \in G\Gamma(p, m, v)$. Similarly as in the proof of the previous theorem, we denote:

$$E_t := \{f + g > (f + g)^*(t)\}, \quad t \in (0, a)$$

Now, for $q \in (0, \infty)$, denote $c_q := 2^{1+\frac{1}{q}}$. By using the previous lemma we obtain

$$\left(\int_0^t (f + g)^*(s)^p \mathrm{d}s \right)^{\frac{1}{p}} = \left(\int_{E_t} (f + g)(s)^p d\mu \right)^{\frac{1}{p}}$$
$$\le c_p \left(\left(\int_{E_t} f(s)^p d\mu \right)^{\frac{1}{p}} + \left(\int_{E_t} g(s)^p d\mu \right)^{\frac{1}{p}} \right)$$
$$\le c_p \left(\left(\int_0^t f^*(s)^p \mathrm{d}s \right)^{\frac{1}{p}} + \left(\int_0^t g^*(s)^p \mathrm{d}s \right)^{\frac{1}{p}} \right).$$

Therefore:

$$\|f + g\|_{G\Gamma(p,m,v)} = \left(\int_0^a \left(\int_0^t (f+g)^*(s)^p \mathrm{d}s \right)^{\frac{m}{p}} v(t)\mathrm{d}t \right)^{\frac{1}{m}}$$

$$= \left(\int_0^a \|(f+g)^*\|_{L^p(0,t)}^m \, v(t)\mathrm{d}t \right)^{\frac{1}{m}}$$

$$\leq c_p \left(\left(\int_0^a (\|f^*\|_{L^p(0,t)} + \|g^*\|_{L^p(0,t)})^m v(t)\mathrm{d}t \right)^{\frac{1}{m}} \right)$$

$$= c_p \left\| \|g^*\|_{L^p(0,t)} + \|f^*\|_{L^p(0,t)} \right\|_{L_w^m(0,a)}$$

$$\leq c_p c_m \left\| \|g^*\|_{L^p(0,t)} \right\|_{L_w^m(0,a)} + \left\| \|f^*\|_{L^p(0,t)} \right\|_{L_w^m(0,a)}$$

$$= c_p c_m (\|f\|_{G\Gamma(p,m,v)} + \|g\|_{G\Gamma(p,m,v)}).$$

Observe that (P2) and (P3) are satisfied and that (P4) can be proved exactly in the same way as in the previous theorem.

$\square$

6.2. **Characterization of $G\Gamma'$.** Now when we have stated the necessary and sufficient conditions of normability and quasinormability, we can move to another important task. Let's try to characterize the associate norm. (i.e. find some more explicit formula that is equivalent to it.)

If we manage to find the optimal constant of embedding $G\Gamma(p, m, v) \hookrightarrow \Lambda^1(g^*)$ then:

$$C_{p,m} = \sup_{f \in G\Gamma(p,m,v)} \frac{\int_0^a f^* g^*}{\|f\|_{G\Gamma(p,m,v)}} = \|g\|_{G\Gamma'(p,m,v)}$$

Note that the supremum is exactly the expression of the dual norm, since $(\mathcal{R}, \mu)$ is resonant (see Lemma 3 and the following remark). Let's try to find this optimal constant.

We have

$$\|f\|_{G\Gamma(p,m,v)} := \left(\int_0^a \left(\int_0^t f^*(s)^p \mathrm{d}s \right)^{\frac{m}{p}} v(t)\mathrm{d}t \right)^{\frac{a}{m}}$$

$$= \int_0^a \left(\left(\frac{1}{t} \int_0^t f^*(s)^p \mathrm{d}s \right)^{\frac{m}{p}} t^{\frac{m}{p}} v(t)\mathrm{d}t \right)^{\frac{1}{m}}$$

$$= \left(\int_0^a (f^p(t)^{**})^{\frac{m}{p}} w(t)\mathrm{d}t \right)^{\frac{1}{m}},$$

where w is defined by $w(t) := t^{\frac{m}{p}} v(t)$.

Now we wonder when and with which optimal constant the inequality

$$\int_0^a f^*(t)u(t)\mathrm{d}t \leq C_{p,m} \left(\int_0^a ((f^p)^{**})^{\frac{m}{p}}(t)w(t)\mathrm{d}t \right)^{\frac{1}{m}}$$

holds. Set $h := f^p$. We wish to know when

$$\int_0^a h^*(t)^{\frac{1}{p}} u(t) \mathrm{d}t \leq C_{p,m} \left(\int_0^a h^{**}(t)^{\frac{m}{p}} w(t) \mathrm{d}t \right)^{\frac{1}{m}}.$$

This happens if and only if

$$\left(\int_0^a h^*(t)^{\frac{1}{p}} u(t) \mathrm{d}t \right)^p \leq C_{p,m}^p \left(\int_0^a h^{**}(t)^{\frac{m}{p}} w(t) \mathrm{d}t \right)^{\frac{p}{m}}.$$

Therefore, we are searching for the optimal constant in the embedding

(6.1) $$\Gamma^{\frac{m}{p}}(w) \hookrightarrow \Lambda^{\frac{1}{p}}(u).$$

But these constants have already been characterized in [2], where explicit equivalent formulas were given.

Theorem 9.   (1) *Let* $P \in (0,\infty), Q \in (0,1), Q \leq P$ *and* φ, ϱ *weight functions. The embedding*

(6.2) $$\Gamma^P(\varrho) \hookrightarrow \Lambda^Q(\varphi)$$

holds if and only if

$$C1 := \left(\int_0^a \frac{\Phi(t)^{\frac{1}{1-Q}} + t^{\frac{Q}{1-Q}} \int_t^a \Phi(s)^{\frac{Q}{1-Q}} \varphi(s) s^{-\frac{Q}{1-Q}} \mathrm{d}s \, \Phi(t)^{\frac{Q}{1-Q}} \varphi(t)}{(O(t) + t^P \int_t^a s^{-P} \varrho(s) \mathrm{d}s)^{\frac{R}{P}}} \mathrm{d}t \right)^{\frac{1}{R}} < \infty.$$

(2) *Let* $0 < P \leq Q < \infty$ *and* $1 \leq Q < \infty$ *then* (6.2) *holds if and only if*

$$C2 := \sup_{t \in (0,a)} \frac{\Phi(t)^{\frac{1}{Q}}}{(O(t) + t^p \int_t^a s^{-P} \varrho(s) \mathrm{d}s)^{\frac{1}{P}}} < \infty.$$

(3) *Let* $1 \leq Q < P < \infty$. *Then* (6.2) *holds if and only if*

$$C3 := \int_0^a \frac{\sup_{y \in (t,a)} (y^{-R} \Phi(y)^{\frac{R}{Q}})(O(t) \int_t^a s^{-p} \varrho(s) \mathrm{d}s)^{\frac{1}{R}}}{(O(t) + t^p \int_t^a s^{-p} \varrho(s) \mathrm{d}s)^{\frac{R}{P}}} \mathrm{d}t^P < \infty.$$

(4) *Let* $0 < P \leq Q < 1$ *then* (6.2) *holds if and only if*

$$C4 := \sup_{t \in (0,a)} \frac{\Phi^{\frac{1}{Q}}(t) + t(\int_t^a \Phi(s)^{\frac{Q}{1-Q}} \varphi(s) s^{-\frac{Q}{1-Q}} \mathrm{d}s)^{\frac{1-Q}{Q}}}{(O(t) + t^P \int_t^a s^{-p} \varrho(s) \mathrm{d}s)^{\frac{1}{P}}} < \infty,$$

where $R := \frac{PQ}{P-Q}$ *and* $O(t) := \int_0^t \varrho(s) \mathrm{d}s$, $\Phi(t) := \int_0^t \varphi(s) \mathrm{d}s$. *Moreover the optimal constant* $C \approx C1$ *(resp.* $C \approx C2$, $C \approx C3$, $C \approx C4$), *where the constant of equivalence depends only on* P *and* Q.

Proof. The proof of the theorem can be found in [2, Theorem 4.2] (the proof there is given for σ-finite measure spaces, however all steps in the proof can be done for space with $\mu(\mathcal{R}) = a$ as well). $\qquad\square$

Corollary 2. *Set* $P := \frac{m}{p}$, $Q := \frac{1}{p}$

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(1) *If $p, m \in (1, \infty)$ then $C_{p,m} \approx C1^{\frac{1}{p}}$.*

(2) *If $p \in (0, 1), m \in (1, \infty)$ then $C_{p,m} \approx C3^{\frac{1}{p}}$.*

(3) *If $p \in (0, 1), m \in (0, 1]$ then $C_{p,m} \approx C2^{\frac{1}{p}}$.*

(4) *If $p \in (1, \infty), m \in (0, 1]$ then $C_{p,m} \approx C4^{\frac{1}{p}}$.*

Proof. All the assumptions follow from the preceding argument and Theorem 9 $\qquad\square$

6.3. **Reflexivity.** Let us investigate for which setting of (p,m,v) the $G\Gamma(p, m, w)$-spaces are reflexive.

Now, equipped with the explicit formula for associate norm, recall theorem 1. We need to show for which setting the space $G\Gamma(p, m, v)$ and it's associate space have absolutely continuous norm.

Theorem 10. *The function space $G\Gamma(p, m, v)$ has absolutely continuous norm for $p, m \in (1, \infty)$ if and only if at least one of the following conditions holds:*

(1) $\mu(\mathcal{R}) = a < \infty$,

(2) $\int_0^a (tv(t))^{\frac{m}{p}} \, dt = \infty$.

Proof. Assume that the condition (1) is satisfied. Let $E_n \downarrow \emptyset$ and $f \in G\Gamma(p, m, v)$. Then

$$\int_0^a \left(\int_0^t (f\chi_{E_n})^*(s)^p \mathrm{d}sv(t) \right)^{\frac{m}{p}} \mathrm{d}t = \int_0^a \left(\int_0^{\min(t, \mu(E_n))} f^*(s)^p \mathrm{d}sv(t) \right)^{\frac{m}{p}} \mathrm{d}t.$$

Due to the finiteness of the measure of $\mathcal{R}$, $E_n \downarrow \emptyset$ implies $\mu(E_n) \downarrow 0$. Therefore,

$$\int_0^{\min(t, \mu(E_n))} f^*(s)^p \mathrm{d}sv(t) \to 0$$

for all $a > t > 0$, so we have the pointwise convergence

$$\left(\int_0^t (f\chi_{E_n})^*(s)^p \mathrm{d}sv(t) \right)^{\frac{m}{p}} \to 0.$$

Now, if we use the dominated convergence theorem with $(\int_0^t f^*(s)^p \mathrm{d}sv(t))^{\frac{m}{p}}$ as a majorant we obtain $\|f\chi_{E_n}\| \to 0$, which was what we wanted.

Assume that (2) is satisfied and $\mu(\mathcal{R}) = \infty$. Then, for every $f \in G\Gamma(p, m, v)$, the sets $F_k := \{f \geq \frac{1}{k}\}$ have finite measure. Let $E_n \downarrow \emptyset$. Set $f_n := f\chi_{E_n}$, $f_{n,k} := f_n \chi_{F_k}$ and choose $\varepsilon > 0$. We have

$$\|f_n\| \leq \|f_n - f_{n,k}\| + \|f_{n,k}\|.$$

Observe that:

$$\|f_n - f_{n,k}\| = \int_0^\infty \left(\int_0^t (|f - f\chi_{F_k}|\chi_{E_n})^*(s)^p \mathrm{d}sv(t) \right)^{\frac{m}{p}} \mathrm{d}t$$

$$\geq \int_0^\infty \left(\int_0^t (|f - f\chi_{F_k}|\chi_{E_{n+1}})^*(s)^p \mathrm{d}sv(t) \right)^{\frac{m}{p}} \mathrm{d}t$$

$$= \|f_{n+1} - f_{n+1,k}\|.$$

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Fix n. We see that:

$$\|f_n - f_{n,k}\|^m \leq \int_0^\infty \left(\int_0^t \left(\min\left(f(s), \frac{1}{k}\right)\right)^* (s)^p \mathrm{d}s v(t) \right)^{\frac{m}{p}} \mathrm{d}t.$$

For every $t > 0$ we have:

$$\left(\int_0^t \left(\min\left(f(s), \frac{1}{k}\right)\right)^p \mathrm{d}s \right)^{\frac{m}{p}} \overset{k \to \infty}{\rightrightarrows} 0.$$

Therefore,

$$\|f_n - f_{n,k}\| \overset{k \to \infty}{\rightrightarrows} 0,$$

by the dominated convergence theorem. Observe that for every $k \in \mathbb{N}$

$$\|f_{n,k}\| \overset{n \to \infty}{\rightrightarrows} 0,$$

which follows from the first part of the proof since F_k have finite measure. Let's choose $k \in \mathbb{N}$ such that $\|f_1 - f_{1,k}\| < \varepsilon$ and n_0, such that $\|f_{n,k}\| < \varepsilon$ for all $n > n_0$. Conclude:

$$\|f_n\| \leq \|f_n - f_{n,k}\| + \|f_{n,k}\| \leq \|f_{1,k} - f_1\| + \|f_{n,k}\| \leq 2\varepsilon.$$

Now, for v with $\int_0^a (tv(t))^{\frac{m}{p}} \mathrm{d}t < \infty$ one readily checks that function $f(x) = 1$, $x \in \mathcal{R}$ is an element of $G\Gamma(p, m, v)$ and doesn't have an absolutely continuous norm. $\qquad \square$

Theorem 11. *Let* $p, m \in (1, \infty)$ *and let* v *be a weight on* $(0, a)$. *Then the associate space to* $G\Gamma(p, m, v)$ *has an absolutely continuous norm.*

Proof. Let $p, m \in (1, \infty)$ and let v be a weight on $(0, a)$. From the first section of this paper we know that

$$\|g\|_{G\Gamma'(p,m,v)} \approx \left(\int_0^a \frac{G(t)^{\frac{1}{1-Q}} + t^{\frac{Q}{Q-1}} \int_t^a G(s)^{\frac{Q}{Q-1}} g(s) s^{\frac{Q}{Q-1}} \mathrm{d}s G(t)^{\frac{Q}{1-Q}} g(t)}{\left(W(t) + t^P \int_t^a s^{-P} w(s) \mathrm{d}s\right)^{\frac{R}{P}}} \mathrm{d}t \right)^{\frac{1}{PR}},$$

where $w(t) := t^{\frac{m}{p}} v(t)$, $G(t) := \int_0^t g(s)\mathrm{d}s$ and $W(t) := \int_0^t w(s)\mathrm{d}s$. Let E_n be a sequence of sets such that $E_n \downarrow \emptyset$. Now denote

$$U_g(t) := G(t)^{\frac{1}{1-Q}} + t^{\frac{Q}{Q-1}} \int_t^a G(s)^{\frac{Q}{Q-1}} g(s) s^{\frac{Q}{Q-1}} \mathrm{d}s G(t)^{\frac{Q}{1-Q}} g(t),$$

$$L(t) := \left(W(t) + t^P \int_t^a s^{-P} w(s) \mathrm{d}s \right)^{\frac{R}{P}}$$

and $g_n := g\chi_{E_n}$. Now we need to show that

$$\int_0^a \frac{U_{g_n}(t)}{L(t)} \mathrm{d}t \overset{n \to \infty}{\rightrightarrows} 0.$$

It's suffices to show that $U_{g_n} \to 0$ pointwise. The rest follows immediately from the dominated convergence theorem.

We have

$$G_n(t) := \int_0^t (g\chi_{E_n})^* \mathrm{d}t \leq \int_0^{\min(t, \mu(E_n))} g^*(t) \mathrm{d}t \overset{n \to \infty}{\rightrightarrows} 0,$$

hence

$$U_{g_n}(t) := G_n(t)^{\frac{1}{1-Q}} + t^{\frac{Q}{Q-1}} \int_t^a G_n(s)^{\frac{Q}{Q-1}} g_n(s) s^{\frac{Q}{Q-1}} \mathrm{d}s G_n(t)^{\frac{Q}{1-Q}} g_n(t) \overset{n \to \infty}{\rightrightarrows} 0.$$

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So we have the pointwise convergence, as needed. $\qquad\square$

Corollary 3. *For $p, m \in (1, \infty)$ the space $G\Gamma(p, m, v)$ is reflexive if and only if at least one of the following conditions holds:*

(1) $\mu(\mathcal{R}) < \infty$

(2) $(tw(t))^{\frac{m}{p}} \notin L^1(\mathbb{R})$

6.4. **Open problems in Gamma spaces.** There are many challenging problems yet to be solved, for instance:

(1) As the continuous and almost compact embeddings between classical Lorentz spaces
$$\Lambda^p(v) \hookrightarrow \Lambda^q(w), \Lambda^p \hookrightarrow \Gamma^q(v), \Gamma^p \hookrightarrow \Gamma^q(v), \Gamma^p \hookrightarrow \Lambda^q(v)$$
$$\Lambda^p(v) \hookrightarrow^* \Lambda^q(w), \Lambda^p \hookrightarrow^* \Gamma^q(v), \Gamma^p \hookrightarrow^* \Gamma^q(v), \Gamma^p \hookrightarrow^* \Lambda^q(v),$$

have been characterized in [2] and [14], one could ask for generalization, considering instead of the space Gamma the space $G\Gamma(p, m, v)$.

(2) In classical spaces the conditions for uniform convexity have been made for Λ spaces in [11]. For $G\Gamma(p, m, v)$ the geometrical properties are still unclassified.

(3) Further generalization of $G\Gamma(p, m, v)$ could be made if we consider space $G\Gamma(p, m, u, v)$ with the norm defined as

$$\|f\|_{G\Gamma(p,m,u,v)} := \left(\left(\int_0^t f^*(s)^p u(s) \mathrm{d}s \right)^{\frac{m}{p}} v(t) \mathrm{d}t \right)^{\frac{1}{m}}.$$

As there are two weights and one is inside the inner integral we have a problem with normability and quasinormability similar to the that for Λ spaces.

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