Charles University in Prague
Faculty of Mathematics and Physics

# MASTER THESIS

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# Skorokompaktní vnoření prostorů funkcí
# Almost compact embeddings of function spaces

Department of Mathematical Analysis

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Abstrakt: Práce se zabývá studiem skorokompaktních vnoření prostorů funkcí, konkrétní zkoumanou třídou jsou klasické a slabé Lorentzovy prostory s normou danou pomocí obecné váhové funkce. Tyto prostory obecně nejsou Banachovy prostory funkcí, skorokompaktní vnoření je proto zavedeno pro obecnější struktury r.i. svazů funkcí (tedy svazů daných prostřednictvím funkcionálu invariantního vůči nerostoucímu přerovnání). Je dokázána obecná charakterizace skorokompaktního vnoření r.i. svazu do Lorentzova prostoru pomocí optimální konstanty jistého spojitého vnoření. Na základě tohoto tvrzení a známých výsledků o spojitých vnořeních jsou následně poskytnuty explicitní charakterizace vzájemných skorokompaktních vnoření všech typů Lorentzových prostorů.

Klíčová slova: skorokompaktní vnoření, Lorentzovy prostory, Banachovy prostory funkcí, r.i. prostory (s normou invariantní vůči nerostoucímu přerovnání).

Title: Almost compact embeddings of function spaces

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Abstract: This work is dealing with almost-compact embeddings of function spaces, in particular, the class of classical and weak Lorentz spaces with a norm given by a general weight fuction is studied. These spaces are not Banach function spaces in general, thus the almost-compact embedding is defined for more general sturctures of rearrangement-invariant lattices. A general characterization of when an r.i. lattice is almost-compactly embedded into a Lorentz space, involving an optimal constant of a certain continuous embedding, is proved. Based on this theorem and appropriate known results about continuous embeddings, explicit characterizations of mutual almost-compact embeddings of all subtypes of Lorentz spaces are obtained.

Keywords: almost-compact embedding, Lorentz spaces, Banach function spaces, rearrangement-invariant spaces.

# Contents

# 1 Introduction

Let us at first introduce the topic in a short and general way. Proper definitions of the objects and relations mentioned in here are then summarized in the Preliminaries' section below.

The almost-compact embedding (denoted by $\overset{*}{\hookrightarrow}$ and also called absolutely continuous ([9])) of function spaces can be naturally understood as an intermediate step between the continuous and compact embedding, being stronger than the former but generally weaker than the latter ([18]). Anyway, above all it represents a valuable tool for investigating compactness of embeddings of function spaces. Indeed, it has been shown in [9] that $W^1X \hookrightarrow\hookrightarrow Y$ is equivalent to $W^1X \overset{*}{\hookrightarrow} Y$ for every pair of Banach function spaces satisfying $X = X_a$ and $Y = Y_a$. $X_a$ here stands for the set of functions with absolute continuous norm (cf. [2]) and $W^1X$ denotes the Sobolev space of functions $\{f \in X;\, \nabla f \in X\}$. That example shows that, under certain conditions, the questions concerning compact embeddings, which are in general difficult to deal with, can be transformed into problems of almost-compact embeddings which we hope to be easier to solve.

However, only a little is still known about this type of embeddings. Some basic or general topics have been recently investigated in [9], [18] but there is no general theory providing a characterization of when $X \overset{*}{\hookrightarrow} Y$ occurs. The exception are some necessary or sufficient conditions ([9], [18]) and characterizations for limited special classes of function spaces (trivial cases of Lebesgue spaces, Orlicz $L_A$ spaces; endpoint Lorentz and Marcinkiewicz spaces in [18]). Therefore, it is reasonable to search for suitable necessary and sufficient conditions for other classes of function spaces.

The presented thesis focuses on almost-compact embeddings of classical and weak Lorentz spaces $\Lambda$ and $\Gamma$. Classical $\Lambda$-spaces were first introduced by Lorentz in [13], $\Gamma$-spaces then by Sawyer in [16]. The weak-type spaces were first used in [7]. Since the class of Lorentz spaces is defined in terms of a general weight function, it covers a large scale of common function spaces, ranging from Lebesgue spaces, over the two-parametric Lorentz spaces $L_{p,q}$ to Lorentz-Zygmund spaces ([1]), Zygmund classes etc.

An almost-compact embedding is usually defined in terms of Banach function spaces in the sense of Luxemburg's definition (see Section 2 below). The Lorentz "spaces" are, however, in general not Banach function spaces (these problems were already studied in [13], a complex insight then can be found in [8]). Therefore, we define the embeddings for a generalized structure of r.i. lattices. This already covers a sufficient amount of usable types of function "spaces" and as well such a generalization does not affect the sense and nature of the embedding.

In Section 3, we state a characterization of an almost-compact embedding of an arbitrary r.i. lattice into a Lorentz space in terms of an optimal constant of certain continuous embeddings. Then we can proceed to the formulation of explicit characterizations of almost-compact embeddings for the case of both the domain and range space (lattice) being a Lorentz space, covering all the possible combinations of the subtypes. For this task we use characterizations of the necessary continuous embeddings which have been already known. Despite this, in some cases, these existing results do not cover the problem in sufficient generality. If this happens, we at first prove the results in the desired form, usually by techniques introduced by the original authors.

## 2 Preliminaries

Throughout the text, the following notation will be used:

We write $F \lesssim G$ if $F \leq CG$ where $C > 0$ is a constant independent of appropriate quantities in $F$ and $G$. If both $F \lesssim G$ and $G \lesssim F$ are true, we write $F \simeq G$.

When $p \in (0,\infty) \setminus \{1\}$, we set $p' := \frac{p}{p-1}$ (the *conjugate exponent* of $p$). Notice that $p'$ thus may take negative values as well.

If we mention $(0,\infty)$ in the sense of a measure space, it will always be meant to be endowed with Lebesgue measure.

The usual symbol $\chi_E$ denotes the characteristic function of a set $E$.

Let $(R,\mu)$ be a $\sigma$-finite measure space. Let the symbol $\mathscr{M}(R,\mu)$ denote the cone of all $\mu$-measurable functions $f\colon R \to [-\infty,\infty]$ which are finite $\mu$-a.e. Moreover, let us denote $\mathscr{M}_+(R,\mu) := \{f \in \mathscr{M}(R,\mu);\ f(R) \subset [0,\infty]\}$.

**Definition 2.1.** A mapping $\varrho : \mathscr{M}_+(R,\mu) \to [0,\infty]$ is called a *Banach function norm* (or simply a *function norm*) if, for all $f, g, f_n \in \mathscr{M}_+(R,\mu), n \in \mathbb{N}$, for all constants $a \geq 0$ and for all $\mu$-measurable subsets $E \subset R$, the following properties hold:

(P1) $\varrho(f) = 0 \ \Leftrightarrow\ f = 0\ \mu\text{-}a.e.,\ \varrho(af) = a\varrho(f),\ \varrho(f+g) \leq \varrho(f) + \varrho(g)$,

(P2) $0 \leq g \leq f\ \mu\text{-}a.e.\ \Rightarrow\ \varrho(g) \leq \varrho(f)$,

(P3) $0 \leq f_n \uparrow f\ \mu\text{-}a.e.\ \Rightarrow\ \varrho(f_n) \uparrow \varrho(f)$,

(P4) $\mu(E) < \infty\ \Rightarrow\ \varrho(\chi_E) < \infty$,

(P5) $\mu(E) < \infty\ \Rightarrow\ \int_E f\,\mathrm{d}\mu \leq C_E\varrho(f)$ for a constant $C_E$ depending on $E$ and $\varrho$ but independent of $f$.

Let $\varrho$ be a function norm. The collection $X = X(\varrho)$ of all $f \in \mathscr{M}(R,\mu)$ such that $\varrho(|f|) < \infty$ is called a *Banach function space* (BFS for short). For each $f \in X$, define $\|f\|_X := \varrho(f)$.

For a given BFS $X$, its *associate space* $X'$ consists of all $g \in \mathscr{M}(R,\mu)$ for which the *associate norm*

$$\|g\|_{X'} := \sup\left\{\int_R |fg|\,\mathrm{d}\mu,\ f \in X,\ \|f\|_X \leq 1\right\}$$

is finite.

It holds that $X'$ is itself a BFS, $X'' = (X')' = X$ and the *Hölder inequality* holds:

$$\int_R |fg|\,\mathrm{d}\mu \leq \|f\|_X \|g\|_{X'}. \tag{1}$$

A typical example of a BFS is the *weighted $L^p$ space* over $(0,\infty)$ defined as follows: Let $1 \leq p \leq \infty$ and let $u$ be a weight on $(0,\infty)$, i.e. a measurable function $u\colon (0,\infty) \to [0,\infty)$. For $f \in \mathscr{M}(0,\infty)$ we define

$$\|f\|_{L^p(u)} := \begin{cases} \left(\int_0^\infty |f(x)|^p u(x)\,\mathrm{d}x\right)^{\frac{1}{p}} & \text{for } p \in [1,\infty), \\ \operatorname*{ess\,sup}_{x\in(0,\infty)} |f(x)u(x)| & \text{for } p = \infty. \end{cases} \tag{2}$$

The space $L^p(u)$ is then defined by $L^p(u) := \{f \in \mathscr{M}(0,\infty);\ \|f\|_{L^p(u)} < \infty\}$. If the weight $u$ is moreover positive, i.e. $u((0,\infty)) \subset (0,\infty)$, then $L^p(u)$ is a BFS, otherwise $\|\cdot\|_{L^p(u)}$ is at least a seminorm. Notice that if $u \equiv 1$, we obtain the standard $L^p$ space over $(0,\infty)$.

**Definition 2.2.** Let $(R,\mu)$ be a $\sigma$-finite measure space. Given $f \in \mathscr{M}(R,\mu)$, we define its *distribution function* by

$$\lambda_f(t) := \mu(\{x \in R;\ |f(x)| > t\}), \quad t > 0,$$

and the *nonincreasing rearrangement* of $f$ by

$$f^*(t) := \inf\{s > 0;\ \lambda_f(s) \le t\}, \quad t \in (0,\mu(R)).$$

Furthermore, if $u$ is a positive weight on $(0,\mu(R))$ and $U(t) = \int_0^t u(s)\,\mathrm{d}s, t \in (0,\mu(R))$, we define

$$f_u^{**}(t) := U^{-1}(t)P(f^*u)(t), \quad t \in (0,\mu(R)),$$

where $P\colon \mathscr{M}_+(R,\mu) \to \mathscr{M}_+(R,\mu)$ is the *Hardy operator* given by

$$Ph(t) := \int_0^t h(s)\,\mathrm{d}s, \quad t \in (0,\mu(R)).$$

If $u \equiv 1$, we denote simply

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s, \quad t \in (0,\mu(R)),$$

which function is called the *maximal function* of $f^*$.

**Remark 2.3.** If $f,g \in \mathscr{M}(R,\mu)$, the *Hardy-Littlewood inequality* holds (cf. [2, Section 2, Theorem 2.2]):

$$\int_R |fg|\,\mathrm{d}\mu \le \int_0^{\mu(R)} f^*(t)g^*(t)\,\mathrm{d}t. \tag{3}$$

For an arbitrary $f \in \mathscr{M}(R,\mu)$, a positive weight $u$ and $t > 0$, it holds

$$f^*(t) = U^{-1}(t)\int_0^t f^*(t)u(x)\,\mathrm{d}x \le U^{-1}(t)\int_0^t f^*(x)u(x)\,\mathrm{d}x = f_u^{**}(t), \tag{4}$$

therefore we obtain from (3) and (4)

$$\int_R |fg|\,\mathrm{d}\mu \le \int_0^{\mu(R)} f_u^{**}(t)g_u^{**}(t)\,\mathrm{d}t \tag{5}$$

for all $f,g \in \mathscr{M}(R,\mu)$ and every fixed positive weight $u$.

**Definition 2.4.** We say that a BFS $X$ is *rearrangement-invariant* (usually abbreviated *r.i.*) if $\|f\|_X = \|g\|_X$ for all $f,g \in X$ which are *equimeasurable*, i.e. for which $f^* = g^*$.

**Definition 2.5.** Let $(R,\mu)$ be a nonatomic $\sigma$-finite measure space and let $X$ be an r.i. BFS over $(R,\mu)$. According to the Luxemburg representation theorem (cf. [2, Section 2, Theorem 4.10]), there exists a not necessarily unique r.i. BFS $\overline{X}$ over $[0,\mu(R))$ such that $\|f\|_X = \|f^*\|_{\overline{X}}$ for every $f \in \mathscr{M}(R,\mu)$. The space $\overline{X}$ is called the *representation space* of $X$.

**Definition 2.6.** Let $(R,\mu)$ be a nonatomic $\sigma$-finite measure space and let $X$ be a set of functions from $\mathscr{M}(R,\mu)$, containing characteristic functions of sets of finite measure, endowed with a homogenous functional $\|\cdot\|_X$ defined for every $f \in \mathscr{M}(R,\mu)$ and such that $f \in X$ if and only if $\|f\|_X < \infty$. We say that $X$ has the *lattice property* if $\|f\|_X \le \|g\|_X$ for all $f, g \in \mathscr{M}(R,\mu)$ such that $0 \le f \le g$. If, moreover, it holds $\|f\|_X = \|g\|_X$ whenever $f^* = g^*$, then we call $X$ to be a *rearrangement-invariant lattice.*

It may be seen easily that since the functional $\|\cdot\|_X$ from above is homogeneous, every r.i. lattice $X$ is a homogeneous set, i.e. it satisfies

$$f \in X \quad \Leftrightarrow \quad cf \in X \quad \text{for all } c \in \mathbb{R}.$$

We may also notice that the weighted $L^p$ space given by (2) is an r.i. BFS if and only if $u$ is a positive constant. However, if we replace $f$ by $f^*$ in the $p \in [1,\infty)$ case of (2) (then we have to change the integration domain to $(0,\mu(R))$, as well), we obtain a $\Lambda^p(u)$ space which will be defined in Section 3. This structure is always an r.i. lattice and under some additional restrictions on $u$ it is even an r.i. BFS. (Details may be found in [8].)

**Definition 2.7.** Let $X$ be an r.i. lattice over $(R,\mu)$. If $\overline{X}$ is an r.i. lattice over $(0,\mu(R))$ such that $\|f\|_X = \|f^*\|_{\overline{X}}$ for every $f \in \mathscr{M}(R,\mu)$, then $\overline{X}$ is called to be a *representation lattice* for $X$.

**Definition 2.8.** Let $X, Y$ be r.i. lattices over $(R,\mu)$. We say that $X$ is *continuously embedded* into $Y$ and write $X \hookrightarrow Y$, if for all $f \in X$ it holds

$$\|f\|_Y \le C\|f\|_X$$

where $C > 0$ is a constant independent of $f$. The constant $C$ is then called *optimal for* $X \hookrightarrow Y$ if

$$\operatorname{Opt}(X,Y) := \sup_{f \in X} \frac{\|f\|_Y}{\|f\|_X} = C.$$

Here we adhere to the convention $0/0 := 0$, $x/0 := \infty$, $x > 0$, thus the definition is correct and corresponds to the notion of a continuous embedding.

**Definition 2.9.** For a given $(R,\mu)$, an arbitrary sequence $\{E_n\}_{n\in\mathbb{N}}$ of $\mu$-measurable subsets of $R$ satisfying $E_n \downarrow \varnothing$ $\mu$-a.e. is called a *test sequence*.

**Definition 2.10.** Let $X, Y$ be r.i. lattices over $(R,\mu)$. We say that $X$ is *almost-compactly embedded* into $Y$ and write $X \overset{*}{\hookrightarrow} Y$ if for every test sequence $\{E_n\}_{n\in\mathbb{N}}$ it holds that

$$\lim_{n\to\infty} \sup_{\|f\|_X \le 1} \|f\chi_{E_n}\|_Y = 0. \tag{6}$$

**Remark 2.11.** Consider the following example: For every measurable $f : (0,1) \to \mathbb{R}$, let us define

$$\|f\|_X := f^*\left(\frac{3}{4}\right)$$

and

$$\|f\|_Y := f^*\left(\frac{1}{2}\right).$$

As usual, $X$ and $Y$ are defined as the sets of all those $f \in \mathscr{M}(0,1)$ for which $\|f\|_X < \infty$ and $\|f\|_Y < \infty$, respectively. Both $X$ and $Y$ are then r.i. lattices. Moreover, it holds that $X \overset{*}{\hookrightarrow} Y$, since $\|f\chi_E\|_Y = 0$ for every $f \in \mathscr{M}(0,1)$ and every $E \subset (0,1)$ such that $|E| < \frac{1}{2}$. On the other hand, we see that $X \not\hookrightarrow Y$ (consider $f_n \coloneqq n\chi_{[0,\frac{2}{3}]}$).

The previous example shows that if $X$ and $Y$ are only r.i. lattices, the implication $X \overset{*}{\hookrightarrow} Y \Rightarrow X \hookrightarrow Y$ is not necessarily true. However, in the sequel we will focus on such r.i. lattices for which it is true. The following proposition shows that it is moreover natural to restrict ourselves to r.i. lattices over a space of finite measure:

**Proposition 2.12.** *Let $(R,\mu)$ be a $\sigma$-finite nonatomic measure space with $\mu(R) = \infty$ and let $X, Y$ be r.i. lattices over it, such that $X \hookrightarrow Y$. Suppose that there exists a $\mu$-measurable set $E \subset R$ such that $\mu(E) \in (0,\infty)$ and $\|\chi_E\|_Y > 0$. Then $X \overset{*}{\hookrightarrow} Y$ if and only if $X = \{0\}$.*

*Proof.* Obviously $\{0\} \overset{*}{\hookrightarrow} Y$. Conversely, assume that $X \neq \{0\}$, therefore also $Y \neq \{0\}$ since $X \hookrightarrow Y$. Using the r.i. property, we get the following: There exists a $\mu$-measurable $E \subset R$ with $\mu(E) = \varepsilon > 0$ such that $\|\chi_E\|_X, \|\chi_E\|_Y \in (0,\infty)$. We can assume $\|\chi_E\|_X = 1$. Moreover, there is a sequence $\{E_n\}$ of pairwise disjoint $\mu$-measurable subsets of $R$ such that $\mu(E_n) = \varepsilon$ for all $n \in \mathbb{N}$. Clearly, $\widetilde{E}_n \coloneqq \bigcup_{k\geq n} E_k$ is a test sequence. Thus, $f_n \coloneqq \chi_{E_n}$ satisfy $\|f_n\|_X = 1$ but $\|f_n\chi_{\widetilde{E}_n}\| \not\to 0$. Thus, $X \overset{*}{\not\hookrightarrow} Y$. □

**Proposition 2.13.** *Let $X, Y$ be r.i. lattices over a nonatomic measure space $(R,\mu)$ such that $\mu(R) < \infty$. Then it is equivalent:*

(i) $X \overset{*}{\hookrightarrow} Y$,

(ii) $\lim\limits_{s\to 0+} \sup\limits_{\|f\|_X\leq 1} \|f^*\chi_{[0,s]}\|_{\overline{Y}} = 0.$

*Proof.* A proof is done in [18, Lemma 5.1] for BFS and it is correct for r.i. lattices as well. Let us show a sketch of another proof:

"(ii)⇒(i)": Let $\{E_n\}$ be a test sequence. For all $f \in \mathscr{M}(R,\mu)$ and $n \in \mathbb{N}$ it holds that

$$(f\chi_{E_n})^* \leq f^*\chi_{[0,\mu(E_n)]}.$$

Thus, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \sup_{\|f\|_X\leq 1} \|f\chi_{E_n}\|_Y = \\ = &\lim_{n\to\infty} \sup_{\|f\|_X\leq 1} \|(f\chi_{E_n})^*\|_{\overline{Y}} \leq \\ \leq &\lim_{n\to\infty} \sup_{\|f\|_X\leq 1} \|f^*\chi_{[0,\mu(E_n)]}\|_{\overline{Y}} \end{aligned}$$

and the last part is equal to zero according to (ii).

"(i)⇒(ii)": It suffices to realize that for every $f \in \mathscr{M}(R,\mu)$ and a given scalar $t \in (0,\mu(R)]$ there exists a $\mu$-measurable set $E \subset R$ such that $\mu(E) = t$ and

$$(f\chi_E)^* = f^*\chi_{[0,t)}.$$

A precise construction which proves this assertion may be found in the proof of [2, Chapter 2, Lemma 2.5]. □

**Definition 2.14.** Let $X$ be an r.i. lattice over a nonatomic measure space $(R,\mu)$. For each finite $t \in [0,\mu(R)]$ let $E$ be a subset of $R$ such that $\mu(E) = t$, and $\varphi_X(t) = \|\chi_E\|_X$. The function $\varphi_X$ so defined is called the *fundamental function* of $X$.

**Remark 2.15.** Notice that the above definition is correct: At first, existence of a $\mu$-measurable set $E \subset R$ such that $\mu(E) = t$ is granted for every $t \in [0,\mu(R)]$ since $(R,\mu)$ is nonatomic. Furthermore, if $\mu(E_1) = \mu(E_2) = t$, then $\chi^*_{E_1} = \chi^*_{E_2}$, thus $\|\chi_{E_1}\|_X = \|\chi_{E_2}\|_X$ from the r.i. property.

**Lemma 2.16.** *Let $X$, $Y$ be r.i. lattices over a measurable space $(R,\mu)$ such that $\mu(R) < \infty$. Assume that $X \overset{*}{\hookrightarrow} Y$. Then it holds*

$$\lim_{t\to 0+} \frac{\varphi_Y(t)}{\varphi_X(t)} = 0. \tag{7}$$

*Proof.* It does no harm to assume that $\varphi_X(t) > 0$ for all $t \in [0,\mu(R)]$ and the same holds for $\varphi_Y$. Indeed, if this is not true, then $\|\cdot\|_X \equiv 0$. Next, let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of $\mu$-measurable subsets of $R$, such that $E_n \to \varnothing$ $\mu$-a.e. We may assume that $\{E_n\}$ is even a test sequence, i.e. the convergence is monotone. (If not, replace $E_n$ by $\bigcup_{k\geq n} E_k$.) Let us denote $t_n := \mu(E_n)$, $n \in \mathbb{N}$. For each function $f_n := \chi_{E_n}/\|\chi_{E_n}\|_X$ $(n \in \mathbb{N})$ it holds $\|f_n\|_X = 1$ and $f_n\chi_{E_n} = f_n$. Hence, from (6) we obtain

$$\lim_{n\to\infty} \frac{\varphi_Y(t_n)}{\varphi_X(t_n)} = \lim_{n\to\infty} \frac{\|\chi_{E_n}\|_Y}{\|\chi_{E_n}\|_X} = 0.$$

Since $t_n$ may be an arbitrary sequence of positive numbers decreasing to zero (recall again that $(R,\mu)$ is nonatomic), (7) must be true. □

## 3 Classical and weak Lorentz spaces

Let $(R,\mu)$ be a nonatomic measure space with $\mu(R) < \infty$, without loss of generality we assume $\mu(R) = 1$. We recall that a weight is a measurable function $v\colon (0,1) \to [0,\infty)$. We denote $V(t) := \int_0^t v(x)\,\mathrm{d}x$ and $V_s(t) := \int_0^t v(x)\chi_{[0,s]}(x)\,\mathrm{d}x$ for $t, s \in (0,1]$.

**Definition 3.1.** We say that a weight $v$ is *admissible* if

$$0 < V(t) < \infty \tag{8}$$

for all $t \in (0,1]$. In particular, every admissible weight is integrable on [0,1].

**Definition 3.2.** For $0 < p < \infty$, an admissible weight $v$ and a positive integrable weight $u$ we define the following function spaces:

$$\begin{aligned}
\Lambda^p(v) &:= \left\{ f \in \mathscr{M}(R,\mu);\ \|f\|_{\Lambda^p(v)} := \left(\int_0^1 (f^*(t))^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} < \infty \right\},\\
\Lambda^{p,\infty}(v) &:= \left\{ f \in \mathscr{M}(R,\mu);\ \|f\|_{\Lambda^{p,\infty}(v)} := \sup_{0<t<1} f^*(t) V^{\frac{1}{p}}(t) < \infty \right\},\\
\Gamma^p_u(v) &:= \left\{ f \in \mathscr{M}(R,\mu);\ \|f\|_{\Gamma^p_u(v)} := \left(\int_0^1 (f^{**}_u(t))^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} < \infty \right\},\\
\Gamma^{p,\infty}_u(v) &:= \left\{ f \in \mathscr{M}(R,\mu);\ \|f\|_{\Gamma^{p,\infty}_u(v)} := \sup_{0<t<1} f^{**}_u(t) V^{\frac{1}{p}}(t) < \infty \right\}.
\end{aligned}$$

The spaces $\Lambda^p(v)$ and $\Gamma^p_u(v)$ are called *classical Lorentz spaces* and the spaces $\Lambda^{p,\infty}(v)$ and $\Gamma^{p,\infty}_u(v)$ are called *weak Lorentz spaces*. Altogether, we will denote by $\mathfrak{L}(R,\mu)$ the family of all the Lorentz spaces of these four types. In case of $\Gamma$-spaces, if the symbol $u$ is omitted and it is written just $\Gamma^p(v), \Gamma^{p,\infty}(v)$, it means that $u \equiv 1$. The weight $v$ appearing in the above definition may be called the *main weight* of the corresponding Lorentz space.

**Remark 3.3.** (i) If $f$ belongs to a $\Gamma$-type Lorentz space $X$ with an admissible weight, then $f \in L^1(R,\mu)$.

(ii) A Lorentz space $X \in \mathfrak{L}(R,\mu)$ does not have to be a Banach function space. In a general case, the functional $\|\cdot\|_X$, as defined above, does not have to be a Banach function norm, even the set $\{f \in \mathscr{M}(R,\mu); \|f\|_X < \infty\}$ does not have to be linear. For examples to this situation and a detailed treatment of the problem, see [8]. However, it can be checked easily that every Lorentz space is at least an r.i. lattice. Anyway, we will use the term "space" even if the particular Lorentz-type structure is not really a space in the standard sense. Similarly, we will always call the functional $\|\cdot\|_X$ a "norm".

(iii) As the Lorentz spaces are defined directly in terms of the nonincreasing rearrangement, the nature of their corresponding representation spaces (precisely, lattices) is obvious. Indeed, for $X \in \mathfrak{L}(R,\mu)$, the corresponding representation space (lattice) $\overline{X}$ is a Lorentz space of the same type over $[0,1]$ (with Lebesgue measure).

Integrability of the main weight ensures that a characteristic function of $R$ lies in the particular Lorentz space, so the latter one is then an r.i. lattice. Now we put some light on why we add the admissibility requirement (in the sense of Definition 3.1).

**Proposition 3.4.** *Let $X$ be an r.i. lattice and $Y$ be a Lorentz space with an admissible main weight $w$. Assume that $X \overset{*}{\hookrightarrow} Y$. Then $X \hookrightarrow Y$.*

*Proof.* For a contradiction, suppose that $X \overset{*}{\hookrightarrow} Y$ but $X \not\hookrightarrow Y$.

Suppose at first that $Y = \Gamma_u^q(w)$ for a $q \in (0,\infty)$ and a positive integrable $u$. Since $X \not\hookrightarrow Y$, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions such that

$$\|f_n\|_X \le 1 \quad \text{and} \quad \int_0^1 ((f_n)_u^{**})^q(x)w(x)\,\mathrm{d}x > n+1$$

for every $n \in \mathbb{N}$. On the other hand, by Proposition 2.13, $X \overset{*}{\hookrightarrow} Y$ yields that there exists $S \in (0,1)$ such that

$$\int_0^S ((f_n)_u^{**})^q(x)w(x)\,\mathrm{d}x \le \|f^*\chi_{[0,S]}\|_{\overline{Y}} \le 1 \tag{9}$$

for all $n \in \mathbb{N}$, hence

$$\int_S^1 ((f_n)_u^{**})^q(x)w(x)\,\mathrm{d}x > n, \quad n \in \mathbb{N}.$$

By the monotonicity of $(f_n)_u^{**}$ we have

$$1 \ge \int_0^S ((f_n)_u^{**})^q(x)w(x)\,\mathrm{d}x \ge ((f_n)_u^{**})^q(S)W(S)$$

and

$$n < \int_S^1 ((f_n)_u^{**})^q(x)w(x)\,\mathrm{d}x \le ((f_n)_u^{**})^q(S)\int_S^1 w(x)\,\mathrm{d}x \le ((f_n)_u^{**})^q(S)W(1).$$

By putting these inequalities together we obtain

$$W(S) < \frac{W(1)}{n} \quad \text{for all } n \in \mathbb{N}.$$

This yields $W(S) = 0$ which contradicts the assumption of admissibility of $w$.

If $Y = \Lambda^q(w)$, the first two expressions in (9) are even equal and the rest is carried out in the same way. Cases of weak-type spaces are analogous. □

Now we will mention an easy result about some continuous embeddings involving weak Lorentz spaces which will be useful in the sequel.

**Proposition 3.5.** *Let $X$ be an r.i. lattice, $p,q \in (0,\infty)$ and let $v,w$ be admissible weights. Let $u$ be a positive integrable weight.*

(i) *The embedding $\Lambda^{p,\infty}(v) \hookrightarrow X$ holds if and only if*

$$A_{(10)} := \|V^{-\frac{1}{p}}\|_X < \infty. \tag{10}$$

*Moreover,* $\operatorname{Opt}(\Lambda^{p,\infty}(v), X) = A_{(10)}$.

(ii) *The embedding* $X \hookrightarrow \Gamma_u^{q,\infty}(w)$ *holds if and only if*

$$A_{(11)} := \sup_{0<t<1} U^{-1}(t) W^{\frac{1}{q}}(t) \operatorname{Opt}(X, \Lambda^1(u\chi_{[0,t]})) < \infty. \tag{11}$$

*Moreover,* $\operatorname{Opt}(X, \Gamma_u^{q,\infty}(w)) = A_{(11)}$.

(iii) *The embedding* $X \hookrightarrow \Lambda^{q,\infty}(w)$ *holds if and only if*

$$A_{(12)} := \sup_{0<t<1} W^{\frac{1}{q}}(t) \varphi_X^{-1}(t) < \infty \tag{12}$$

*where* $\varphi_X$ *denotes the fundamental function of* $X$ (see Definition 2.14)*. Moreover,* $\operatorname{Opt}(X, \Lambda^{q,\infty}(w)) = A_{(12)}$.

*Proof.* (i) This is a particular case of [19, Proposition 2.7], it can be also found in [5, Theorem 2.6(i)].

(ii) By changing the suprema, we obtain

$$\begin{aligned}
\operatorname{Opt}(X, \Gamma_u^{q,\infty}(w)) &= \sup_{\|f\|_X \le 1} \sup_{0<t<1} f_u^{**}(t) W^{\frac{1}{q}}(t) = \\
&= \sup_{0<t<1} U^{-1}(t) W^{\frac{1}{q}}(t) \sup_{\|f\|_X \le 1} \int_0^t f^*(s) u(s)\,\mathrm{d}s = \\
&= A_{(11)}.
\end{aligned}$$

(iii) This part is proved in [5, Theorem 2.6(iii)]. □

Now we proceed to stating the essential result of this chapter. In the case of the "range" space (lattice) being a Lorentz space, it transforms the problem of the almost-compact embedding just into a question of an optimal constant which is then quite easy to deal with.

**Theorem 3.6.** *Let* $X$ *be an r.i. lattice and let* $Y \in \mathfrak{L}(R,\mu)$*. Let* $w$ *be an admissible weight and let* $u$ *be a positive weight. Then*

(i) *If* $Y = \Lambda^q(w)$ *then* $X \overset{*}{\hookrightarrow} Y$ *if and only if*

$$\lim_{s \to 0+} \operatorname{Opt}(X, \Lambda^q(w\chi_{[0,s]})) = 0. \tag{13}$$

(ii) *If* $Y = \Lambda^{q,\infty}(w)$ *then* $X \overset{*}{\hookrightarrow} Y$ *if and only if*

$$\lim_{s \to 0+} \operatorname{Opt}(X, \Lambda^{q,\infty}(w\chi_{[0,s]})) = 0. \tag{14}$$

(iii) *If* $Y = \Gamma_u^q(w)$ *then* $X \overset{*}{\hookrightarrow} Y$ *if and only if*

$$\lim_{s \to 0+} \operatorname{Opt}(X, \Gamma_u^q(w\chi_{[0,s]})) = 0 \tag{15}$$

*and*

$$\lim_{s \to 0+} \operatorname{Opt}(X, \Lambda^1(u\chi_{[0,s]})) \cdot \left( \int_s^1 U^{-q}(x) w(x)\,\mathrm{d}x \right)^{\frac{1}{q}} = 0. \tag{16}$$

(iv) *If* $Y = \Gamma_u^{q,\infty}(w)$ *then* $X \overset{*}{\hookrightarrow} Y$ *if and only if*

$$\lim_{s\to 0+} \operatorname{Opt}\left(X, \Gamma_u^{q,\infty}(w\chi_{[0,s]})\right) = 0 \tag{17}$$

*and*

$$\lim_{s\to 0+} \operatorname{Opt}\left(X, \Lambda^1(u\chi_{[0,s]})\right) \sup_{s<t<1} W^{\frac{1}{q}}(t)U^{-1}(t) = 0. \tag{18}$$

*Proof.* (i) Assume $Y = \Lambda^q(w)$. By the definition, $X \overset{*}{\hookrightarrow} Y$ means that, for every test sequence $\{E_n\}$, it holds

$$\lim_{n\to\infty} \sup_{\|f\|_X \le 1} \|f\chi_{E_n}\|_Y = 0. \tag{19}$$

This is further equivalent to

$$\lim_{s\to 0+} \sup_{\|f\|_X \le 1} \|f^*\chi_{[0,s]}\|_{\overline{Y}} = 0 \tag{20}$$

according to Proposition 2.13. Furthermore, (20) clearly holds if and only if

$$\lim_{s\to 0+} \sup_{f\in X} \frac{\|f^*\chi_{[0,s]}\|_{\overline{Y}}}{\|f\|_X} = 0. \tag{21}$$

Here we adhere to the convention $0/0 := 0$, $x/0 := \infty$ for $x > 0$. The supremum in (21) is an optimal constant for a certain continuous embedding. Precisely, for every $f \in Y$ and $s \in (0,1)$ it holds

$$\|f^*\chi_{[0,s]}\|_{\overline{Y}} = \|f\|_{\Lambda^q(w\chi_{[0,s]})}, \tag{22}$$

therefore

$$\sup_{f\in X} \frac{\|f^*\chi_{[0,s]}\|_{\overline{Y}}}{\|f\|_X} = \operatorname{Opt}\left(X, \Lambda^q(w\chi_{[0,s]})\right)$$

and the claim is proved.

(ii) Suppose that $f \in \mathscr{M}(R,\mu)$ and $0 < s < 1$. Since $f^*$ is nonincreasing and $W$ is continuous, it holds

$$\sup_{t\in[s,1)} f^*(t)W_s^{\frac{1}{q}}(t) = \sup_{t\in[s,1)} f^*(t)W_s^{\frac{1}{q}}(s) = f^*(s)W_s^{\frac{1}{q}}(s) \le \sup_{t\in(0,s)} f^*(t)W_s^{\frac{1}{q}}(t),$$

hence we obtain

$$\sup_{t\in(0,1)} f^*(t)W_s^{\frac{1}{q}}(t) = \sup_{t\in(0,s)} f^*(t)W_s^{\frac{1}{q}}(t) = \sup_{t\in(0,s)} f^*(t)W^{\frac{1}{q}}(t).$$

Therefore, it holds

$$\|f^*\chi_{[0,s]}\|_{\overline{\Lambda^{q,\infty}(w)}} = \sup_{0<t<s} f^*(t)W^{\frac{1}{q}}(t) = \sup_{0<t<1} f^*(t)W_s^{\frac{1}{q}}(t) = \|f^*\chi_{[0,s)}\|_{\overline{\Lambda^{q,\infty}(w\chi_{[0,s]})}}.$$

The rest of the proof is then the same as in (i).

(iii) Now $Y = \Gamma_u^q(w)$. As we observe, for $0 < s < t < 1$ and $f \in L^1(R,\mu)$ it holds

$$(f^*\chi_{[0,s]})_u^{**}(t) = U^{-1}(t)\int_0^s f^*(x)u(x)\,\mathrm{d}x = f_u^{**}(s)U(s)U^{-1}(t).$$

Thus, for each $f \in \Gamma_u^q(w)$ we obtain

$$\|f^*\chi_{[0,s]}\|_{\overline{\Gamma_u^q(w)}}^q = \int_0^s (f_u^{**}(x))^q w(x)\,\mathrm{d}x + \int_s^1 [f_u^{**}(s)U(s)]^q\, U^{-q}(x)w(x)\,\mathrm{d}x \simeq$$

$$\simeq \|f\|_{\Gamma_u^q(w\chi_{[0,s]})}^q + [f_u^{**}(s)U(s)]^q \int_s^1 U^{-q}(x)w(x)\,\mathrm{d}x$$

which yields

$$\|f^*\chi_{[0,s]}\|_{\overline{\Gamma_u^q(w)}} \simeq \|f\|_{\Gamma_u^q(w\chi_{[0,s]})} + f_u^{**}(s)U(s)\left(\int_s^1 U^{-q}(x)w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}. \tag{23}$$

We recall

$$f_u^{**}(s)U(s) = \int_0^s f^*(x)u(x)\,\mathrm{d}x = \|f\|_{\Lambda^1(u\chi_{[0,s]})},$$

so we finally obtain

$$\sup_{\|f\|_X \le 1} \|f\|_{\Gamma_u^q(w\chi_{[0,s]})} = \sup_{f\in X} \frac{\|f\|_{\Gamma^q(w\chi_{[0,s]})}}{\|f\|_X} = \mathrm{Opt}\left(X, \Gamma_u^q(w\chi_{[0,s]})\right)$$

and

$$\sup_{\|f\|_X \le 1} f_u^{**}(s)U(s)\left(\int_s^1 U^{-q}(t)w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} = \sup_{f\in X} \frac{\|f\|_{\Lambda^1(u\chi_{[0,s]})}}{\|f\|_X}\left(\int_s^1 U^{-q}(t)w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} =$$

$$= \mathrm{Opt}\left(X, \Lambda^1(u\chi_{[0,s]})\right)\cdot\left(\int_s^1 U^{-q}(t)w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}$$

which together with (23) gives the result.

(iv) This part is proved analogously as (iii), using the relation

$$\|f^*\chi_{[0,s]}\|_{\overline{\Gamma_u^{q,\infty}(w)}} = \max\left\{\|f\|_{\Gamma_u^{q,\infty}(w\chi_{[0,s]})};\ f_u^{**}(s)U(s)\sup_{s<t<1} W^{\frac{1}{q}}(t)U^{-1}(t)\right\}. \tag{24}$$

□

**Remark 3.7.** Let us show how Theorem 3.6 will be usually applied. Let $X$ be an r.i. lattice, let $0 < q < \infty$ and denote by $A$ the set of all admissible weights. Suppose there exists a functional $\Phi\colon A \to [0,\infty]$ such that, for $w \in A$, $X \hookrightarrow \Lambda^q(w)$ if and only if $\Phi(w) < \infty$ and it holds $\mathrm{Opt}\,(X, \Lambda^q(w)) \simeq \Phi(w)$. Then Theorem 3.6(i) yields that $X \overset{*}{\hookrightarrow} \Lambda^q(w)$ occurs if and only if

$$\lim_{s\to 0+} \Phi(w\chi_{[0,s]}) = 0.$$

When dealing with the other types of Lorentz spaces, the same approach can be adopted.

Theorem 3.6 provides us with a useful tool for describing almost-compact embeddings of the Lorentz spaces. In fact, mutual continuous embeddings of all Lorentz type spaces are characterized using such functional $\Phi$ as seen above in Remark 3.7, therefore we obtain simple characterizations of the corresponding almost-compact embeddings.

In the following part we are going to use known results about continuous embeddings. Most of them are contained in the survey paper [5] but partly in an incomplete or a bit outdated form. Some of these were later significantly improved and extended in [11]. One missing case is also solved in [4]. As we will see, some cases will also need to be re-proved in a generalized version at first.

### 3.1 Embeddings of type $\Lambda \overset{*}{\hookrightarrow} \Lambda$

**Lemma 3.8.** *Let $v, w$ be admissible weights and $p, q \in (0, \infty)$.*

(i) *Let $0 < p \le q < \infty$. Then $\Lambda^p(v) \hookrightarrow \Lambda^q(w)$ if and only if $A_{(25)} < \infty$ where*

$$A_{(25)} := \sup_{0<t<1} W^{\frac{1}{q}}(t) V^{-\frac{1}{p}}(t). \tag{25}$$

*Moreover,* $\operatorname{Opt}(\Lambda^p(v), \Lambda^q(w)) = A_{(25)}$.

(ii) *Let $0 < q < p < \infty$. Then $\Lambda^p(v) \hookrightarrow \Lambda^q(w)$ if and only if $A_{(26)} < \infty$ where*

$$A_{(26)} := \left( \int_0^1 W^{\frac{r}{p}}(x) V^{-\frac{r}{p}}(x) w(x) \, \mathrm{d}x \right)^{\frac{1}{r}} \tag{26}$$

*where $r = \frac{pq}{p-q}$. Moreover,* $\operatorname{Opt}(\Lambda^p(v), \Lambda^q(w)) \simeq A_{(26)}$.

This lemma is a direct consequence of the result below. Here comes also one of the occasions on which we use weights defined on $(0, \infty)$.

**Lemma 3.9.** *Suppose that $v, w \in L^1(0, \infty)$ are nonnegative and satisfy $V(t), W(t) > 0$ for all $t > 0$. Let $p, q \in (0, \infty)$. Just for purpose of this lemma, suppose that $\Lambda^p(v)$ and $\Lambda^q(w)$ are defined over a $\sigma$-finite nonatomic measure space $(S, \nu)$ with $\nu(S) = \infty$, i.e.*

$$\Lambda^p(v) := \left\{ f \in \mathscr{M}(S, \nu);\ \|f\|_{\Lambda^p(v)} := \left( \int_0^\infty (f^*(t))^p v(t) \, \mathrm{d}t \right)^{\frac{1}{p}} < \infty \right\},$$

*similarly for $\Lambda^q(w)$.*

(i) *Let $0 < p \le q < \infty$. Then $\Lambda^p(v) \hookrightarrow \Lambda^q(w)$ if and only if $\widetilde{A}_{(25)} < \infty$ where*

$$\widetilde{A}_{(25)} := \sup_{t>0} W^{\frac{1}{q}}(t) V^{-\frac{1}{p}}(t).$$

*Moreover,* $\operatorname{Opt}(\Lambda^p(v), \Lambda^q(w)) = \widetilde{A}_{(25)}$.

(ii) *Let $0 < q < p < \infty$. Then $\Lambda^p(v) \hookrightarrow \Lambda^q(w)$ if and only if $\widetilde{A}_{(26)} < \infty$ where*

$$\widetilde{A}_{(26)} := \left( \int_0^\infty W^{\frac{r}{p}}(x) V^{-\frac{r}{p}}(x) w(x) \, \mathrm{d}x \right)^{\frac{1}{r}}.$$

*where $r = \frac{pq}{p-q}$. Moreover,* $\operatorname{Opt}(\Lambda^p(v), \Lambda^q(w)) \simeq \widetilde{A}_{(26)}$.

*Proof.* This result is proved in [16, Remark (i), p. 148] for $1 < p, q < \infty$ and in [20, Proposition 1] for $0 < p, q < \infty$. $\square$

For a while, let us keep the setting from the previous lemma but consider the expression $\widetilde{A}_{(25)}$ for $q < p$ as well. Then, in fact, it holds $\widetilde{A}_{(25)} \lesssim \widetilde{A}_{(26)}$. We summarize this in the proposition below. The result appears also in [20, Lemma, p. 176].

**Proposition 3.10.** *Let $v, w$ be weights over $(0,\infty)$ such that (8) holds for every $t > 0$. Assume that $0 < q < p < \infty$. Set $r := \frac{pq}{p-q}$. Then it holds*

$$\begin{aligned} \widetilde{A}^r_{(25)} = \sup_{t>0} W^{\frac{r}{q}}(t) V^{-\frac{r}{p}}(t) \ &\lesssim \ \widetilde{A}^r_{(26)} = \int_0^\infty W^{\frac{r}{p}}(x) V^{-\frac{r}{p}}(x) w(x)\,\mathrm{d}x \ \simeq \\ &\simeq \ \int_0^\infty W^{\frac{r}{q}}(x) V^{-\frac{r}{q}}(x) v(x)\,\mathrm{d}x. \end{aligned} \tag{27}$$

*Proof.* Since $\frac{r}{q} - 1 = \frac{r}{p}$, for a fixed $t > 0$ it holds

$$\begin{aligned} W^{\frac{r}{q}}(t) V^{-\frac{r}{p}}(t) \ &= \ V^{-\frac{r}{p}}(t) \int_0^t (W^{\frac{r}{q}}(x))'\,\mathrm{d}x \le \\ &\le \ \int_0^t (W^{\frac{r}{q}}(x))' V^{-\frac{r}{p}}(x)\,\mathrm{d}x \lesssim \int_0^\infty W^{\frac{r}{p}}(x) V^{-\frac{r}{p}}(x) w(x)\,\mathrm{d}x. \end{aligned}$$

By passing to the supremum over $t > 0$, we get the first inequality in (27). Furthermore, applying integration by parts, we get

$$\int_0^\infty ((W^{\frac{r}{q}}(x))' V^{-\frac{r}{p}}(x)\,\mathrm{d}x \lesssim \int_0^\infty W^{\frac{r}{q}}(x) V^{-\frac{r}{q}}(x) v(x)\,\mathrm{d}x.$$

The converse inequality is proved in the same way. $\square$

These "infinite-measure-domain results" will be used later. Now we return to the Lorentz spaces over $(R, \mu)$, i.e. given by Definition 3.2. Weights are thus defined again on $(0, 1)$. At first we recall the following basic fact:

**Remark 3.11.** Let $\psi\colon (0,1) \to (0,\infty)$ be a function. Then

$$\lim_{t\to 0+} \psi(t) = 0 \quad \Leftrightarrow \quad \lim_{s\to 0+} \sup_{0<t<s} \psi(t) = 0.$$

Now we state a theorem about an almost-compact embedding involving the situation from Lemma 3.8.

**Theorem 3.12** (The case $\Lambda^p(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$)**.** *Let $v, w$ be admissible weights and $p, q \in (0,\infty)$. When $p > q$, we set $r = \frac{pq}{p-q}$.*

(i) *For $0 < p \le q < \infty$, $\Lambda^p(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$ if and only if*

$$\lim_{t\to 0+} W(t)^{\frac{1}{q}} V(t)^{-\frac{1}{p}} = 0. \tag{28}$$

(ii) *For $0 < q < p < \infty$, $\Lambda^p(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$ if and only if $\Lambda^p(v) \hookrightarrow \Lambda^q(w)$, i.e. if and only if*

$$\int_0^1 W^{\frac{r}{p}}(x) V^{-\frac{r}{p}}(x) w(x)\,\mathrm{d}x < \infty. \tag{29}$$

*Proof.* (i) By Theorem 3.6 we immediately obtain the characterization of $\Lambda^p(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$ in the form

$$\lim_{s\to 0+} \sup_{0<t<1} W_s(t)^{\frac{1}{q}} V(t)^{-\frac{1}{p}} = 0.$$

By monotonicity of $V$, this is obviously equivalent to

$$\lim_{s\to 0+} \sup_{0<t<s} W(t)^{\frac{1}{q}} V(t)^{-\frac{1}{p}} = 0$$

and, by Remark 3.11, to (28).

(ii) Similarly as in the previous case, we obtain that $\Lambda^p(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$ if and only if

$$\lim_{s\to 0+} \left( \int_0^s W^{\frac{r}{p}}(x) V^{-\frac{r}{p}}(x) w(x)\,\mathrm{d}x \right)^{\frac{1}{r}} = 0. \tag{30}$$

However, $r$ is positive, thus the absolute continuity of the integral assures that (29) implies (30). For the converse, $W^{\frac{r}{p}} V^{-\frac{r}{p}}$ is a continuous function from $(0,1]$ to $(0,\infty)$, hence it is bounded on every interval $[s,1]$, $s \in (0,1)$. Combined with the integrability of $w$ this shows that (30) implies (29). □

**Lemma 3.13.** *Let $v, w$ be admissible weights and $p, q \in (0,\infty)$. Then $\Lambda^p(v) \hookrightarrow \Lambda^{q,\infty}(w)$ if and only if $A_{(25)} < \infty$. Moreover,* $\operatorname{Opt}(\Lambda^p(v), \Lambda^{q,\infty}(w)) = A_{(25)}$.

*Proof.* See [5, Theorem 3.2]. □

**Theorem 3.14** (The case $\Lambda^p(v) \overset{*}{\hookrightarrow} \Lambda^{q,\infty}(w)$)**.** *Let $v, w$ be admissible weights and $p, q \in (0,\infty)$. Then $\Lambda^p(v) \overset{*}{\hookrightarrow} \Lambda^{q,\infty}(w)$ if and only if (28) holds.*

*Proof.* This result follows from Theorem 3.6 and Lemma 3.13. □

**Lemma 3.15.** *Let $v, w$ be admissible weights and $p, q \in (0,\infty)$. Then $\Lambda^{p,\infty}(v) \hookrightarrow \Lambda^q(w)$ if and only if $A_{(31)} < \infty$ where*

$$A_{(31)} := \left( \int_0^1 V^{-\frac{q}{p}}(t) w(t)\,\mathrm{d}t \right)^{\frac{1}{q}}. \tag{31}$$

*Moreover,* $\operatorname{Opt}(\Lambda^{p,\infty}(v), \Lambda^q(w)) = A_{(31)}$.

*Proof.* See [5, Theorem 3.3]. □

**Theorem 3.16** (The case $\Lambda^{p,\infty}(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$)**.** *Let $v, w$ be admissible weights and $p, q \in (0,\infty)$. Then $\Lambda^{p,\infty}(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$ if and only if*

$$\lim_{s\to 0+} \left( \int_0^s V^{-\frac{q}{p}}(t) w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} = 0.$$

*Proof.* The statement follows from Theorem 3.6 and Lemma 3.15. □

**Lemma 3.17.** *Let $v, w$ be admissible weights and $p, q \in (0,\infty)$. Then $\Lambda^{p,\infty}(v) \hookrightarrow \Lambda^{q,\infty}(w)$ if and only if $A_{(25)} < \infty$. Moreover,* $\operatorname{Opt}(\Lambda^{p,\infty}(v), \Lambda^{q,\infty}(w)) = A_{(25)}$.

*Proof.* See [5, Theorem 3.4]. □

**Theorem 3.18** (The case $\Lambda^{p,\infty}(v) \overset{*}{\hookrightarrow} \Lambda^{q,\infty}(w)$). *Let $v, w$ be admissible weights and $p, q \in (0, \infty)$. Then $\Lambda^{p,\infty}(v) \overset{*}{\hookrightarrow} \Lambda^{q,\infty}(w)$ if and only if (28) holds.*

*Proof.* This result is a direct consequence of Theorem 3.6 and Lemma 3.17. □

### 3.2 Embeddings of type $\Lambda \overset{*}{\hookrightarrow} \Gamma$

We are now about to state another lemma about a particular continuous embedding of Lorentz spaces. However, we want the result to involve a general positive and integrable Hardy weight $u$, but the existing results have been formulated and proved just for $u \equiv 1$. (An almost complete survey of this special case can be found in [5, Theorem 4.1].) Therefore, this time we will even prove the lemma in order to obtain the result in the desired generality. We will also make another exception this time and formulate the lemma for Lorentz spaces over a domain with infinite measure (although they will be denoted still by $\Lambda^p(v), \Gamma^q_u(w)$, etc. as in Lemma 3.9). The particular result for a finite-measure space, which we need for the almost-compact embedding characterization, will then follow by an extension of particular weights over $[0,1]$ by zero on $(1,\infty)$ and applying the wider result for weights over $[0,\infty)$.

**Lemma 3.19.** *Let $v, w$ be integrable weights over $(0,\infty)$ such that (8) holds for all $t > 0$. Let $u$ be a positive integrable weight on $(0,\infty)$. Assume that $p, q \in (0,\infty)$. When $p > q$, we set $r = \frac{pq}{p-q}$.*

(i) *Let $1 < p \le q < \infty$. Then $\Lambda^p(v) \hookrightarrow \Gamma^q_u(w)$ if and only if $\widetilde{A}_{(25)} + A_{(32)} < \infty$ where*

$$A_{(32)} := \sup_{t>0} \left( \int_t^\infty U^{-q}(x) w(x)\,\mathrm{d}x \right)^{\frac{1}{q}} \left( \int_0^t U^{p'}(x) V^{-p'}(x) v(x)\,\mathrm{d}x \right)^{\frac{1}{p'}}. \tag{32}$$

*Moreover,* $\operatorname{Opt}(\Lambda^p(v), \Gamma^q_u(w)) \simeq \widetilde{A}_{(25)} + A_{(32)}$.

(ii) *Let $0 < p \le 1$, $0 < p \le q < \infty$. Then $\Lambda^p(v) \hookrightarrow \Gamma^q_u(w)$ if and only if $\widetilde{A}_{(25)} + A_{(33)} < \infty$ where*

$$A_{(33)} := \sup_{t>0} U(t) V^{-\frac{1}{p}}(t) \left( \int_t^\infty U^{-q}(x) w(x)\,\mathrm{d}x \right)^{\frac{1}{q}}. \tag{33}$$

*Moreover,* $\operatorname{Opt}(\Lambda^p(v), \Gamma^q_u(w)) \simeq \widetilde{A}_{(25)} + A_{(33)}$.

(iii) *Let $1 < p < \infty$, $0 < q < p < \infty$, $q \ne 1$. Then $\Lambda^p(v) \hookrightarrow \Gamma^q_u(w)$ if and only if $\widetilde{A}_{(26)} + A_{(34)} < \infty$ where*

$$A_{(34)} := \Bigg( \int_0^\infty \left( \int_t^\infty U^{-q}(x) w(x)\,\mathrm{d}x \right)^{\frac{r}{q}} \times \\ \times \left( \int_0^t U^{p'}(x) V^{-p'}(x) v(x)\,\mathrm{d}x \right)^{\frac{r}{q'}} U^{p'}(t) V^{-p'}(t) v(t)\,\mathrm{d}t \Bigg)^{\frac{1}{r}}. \tag{34}$$

*Moreover,* $\operatorname{Opt}(\Lambda^p(v), \Gamma^q_u(w)) \simeq \widetilde{A}_{(26)} + A_{(34)}$.

(iv) *Let* $1 = q < p < \infty$. *Then* $\Lambda^p(v) \hookrightarrow \Gamma^q_u(w)$ *if and only if* $A_{(35)} < \infty$ *where*

$$A_{(35)} := \left(\int_0^\infty \left(W(t)+U(t)\int_t^\infty U^{-1}(x)w(x)\,\mathrm{d}x\right)^{p'-1} \times \right.$$
$$\left. \times\, V^{1-p'}(t)\,u(t)\left(\int_t^\infty U^{-1}(x)w(x)\,\mathrm{d}x\right)\mathrm{d}t\right)^{-\frac{1}{p'}}. \quad (35)$$

*Moreover,* $\operatorname{Opt}(\Lambda^p(v), \Gamma^q_u(w)) \simeq A_{(35)}$.

(v) *Let* $0 < q < p \le 1$. *Then* $\Lambda^p(v) \hookrightarrow \Gamma^q_u(w)$ *if and only if* $\widetilde{A}^p_{(26)} + A_{(36)} < \infty$ *where*

$$A_{(36)} := \left(\int_0^\infty \sup_{0<x\le t} U^r(x)V^{-\frac{r}{p}}(x)\left(\int_t^\infty U^{-q}(y)w(y)\,\mathrm{d}y\right)^{\frac{r}{p}} U^{-q}(t)w(t)\,\mathrm{d}t\right)^{\frac{p}{r}}. \quad (36)$$

*Moreover,* $\operatorname{Opt}(\Lambda^p(v), \Gamma^q_u(w)) \simeq \widetilde{A}^p_{(26)} + A_{(36)}$.

*Proof.* (i) For a function $f \in \mathscr{M}(0,\infty)$, let us define

$$Tf(x) := U^{-1}(x)\int_0^x f(y)u(y)\,\mathrm{d}y, \quad x \in (0,\infty).$$

Furthermore, the adjoint operator $T^*$ (under the pairing $\int_0^\infty fg$) is given by

$$T^*g(y) := u(y)\int_y^\infty g(x)U^{-1}(x)\,\mathrm{d}x, \quad y \in (0,\infty),$$

where $g \in \mathscr{M}(0,1)$. Under this setting, by [16, §1 and Theorem 1] we have the following: There exists a constant $C > 0$ such that

$$\left(\int_0^\infty (Tf(x))^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \quad (37)$$

holds for all nonincreasing and nonnegative $f \in \mathscr{M}(0,\infty)$ if and only if

$$\left(\int_0^\infty \left(\int_0^x T^*g\right)^{p'} v(x)V^{-p'}(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} + V^{\frac{1}{p}}(\infty)\int_0^\infty T^*g \le$$
$$\le C\left(\int_0^\infty g^{q'}(x)w^{1-q'}(x)\,\mathrm{d}x\right)^{\frac{1}{q'}} \quad (38)$$

holds for all nonnegative $g \in \mathscr{M}(0,\infty)$. Moreover, $C$ is the least constant such that (37) holds if and only if it is the least constant for (38). Next, for $g \in \mathscr{M}(0,\infty)$ nonnegative and $t \in (0,\infty)$ we have, by Fubini theorem,

$$\int_0^t T^*g(x)\,\mathrm{d}x = \int_0^t u(y)\left(\int_y^\infty g(x)U^{-1}(x)\,\mathrm{d}x\right)\mathrm{d}y =$$
$$= \int_0^t g(x)\,\mathrm{d}x + U(t)\int_t^\infty g(x)U^{-1}(x)\,\mathrm{d}x = Pg(t) + U(t)P^*(gU^{-1})(t)$$

where $P$ is the Hardy operator (see Definition 2.2) and $P^*$ stands for its adjoint. By [14], $P$ is bounded from $L^{q'}(w^{1-q'})$ to $L^{p'}(V^{-p'}v)$ if and only if

$$\sup_{t>0}\left(\int_t^\infty V^{-p'}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} W^{\frac{1}{q}}(t) = A < \infty \tag{39}$$

and in here it holds $\|P\| \simeq A$ where $\|P\|$ is the corresponding operator norm. Similarly, $P^*$ is bounded from $L^{q'}(w^{1-q'})$ to $L^{p'}(U^{p'}V^{-p'}v)$ if and only if

$$\sup_{t>0}\left(\int_0^t U^{p'}(x)V^{p'}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}\left(\int_t^\infty U^{-q}(x)w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} = B < \infty \tag{40}$$

and, moreover, $\|P^*\| \simeq B$. Furthermore, by the reverse Hölder inequality (cf. [16, equation (1.6)]),

$$\int_0^\infty T^*g(x)\,\mathrm{d}x = \int_0^\infty g(x)\,\mathrm{d}x \le MV^{\frac{1}{p}}(\infty)\left(\int_0^\infty g^{q'}(x)w^{1-q'}(x)\,\mathrm{d}x\right)^{\frac{1}{q'}}$$

for all nonnegative $g$ with the least such $M > 0$ if and only if

$$W^{\frac{1}{q}}(\infty) = MV^{\frac{1}{p}}(\infty). \tag{41}$$

We observe that

$$(p'-1)\int_t^\infty V^{-p'}(x)v(x)\,\mathrm{d}x = V^{1-p'}(t) - V^{1-p'}(\infty),$$

so (39) is rewritten as

$$\sup_{t>0}\left[V^{1-p'}(t) - V^{1-p'}(\infty)\right]^{\frac{1}{p'}} W^{\frac{1}{q}}(t) = A(p'-1)^{\frac{1}{p'}} < \infty. \tag{42}$$

Now we are going to prove that (42) together with (41) are equivalent to

$$\sup_{t>0} W^{\frac{1}{q}}(t)V^{-\frac{1}{p}}(t) = K < \infty. \tag{43}$$

Suppose at first that (42) and (41) hold. We can find $T > 0$ such that $V(T) = 2^{1-p}V(\infty)$. Then

$$\begin{aligned} W^{\frac{1}{q}}(t)V^{-\frac{1}{p}}(t) &= 2^{\frac{1}{p'}}W^{\frac{1}{q}}(t)\left(V^{1-p'}(t) - \frac{1}{2}V^{1-p'}(t)\right)^{\frac{1}{p'}} \le \\ &\le 2^{\frac{1}{p'}}W^{\frac{1}{q}}(t)\left(V^{1-p'}(t) - \frac{1}{2}V^{1-p'}(T)\right)^{\frac{1}{p'}} = \\ &= 2^{\frac{1}{p'}}W^{\frac{1}{q}}(t)\left(V^{1-p'}(t) - V^{1-p'}(\infty)\right)^{\frac{1}{p'}} = 2^{\frac{1}{p'}}(p'-1)^{\frac{1}{p'}}A \end{aligned}$$

for all $t \in (0, T]$ and

$$W^{\frac{1}{q}}(t)V^{-\frac{1}{p}}(t) \le W^{\frac{1}{q}}(\infty)V^{-\frac{1}{p}}(T) \le 2^{\frac{1}{p'}}W^{\frac{1}{q}}(\infty)V^{-\frac{1}{p}}(\infty) = 2^{\frac{1}{p'}}M$$

for all $t \in [T,\infty)$. By now we see that $K \lesssim A + M$. Now suppose that (43) holds. On taking limit for $t \to \infty$ in (43) we obtain that (41) holds with $M \le K$. Besides that, it holds

$$2^{\frac{1}{p'}} A(p'-1)^{\frac{1}{p'}} \le \sup_{t>0} W^{\frac{1}{q}}(t)\left[V^{1-p'}(t) - V^{1-p'}(\infty)\right]^{\frac{1}{p'}} \le \sup_{t>0} W^{\frac{1}{q}}(t)V^{-\frac{1}{p}}(t) = K,$$

hence $A + M \simeq K$ and, in particular, the desired equivalence is proved. Moreover, we recall that $B$ (from (40)) is equal to $A_{(32)}$, thus the result follows, including the statement about the optimal embedding constant.

While solving the remaining cases, we will now no longer explicitly prove the estimates for the optimal constants as we did in above. However, if needed, the precise estimates can be obtained by treating the inequalities in the following parts in such a detail as in the case (i).

(ii) The necessity part is very easy: Testing the embedding by characteristic functions of intervals $[0,t]$, $t > 0$, gives us

$$\sup_{t>0} V^{-\frac{1}{p}}\left(W(t) + \int_t^\infty U^{-q}(x)w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} < \infty,$$

which, with help of integrability of $u$, yields that $A_{(33)} < \infty$. Obviously, $\widetilde{A}_{(25)}$ has to be true as well since $\Gamma_u^q(w) \hookrightarrow \Lambda^q(w)$.

Now we prove sufficiency. Suppose that $\widetilde{A}_{(25)} + A_{(33)} < \infty$. Let $f \in \mathscr{M}_+(0,\infty)$ be a nonincreasing function (then $f = f^*$ a.e.). If $\int_0^\infty f^p v = \infty$, then

$$\left(\int_0^\infty (f_u^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (f^*(x))^p v(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{44}$$

is trivially satisfied. Otherwise, we have to apply the following "Stepanov's method": There exists a sequence $\{f_n\}$ of nonincreasing functions such that $f_n \in C_A^1(0,\infty)$, $n \in \mathbb{N}$, and $f_n(x) \uparrow f^*(x)$ for a.e. $x \in (0,\infty)$. The symbol $C_A^1$ we have used has the following meaning:

$$C_A^1(0,\infty) := \left\{g \in \mathscr{M}(0,\infty);\ g' \in C(0,\infty)\ \&\ \lim_{x\to\infty} g(x) = 0\right\}.$$

Suppose that

$$\left(\int_0^\infty ((f_n)_u^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (f_n^*(x))^p v(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{45}$$

holds for all $n \in \mathbb{N}$. Then, by applying the Levi monotone convergence theorem on (45), we obtain that (44) holds for all $f \in \mathscr{M}_+(0,\infty)$ if and only if it holds for all nonincreasing $f \in C_A^1(0,\infty)$. Hence, in (44) it suffices to consider $f \in C_A^1(0,\infty)$ and nonincreasing. For every such $f$ there exists a function $h \in \mathscr{M}_+(0,\infty)$ such that

$$f(x) = \int_x^\infty h(s)\,\mathrm{d}s, \quad x \in (0,\infty). \tag{46}$$

For this $f$ and the corresponding $h$, we denote

$$g(y) := \left(\int_y^\infty h\right)^{p-1} h(y) = -p^{-1}\frac{\mathrm{d}}{\mathrm{d}y}\left(\int_y^\infty h\right)^p = -p^{-1}(f^p)'(y), \quad y \in (0,\infty).$$

Thus, for every fixed $x \in (0,\infty)$ we have $f(x) = \left(p\int_x^\infty g(y)\,\mathrm{d}y\right)^{\frac{1}{p}}$ and we write

$$
\begin{aligned}
\int_0^x f(s)u(s)\,\mathrm{d}s &= p^{\frac{1}{p}}\int_0^x \left[\int_s^\infty g(y)\,\mathrm{d}y\right]^{\frac{1}{p}} u(s)\,\mathrm{d}s \le \\
&\le 2^{\frac{1}{p}}p^{\frac{1}{p}}\left[\int_0^x\left(\int_s^x g(y)\,\mathrm{d}y\right)^{\frac{1}{p}} u(s)\,\mathrm{d}s + \int_0^x\left(\int_x^\infty g(y)\,\mathrm{d}y\right)^{\frac{1}{p}} u(s)\,\mathrm{d}s\right] \le \\
&\le 2^{\frac{1}{p}}p^{\frac{1}{p}}\int_0^x\left(\int_s^x g(y)\,\mathrm{d}y\right)^{\frac{1}{p}} u(s)\,\mathrm{d}s + 2^{\frac{1}{p}}U(x)f(x). \qquad (47)
\end{aligned}
$$

From the Minkowski integral inequality (see [15, Chapter 0, Proposition 3.2]) we get

$$
\begin{aligned}
&\int_0^x\left(\int_s^x g(y)\,\mathrm{d}y\right)^{\frac{1}{p}} u(s)\,\mathrm{d}s = \int_0^x\left(\int_0^x \chi_{(s,x)}(y)g(y)\,\mathrm{d}y\right)^{\frac{1}{p}} u(s)\,\mathrm{d}s \le \\
&\qquad \le \left[\int_0^x\left(\int_0^x \left(\chi_{(s,x)}(y)g(y)\right)^{\frac{1}{p}} u(s)\,\mathrm{d}s\right)^p \mathrm{d}y\right]^{\frac{1}{p}} = \left(\int_0^x g(y)U^p(y)\,\mathrm{d}y\right)^{\frac{1}{p}}
\end{aligned}
$$

which together with (47) yields

$$\int_0^x f(s)u(s)\,\mathrm{d}s \lesssim \left(\int_0^x g(y)U^p(y)\,\mathrm{d}y\right)^{\frac{1}{p}} + U(x)f(x).$$

It follows that

$$
\begin{aligned}
&\left[\int_0^\infty\left(U^{-1}(x)\int_0^x f(s)u(s)\,\mathrm{d}s\right)^q w(x)\,\mathrm{d}x\right]^{\frac{1}{q}} \lesssim \\
&\lesssim \left[\int_0^\infty U^{-q}(x)\left(\int_0^x g(y)U^p(y)\,\mathrm{d}y\right)^{\frac{q}{p}} w(x)\,\mathrm{d}x\right]^{\frac{1}{q}} + \left(\int_0^\infty f^q(x)w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} =: I_1 + I_2.
\end{aligned}
$$

Recalling that $f$ is nonincreasing, from Lemma 3.9(i) we obtain

$$I_2 \lesssim \widetilde{A}_{(25)}\left(\int_0^\infty f^p(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

As for the rest, using the Minkowski integral inequality (for the $L^{\frac{q}{p}}$ norm) again, we get

$$
\begin{aligned}
I_1 &= \left[\int_0^\infty\left(\int_0^\infty \chi_{(0,x)}(y)g(y)U^p(y)U^{-p}(x)w^{\frac{p}{q}}(x)\,\mathrm{d}y\right)^{\frac{q}{p}}\mathrm{d}x\right]^{\frac{1}{q}} \le \\
&\le \left[\int_0^\infty g(y)U^p(y)\left(\int_0^\infty \chi_{(0,x)}(y)U^{-q}(x)w(x)\,\mathrm{d}x\right)^{\frac{p}{q}}\mathrm{d}y\right]^{\frac{1}{p}} = \\
&= \left[\int_0^\infty g(y)U^p(y)\left(\int_y^\infty U^{-q}(x)w(x)\,\mathrm{d}x\right)^{\frac{p}{q}}\mathrm{d}y\right]^{\frac{1}{p}} = \\
&= \left(\int_0^\infty g(y)\left[U(y)\left(\int_y^\infty U^{-q}(x)w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(y)\right]^p V(y)\,\mathrm{d}y\right)^{\frac{1}{p}} \le \\
&\le A_{(33)}\left(\int_0^\infty g(y)V(y)\,\mathrm{d}y\right)^{\frac{1}{p}} \lesssim A_{(33)}\left(\int_0^\infty f^p(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}
$$

The last inequality follows from integration by parts. Indeed, since $f \in C_A^1(0,\infty)$, we have

$$\begin{aligned}\int_0^\infty g(y)V(y)\,\mathrm{d}y &\simeq \int_0^\infty f^p(x)v(x)\,\mathrm{d}x + [-f^pV]_0^\infty = \\ &= \int_0^\infty f^p(x)v(x)\,\mathrm{d}x + \lim_{y\to 0+} f^p(y)V(y) \le \\ &\le \int_0^\infty f^p(x)v(x)\,\mathrm{d}x + \lim_{y\to 0+} \int_0^y f^p(x)v(x)\,\mathrm{d}x = \\ &= \int_0^\infty f^p(x)v(x)\,\mathrm{d}x.\end{aligned}$$

Altogether, the proof of this case is now complete.

(iii) Let us take $0 < q < 1 < p$ and start with the sufficiency. Assume that $\widetilde{A}_{(26)}, A_{(34)} < \infty$. By the standard argument (cf. (46) and the comments before it), we need to prove that (44) holds for all nonincreasing $f \in C_A^1(0,\infty)$. So, let us Let $f$ be such function, then there is a function $h \in \mathscr{M}_+(0,\infty)$ such that (46) holds. Take a fixed $t > 0$. By changing the order of integration, we get

$$U^{-1}(t)\int_0^t f(x)u(x)\,\mathrm{d}x = U^{-1}(t)\int_0^t h(x)U(x)\,\mathrm{d}x + \int_t^\infty h(x)\,\mathrm{d}x.$$

Hence,

$$\begin{aligned}&\left(\int_0^\infty U^{-q}(t)\left[\int_0^t f(x)u(x)\,\mathrm{d}x\right]^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \lesssim \\ &\qquad\lesssim \left(\int_0^\infty \left[\int_0^t h(x)U(x)\,\mathrm{d}x\right]^q U^{-q}(t)w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} + \left(\int_0^\infty f^q(t)w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned}$$

In the last factor we have used the identity from (46). Since $\widetilde{A}_{(26)} < \infty$, Lemma 3.9(ii) yields $\left(\int_0^\infty f^q w\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty f^p v\right)^{\frac{1}{p}}$, so it remains to find an upper bound for

$$I_1 := \left(\int_0^\infty \left[\int_0^t h(x)U(x)\,\mathrm{d}x\right]^q U^{-q}(t)w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}.$$

Let us denote $\Phi(y) := h(y)V(y)$, $\Psi(y) := \int_0^y \Phi(s)\,\mathrm{d}s$, $y \in (0,\infty]$. Integration by parts gives

$$\int_0^t U(x)\Phi(x)V^{-1}(x)\,\mathrm{d}x \le U(t)\Psi(t)V^{-1}(t) + \int_0^t U(x)\Psi(x)V^{-2}(x)v(x)\,\mathrm{d}x,$$

therefore,

$$I_1^q \lesssim \int_0^\infty \frac{\Psi^q(t)}{V^q(t)}w(t)\,\mathrm{d}t + \int_0^\infty \left(\int_0^t \frac{U(x)\Psi(x)}{V^2(x)}v(x)\,\mathrm{d}x\right)^q U^{-q}(t)w(t)\,\mathrm{d}t =: I_2^q + I_3^q.$$

By integration by parts, we get

$$I_2^q \lesssim \Psi^q(\infty)W(\infty)V^{-q}(\infty) + \int_0^\infty \Psi^q(t)W(t)V^{-q-1}(t)v(t)\,\mathrm{d}t =: I_4^q + I_5^q.$$

Hölder inequality yields

$$\Psi(\infty) = \int_0^\infty f(t)v(t)\,\mathrm{d}t \le \left(\int_0^\infty f^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} V^{\frac{1}{p'}}(\infty),$$

thus

$$I_4 \le \left(\int_0^\infty f^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} V^{-\frac{1}{p}}(\infty)W^{\frac{1}{q}}(\infty) \le \left(\int_0^\infty f^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \widetilde{A}_{(26)}$$

where the last step is obtained from Proposition 3.10. Next, since

$$\sup_{t>0} \left(\int_0^t V^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_0^t \left[v^{1-p}(x)\right]^{1-p'}\,\mathrm{d}x\right)^{\frac{1}{p'}} = \sup_{t>0} V^{-\frac{1}{p'}}(t)V^{\frac{1}{p'}}(t) = 1 < \infty,$$

by the weighted Hardy inequality for nonnegative functions ([20, Theorem 1(a)], see also [14]), it holds

$$\int_0^\infty \left(\int_0^t f(x)v(x)\,\mathrm{d}x\right)^p V^{-p}(t)v(t)\,\mathrm{d}t \lesssim \int_0^\infty f^p(t)v(t)\,\mathrm{d}t. \tag{48}$$

Fubini theorem and (46) yield

$$\begin{aligned}\int_0^t h(s)V(s)\,\mathrm{d}s \ &\le \ \int_0^t h(s)V(s)\,\mathrm{d}s + \int_t^\infty h(s)V(t)\,\mathrm{d}s = \\ &= \ \int_0^t \left(\int_x^\infty h(s)\,\mathrm{d}s\right) v(x)\,\mathrm{d}x = \int_0^t f(x)v(x)\,\mathrm{d}x,\end{aligned}$$

so we get

$$\int_0^\infty \Psi^p(t)V^{-p}(t)v(t)\,\mathrm{d}t \lesssim \int_0^\infty f^p(t)v(t)\,\mathrm{d}t. \tag{49}$$

Hence, subsequently applying Hölder inequality, Proposition 3.10, (45) and (48), we get

$$\begin{aligned}I_5 &\le \left(\int_0^\infty W^{\frac{r}{q}}(t)V^{-\frac{r}{q}}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{r}} \left(\int_0^\infty \Psi^p(t)V^{-p}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \simeq \\ &\simeq \widetilde{A}_{(26)} \left(\int_0^\infty \Psi^p(t)V^{-p}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \lesssim \widetilde{A}_{(26)} \left(\int_0^\infty f^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}.\end{aligned}$$

Another application of the appropriate type of Hardy inequality ([20, Theorem 1(b)]) and (48) yields

$$I_3 \lesssim \widetilde{A}_{(26)} \left(\int_0^\infty \Psi^p(t)V^{-p}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \lesssim \widetilde{A}_{(26)} \left(\int_0^\infty f^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}$$

which was the last required estimate.

Now we turn to the necessity question. For $x > 0$, let us denote

$$\vartheta(x) := U^{p'}(x)V^{-p'}(x)v(x); \quad \Theta(x) := \int_0^x \vartheta(s)\,\mathrm{d}s.$$

Suppose that $\Lambda^p(v) \hookrightarrow \Gamma_u^q(w)$ with $C := \operatorname{Opt}(\Lambda^p(v), \Gamma_u^q(w))$. At first, we will show that $A_{(34)} \simeq A_{(50)}$, where

$$A_{(50)} := \left( \int_0^\infty \left( \int_t^\infty U^{-q}(x) w(x) \, \mathrm{d}x \right)^{\frac{r}{p}} \Theta^{\frac{r}{p'}}(t) U^{-q}(t) w(t) \, \mathrm{d}t \right)^{\frac{1}{r}}. \tag{50}$$

For $n \in \mathbb{N}$ consider the functions $v_n := v\chi_{[\frac{1}{n},\infty]}$ and $w_n := w\chi_{[0,n]}$. In addition, let us denote $\vartheta_n(x) := U^{p'}(x) V_n^{-p'}(x) v_n(x)$ and $\Theta_n(x) := \int_0^x \vartheta_n(s) \, \mathrm{d}s$. Integration by parts gives

$$\begin{aligned} &\int_0^\infty \left( \int_t^\infty U^{-q}(x) w_n(x) \, \mathrm{d}x \right)^{\frac{r}{p}} U^{-q}(t) w_n(t) \Theta_n^{\frac{r}{p'}}(t) \, \mathrm{d}t = \\ = \ &\frac{q}{r} \int_0^\infty \left( \int_t^\infty U^{-q}(x) w_n(x) \, \mathrm{d}x \right)^{\frac{r}{q}} \Theta_n^{\frac{r}{q'}}(t) \vartheta_n(t) \, \mathrm{d}t. \end{aligned}$$

Taking the limit for $n \to \infty$ on both sides of the equation and using the Levi theorem gives $A_{(50)} \simeq A_{(34)}$. This relation will be used later. Now, it holds

$$\begin{aligned} A_{(34)}^r &= \int_0^\infty \left( \int_t^\infty U^{-q}(x) w(x) \, \mathrm{d}x \right)^{\frac{r}{p}} \Theta^{\frac{r}{q'}}(t) \vartheta(t) V^{-1}(t) V(t) \, \mathrm{d}t = \\ &= \int_0^\infty \left[ \int_y^\infty \left( \int_t^\infty U^{-q}(x) w(x) \, \mathrm{d}x \right)^{\frac{r}{q}} \Theta^{\frac{r}{q'}}(t) \vartheta(t) V^{-1}(t) \, \mathrm{d}t \right] v(y) \, \mathrm{d}y =: \int_0^\infty g^p(x) v(x) \, \mathrm{d}x. \end{aligned}$$

We may assume that

$$\int_t^\infty U^{-q}(x) w(x) \, \mathrm{d}x \quad \text{for every } t > 0 \tag{51}$$

(otherwise $\Gamma_u^q(w) = \{0\}$). The function $g$ then takes finite values and is nonincreasing. The embedding then yields

$$C A_{(34)}^{\frac{r}{p}} = C \left( \int_0^\infty g^p v \right)^{\frac{1}{p}} \gtrsim \left( \int_0^\infty U^{-q}(s) \left( \int_0^s g u \right)^q w(s) \, \mathrm{d}s \right)^{\frac{1}{q}}.$$

Let us assume for a while that $v$ is just locally integrable on $(0,\infty)$ but $V(\infty) = \infty$. Then, for $s > 0$ we have

$$\int_0^s g u \ge \int_0^s \left( \int_y^s \left( \int_t^\infty U^{-q} w \right)^{\frac{r}{q}} \Theta^{\frac{r}{q'}}(t) \vartheta(t) V^{-1}(t) \, \mathrm{d}t \right)^{\frac{1}{p}} u(y) \, \mathrm{d}y \ge \left( \int_s^\infty U^{-q} w \right)^{\frac{r}{qp}} J_1(s),$$

where

$$J_1(s) := \int_0^s \left( \int_y^s \Theta^{\frac{r}{q'}}(t) \vartheta(t) V^{-1}(t) \, \mathrm{d}t \right)^{\frac{1}{p}} u(y) \, \mathrm{d}y \ge \Theta^{\frac{r}{q'p}}(s) J_2(s),$$

where, finally,

$$J_2(s) := \int_0^s \left( \int_y^s \vartheta(t) V^{-1}(t) \, \mathrm{d}t \right)^{\frac{1}{p}} u(y) \, \mathrm{d}y.$$

Now we continue estimating $J_2$:

$$\begin{aligned}
J_2(s) &= \int_0^s \left(\int_y^s U^{p'}(x)V^{-1-p'}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}} u(y)\,\mathrm{d}y \geq \\
&\geq \int_0^s U^{\frac{p'}{p}}(y)\left(\int_y^s V^{-1-p'}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}} u(y)\,\mathrm{d}y = \\
&= \frac{1}{p}\int_0^s U^{\frac{p'}{p}}(y)u(y)\int_y^s \left(\int_x^s V^{-1-p'}v\right)^{-\frac{1}{p'}} V^{-1-p'}(x)v(x)\,\mathrm{d}x\,\mathrm{d}y \simeq \\
&\simeq \int_0^s \int_0^x U^{\frac{p'}{p}}(y)u(y)\left(\int_x^s V^{-1-p'}v\right)^{-\frac{1}{p'}} V^{-1-p'}(x)v(x)\,\mathrm{d}y\,\mathrm{d}x \geq \\
&\geq \int_0^s V^{-1-p'}(x)v(x)\left(\int_x^\infty V^{-1-p'}v\right)^{-\frac{1}{p'}} \int_0^x U^{\frac{p'}{p}}(y)u(y)\,\mathrm{d}y\,\mathrm{d}x \simeq \\
&\simeq \int_0^s V^{-p'}(x)v(x)\int_0^x U^{p'-1}(y)u(y)\,\mathrm{d}y\,\mathrm{d}x \simeq \\
&\simeq \int_0^s U^{p'}(x)V^{-p'}(x)v(x)\,\mathrm{d}x = \Theta(s).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
CA_{(34)}^{\frac{r}{p}} &\geq \left(\int_0^\infty \left(\int_s^\infty U^{-q}w\right)^{\frac{r}{p}} J_1^q(s)U^{-q}(s)w(s)\,\mathrm{d}s\right)^{\frac{1}{p}} \geq \\
&\geq \left(\int_0^\infty \left(\int_s^\infty U^{-q}w\right)^{\frac{r}{p}} \Theta^{\frac{r(q-1)}{p}}(s)J_2^q(s)U^{-q}(s)w(s)\,\mathrm{d}s\right)^{\frac{1}{q}} \gtrsim \\
&\gtrsim \left(\int_0^\infty \left(\int_s^\infty U^{-q}w\right)^{\frac{r}{p}} \Theta^{\frac{r}{p'}}(s)U^{-q}(s)w(s)\,\mathrm{d}s\right)^{\frac{1}{q}} = A_{(50)}^{\frac{r}{q}} \gtrsim A_{(34)}^{\frac{r}{q}}.
\end{aligned}$$

If $A_{(34)} < \infty$, then we already have $C \gtrsim A_{(34)}^{\frac{r}{q}-\frac{r}{p}} = A_{(34)}$. Now assume that, on the contrary, $A_{(34)} = \infty$. Recalling that (51) holds, we see that

$$\infty > A_n^r := \int_0^\infty \min\left\{n, \left(\int_t^\infty U^{-q}w\right)^{\frac{r}{q}}\right\} \Theta^{\frac{r}{q'}}(t)\vartheta t\chi_{[0,n]}(t)\,\mathrm{d}t$$

for all $n \in \mathbb{N}$ and $A_n \uparrow \infty$ for $n \to \infty$. We can carry out the whole procedure above to find out that $C \gtrsim A_n \uparrow \infty$, hence $C = \infty$ as well. Therefore, the initial assumption of $C$ being finite already assures that $A_{(34)} < \infty$. Now, to get rid of the assumption of $V(\infty) = \infty$, if $V(\infty) < \infty$, which is the case that interests us, we consider a function $v_\varepsilon := v + \varepsilon$ for an $\varepsilon > 0$ and observe that $\|\cdot\|_{\Gamma_u^q(w)} \leq C\|\cdot\|_{\Lambda^p(v_\varepsilon)}$ holds, with $C$ still being equal to $\operatorname{Opt}(\Lambda^p(v), \Gamma_u^q(w))$. Therefore, proceeding just as before, we obtain that $C \gtrsim A_\varepsilon$, where $A_\varepsilon$ is given by (34) with $v$ replaced by $v_\varepsilon$. Fatou lemma (letting $\varepsilon \to 0+$) then yields $C \gtrsim A_{(34)}$. Finally we note that since $\Gamma_u^q(w) \hookrightarrow \Lambda^q(w)$, we have $\widetilde{A}_{(26)} = \operatorname{Opt}(\Lambda^p(v), \Lambda^q(w)) \leq \operatorname{Opt}(\Lambda^p(v), \Gamma_u^q(w)) = C$, hence we obtain the desired inequality $C \gtrsim A_{(34)} + \widetilde{A}_{(26)}$.

The remaining case $1 < q < p < \infty$ is treated by methods similar to those of (i), so we omit this part. The reader may as well refer to the particular part of [16, Theorem 1(b)], where the proof for $u \equiv 1$ is contained.

(iv) Fubini theorem yields $\Gamma_u^1(w) = \Lambda^1(z)$ where

$$z(x) = u(x)\int_x^\infty U^{-1}(t)w(t)\,\mathrm{d}t, \quad x > 0.$$

Using Fubini theorem again, we also get

$$Z(x) = W(x) + U(x)\int_x^\infty U^{-1}(t)w(t)\,\mathrm{d}t, \quad x > 0.$$

From Lemma 3.9(ii) it follows that $\Lambda^p(v) \hookrightarrow \Lambda^1(z)$ if and only if $A_{(35)} < \infty$ and then, moreover, $\operatorname{Opt}(\Lambda^p(v), \Lambda^1(z)) \simeq A_{(35)}$.

(v) First assume that (44) holds for all $f$ such that $f \in \mathscr{M}(R,\mu)$. Since

$$\sup_{0<s\le t} U(s)f^*(s) \le \int_0^t f^*(s)u(s)\,\mathrm{d}s, \quad t > 0,$$

it also holds that

$$\left(\int_0^\infty \left[\sup_{0<s\le t} U(s)f^*(s)\right]^q U^{-q}(t)w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty (f^*(t))^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \tag{52}$$

for all $f \in \mathscr{M}(R,\mu)$. In particular, (52) is satisfied for every function $f$ whose nonincreasing rearrangement can be represented as

$$f^*(t) = \left(\int_t^\infty h(s)\,\mathrm{d}s\right)^{\frac{1}{p}}$$

where $h$ is a nonnegative function on $(0,\infty)$. Fubini theorem then yields that

$$\left(\int_0^\infty \left[\sup_{0<s\le t} U^p(s)\int_s^\infty h(x)\,\mathrm{d}x\right]^{\frac{q}{p}} U^{-q}(t)w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty h(t)V(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \tag{53}$$

holds for every $h \in \mathscr{M}_+(0,\infty)$. We define the operator $Q\colon \mathscr{M}_+(0,\infty) \to \mathscr{M}_+(0,\infty)$ by

$$Qg(s) := \int_s^\infty U^{-1}(x)g(x)\,\mathrm{d}x, \quad g \in \mathscr{M}_+(0,\infty), s \in (0,\infty),$$

so, by substituting $h(x) \leftrightarrow U^{-1}(x)g(x)$ in (53), we obtain

$$\left(\int_0^\infty \left[\sup_{0<s\le t} U^p(s)Qg(s)\right]^{\frac{q}{p}} U^{-q}(t)w(t)\,\mathrm{d}t\right)^{\frac{p}{q}} \lesssim \int_0^\infty g(t)U^{-1}(t)V(t)\,\mathrm{d}t \tag{54}$$

for all $h \in \mathscr{M}_+(0,\infty)$. Carrying out subsequent substitutions, we get

$$\begin{aligned}
&\int_0^\infty \left[\sup_{0<s\le t} U^p(s)Qg(s)\right]^{\frac{q}{p}} U^{-q}(t)w(t)\,\mathrm{d}t = \\
= \quad &\int_0^\infty \left[\sup_{0<s\le t} U^p(s)\int_0^{\frac{1}{s}} g\left(\frac{1}{x}\right)U\left(\frac{1}{x}\right)x^{-2}\,\mathrm{d}x\right]^{\frac{q}{p}} U^{-q}(t)w(t)\,\mathrm{d}t = \\
= \quad &\int_0^\infty \left[\sup_{\frac{1}{t}\le s<\infty} U^p\left(\frac{1}{s}\right)\int_0^s g\left(\frac{1}{x}\right)U\left(\frac{1}{x}\right)x^{-2}\,\mathrm{d}x\right]^{\frac{q}{p}} U^{-q}(t)w(t)\,\mathrm{d}t = \\
= \quad &\int_0^\infty \left[\sup_{t\le s<\infty} U^p\left(\frac{1}{s}\right)\int_0^s g\left(\frac{1}{x}\right)U\left(\frac{1}{x}\right)x^{-2}\,\mathrm{d}x\right]^{\frac{q}{p}} U^{-q}\left(\frac{1}{t}\right)w\left(\frac{1}{t}\right)t^{-2}\,\mathrm{d}t.
\end{aligned}$$

Therefore, (54) yields

$$\left(\int_0^\infty \left[\sup_{t\le s<\infty} U^p\left(\frac{1}{s}\right)\int_0^s h(x)\,\mathrm{d}x\right]^{\frac{q}{p}} U^{-q}\left(\frac{1}{t}\right) w\left(\frac{1}{t}\right) t^{-2}\,\mathrm{d}t\right)^{\frac{p}{q}} \lesssim \int_0^\infty h(x)V\left(\frac{1}{x}\right)\mathrm{d}x \quad (55)$$

for all $g \in \mathscr{M}_+(0,\infty)$. At this point we will use [10, Theorem 4.4], with the following setting: $q = \frac{q}{p}$, $p = 1$, $u(t) = U^p(t)$, $w(t) = U^{-q}\left(\frac{1}{t}\right) w\left(\frac{1}{t}\right) t^{-2}$, $v(t) = V\left(\frac{1}{t}\right)$, hence $r = \frac{r}{p}$, $\overline{u}(t) = U^p\left(\frac{1}{t}\right)$ and $\sigma_p(0,t) = V^{-1}\left(\frac{1}{t}\right)$. (All the left-hand-side objects refer to those appearing in the theorem's statement, the right-hand-side ones are those we have been working with.) The theorem yields that (55) holds for all $g \in \mathscr{M}_+(0,\infty)$ if and only if $\widetilde{A}^p_{(26)}$ and $A_{(36)}$ are finite.

For the converse implication (sufficiency), suppose that $\widetilde{A}^p_{(26)}, A_{(36)} < \infty$. We need the inequality

$$\left(\int_0^\infty \left[U^{-1}(t)\int_0^t f^*(s)u(s)\,\mathrm{d}s\right]^q w(t)\,\mathrm{d}t\right)^{\frac{p}{q}} \lesssim \int_0^\infty f^*(t)v(x) \quad (56)$$

to hold for all $f \in \mathscr{M}(R,\mu)$. For every such $f$, as a particular case of [6, Theorem 3.2] we have

$$\left(\int_0^t (f^*(s))^{\frac{1}{p}} u(s)\,\mathrm{d}s\right)^p \le p\int_0^t f^*(s)U^{p-1}(s)u(s)\,\mathrm{d}s, \quad t>0.$$

Hence, to prove (56), we will show that

$$\left(\int_0^\infty \left[U^{-p}(t)\int_0^t f^*(s)U^{p-1}(s)u(s)\,\mathrm{d}s\right]^{\frac{q}{p}} w(t)\,\mathrm{d}t\right)^{\frac{p}{q}} \lesssim \int_0^\infty f^*(t)v(t)\,\mathrm{d}t \quad (57)$$

holds for all $f \in \mathscr{M}(R,\mu)$. Considering Stepanov's method (see the case (ii) for explanation), we know that it suffices to prove that (57) holds for every $f \in \mathscr{M}(R,\mu)$ for which there exists a function $h \in \mathscr{M}_+(0,\infty)$ such that $f^*(t) = \int_t^\infty h(s)\,\mathrm{d}s$ for every $t>0$. By Fubini theorem, for such pair $f, h$ we have

$$U^{-p}(t)\int_0^t f^*(s)U^{p-1}(s)u(s)\,\mathrm{d}s \simeq \int_t^\infty h(s)\,\mathrm{d}s + U^{-p}(t)\int_0^t U^p(s)h(s)\,\mathrm{d}s$$

and

$$\int_0^\infty f^*(t)v(t)\,\mathrm{d}s = \int_0^\infty h(t)V(t)\,\mathrm{d}t.$$

Thus, we obtain that (57) holds if

$$\left(\int_0^\infty \left[\int_t^\infty h(s)\,\mathrm{d}s\right]^{\frac{q}{p}} w(t)\,\mathrm{d}t\right)^{\frac{p}{q}} \lesssim \int_0^\infty h(t)V(t)\,\mathrm{d}t \quad (58)$$

and

$$\left(\int_0^\infty \left[U^{-p}(t)\int_0^t U^p(s)h(s)\,\mathrm{d}s\right]^{\frac{q}{p}} w(t)\,\mathrm{d}t\right)^{\frac{p}{q}} \lesssim \int_0^\infty h(t)V(t)\,\mathrm{d}t \quad (59)$$

hold for all $h \in \mathscr{M}_+(0,\infty)$. An application of [17, Theorem 3.3] (in the setting $f = hU$, $v = U^{-q}w$, $u = VU^{-1}$) and its analogue for $\int_t^\infty$ in place of $\int_0^t$ gives that (58) and (59) hold if and only if $\widetilde{A}^p_{(26)}, A_{(36)} < \infty$ which was our initial assumption. □

From now on, we will return to the Lorentz spaces of functions over a finite-measure domain. Thus, weights will be again defined just on $[0,1]$.

**Theorem 3.20** (The case $\Lambda^p(v) \overset{*}{\hookrightarrow} \Gamma^q_u(w)$). *Let $v, w$ be admissible weights and $p, q \in (0,\infty)$. Let $u$ be a positive integrable weight.*

(i) *Let $1 < p \le q < \infty$. Then $\Lambda^p(v) \overset{*}{\hookrightarrow} \Gamma^q_u(w)$ if and only if both*

$$\lim_{t\to 0+} \left( \int_t^1 U^{-q}(x) w(x)\,\mathrm{d}x \right)^{\frac{1}{q}} \left( \int_0^t U^{p'}(x) V^{-p'}(x) v(x)\,\mathrm{d}x \right)^{\frac{1}{p'}} = 0 \tag{60}$$

*and*

$$\lim_{t\to 0+} W^{\frac{1}{q}}(t) V^{-\frac{1}{p}}(t) = 0. \tag{61}$$

*hold true.*

(ii) *Let $0 < p \le 1,\ 0 < p \le q < \infty$. Then $\Lambda^p(v) \overset{*}{\hookrightarrow} \Gamma^q_u(w)$ if and only if both (61) and*

$$\lim_{t\to 0+} U(t) V^{-\frac{1}{p}}(t) \left( \int_t^1 U^{-q}(x) w(x)\,\mathrm{d}x \right)^{\frac{1}{q}} = 0 \tag{62}$$

*hold true.*

(iii) *Let $1 < p < \infty,\ 0 < q < p < \infty,\ q \ne 1$. Then $\Lambda^p(v) \overset{*}{\hookrightarrow} \Gamma^q_u(w)$ if and only if $A_{(26)} < \infty$,*

$$\int_0^1 \left( \int_t^1 U^{-q}(x) w(x)\,\mathrm{d}x \right)^{\frac{r}{q}} \times \\ \times \left( \int_0^t U^{p'}(x) V^{-p'}(x) v(x)\,\mathrm{d}x \right)^{\frac{r}{q'}} U^{p'}(t) V^{-p'}(t) v(t)\,\mathrm{d}t < \infty \tag{63}$$

*and (60) holds.*

(iv) *Let $1 = q < p < \infty$. Then $\Lambda^p(v) \overset{*}{\hookrightarrow} \Gamma^q_u(w)$ if and only if*

$$\int_0^\infty \left( W(t) + U(t) \int_t^\infty \frac{w(x)}{U(x)}\,\mathrm{d}x \right)^{p'-1} V^{1-p'}(t)\, u(t) \left( \int_t^\infty \frac{w(x)}{U(x)}\,\mathrm{d}x \right) \mathrm{d}t < \infty$$

*and (60) holds.*

(v) *Let $0 < q < p \le 1$. Then $\Lambda^p(v) \overset{*}{\hookrightarrow} \Gamma^q_u(w)$ if and only if $A_{(26)} < \infty$,*

$$\int_0^\infty \sup_{0<x\le t} U^r(x) V^{-\frac{r}{p}}(x) \left( \int_t^\infty U^{-q}(x) w(x)\,\mathrm{d}x \right)^{\frac{r}{p}} U^{-q}(t) w(t)\,\mathrm{d}t < \infty$$

*and (62) holds.*

*Proof.* (i) Let $s \in (0,1)$ be fixed. From Lemma 3.19(i) we have

$$\operatorname{Opt}(\Lambda^p(v), \Gamma_u^q(w\chi_{[0,s]})) \simeq$$
$$\simeq \sup_{0<t<s} \left(\int_t^s U(x)^{-q} w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \left(\int_0^t U(x)^{p'} V^{-p'}(x) v(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} + \sup_{0<t<s} W^{\frac{1}{q}}(t) V^{-\frac{1}{p}}(t).$$

Next, Lemma 3.8 (ii) provides

$$\begin{aligned}\operatorname{Opt}(\Lambda^p(v), \Lambda^1(u\chi_{[0,s]})) &= \left(\int_0^s U^{p'-1}(x) u(x) V^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} = \\ &= \left(\int_0^s U^{p'-1}(x) u(x) V_s^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}. \end{aligned} \tag{64}$$

Applying Proposition 3.10 (with $w := u\chi_{[0,s]}$ and $v := v\chi_{[0,s]}$) shows that

$$\left(\int_0^s U^{p'-1}(x) u(x) V_s^{1-p'}(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} \simeq \left(\int_0^s U^{p'}(x) V^{-p'}(x) v(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}. \tag{65}$$

From Theorem 3.6(iii) now follows the characterization in the form of the following conditions: (60),

$$\lim_{s\to 0+} \sup_{0<t<s} \left(\int_t^s U^{-q}(x) w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \left(\int_0^t U^{p'}(x) V^{-p'}(x) v(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} = 0 \tag{66}$$

and

$$\lim_{s\to 0+} \sup_{0<t<s} W^{\frac{1}{q}}(t) V^{-\frac{1}{p}}(t) = 0. \tag{67}$$

These conditions may be simplified: For $s \in (0,1)$ it holds

$$\sup_{0<t<s} \left(\int_t^s U^{-q}(x) w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \left(\int_0^t U^{p'}(x) V^{-p'}(x) v(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} \le$$
$$\le \sup_{0<t<s} \left(\int_t^1 U^{-q}(x) w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \left(\int_0^t U^{p'}(x) V^{-p'}(x) v(x)\,\mathrm{d}x\right)^{\frac{1}{p'}},$$

hence, following Remark 3.11, the condition (60) implies (66) and, furthermore, (67) is equivalent to (61).

(ii) Similarly as in (i), Theorem 3.6(iii), Lemma 3.19(ii) and Lemma 3.8(i) provide that $\Lambda^p(v) \overset{*}{\hookrightarrow} \Gamma_u^q(w)$ if and only if

$$\lim_{s\to 0+} \sup_{0<t<s} U(t) V^{-\frac{1}{p}}(t) \left(\int_t^s U^{-q}(x) w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} + \sup_{0<t<s} W^{\frac{1}{q}}(t) V^{-\frac{1}{p}}(t) = 0$$

and

$$\lim_{s\to 0+} \sup_{0<t<s} U(t) V^{-\frac{1}{p}}(t) \left(\int_s^1 U^{-q}(x) w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} = 0.$$

Using Remark 3.11, we deduce that these conditions together are equivalent to (61) and (62).

(iii) By Theorem 3.6(iii), Lemma 3.19(iii) and Lemma 3.8(ii) and having considered also (64) and (65), in this case we obtain the characterization by

$$\lim_{s\to 0+} \int_0^s W^{\frac{r}{p}}(x) V^{-\frac{r}{p}}(x) w(x)\,\mathrm{d}x = 0,$$

$$\lim_{s\to 0+} \int_0^s \left(\int_t^s U^{-q} w\right)^{\frac{r}{q}} \left(\int_0^t U^{p'} V^{-p'} v\right)^{\frac{r}{q'}} U^{p'}(t) V^{-p'}(t) v(t)\,\mathrm{d}t = 0$$

and (60). As usual, the first two conditions are implied by (63) and the fact that $A_{(26)} < \infty$, thanks to the absolute continuity of the integral. The converse implication is also true (a simple reason is given by Proposition 3.4).

The remaining cases (iv) and (v) are analogous. □

**Lemma 3.21.** *Let $v, w$ be admissible weights and $p, q \in (0,\infty)$. Let $u$ be a positive integrable weight.*

(i) *Let $0 < p \le 1$. Then $\Lambda^p(v) \hookrightarrow \Gamma_u^{q,\infty}(w)$ if and only if $A_{(68)} < \infty$ where*

$$A_{(68)} := \sup_{0<\tau<t<1} U(\tau) U^{-1}(t) W^{\frac{1}{q}}(t) V^{-\frac{1}{p}}(\tau). \tag{68}$$

*Moreover,* $\operatorname{Opt}(\Lambda^p(v), \Gamma_u^{q,\infty}(w)) = A_{(68)}$.

(ii) *Let $1 < p < \infty$. Then $\Lambda^p(v) \hookrightarrow \Gamma_u^{q,\infty}(w)$ if and only if $A_{(69)} < \infty$ where*

$$A_{(69)} := \sup_{0<t<1} U^{-1}(t) W^{\frac{1}{q}}(t) \left(\int_0^t U^{p'-1}(x) V^{1-p'}(x) u(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}. \tag{69}$$

*Moreover,* $\operatorname{Opt}(\Lambda^p(v), \Gamma_u^{q,\infty}(w)) \simeq A_{(69)}$.

*Proof.* Both cases follow from Proposition 3.5(ii) and Lemma 3.8. □

**Theorem 3.22** (The case $\Lambda^p(v) \overset{*}{\hookrightarrow} \Gamma_u^{q,\infty}(w)$)**.** *Let $v, w$ be admissible weights and $p, q \in (0,\infty)$. Let $u$ be a positive integrable weight.*

(i) *Let $0 < p \le 1$. Then $\Lambda^p(v) \overset{*}{\hookrightarrow} \Gamma_u^{q,\infty}(w)$ if and only if*

$$\lim_{s\to 0+} \sup_{\substack{0<\tau<s \\ \tau\le t<1}} U(\tau) U^{-1}(t) W^{\frac{1}{q}}(t) V^{-\frac{1}{p}}(\tau) = 0. \tag{70}$$

(ii) *Let $1 < p < \infty$. Then $\Lambda^p(v) \overset{*}{\hookrightarrow} \Gamma_u^{q,\infty}(w)$ if and only if*

$$\lim_{s\to 0+} \sup_{0<t<1} U^{-1}(t) W^{\frac{1}{q}}(t) \left(\int_0^{\min\{t,s\}} U^{p'-1}(x) V^{1-p'}(x) u(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} = 0. \tag{71}$$

*Proof.* (i) By Theorem 3.6(iv), the characterization of $\Lambda^p(v) \overset{*}{\hookrightarrow} \Gamma_u^{q,\infty}(w)$ is given by (17) and (18) with $X := \Lambda^p(v)$. We begin with the necessity of (70). By Lemma 3.8(i), (18) occurs if and only if

$$\lim_{s\to 0+} \sup_{0<\tau<s<t<1} U(\tau)U^{-1}(t)W^{\frac{1}{q}}(t)V^{-\frac{1}{p}}(\tau) = 0. \tag{72}$$

By Lemma 3.21, (17) holds if and only if

$$\lim_{s\to 0+} \sup_{0<\tau<t<1} U(\tau)U^{-1}(t)W_s^{\frac{1}{q}}(t)V^{-\frac{1}{p}}(\tau) = 0. \tag{73}$$

Obviously, (73) implies

$$\lim_{s\to 0+} \sup_{0<\tau<t<s} U(\tau)U^{-1}(t)W^{\frac{1}{q}}(t)V^{-\frac{1}{p}}(\tau) = 0 \tag{74}$$

which together with (72) gives (70).

As for the sufficiency, it remains to show (70) implies (73). Assume (70) holds. Thus, there exists $s \in (0,1)$ such that

$$\sup_{\substack{0<\tau<s\\ \tau<t<1}} U(\tau)U^{-1}(t)W^{\frac{1}{q}}(t)V^{-\frac{1}{p}}(\tau) < \infty. \tag{75}$$

Take this fixed $s$. The functions $u, v, w$ are admissible, thus there are constants $a, b > 0$ such that

$$a < \min\{U(t), V(t), W(t)\} \le \max\{U(t), V(t), W(t)\} < b$$

for all $t \in (s,1]$, therefore

$$\sup_{s<\tau<t<1} U(\tau)U^{-1}(t)W^{\frac{1}{q}}(t)V^{-\frac{1}{p}}(\tau) < \infty. \tag{76}$$

After combining (75) with (76), we observe that $A_{(68)}$ is finite. (This could be also obtained from Proposition 3.4.) Now take an arbitrary $\varepsilon \in (0, A_{(68)}]$. The function $W$ is nonnegative, nondecreasing, continuous and $\lim_{s\to 0+} W(s) = 0$. Hence, we can find an $S \in (0,1]$ such that $W^{\frac{1}{q}}(S)W^{-\frac{1}{q}}(1) \le \varepsilon A_{(68)}^{-1}$. Now we observe that for every $s \in (0,S)$ there exists a $\delta = \delta(\varepsilon, s) > 0$ such that for every $t \in [\delta, 1]$ it holds $W_s(t)W^{-1}(t) \le \varepsilon A_{(68)}^{-1}$ and $\delta$ is the least number with this property. Thus, for every fixed $s \in (0,S)$ it holds

$$\begin{aligned}
\sup_{0<\tau<t<1} \frac{U(\tau)W_s^{\frac{1}{q}}(t)}{U(t)V^{\frac{1}{p}}(\tau)} &= \max\left\{ \sup_{0<\tau<t<\delta} \frac{U(\tau)W_s^{\frac{1}{q}}(t)}{U(t)V^{\frac{1}{p}}(\tau)};\ \sup_{\substack{0<\tau<t<1\\ \delta\le t}} \frac{U(\tau)W_s^{\frac{1}{q}}(t)}{U(t)V^{\frac{1}{p}}(\tau)} \right\} = \\
&= \max\left\{ \sup_{0<\tau<t<\delta} \frac{U(\tau)W_s^{\frac{1}{q}}(t)}{U(t)V^{\frac{1}{p}}(\tau)};\ \sup_{\substack{0<\tau<t<1\\ \delta\le t}} \frac{U(\tau)W_s^{\frac{1}{q}}(t)W^{\frac{1}{q}}(t)}{U(t)W^{\frac{1}{q}}(t)V^{\frac{1}{p}}(\tau)} \right\} \le \\
&\le \max\left\{ \sup_{0<\tau<t<\delta} \frac{U(\tau)W_s^{\frac{1}{q}}(t)}{U(t)V^{\frac{1}{p}}(\tau)};\ \varepsilon A_{(68)}^{-1} \sup_{\substack{0<\tau<t<1\\ \delta\le t}} \frac{U(\tau)W^{\frac{1}{q}}(t)}{U(t)V^{\frac{1}{p}}(\tau)} \right\} \le \\
&\le \max\left\{ \sup_{0<\tau<t<\delta} \frac{U(\tau)W_s^{\frac{1}{q}}(t)}{U(t)V^{\frac{1}{p}}(\tau)};\ \varepsilon \right\} = \max\{a(s), \varepsilon\},
\end{aligned}$$

say. Moreover, we observe that $\delta(\varepsilon,s)$ vanishes as $s\to 0+$. Therefore, since $W_s \le W$, from (70) we obtain that $a(s)\to 0$ for $s\to 0+$ as well. Hence, for a fixed $\varepsilon>0$ we were able to find an $s$ such that

$$\sup_{0<\tau<t<1} U(\tau)W_s^{\frac{1}{q}}(t)U^{-1}(t)V^{-\frac{1}{p}}(\tau)\le\varepsilon.$$

This implies the validity of (73) and the proof of this part is complete.

(ii) We just proceed similarly as we did in (i). Again, we use Theorem 3.6(iv) with $X:=\Lambda^p(v)$. So, according to Lemma 3.8(ii), we see that (18) is equivalent to

$$\lim_{s\to 0+}\ \sup_{s\le t<1}\ U^{-1}(t)W^{\frac{1}{q}}(t)\left(\int_0^s U^{p'-1}(x)V^{1-p'}(x)u(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}=0, \tag{77}$$

while (17) is rewritten as

$$\lim_{s\to 0+}\ \sup_{0<t<1}\ U^{-1}(t)W_s^{\frac{1}{q}}(t)\left(\int_0^t U^{p'-1}(x)V^{1-p'}(x)u(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}=0. \tag{78}$$

The condition (71) is a combination of (77) and

$$\lim_{s\to 0+}\ \sup_{0<t<s}\ U^{-1}(t)W^{\frac{1}{q}}(t)\left(\int_0^t U^{p'-1}(x)V^{1-p'}(x)u(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}=0, \tag{79}$$

hence it remains to show that (79) implies (78). So, assume that (79) holds. It is no problem to show that $A_{(69)}<\infty$. Consider a fixed $\varepsilon\in(0,A_{(69)}]$. Again, we find $S\in(0,1]$ such that $W^{\frac{1}{q}}(S)W^{-\frac{1}{q}}(1)\le\varepsilon A_{(69)}^{-1}$. Then for every $s\in(0,S)$ exists the least $\delta(\varepsilon,s)>0$ such that for every $t\in[\delta,1]$ it holds $W_s(t)W^{-1}(t)\le\varepsilon A_{(69)}^{-1}$. For every fixed $s\in(0,S)$ and every $t\in(0,1)$ it holds

$$\begin{aligned}&U^{-1}(t)W_s^{\frac{1}{q}}(t)\left(\int_0^t U^{p'-1}(x)V^{1-p'}(x)u(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}\le\\ &\qquad\max\left\{\sup_{0<t<\delta(\varepsilon,s)} U^{-1}(t)W_s^{\frac{1}{q}}(t)\left(\int_0^t U^{p'-1}(x)V^{1-p'}(x)u(x)\,\mathrm{d}x\right)^{\frac{1}{p'}};\ \varepsilon\right\}\end{aligned}$$

which yields (78). Details are omitted in here as they can be found in part (i). □

**Lemma 3.23.** *Let $v,w$ be admissible weights and $p,q\in(0,\infty)$. Let $u$ be a positive integrable weight. Then $\Lambda^{p,\infty}(v)\hookrightarrow\Gamma_u^q(w)$ if and only if $A_{(80)}<\infty$, where*

$$A_{(80)}:=\left(\int_0^1\left[\int_0^t V^{-\frac{1}{p}}(x)u(x)\,\mathrm{d}x\right]^q U^{-q}(t)w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}. \tag{80}$$

*Moreover,* $\operatorname{Opt}(\Lambda^{p,\infty}(v),\Gamma_u^q(w))\simeq A_{(80)}$.

*Proof.* The result follows from Proposition 3.5(i). □

**Theorem 3.24** (The case $\Lambda^{p,\infty}(v) \overset{*}{\hookrightarrow} \Gamma^q_u(w)$)**.** *Let $v, w$ be admissible weights and $p, q \in (0,\infty)$. Let $u$ be a positive integrable weight. Then $\Lambda^{p,\infty}(v) \overset{*}{\hookrightarrow} \Gamma^q_u(w)$ if and only if*

$$\lim_{s\to 0+} \int_0^1 \left( \int_0^{\min\{t,s\}} V^{-\frac{1}{p}}(x)u(x)\,\mathrm{d}x \right)^q U^{-q}(t)w(t)\,\mathrm{d}t = 0. \tag{81}$$

*Proof.* By Theorem 3.6(iii) and Lemmas 3.15 and 3.23 we get the characterization by

$$\lim_{s\to 0+} \int_0^s \left( \int_0^t V^{-\frac{1}{p}}(x)u(x)\,\mathrm{d}x \right)^q U^{-q}(t)w(t)\,\mathrm{d}t = 0 \tag{82}$$

and

$$\lim_{s\to 0+} \left( \int_s^1 U^{-q}(t)w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} \int_0^s V^{-\frac{1}{p}}(x)u(x)\,\mathrm{d}x = 0.$$

The latter condition can be obviously rewritten by

$$\lim_{s\to 0+} \int_s^1 \left( \int_0^s V^{-\frac{1}{p}}(x)u(x)\,\mathrm{d}x \right)^q U^{-q}(t)w(t)\,\mathrm{d}t = 0.$$

A combination of this condition and (82) is (81). □

**Lemma 3.25.** *Let $v, w$ be admissible weights and $p, q \in (0,\infty)$. Let $u$ be a positive integrable weight. Then $\Lambda^{p,\infty}(v) \hookrightarrow \Gamma^{q,\infty}_u(w)$ if and only if $A_{(83)} < \infty$, where*

$$A_{(83)} := \sup_{0<t<1} U^{-1}(t)W^{\frac{1}{q}}(t) \int_0^t V^{-\frac{1}{p}}(x)u(x)\,\mathrm{d}x. \tag{83}$$

*Moreover,* $\operatorname{Opt}(\Lambda^{p,\infty}(v), \Gamma^{q,\infty}_u(w)) = A_{(83)}$.

*Proof.* See Proposition 3.5(i). □

**Theorem 3.26** (The case $\Lambda^{p,\infty}(v) \overset{*}{\hookrightarrow} \Gamma^{q,\infty}_u(w)$)**.** *Let $v, w$ be admissible weights and $p, q \in (0,\infty)$. Let $u$ be a positive integrable weight. Then $\Lambda^{p,\infty}(v) \overset{*}{\hookrightarrow} \Gamma^{q,\infty}_u(w)$ if and only if*

$$\lim_{s\to 0+} \sup_{0<t<1} U^{-1}(t)W^{\frac{1}{q}}(t) \int_0^{\min\{t,s\}} V^{-\frac{1}{p}}(x)u(x)\,\mathrm{d}x = 0.$$

*Proof.* From Theorem 3.6(iv) and Lemmas 3.15 and 3.25 we have the conditions

$$\lim_{s\to 0+} \sup_{s<t<1} U^{-1}(t)W^{\frac{1}{q}}(t) \int_0^s V^{-\frac{1}{p}}(x)u(x)\,\mathrm{d}x = 0$$

and

$$\lim_{s\to 0+} \sup_{0<t<1} U^{-1}(t)W_s^{\frac{1}{q}}(t) \int_0^t V^{-\frac{1}{p}}(x)u(x)\,\mathrm{d}x = 0.$$

Since the function $U^{-1}(t)\int_0^t V^{-\frac{1}{p}}(x)u(x)\,\mathrm{d}x = (V^{-\frac{1}{p}})^{**}_u(t)$ is nonincreasing in $t$, the latter expression is rewritten by

$$\lim_{s\to 0+} \sup_{0<t\le s} U^{-1}(t)W^{\frac{1}{q}}(t) \int_0^t V^{-\frac{1}{p}}(x)u(x)\,\mathrm{d}x = 0$$

and the rest is obvious. □

### 3.3 Embeddings of type $\Gamma \overset{*}{\hookrightarrow} \Lambda$

**Definition 3.27.** Let $\varphi: [0,1] \to [0,\infty)$ be a continuous strictly increasing function such that $\varphi(0) = 0$. Let $\nu$ be a nonnegative Borel measure on $[0,1]$. Then $\nu$ is called *nondegenerate with respect to* $\varphi$ if the following conditions are satisfied for every $t \in (0,1)$:

$$\int_{[0,1]} \frac{\mathrm{d}\nu(s)}{\varphi(s)+\varphi(t)} < \infty, \qquad \int_{[0,1]} \frac{\mathrm{d}\nu(s)}{\varphi(s)} = \infty. \tag{84}$$

In the following, the symbol $\mathrm{d}(U^p(t))$ denotes the Lebesgue-Stieltjes integration.

**Lemma 3.28.** *Let $v,w$ be admissible weights. Let $u$ be a positive integrable weight and $0 < p,q < \infty$. When $p > q$, we set $r = \frac{pq}{p-q}$. Assume that the measure $v(t)\,\mathrm{d}t$ is nondegenerate with respect to $U^p$.*

(i) *If $0 < p \le q < \infty$ and $1 \le q < \infty$, then $\Gamma^p_u(v) \hookrightarrow \Lambda^q(w)$ if and only if $A_{(85)} < \infty$ where*

$$A_{(85)} := \sup_{0<t<1} \frac{W^{\frac{1}{q}}(t)}{\left(V(t) + U^p(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}}. \tag{85}$$

(ii) *If $1 \le q < p < \infty$, then $\Gamma^p_u(v) \hookrightarrow \Lambda^q(w)$ if and only if $A_{(86)} < \infty$ where*

$$A_{(86)} := \left( \int_0^1 \frac{U^r(t)\left[\sup_{y\in[t,1]} U^{-r}(y)W^{\frac{r}{q}}(y)\right]}{\left(V(t) + U^p(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{r}{p}+2}} \times \right.$$
$$\left. \times\, V(t)\left(\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right) \mathrm{d}(U^p(t)) \right)^{\frac{1}{r}}. \tag{86}$$

(iii) *If $0 < p \le q < 1$, then $\Gamma^p_u(v) \hookrightarrow \Lambda^q(w)$ if and only if $A_{(87)} < \infty$ where*

$$A_{(87)} := \sup_{0<t<1} \frac{W^{\frac{1}{q}}(t) + U(t)\left(\int_t^1 U^{q'}(x)W^{-q'}(x)w(x)\,\mathrm{d}x\right)^{-\frac{1}{q'}}}{\left(V(t) + U^p(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}}. \tag{87}$$

(iv) *If $0 < q < 1$ and $0 < q < p$, then $\Gamma^p_u(v) \hookrightarrow \Lambda^q(w)$ if and only if $A_{(88)} < \infty$ where*

$$A_{(88)} := \left( \int_0^1 \frac{\left[W^{1-q'}(t) + U^{-q'}(t)\int_t^1 U^{q'}(x)W^{-q'}(x)w(x)\,\mathrm{d}x\right]^{-\frac{r}{q'}-1}}{\left(V(t) + U^p(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{r}{p}}} \times \right.$$
$$\left. \times\, W^{-q'}(t)w(t)\,\mathrm{d}t \right)^{\frac{1}{r}}. \tag{88}$$

*Moreover, in each this case, for the appropriate $A \in \{A_{(85)}, A_{(86)}, A_{(87)}, A_{(88)}\}$ it holds $\operatorname{Opt}(\Gamma^p_u(v), \Lambda^q(w)) \simeq A$.*

*Proof.* This result follows from [11, Theorem 4.2]. However, that theorem in the original paper contains some stronger restrictions on $v$ since it is stated for spaces over $[0,\infty)$. Anyway, if we consider spaces over $[0,1]$, the proof of the original theorem and all the supporting results can be re-done based just on our setup of Lemma 3.28. In fact, we just ignore the parts of the proof which cover the interval $(1,\infty)$. We omit the details as they contain no new ideas compared to [11]. $\square$

**Theorem 3.29** (The case $\Gamma^p_u(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$)**.** *Let $v, w$ be admissible weights. Let $u$ be a positive integrable weight and $0 < p, q < \infty$. When $p > q$, we set $r = \frac{pq}{p-q}$. Assume that the measure $v(t)\,\mathrm{d}t$ is nondegenerate with respect to $U^p$.*

(i) *If $0 < p \le q < \infty$ and $1 \le q < \infty$, then $\Gamma^p_u(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$ if and only if*

$$\lim_{t\to 0+} \frac{W^{\frac{1}{q}}(t)}{\left(V(t) + U^p(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}} = 0. \tag{89}$$

(ii) *If $1 \le q < p < \infty$, then $\Gamma^p_u(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$ if and only if*

$$\int_0^1 \frac{U^r(t)\left[\sup_{y\in[t,1]} U^{-r}(y)W^{\frac{r}{q}}(y)\right] V(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x}{\left(V(t) + U^p(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{r}{p}+2}}\,\mathrm{d}(U^p(t)) < \infty. \tag{90}$$

(iii) *If $0 < p \le q < 1$, then $\Gamma^p_u(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$ if and only if*

$$\lim_{t\to 0+} \frac{W^{\frac{1}{q}}(t) + U(t)\left(\int_t^1 U^{q'}(x)W^{-q'}(x)w(x)\,\mathrm{d}x\right)^{-\frac{1}{q'}}}{\left(V(t) + U^p(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}} = 0.$$

(iv) *If $0 < q < 1$ and $0 < q < p$, then $\Gamma^p_u(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$ if and only if*

$$\int_0^1 \frac{\left[W^{1-q'}(t) + U^{-q'}(t)\int_t^1 U^{q'}(x)W^{-q'}(x)w(x)\,\mathrm{d}x\right]^{-\frac{r}{q'}-1} W^{-q'}(t)w(t)}{\left(V(t) + U^p(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{r}{p}}}\,\mathrm{d}t < \infty.$$

*Proof.* (i) Theorem 3.6 and Lemma 3.28 imply that $\Gamma^p_u(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$ if and only if

$$\lim_{s\to 0+} \sup_{0<t<1} \frac{W_s^{\frac{1}{q}}(t)}{\left(V(t) + U^p(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}}. \tag{91}$$

One can easily check that for $t \in (0,1)$,

$$\left(V(t) + U^p(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}} = \varphi_{\Gamma^p_u(v)}(t), \tag{92}$$

where $\varphi_{\Gamma^p_u(v)}$ denotes the fundamental function of $\Gamma^p_u(v)$ (see Definition 2.14). Thus, the denominator in (91) is nondecreasing in $t$, hence (91) is equivalent to (89).

(ii) Let us denote

$$a(t) := \frac{V(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x}{\left(V(t)+U^p(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{r}{p}+2}}, \quad t\in(0,1).$$

Theorem 3.6(i) provides that $\Gamma^p_u(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$ if and only if

$$\lim_{s\to 0+}\int_0^1 U^r(t)\Big[\sup_{y\in[t,1]} U^{-r}(y)W_s^{\frac{r}{q}}(y)\Big]a(t)\,\mathrm{d}(U^p(t)) = 0. \tag{93}$$

It is easy to show that (93) implies (90) (either directly or by Proposition 3.4). Now we prove the converse, so we assume (90) and proceed to show (93). For a fixed $s\in(0,1)$ we have $W_s(t)=W(s)$, $t\in[s,1)$, hence

$$\begin{aligned}\int_0^1 U^r(t)\Big[\sup_{y\in[t,1]} U^{-r}(y)W_s^{\frac{r}{q}}(y)\Big]a(t)\,\mathrm{d}(U^p(t)) &=\\ = \int_0^s U^r(t)\Big[\sup_{y\in[t,1]} U^{-r}(y)W_s^{\frac{r}{q}}(y)\Big]a(t)\,\mathrm{d}(U^p(t)) &+\\ + \int_s^1 U^r(t)\Big[\sup_{y\in[t,1]} U^{-r}(y)W^{\frac{r}{q}}(s)\Big]a(t)\,\mathrm{d}(U^p(t))&\end{aligned}$$

and this is equal to

$$\begin{aligned}\int_0^1 U^r(t)\Big[\sup_{y\in[t,1]} U^{-r}(y)W_s^{\frac{r}{q}}(y)\Big]a(t)\chi_{[0,s]}(t)\,\mathrm{d}(U^p(t))&+\\ + \int_0^1 W^{\frac{r}{q}}(s)\,a(t)\chi_{(s,1]}(t)\,\mathrm{d}(U^p(t)).&\end{aligned} \tag{94}$$

We have also used the fact that $U^{-r}$ is decreasing. We observe that the integrands of the both parts of (94) are estimated from above by the integrand of (90), therefore the former summand converges to zero for $s\to 0+$ following the absolute continuity of the integral and the latter one does the same thanks to the Lebesgue dominated convergence theorem, because $W^{\frac{r}{q}}(s) \xrightarrow{s\to 0+} 0$. Hence, (93) is satisfied.

(iii) It suffices to realize that $U(t)\left(\int_t^1 U^{q'}(x)W_s^{-q'}(x)w(x)\chi_{[0,s]}(x)\,\mathrm{d}x\right)^{-\frac{1}{q'}} = 0$ for $t>s$ (remember that $q'<0$) and the rest is done in the same way as (i).

(iv) We use Theorem 3.6(i), Lemma 3.28 and the absolute continuity of the integral. □

**Lemma 3.30.** *Let $v, w$ be admissible weights. Let $u$ be a positive integrable weight and $0<p,q<\infty$. Then $\Gamma^p_u(v)\hookrightarrow\Lambda^{q,\infty}(w)$ if and only if $A_{(85)}<\infty$. Moreover,* $\operatorname{Opt}(\Gamma^p_u(v),\Lambda^{q,\infty}(w)) = A_{(85)}$.

*Proof.* The lemma is a direct consequence of Proposition 3.5(iii). □

**Theorem 3.31** (The case $\Gamma_u^p(v) \overset{*}{\hookrightarrow} \Lambda^{q,\infty}(w)$)**.** *Let $v, w$ be admissible weights. Let $u$ be a positive integrable weight. Suppose that $0 < p, q < \infty$. Then $\Gamma_u^p(v) \overset{*}{\hookrightarrow} \Lambda^{q,\infty}(w)$ if and only if* (89) *holds.*

*Proof.* The result follows by using Theorem 3.6(ii), Lemma 3.30 and the monotonicity of the fundamental function. The procedure is the same as that of the proof of Theorem 3.29(i). □

For details of the objects appearing in the following lemma, see [12, Lemma 1.5, Theorem 1.8] and [11, Remark 2.10(ii)].

**Lemma 3.32.** *Let $v, w$ be admissible weights, let $u$ be a positive integrable weight. Let $0 < p, q < \infty$. Let the function $\varphi$, defined by*

$$\varphi(t) := U(t) \sup_{x\in(t,1)} U^{-1}(x)V^{\frac{1}{p}}(x), \quad t \in (0,1),$$

*satisfy*

$$\lim_{t\to0+} \varphi(t) = \lim_{t\to0+} \inf_{x\in(t,1)} U(x)V^{-\frac{1}{p}}(x) = 0.$$

*Let $\nu$ be the representation measure of $U^q\varphi^{-q}$ with respect to $U^q$, i.e. a nonnegative Borel measure such that*

$$\inf_{x\in(t,1)} U^q(x)V^{-\frac{q}{p}}(x) = U^q(t) \int_{[0,1]} \frac{\mathrm{d}\nu(x)}{U^q(x) + U^q(t)}, \quad t \in (0,1).$$

(i) *If $1 \le q < \infty$ then $\Gamma_u^{p,\infty}(v) \hookrightarrow \Lambda^q(w)$ if and only if $A_{(95)} < \infty$ where*

$$A_{(95)} := \left( \int_0^1 \sup_{x\in(t,1)} W(x)U^{-q}(x)\,\mathrm{d}\nu(t) \right)^{\frac{1}{q}}. \tag{95}$$

*Moreover,* $\operatorname{Opt}(\Gamma_u^{p,\infty}(v), \Lambda^q(w)) \simeq A_{(95)}$.

(ii) *If $0 < q < 1$ then $\Gamma_u^{p,\infty}(v) \hookrightarrow \Lambda^q(w)$ if and only if $A_{(96)} < \infty$ where*

$$A_{(96)} := \left( \int_0^1 U^{-q}(t)W(t) + \left[ \int_t^1 U^{q'}(x)W^{q'}(x)w(x)\,\mathrm{d}x \right]^{1-q} \mathrm{d}\nu(t) \right)^{\frac{1}{q}} < \infty. \tag{96}$$

*Moreover,* $\operatorname{Opt}(\Gamma_u^{p,\infty}(v), \Lambda^q(w)) \simeq A_{(96)}$.

*Proof.* The result is proved in [12, Theorem 1.8] for spaces over a domain with infinite measure. The construction can be however restricted to a finite-measure domain as well (see the comments in the proof of Lemma 3.28). □

**Theorem 3.33** (The case $\Gamma_u^{p,\infty}(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$)**.** *Assume that $u, v, w, p, q, \varphi, \nu$ are as in Lemma 3.32. Recall that we denote $W_s(t) = \int_0^t w(x)\chi_{[0,s]}(x)\,\mathrm{d}x$ for $s, t \in (0,1]$.*

(i) *If $1 \le q < \infty$ then $\Gamma_u^{p,\infty}(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$ if and only if*

$$\lim_{s\to0+} \int_0^1 \sup_{x\in(t,1)} W_s(x)U^{-q}(x)\,\mathrm{d}\nu(t) = 0.$$

(ii) *If* $0 < q < 1$ *then* $\Gamma_u^{p,\infty}(v) \overset{*}{\hookrightarrow} \Lambda^q(w)$ *if and only if*

$$\lim_{s\to 0+} \int_0^1 U^{-q}(t)W_s(t) + \left[\int_t^1 U^{q'}(x)W^{q'}(x)w(x)\chi_{[0,s]}(x)\,\mathrm{d}x\right]^{1-q} \mathrm{d}\nu(t) = 0.$$

*Proof.* The theorem follows from Theorem 3.6(ii) and Lemma 3.32. □

**Lemma 3.34.** *Let* $v, w$ *be admissible weights. Let* $u$ *be a positive integrable weight and* $0 < p, q < \infty$. *Then* $\Gamma_u^{p,\infty}(v) \hookrightarrow \Lambda^{q,\infty}(w)$ *if and only if* $A_{(97)} < \infty$ *where*

$$A_{(97)} := \sup_{0<t<1} W^{\frac{1}{q}}(t)U^{-1}(t)\left(\sup_{t<x<1} U^{-1}(x)V^{\frac{1}{p}}(x)\right)^{-1}. \tag{97}$$

*Moreover,* $\operatorname{Opt}(\Gamma_u^{p,\infty}(v), \Lambda^{q,\infty}(w)) = A_{(97)}$.

*Proof.* The assertion follows by Proposition 3.5(iii) since, for $t \in (0,1)$,

$$\begin{aligned} \varphi_{\Gamma_u^{p,\infty}(v)}(t) &= \max\left\{\sup_{0<x<t} V^{\frac{1}{p}}(x); \sup_{t<x<1} U(t)U^{-1}(x)V^{\frac{1}{p}}(x)\right\} = \\ &= U(t)\sup_{t<x<1} U^{-1}(x)V^{\frac{1}{p}}(x). \end{aligned} \tag{98}$$

□

**Theorem 3.35** (The case $\Gamma_u^{p,\infty}(v) \overset{*}{\hookrightarrow} \Lambda^{q,\infty}(w)$). *Let* $v, w$ *be admissible weights. Let* $u$ *be a positive integrable weight and* $0 < p, q < \infty$. *Then* $\Gamma_u^{p,\infty}(v) \overset{*}{\hookrightarrow} \Lambda^{q,\infty}(w)$ *if and only if*

$$\lim_{s\to 0+} W^{\frac{1}{q}}(t)U^{-1}(t)\left(\sup_{t<x<1} U^{-1}(x)V^{\frac{1}{p}}(x)\right)^{-1} = 0. \tag{99}$$

*Proof.* Applying Theorem 3.6(ii) and Lemma 3.34, we obtain the characterization by

$$\lim_{s\to 0+}\sup_{0<t<1} W_s^{\frac{1}{q}}(t)U^{-1}(t)\left(\sup_{t<x<1} U^{-1}(x)V^{\frac{1}{p}}(x)\right)^{-1} = 0.$$

Equivalence of the previous and (99) is then, as usual, granted by the monotonicity of the fundamental function (see (98)) and Remark 3.11. □

### 3.4 Embeddings of type $\Gamma \overset{*}{\hookrightarrow} \Gamma$

For the definition of nondegeneracy of a measure, which term appears in the following, see Definition 3.27.

**Lemma 3.36.** *Let $v, w$ be admissible weights, let $u$ be a positive integrable weight. Let $0 < p, q < \infty$. When $p > q$, we set $r = \frac{pq}{p-q}$. Assume that the measure $v(t)\,\mathrm{d}t$ is nondegenerate with respect to $U^p$.*

(i) *If $0 < p \le q < \infty$, then $\Gamma^p_u(v) \hookrightarrow \Gamma^q_u(w)$ if and only if $A_{(100)} < \infty$ where*

$$A_{(100)} := \sup_{0<t<1} \frac{\left(W(t) + U^q(t)\int_t^1 U^{-q}(x)w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}}{\left(V(t) + U^p(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}}. \tag{100}$$

*Moreover,* $\operatorname{Opt}\left(\Gamma^p_u(v), \Gamma^q_u(w)\right) \simeq A_{(100)}$.

(ii) *If $0 \le q < p < \infty$, then $\Gamma^p_u(v) \hookrightarrow \Gamma^q_u(w)$ if and only if $A_{(101)} < \infty$ where*

$$A_{(101)} := \left(\int_0^1 \frac{\left(W(t) + U^q(t)\int_t^1 U^{-q}(x)w(x)\,\mathrm{d}x\right)^{\frac{r}{q}-1} w(t)}{\left(V(t) + U^p(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{r}{p}}}\,\mathrm{d}t\right)^{\frac{1}{p}}. \tag{101}$$

*Moreover,* $\operatorname{Opt}\left(\Gamma^p_u(v), \Gamma^q_u(w)\right) \simeq A_{(101)}$.

*Proof.* See [11, Theorem 5.1] and the comments in the proof of Lemma 3.28. □

**Theorem 3.37** ($\Gamma^p_u(v) \overset{*}{\hookrightarrow} \Gamma^q_u(w)$)**.** *Let $v, w$ be admissible weights, let $u$ be a positive integrable weight. Let $0 < p, q < \infty$. When $p > q$, we set $r = \frac{pq}{p-q}$. Assume that the measure $v(t)\,\mathrm{d}t$ is nondegenerate with respect to $U^p$.*

(i) *If $0 < p \le q < \infty$, then $\Gamma^p_u(v) \overset{*}{\hookrightarrow} \Gamma^q_u(w)$ if and only if*

$$\lim_{t\to 0+} \frac{\left(W(t) + U^q(t)\int_t^1 U^{-q}(x)w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}}{\left(V(t) + U^p(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}} = 0. \tag{102}$$

(ii) *If $0 \le q < p < \infty$, then $\Gamma^p_u(v) \overset{*}{\hookrightarrow} \Gamma^q_u(w)$ if and only if both*

$$\int_0^1 \frac{\left(W(t) + U^q(t)\int_t^1 U^{-q}(x)w(x)\,\mathrm{d}x\right)^{\frac{r}{q}-1} w(t)}{\left(V(t) + U^p(t)\int_t^1 U^{-p}(x)v(x)\,\mathrm{d}x\right)^{\frac{r}{p}}}\,\mathrm{d}t < \infty \tag{103}$$

*and* (102) *hold true.*

*Proof.* Let us prove (ii). Necessity of (103) follows from Proposition 3.4 and Lemma 3.36(ii). We recall the form of $\varphi_{\Gamma^p_u(v)}$ from (92). We may get an analogous expression for $\varphi_{\Gamma^q_u(w)}$. Hence, necessity of (102) follows from Proposition 2.16.

As for the sufficiency, this time we will adopt a slightly different approach. Assume that both (102) and (103) hold. We know that (23) holds for an arbitrary $f \in \Gamma^q_u(w)$ and $s \in (0,1)$. Using Lemma 3.36(ii) and the absolute continuity of the integral, we obtain that (103) implies

$$\lim_{s\to 0+} \sup_{\|f\|_{\Gamma^p_u(v)}\le 1} \|f^*\|_{\Gamma^q_u(w\chi_{[0,s]})} = 0.$$

Hence, now it suffices to prove that $\sup_{\|f\|_{\Gamma^p_u(v)}\le 1} U(s) f^{**}_u(s) \left(\int_s^1 U^{-q} w\right)^{\frac{1}{q}}$ vanishes as $s \to 0+$. We see that

$$\begin{aligned}
&\sup_{\|f\|_{\Gamma^p_u(v)}\le 1} U(s) f^{**}_u(s) \left(\int_s^1 U^{-q} w\right)^{\frac{1}{q}} = \\
&= \sup_{\|f\|_{\Gamma^p_u(v)}\le 1} f^{**}_u(s) \left(V(s) + U^p(s)\int_s^1 U^{-p} v\right)^{\frac{1}{p}} \frac{U(s)(\int_s^1 U^{-q} w)^{\frac{1}{q}}}{\left(V(s) + U^p(s)\int_s^1 U^{-p} v\right)^{\frac{1}{p}}} \le \\
&\le \sup_{\|f\|_{\Gamma^p_u(v)}\le 1} \left(\int_0^s (f^{**}_u(x))^p v(x)\,\mathrm{d}x + (U(s) f^{**}_u(s))^p \int_s^1 U^{-p} v\right)^{\frac{1}{p}} \frac{U(s)(\int_s^1 U^{-q} w)^{\frac{1}{q}}}{\left(V(s) + U^p(s)\int_s^1 U^{-p} v\right)^{\frac{1}{p}}} = \\
&= \sup_{\|f\|_{\Gamma^p_u(v)}\le 1} \|f^*\chi_{[0,s]}\|_{\overline{\Gamma^p_u(v)}} \cdot \frac{U(s)(\int_s^1 U^{-q} w)^{\frac{1}{q}}}{\left(V(s) + U^p(s)\int_s^1 U^{-p} v\right)^{\frac{1}{p}}} \le \\
&\le \sup_{\|f\|_{\Gamma^p_u(v)}\le 1} \|f\|_{\Gamma^p_u(v)} \frac{U(s)(\int_s^1 U^{-q} w)^{\frac{1}{q}}}{\left(V(s) + U^p(s)\int_s^1 U^{-p} v\right)^{\frac{1}{p}}} \le \\
&\le \sup_{\|f\|_{\Gamma^p_u(v)}\le 1} \|f\|_{\Gamma^p_u(v)} \frac{\left(W(s) + U^q(s)\int_s^1 U^{-q} w\right)^{\frac{1}{q}}}{\left(V(s) + U^p(s)\int_s^1 U^{-p} v\right)^{\frac{1}{p}}}
\end{aligned}$$

and the last part vanishes as $s \to 0+$ thanks to (102).

The proof of part (i) is analogous. □

**Remark 3.38.** The reader may notice that, in the proof above, we could use Theorem 3.6(iii) in the full form as well, just as we did in the other similar situations throughout this work. This would provide us with conditions similar to (102). However, the alternative way we followed above provides the condition in a simpler form, without need of further estimating.

**Lemma 3.39.** *Let $v, w$ be admissible weights and let $u$ be a positive integrable weight. Assume that $p, q \in (0,\infty)$. Then $\Gamma^p_u(v) \hookrightarrow \Gamma^{q,\infty}_u(w)$ if and only if $A_{(85)} < \infty$. Moreover,* $\operatorname{Opt}(\Gamma^p_u(v), \Gamma^{q,\infty}_u(w)) \simeq A_{(85)}$.

*Proof.* Consider a function $f \in \Gamma_u^p(v)$ and a fixed $t \in (0,1)$. Then we have

$$\begin{aligned} f_u^{**}(t) W^{\frac{1}{q}}(t) &\le A_{(85)} \left( (f_u^{**}(t))^p V(t) + \int_t^1 (f_u^{**}(t))^p U^p(t) U^{-p}(x) v(x)\,\mathrm{d}x \right)^{\frac{1}{p}} \le \\ &\le A_{(85)} \left( \int_0^t (f_u^{**}(x))^p v(x)\,\mathrm{d}x + \int_t^1 (f_u^{**}(x))^p v(x)\,\mathrm{d}x \right)^{\frac{1}{p}} = A_{(85)} \|f\|_{\Gamma_u^p(v)}, \end{aligned}$$

hence passing to supremum over $t$ and then over $f$ gives $\operatorname{Opt}(\Gamma_u^p(v), \Gamma_u^{q,\infty}(w)) \le A_{(85)}$. For the converse inequality, since $\|f\|_{\Lambda^{q,\infty}(w)} \le \|f\|_{\Gamma_u^{q,\infty}(w)}$ (cf. (4)), we have

$$\operatorname{Opt}(\Gamma_u^p(v), \Gamma_u^{q,\infty}(w)) = \sup_{f \in \Gamma_u^p(v)} \frac{\|f\|_{\Gamma_u^{q,\infty}(w)}}{\|f\|_{\Lambda^{q,\infty}(w)}} \cdot \frac{\|f\|_{\Lambda^{q,\infty}(w)}}{\|f\|_{\Gamma_u^p(v)}} \gtrsim A_{(85)}.$$

□

**Theorem 3.40** (The case $\Gamma_u^p(v) \overset{*}{\hookrightarrow} \Gamma_u^{q,\infty}(w)$)**.** *Let $v, w$ be admissible weights, $u$ a positive integrable weight and $p, q \in (0,\infty)$. Then $\Gamma_u^p(v) \overset{*}{\hookrightarrow} \Gamma_u^{q,\infty}(w)$ if and only if (89) holds and*

$$\lim_{s \to 0+} \sup_{t \in (s,1)} \frac{U(s) U^{-1}(t) W^{\frac{1}{q}}(t)}{\left( V(s) + U^p(s) \int_s^1 U^{-p}(x) v(x)\,\mathrm{d}x \right)^{\frac{1}{p}}} = 0.$$

*Proof.* The proof is similar to that of Theorem 3.37. This time we just use the decomposition (24). □

**Lemma 3.41.** *Let $v, w$ be admissible weights and let $u$ be a positive integrable weight. Assume that $p, q \in (0,\infty)$. Then $\Gamma_u^{p,\infty}(v) \hookrightarrow \Gamma_u^q(w)$ if and only if $A_{(104)} < \infty$ where*

$$A_{(104)} := \left( \int_0^1 \frac{w(t)}{U^q(t) \sup_{x \in (t,1)} U^{-q}(x) V^{\frac{q}{p}}(x)}\,\mathrm{d}t \right)^{\frac{1}{q}}. \tag{104}$$

*Moreover,* $\operatorname{Opt}(\Gamma_u^{p,\infty}(v), \Gamma_u^q(w)) \simeq A_{(104)}$.

*Proof.* Take a fixed $t \in (0,1)$ and a function $f \in \Gamma_u^{p,\infty}(v)$. Then it holds

$$\begin{aligned} (f_u^{**}(t))^q w(t) &= (f_u^{**}(t))^q w(t) \cdot \frac{U^q(t) \sup_{x \in (t,1)} U^{-q}(x) V^{\frac{q}{p}}(x)}{U^q(t) \sup_{x \in (t,1)} U^{-q}(x) V^{\frac{q}{p}}(x)} \le \\ &\le \frac{w(t) \sup_{x \in (t,1)} \left( \int_0^t f^*(y) u(y)\,\mathrm{d}y \right)^q U^{-q}(x) V^{\frac{q}{p}}(x)}{U^q(t) \sup_{x \in (t,1)} U^{-q}(x) V^{\frac{q}{p}}(x)} \le \\ &\le \frac{w(t) \sup_{x \in (t,1)} (f_u^{**}(x))^q V^{\frac{q}{p}}(x)}{U^q(t) \sup_{x \in (t,1)} U^{-q}(x) V^{\frac{q}{p}}(x)} \le \\ &\le \|f\|^q_{\Gamma_u^{p,\infty}(v)} \frac{w(t)}{U^q(t) \sup_{x \in (t,1)} U^{-q}(x) V^{\frac{q}{p}}(x)}. \end{aligned}$$

Integrating the first and last factor over $t \in (0,1)$ and taking the $q$-th root of the result gives us $A_{(104)} \geq \operatorname{Opt}(\Gamma_u^{p,\infty}(v), \Gamma_u^q(w))$.

Now we are going to prove the converse estimate. Assume that $\Gamma_u^{p,\infty}(v) \hookrightarrow \Gamma_u^q(w)$. The function $\omega : t \mapsto (\sup_{x\in(t,1)} U^{-1}(x)V^{\frac{1}{p}}(x))^{-1}$ is $U$-quasiconcave. This means that $\omega$ is equivalent to a nondecreasing function on $[0,1]$ and $\frac{\omega}{U}$ is equivalent to a nonincreasing function on $[0,1]$ (cf. also [11, Definition 2.2]). Indeed, $\omega$ is obviously nondecreasing and $\frac{\omega}{U}$ is nonincreasing by (98), since a fundamental function is always nondecreasing. By [11, Lemma 2.8(ii)], there exists a nonincreasing function $h \in \mathscr{M}_+(0,1)$ and a constant $\lambda \geq 0$ such that $\int_0^t hu < \infty$ for all $t \in [0,1]$ and

$$\Big(\sup_{x\in(t,1)} U^{-1}(x)V^{\frac{1}{p}}(x)\Big)^{-1} \simeq \lambda + \int_0^t h(x)u(x)\,\mathrm{d}x, \qquad t \in [0,1]. \tag{105}$$

Next, a simple computation gives

$$\|f\|_{\Gamma_u^{p,\infty}(v)} = \sup_{t\in(0,1)} \left(\int_0^t f^*(x)u(x)\,\mathrm{d}x\right) \sup_{x\in(t,1)} U^{-1}(x)V^{\frac{1}{p}}(x) \tag{106}$$

for every $f \in \mathscr{M}(R,\mu)$. Having recalled that $(R,\mu)$ is nonatomic, we find a function $f \in \mathscr{M}(R,\mu)$ such that $f^* = h$ a.e. Therefore, applying (105) and (106) on this $f$, we obtain

$$\|f\|_{\Gamma_u^{p,\infty}(v)} = \sup_{t\in(0,1)} \left(\int_0^t h(x)u(x)\,\mathrm{d}x\right) \sup_{x\in(t,1)} U^{-1}(x)V^{\frac{1}{p}}(x) \lesssim \sup_{t\in(0,1)} \frac{\int_0^t h(x)u(x)\,\mathrm{d}x}{\lambda + \int_0^1 h(x)u(x)\,\mathrm{d}x} \leq 1.$$

Now, using $\Gamma_u^{p,\infty}(v) \hookrightarrow \Gamma_u^q(w)$, we have

$$\begin{aligned} A_{(104)} &= \left(\int_0^1 \left[\Big(\sup_{x\in(t,1)} U^{-1}(x)V^{\frac{1}{p}}(x)\Big)^{-1} - \lambda + \lambda\right]^q U^{-q}(t)w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \lesssim \\ &\lesssim \left(\int_0^1 \left[\int_0^t h(x)u(x)\,\mathrm{d}x + \lambda\right]^q U^{-q}(t)w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \lesssim \\ &\lesssim \|f\|_{\Gamma_u^q(w)} + \lambda\left(\int_0^1 U^{-q}(t)w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \lesssim \\ &\lesssim \operatorname{Opt}(\Gamma_u^{p,\infty}(v), \Gamma_u^q(w)) + \lambda\left(\int_0^1 U^{-q}(t)w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}. \end{aligned}$$

If $\lambda = 0$, we are done. Assume that $\lambda > 0$. On taking the limit for $t \to 0+$ in (105), we obtain

$$\sup_{x\in(0,1)} U^{-1}(x)V^{\frac{1}{p}}(x) \simeq \frac{1}{\lambda}.$$

Now, take a fixed $\varepsilon \in (0,1)$ and find a function $g$ such that $g^* = \lambda U^{-1}(\varepsilon)\chi_{[0,\varepsilon]}$. Then

$$\begin{aligned}
\|g\|_{\Gamma_u^{p,\infty}(v)} &= \sup_{x\in(0,1)} \left(\int_0^x g^*(y)u(y)\,\mathrm{d}y\right) U^{-1}(x)V^{\frac{1}{p}}(x) \leq \\
&\leq \lambda \max\left\{ \sup_{x\in(0,\varepsilon)} U^{-1}(\varepsilon)V^{\frac{1}{p}}(x) + \sup_{x\in(\varepsilon,1)} U^{-1}(x)V^{\frac{1}{p}}(x) \right\} = \\
&= \lambda \sup_{x\in(\varepsilon,1)} U^{-1}(x)V^{\frac{1}{p}}(x) \leq \\
&\leq \lambda \sup_{x\in(0,1)} U^{-1}(x)V^{\frac{1}{p}}(x) \simeq 1.
\end{aligned}$$

Next, observe that $g_u^{**}(x) \geq \lambda U^{-1}(x)\chi_{(\varepsilon,1)}(x)$ for $x \in (0,1)$. Finally, we get

$$\begin{aligned}
\lambda\left(\int_\varepsilon^1 U^{-q}(x)w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} &\leq \|g\|_{\Gamma_u^q(w)} \leq \\
&\leq \|g\|_{\Gamma_u^{p,\infty}(v)} \operatorname{Opt}\left(\Gamma_u^{p,\infty}(v), \Gamma_u^q(w)\right) \lesssim \\
&\lesssim \operatorname{Opt}\left(\Gamma_u^{p,\infty}(v), \Gamma_u^q(w)\right),
\end{aligned}$$

thus, by letting $\varepsilon \to 0+$, we obtain the final result. □

**Theorem 3.42** (The case $\Gamma_u^{p,\infty}(v) \overset{*}{\hookrightarrow} \Gamma_u^q(w)$)**.** *Let $v, w$ be admissible weights and let $u$ be a positive integrable weight. Assume that $p, q \in (0,\infty)$. Then $\Gamma_u^{p,\infty}(v) \overset{*}{\hookrightarrow} \Gamma_u^q(w)$ if and only if $A_{(104)} < \infty$ and*

$$\lim_{s\to 0+} \frac{\left(\int_s^1 U^{-q}(x)w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}}{\sup_{x\in(s,1)} U^{-1}(x)V^{\frac{1}{p}}(x)} = 0.$$

*Proof.* Again, this result is obtained by a similar method as Theorem 3.37. □

**Lemma 3.43.** *Let $v, w$ be admissible weights and let $u$ be a positive integrable weight. Assume that $p, q \in (0,\infty)$. Then $\Gamma_u^{p,\infty}(v) \hookrightarrow \Gamma_u^{q,\infty}(w)$ if and only if $A_{(107)} < \infty$ where*

$$A_{(107)} := \sup_{0<t<1} \frac{W^{\frac{1}{q}}(t)}{U(t)\sup_{x\in(t,1)} U^{-1}(x)V^{\frac{1}{p}}(x)} < \infty. \tag{107}$$

*Moreover,* $\operatorname{Opt}\left(\Gamma_u^{p,\infty}(v), \Gamma_u^{q,\infty}(w)\right) \simeq A_{(107)}$.

*Proof.* At first, we take a fixed $t \in (0,1)$ and notice that

$$V^{\frac{1}{p}}(t) \leq \sup_{x\in(t,1)} U(t)U^{-1}(x)V^{\frac{1}{p}}(x) \tag{108}$$

and the same holds for $W^{\frac{1}{q}}$ in place of $V^{\frac{1}{p}}$. Let $f \in \Gamma_u^{p,\infty}(v)$. Then

$$f_u^{**}(t)W^{\frac{1}{q}}(t) \leq A_{(107)} f_u^{**}(t)U(t) \sup_{x\in(t,1)} U^{-1}(x)V^{\frac{1}{p}}(x) \leq A_{(107)}\|f\|_{\Gamma_u^{p,\infty}(v)},$$

thus, by taking the supremum over $t \in (0,1)$, we obtain $\operatorname{Opt}(\Gamma_u^{p,\infty}(v), \Gamma_u^{q,\infty}(w)) \le A_{(107)}$.

As for the converse estimate, testing the embedding inequality by the characteristic function $\chi_{[0,t]}$ for a fixed $t \in (0,1)$ gives

$$\operatorname{Opt}(\Gamma_u^{p,\infty}(v), \Gamma_u^{q,\infty}(w)) \ge \frac{U(t) \sup\limits_{x\in(t,1)} U^{-1}(x) W^{\frac{1}{q}}(x)}{U(t) \sup\limits_{x\in(t,1)} U^{-1}(x) V^{\frac{1}{p}}(x)} \ge \frac{W^{\frac{1}{q}}(t)}{U(t) \sup\limits_{x\in(t,1)} U^{-1}(x) V^{\frac{1}{p}}(x)}.$$

Taking the supremum over $t \in (0,1)$, we get the desired estimate. □

**Theorem 3.44** (The case $\Gamma_u^{p,\infty}(v) \overset{*}{\hookrightarrow} \Gamma_u^{q,\infty}(w)$)**.** *Let $v, w$ be admissible weights and let $u$ be a positive integrable weight. Assume that $p, q \in (0,\infty)$. Then $\Gamma_u^{p,\infty}(v) \overset{*}{\hookrightarrow} \Gamma_u^{q,\infty}(w)$ if and only if*

$$\lim_{s\to 0+} \frac{\sup_{t\in(s,1)} U^{-1}(t) W^{\frac{1}{q}}(t)}{\sup_{t\in(s,1)} U^{-1}(t) V^{\frac{1}{p}}(t)} = 0. \tag{109}$$

*Proof.* Thanks to (108) it is clear that (109) implies

$$\lim_{s\to 0+} \frac{W^{\frac{1}{q}}(s)}{U(s) \sup_{t\in(s,1)} U^{-1}(t) V^{\frac{1}{p}}(t)} = 0.$$

According to Lemma 3.43 and Remark 3.11, this condition is equivalent to

$$\lim_{s\to 0+} \sup_{\|f\|_{\Gamma_u^{p,\infty}(v)} \le 1} \|f^*\|_{\Gamma_u^{q,\infty}(w\chi_{[0,s]})} = 0.$$

We recall (24) and (98) and then proceed similarly as in the proof of Theorem 3.37. □

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