Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

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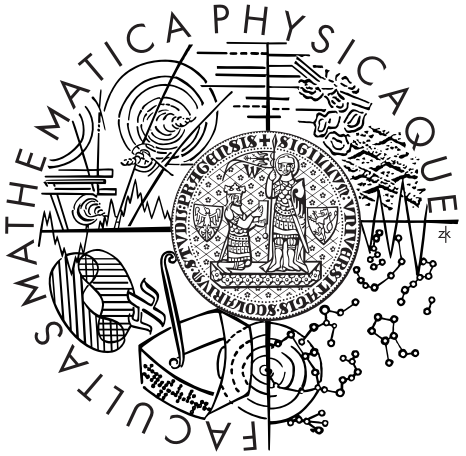

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Obecná enumerace číselných rozkladů

Katedra aplikované matematiky

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Studijní program: Matematika
Studijní obor: Matematická analýza

Praha 2011

Děkuji vedoucímu diplomové práce panu docentu Martinu Klazarovi za jeho vedení, poskytnutí doporučené literatury a cenné rady, které mi v průběhu řešení práce poskytl, mojí rodině za psychickou podporu a Miloslavu Holíkovi za technické zázemí.

Prohlašuji, že jsem tuto diplomovou práci vypracoval samostatně a výhradně s použitím citovaných pramenů, literatury a dalších odborných zdrojů.

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Název práce: Obecná enumerace číselných rozkladů

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Katedra: Katedra aplikované matematiky

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Abstrakt: Předložená diplomová práce se zabývá asymptotikami počítacích funkcí ideálů číselných rozkladů. Jejím hlavním cílem je zjistit největší možný asymptotický růst počítací funkce rozkladového ideálu, která je nekonečněkrát rovna nule. Autor se na základě znalosti asymptotik vybraných rozkladových ideálů snaží pomocí kombinatorických a základních analytických metod odvodit odhady hledané asymptotiky. Výsledkem je za prvé slabší horní odhad, za druhé poměrně silný dolní odhad a za třetí, pro speciální třídu rozkladových ideálů je nalezen největší asymptotický růst.

Klíčová slova: Číselné rozklady, asymptotika rozkladů, rozkladové ideály, počítací funkce, kombinatorická enumerace.

Title: General enumeration of integer partitions

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Abstract: Submitted thesis concerns with asymptotics of counting functions of ideals of integer partitions. Its main aim is to find the fastest growth of counting function of partition ideal, which is infinitely many times equal to zero. Using mostly combinatorial and fundamental analytical methods author tries to deduce estimates for such a counting function. As a result, firstly, there is found an upper bound, secondly, we have strong lower bound and thirdly, there is an exact asymptotic growth for the special family of partition ideals.

Keywords: Integer partitions, asymptotics of partitions, partition ideals, counting function, combinatorial enumeration.

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Preface

The goal of this thesis is to study the number of partitions with a given property. More precisely, we introduce special sets of partitions called partition ideals and we are concerned about counting functions of these sets. The main aim is to determine the maximal growth of the counting function of some partition ideal, which is infinitely many times equal to zero.

In the first chapter we introduce integer partitions and give several basic theorems. We present both combinatorial and analytical ways to study partitions and show their advantages. After that, we give a brief summary of partition identities with stress on famous Euler's identity. The rest of the chapter is devoted to Cohen–Remmel theorem, with easy combinatorial proof and a number of corollaries.

The second chapter gathers results about asymptotics of counting functions. At the beginning we formulate the most important Hardy–Ramanujan asymptotics for the counting function $p(n)$ of partitions without restriction with a proof of two estimates. Afterwards we jump to the Schur's theorem about asymptotics for the number of partitions with parts in a given finite set A of positive integers, which implies two nice statements. Next sections survey asymptotics of counting function of other famous partition ideals and shows that there is also a explicit formula for $p(n)$.

Main and original theorems can be found in the third chapter. In introduction, we introduce partition ideals and their property (P), which demands counting function to be infinitely many times equal to zero. Second section brings out partition ideal satisfying (P), whose counting function grows very fast. Consequently, in Theorem 3.3.2 we present exact upper bound for asymptotics of counting functions of a special family of partition ideals. Main Theorem 3.4.4 bounds fastest counting function with property (P). Final results of this chapter can be found in Theorem 3.4.3, which characterizes fastest counting functions of partition ideals. All this work indicates that oscillations for the counting functions are really large.

The last chapter gives us a summary of asymptotics and oscillations of counting functions of similar structures like partitions. We bring out the framework of counting functions of hereditary sets in graphs, permutations and words and show that oscillations in these structures are very bounded and sometimes even impossible, which makes our results more valuable.

1. Introduction to the partitions

At the beginning of 17^{th} century G. W. Leibniz wrote a letter to J. Bernoulli, where he asked him in how many ways can one write a positive integer as a sum of positive integers. From this moment on many mathematicians studied this question and developed new branch of mathematics called the partition theory.

1.1 Basic definitions

Definition 1.1.1 *Let n, k be positive integers and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ be a finite non-increasing sequence of positive integers. We call $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ a partition of a positive integer n, if*

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k.$$

The positive integers λ_i, where $i \in [k]$, are called the parts of the partition λ.

We denote the fact that λ is partition of n by $\lambda \vdash n$. Number n is also called the absolute value or the norm of the partition λ and denoted by $|\lambda|$.

Second notation we use is multisets. Multiset is a pair $\lambda = (A, f)$, where A is a set of positive integers and $f : \mathbb{N} \to \mathbb{N}_0$ is a multiplicity (multiset) mapping. The value $f(t)$ denotes how many times t appears in the multiset λ, which means $f(t) = 0$ for $t \notin A$. So we can write

$$\lambda = (1^{f(1)} 2^{f(2)} 3^{f(3)} \ldots),$$

where the exponent does not mean the power, but the multiplicity of part in the partition λ.

Definition 1.1.2 *Let $\mathcal{P}$ be the set of all partitions and $\mathcal{P}_n$ be the set of partitions of n. Let $p(n) = |\mathcal{P}_n|$ be the number of partitions of n, usually called the partition function.*

For a negative n we define $p(n) = 0$ and we set $p(0) = 1$, because the empty set is the only partition of zero. We call that partition the empty partition and mark it θ.

Example 1.1.3. Let us list all partitions of number 6. Because

$$\begin{array}{lll} 6 = 6 & = 5 + 1 & = 4 + 2 \\ = 4 + 1 + 1 & = 3 + 3 & = 3 + 2 + 1 \\ = 3 + 1 + 1 + 1 & = 2 + 2 + 2 & = 2 + 2 + 1 + 1 \\ = 2 + 1 + 1 + 1 + 1 & = 1 + 1 + 1 + 1 + 1 + 1, & \end{array}$$

therefore $p(6) = 11$.

Definition 1.1.4 *Let $S \subset \mathcal{P}$ be a set of partitions. We denote*

$$p(n, S) = |\mathcal{P}_n \cap S|$$

the counting function of the set S.

When A is a set of positive integers, then the counting function $p_A(n)$ counts the number of partitions with parts from the set A.

One of the most important problem of the theory of partitions is to evaluate the counting function of some sets S. This is not difficult for finite S, but even for $S = \mathcal{P}$ it took nearly 300 years to find an asymptotic formula.

Definition 1.1.5 *Suppose we have two partitions $\lambda, \mu \in \mathcal{P}$. We say that λ is a subpartition of μ, if no part λ_i of partition λ has more occurrences inλ than in μ. We use the notation $\lambda \prec \mu$.*

The set $\mathcal{P}$ with relation "to be a subpartition" is a partially ordered set. In the language of multisets, multiset $\lambda = (A, f)$ is a subset of the multiset $\mu = (B, g)$, which means that $f(n) \leq g(n)$ for every positive integer n.

Definition 1.1.6 *Let $\lambda^1, \lambda^2, \ldots \lambda^k$ be partitions in $\mathcal{P}$. Their union $\lambda^1 \cup \ldots \cup \lambda^k$ is the smallest (i.e. with the smallest absolute value) partition μ satisfying $\mu \succ \lambda^i$ for all $i \in [k]$.*

Similarly their intersection $\lambda^1 \cap \ldots \cap \lambda^k$ is the greatest (i.e. with the greatest absolute value) partition μ such that $\mu \prec \lambda^i$ for all $i \in [k]$.

Their sum $\lambda^1 + \ldots + \lambda^k$ is the partition μ of the positive integer $|\lambda^1| + \cdots + |\lambda^k|$ such that the multiplicity of $i \in \mathbb{N}$ in μ is the sum of multiplicities of i in $\lambda^1, \ldots, \lambda^k$.

Finally, for two partitions $\lambda \prec \mu$ define the difference $\mu - \lambda$ as a partition with parts same as partition μ and their multiplicity equal to the difference of their multiplicities in the partitions μ and λ. In the case $\lambda \not\prec \mu$ we define the difference $\mu - \lambda$ as a partition $\mu - (\lambda \cap \mu)$.

To illustrate these definitions in the multiset notation, consider multisets $\lambda^i = (A_i, f_i)$ for $i \in [k]$. Then their union, intersection and sum is

$$\begin{aligned}
\lambda^1 \cup \ldots \cup \lambda^k &= (A_1 \cup \ldots \cup A_k, \max f_i) \\
\lambda^1 \cap \ldots \cap \lambda^k &= (A_1 \cap \ldots \cap A_k, \min f_i) \\
\lambda^1 + \ldots + \lambda^k &= (A_1 \cup \ldots \cup A_k, \textstyle\sum f_i) \\
\lambda^1 - \lambda^2 &= (A_1, \max\{0, f_1 - f_2\}).
\end{aligned}$$

Example 1.1.7. Set $\lambda = (5, 2, 2, 1, 1)$ and $\mu = (7, 5, 3, 1)$. Then their union is the partition $\lambda \cup \mu = (7, 5, 3, 2, 2, 1, 1)$, intersection is the partition $\lambda \cap \mu = (5, 1)$, their sum is the partition $\lambda + \mu = (7, 5, 5, 3, 2, 2, 1, 1, 1)$ and difference $\lambda - \mu$ is the partition $(2, 2, 1)$.

1.2 Analytical and combinatorial view

There are two major ways how to describe partitions, analytical and combinatorial. Whilst combinatorial approach is more illustrative, the analytical approach works with generating functions and often gets better results. We introduce both methods in this section.

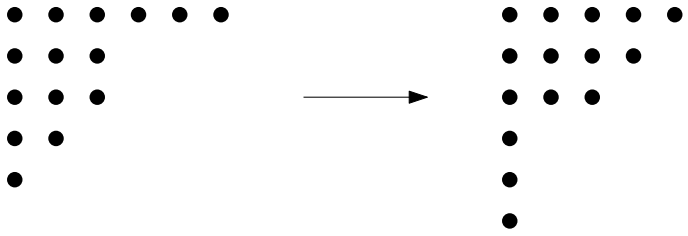
The combinatorial approach represents partition as a set of lattice points in the plane and works with the graphical properties of diagrams.

Definition 1.2.1 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be the partition from $\mathcal{P}$. Then the Ferrers diagram $\mathfrak{F}_\lambda$ of λ is the set of points with integer coordinates $(j, -i)$ such that $0 < i \leq k$ and $0 < j \leq \lambda_i$. Sometimes instead of the lattice points we draw unit squares centered in the lattice points.*

Example 1.2.2. We demonstrate the Ferrers diagram $\mathfrak{F}_\lambda$ for $\lambda = (5,4,4,2)$ in the following figure, first by the lattice points, then by unit squares.

Definition 1.2.3 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition. We define a conjugate partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{k'})$ of λ by setting λ'_i to be the number of parts of λ that are greater or equal i.*

Example 1.2.4. Take $\lambda = (6,3,3,2,1)$. Then the conjugate λ' is the partition $\lambda' = (5,4,3,1,1,1)$. The following Figure visualizes process of conjugation:



The diagrams $\mathfrak{F}_\lambda$ and $\mathfrak{F}_{\lambda'}$ are in coordinate system images of one another in the axial symmetry with the axis passing through all points of the form $(i,-i)$, $i \in \mathbb{Z}$.

That is why some partitions are self-conjugates (e.g. partition $(5,3,3,1,1)$) and the mapping "conjugation" $C : \mathcal{P} \to \mathcal{P}$, which maps λ to λ' is a bijection.

We demonstrate the strength of this observation by the following theorem.

Theorem 1.2.5 Let n, m be positive integers. Then the number of all partitions of n with all parts not exceeding m is equal to the number of partitions of n with at most m parts.

Proof. We use the one-to-one correspondence between partitions and their conjugates. The fact that partition λ has at most m parts means that the parts of the conjugate partition λ' are at most m and vice versa. This conjugation maps partitions of n with all parts not exceeding m to the partitions of n with at most m parts, which concludes the proof. □

The analytical approach uses generating functions and mathematical analysis. First we introduce the generating function.

Definition 1.2.6 *Let $a_0, a_1, a_2, a_3, \ldots$ be sequence of positive integers. Then the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where $x \in \mathbb{C}$, is called the generating function of the sequence $\{a_n\}_{n=0}^{\infty}$.

Definition 1.2.7 *Let A be a set of positive integers. Denote by $p_A(n)$ the number of partitions, whose parts lie in A and by $p_{A,k}(n)$ the number of partitions λ, whose parts lie in A and their multiplicity in λ is at most k.*

Example 1.2.8. Take $A = \{1, 2, 7\}$. Then

$$p_A(4) = 3 \quad \text{and} \quad p_{A,2}(4) = 2,$$

because in both cases the partitions $4 = 2+2 = 2+1+1$ are acceptable and moreover in the first case $4 = 1+1+1+1$ is possible.

Theorem 1.2.9 Let A be the set of positive integers. Set

$$f(x) = \sum_{n=0}^{\infty} p_A(n)x^n \quad \text{and} \quad f_k(x) = \sum_{n=0}^{\infty} p_{A,k}(n)x^n.$$

Then for $|x| < 1$ we have

$$f(x) = \prod_{a\in A} \frac{1}{1-x^a},$$
$$f_k(x) = \prod_{a\in A} (1 + x^a + x^{2a} + \cdots + x^{ka}) = \prod_{a\in A} \frac{1-x^{(k+1)a}}{1-x^a}.$$

Proof. Let us start with the power series

$$\frac{1}{1-x^a} = 1 + x^a + x^{2a} + x^{3a} + \dots$$

which converges absolutely for $|x| < 1$. If $A = \{a_1, a_2, \dots, a_k\}$ is a finite set, then just multiplying the parentheses we obtain

$$\begin{aligned}\prod_{a\in A} \frac{1}{1-x^a} &= \prod_{a\in A} (1 + x^a + x^{2a} + \dots) \\ &= (1 + x^{a_1} + x^{2a_1} + \dots) \cdot \dots \cdot (1 + x^{a_k} + x^{2a_k} + \dots) \\ &= \sum_{n=0}^{\infty} p_A(n)x^n,\end{aligned}$$

because any partition of n is of the form $t_1a_1 + t_2a_2 + \cdots + t_ka_k$, where $t_1, \dots, t_k$ are some non-negative integers. This means that every partition of n contributes to the coefficient $p(A, n)$ of x^n precisely one times and we are done.

If A is infinite, denote $A = \{a_1, a_2, a_3, \dots\}$. Since

$$\sum_{a\in A} \sum_{i=1}^{\infty} |x|^{ia} = \sum_{a\in A} \frac{|x|^a}{1-|x|^a} \le \sum_{a\in A} \frac{|x|^a}{1-|x|} \le \frac{1}{1-|x|} \sum_{a=1}^{\infty} |x|^a = \frac{|x|}{(1-|x|)^2} < \infty,$$

our infinite product converges absolutely and we can multiply the parentheses in the same way to get

$$f(x) = \sum_{n=0}^{\infty} p_A(n)x^n = \prod_{a\in A} \frac{1}{1-x^a}.$$

The second identity uses the finite geometric progression instead of the infinite geometric progression, therefore the justification of the convergence will be the same and we get

$$\begin{aligned}\prod_{a\in A}\frac{1-x^{(k+1)a}}{1-x^a} &= \prod_{a\in A}(1+x^a+x^{2a}+\cdots+x^{ka})\\ &= (1+x^{a_1}+\cdots+x^{ka_1})\cdot(1+x^{a_2}+\cdots+x^{ka_2})\cdot\ldots\\ &= \sum_{n=0}^{\infty}p_{A,k}(n)x^n,\end{aligned}$$

which completes the proof. □

1.3 Partition identities

This section is devoted to the identities of restricted counting functions. That means the counting function of the set of partitions satisfies given condition.

The best example is the famous identity of L. Euler, who discovered and proved it in 1748 in [14]. It was the first known theorem asserting the identity of two apparently different counting functions. We introduce two proofs of this theorem, first by combinatorial methods, second by generating functions.

Theorem 1.3.1 (Euler identity) For every positive integer n the number of partitions of n with mutually distinct parts is equal to the number of partitions of n into odd parts.

First proof. Let us construct the one-to-one correspondence between the partitions with distinct parts and the partitions with odd parts. Such a construction proves the statement of theorem.

Take any partition λ with mutually distinct parts λ_i, $i \in [k]$. Write λ_i in the form $\lambda_i = 2^{a_i}b_i$, where a_i is a non-negative integer and b_i is an odd positive integer. This form is unique. Define mapping which maps part λ_i to 2^{a_i} parts b_i. Doing this with every part of λ, we get partition μ. Clearly all parts of μ are odd.

Now consider partition μ with only odd parts. Take all of its parts with the same absolute value b, and denote their number by m. Write m in 2-adic notation

$$m = 2^{c_0}+2^{c_1}+2^{c_2}+\cdots+2^{c_{l-1}},$$

where $c_0, c_1, \ldots, c_{l-1}$ are distinct positive integers. This notation is also unique. Define mapping which maps all m parts of μ with length b to the l parts $2^{c_0}b$, $2^{c_1}b$, $\ldots$, $2^{c_{l-1}}b$. Repeating this with all parts of μ we get the partition λ. The parts of λ are different, because the 2-adic notation is unique.

Both mappings defined above are inverses of one another, so they define bijection, which concludes the proof. □

Second proof. Denote by $p_O(n)$ the counting function of the partitions with odd parts and by $p_D(n)$ the counting function of the partitions with distinct parts. Then Theorem 1.2.9 states

$$\sum_{n=0}^{\infty}p_D(n)x^n = \prod_{n=1}^{\infty}(1+x^n)$$

and

$$\sum_{n=0}^{\infty} p_O(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}}.$$

Simple algebraic manipulation gives

$$\prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty} \frac{1-x^{2n}}{1-x^n} = \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}}.$$

□

Let us show one example of the bijection described in the first proof. All partitions of 7 with distinct parts are paired with the partitions of 7 with only odd parts:

$$\begin{array}{lll} 7 & = 7 & = 7 \\ & = 6+1 & = 3+3+1 \\ & = 5+2 & = 5+1+1 \\ & = 4+3 & = 3+1+1+1+1 \\ & = 4+2+1 & = 1+1+1+1+1+1+1 \end{array}$$

Theorems such as this are called partition identities. There are many of them, but their proofs are usually not as simple as in the case of Euler's identity. The progress in this branch of mathematics came at the beginning of 20$^{\text{th}}$ century with the generating functions. Here we mention Rogers–Ramanujan identities, which were proved by Rogers and Ramanujan [29] in 1919 and independently by Schur [31] in 1917.

Theorem 1.3.2 The number of partitions of n in which the minimal difference between parts is at least 2 is equal to the number of partitions of n with the parts of the form $5k+1$ and $5k+4$.

Theorem 1.3.3 The number of partitions of n in which the minimal difference between parts is at least 2 and 1 does not appear is equal to the number of partitions of n with the parts of the form $5k+2$ and $5k+3$.

In 1894 Rogers [28] proved the identities

$$1+\sum_{n=1}^{\infty} \frac{x^{n^2}}{(1-x)(1-x^2)\dots(1-x^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-x^{5n+1})(1-x^{5n+4})}$$
$$1+\sum_{n=1}^{\infty} \frac{x^{n^2+n}}{(1-x)(1-x^2)\dots(1-x^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-x^{5n+2})(1-x^{5n+3})},$$

from which it is possible to deduce Theorems 1.3.2 and 1.3.3. Nevertheless Rogers did not find the connection, which was revealed by Ramanujan in 1913. However, Ramanujan was not able to find the proof of the analytic identities. Therefore the desired proof was made in 1919 and this theorem is named after Rogers and Ramanujan, although Schur proved it earlier.

Nowadays we known several more general theorems including both preceding theorems as a special case. Their formulation can be found in Gordon's [16] or in Andrew's paper [1]. Very good but somewhat dated summary can be found in the Andrew's review [2].

1.4 The Cohen–Remmel theorem

This section is devoted to the remarkable general identity of D. Cohen and J. Remmel published by Cohen [12] in 1981 and by Remmel [27] one year later. The charm of this theorem is that it needs only the principle of inclusion and exclusion to prove and is so strong, that many partition identities, including Euler identity, can be obtained from it as special cases.

Before its formulation we prepare for the proof by two theorems. The first one is the well-known principle of inclusion and exclusion, whose proof is only a technical exercise which can be found in many textbooks, therefore we skip it. After that we formulate immediate corollary.

Theorem 1.4.1 (Principle of inclusion and exclusion (PIE)) Let $X_1, X_2, \dots X_k$ be finite sets, all subsets of a set X. Then

$$|X \setminus \bigcup_{i=1}^{k} X_i| = \sum_{I \subset [k]} (-1)^{|I|} |\bigcap_{i \in I} X_i|.$$

Corollary 1.4.2 Let $X_1, X_2, \dots, X_k$ and $Y_1, Y_2, \dots, Y_k$ be finite sets, all subsets of X. Suppose that for all $I \subset [k]$ one has

$$|\bigcap_{i \in I} X_i| = |\bigcap_{i \in I} Y_i|.$$

Then

$$|X \setminus \bigcup_{i=1}^{k} X_i| = |X \setminus \bigcup_{i=1}^{k} Y_i|.$$

Proof. The condition on intersections substituted in the sum in the PIE gives desired equality. □

Theorem 1.4.3 (Cohen–Remmel) Let $\Lambda = \{\lambda^1, \lambda^2, \dots\}$ and $\Gamma = \{\gamma^1, \gamma^2, \dots\}$ be two (infinite) sequences of partitions, which for every finite $I \subset \mathbb{N}$ satisfy the condition

$$|\bigcup_{i \in I} \lambda^i| = |\bigcup_{i \in I} \gamma^i|.$$

Then for every positive integer n the number of partitions of n not containing any partition from Λ is equal to the number of partitions of n not containing any partition from Γ.

Proof. Fix a positive integer n. For every positive integer i set

$$X_i = \{\mu \in \mathcal{P}_n : \mu \succ \lambda^i\} \quad \text{and} \quad Y_i = \{\mu \in \mathcal{P}_n : \mu \succ \gamma^i\}.$$

Take a finite set of indices $1 \le i_1 \le i_2 \le \dots \le i_k$ and construct the sets

$$X = X_{i_1} \cap \dots \cap X_{i_k} \quad \text{and} \quad Y = Y_{i_1} \cap \dots \cap Y_{i_k}.$$

Our aim is to prove $|X| = |Y|$. Observe that X is the set of partitions μ from $\mathcal{P}_n$ such that $\mu \succ \lambda^{i_j}$ for all $j \in [k]$. That equivalently means $\mu \succ \lambda^{i_1} \cup \dots \cup \lambda^{i_k}$. Likewise Y is the set of partitions μ of $\mathcal{P}_n$ satisfying $\mu \succ \gamma^{i_1} \cup \dots \cup \gamma^{i_k}$. This makes the characterization of X and Y.

For every partition $\mu \in X$ define the partition

$$\mu' = \left(\mu - \lambda^{i_1} \cup \ldots \cup \lambda^{i_k}\right) + \gamma^{i_1} \cup \ldots \cup \gamma^{i_k},$$

which lies in Y, because of the characterization of X. Analogically for $\nu \in Y$ define the partition

$$\nu' = \left(\nu - \gamma^{i_1} \cup \ldots \cup \gamma^{i_k}\right) + \lambda^{i_1} \cup \ldots \cup \lambda^{i_k},$$

which lies in X. The only problem is whether the images μ' and ν' are really in $\mathcal{P}_n$. They are, because the condition from the statement says $|\lambda^{i_1} \cup \ldots \cup \lambda^{i_k}| = |\gamma^{i_1} \cup \ldots \cup \gamma^{i_k}|$, which implies $|\mu| = |\mu'|$ and $|\nu| = |\nu'|$.

That means the mappings $\mu \mapsto \mu'$ and $\nu \mapsto \nu'$ are inverses of one another and establish one-to-one correspondence between X and Y. Therefore $|X| = |Y|$. Let k_0 be the smallest k such that the absolute value of the partitions $\lambda^k, \lambda^{k+1}, \ldots$ and $\gamma^k, \gamma^{k+1}, \ldots$ is greater than n.

Now using the previous Corollary 1.4.2 we get

$$|\mathcal{P}_n \backslash (X_1 \cup X_2 \cup \ldots \cup X_{k_0-1})| = |\mathcal{P}_n \backslash (Y_1 \cup Y_2 \cup \ldots \cup Y_{k_0-1})|.$$

Because the lack of the other partitions $\lambda^{k_0}, \lambda^{k_0+1}, \ldots$ and $\gamma^{k_0}, \gamma^{k_0+1}, \ldots$ does not influence any partition of n, we get desired statement. □

This theorem produces many partition identities. It suffices to choose the sets $\Lambda = \{\lambda^1, \lambda^2, \ldots\}$ and $\Gamma = \{\gamma^1, \gamma^2, \ldots\}$ with the condition on the cardinality of the unions. One (but not the only one) way how to fulfill it is to take Λ and Γ such that the partitions $\lambda^1, \lambda^2, \ldots$ are independent, the partitions $\gamma^1, \gamma^2, \ldots$ are independent and $|\lambda^i| = |\gamma^i|$ for all positive integers i.

Corollary 1.4.4 (Glaisher's identity) Let d be a positive integer. Then every positive integer n has as many partitions not containing any multiple of d as partitions in which no part appears more than $d-1$ times.

Proof. Denote

$$\Lambda = \{(d), (2d), (3d), \ldots\}$$
$$\Gamma = \{(1, 1, \ldots, 1), (2, 2, \ldots, 2), (3, 3, \ldots, 3), \ldots\},$$

where every partition in Γ has exactly d parts, and use Cohen–Remmel theorem 1.4.3. □

Observe, that for $d = 2$ we derive Euler identity 1.3.1.

Corollary 1.4.5 (Schur's identity) The number of partitions of positive integer n whose parts are congruent to $\pm 1 \pmod 6$ equals to the number of partitions with distinct parts congruent to $\pm 1 \pmod 3$.

Proof. It suffices to use Cohen–Remmel Theorem 1.4.3 with the sets

$$\Lambda = \{(2), (3), (4), (6), (8), (9), (10), (12), (14), (15), \ldots\}$$
$$\Gamma = \{(1, 1), (3), (2, 2), (6), (4, 4), (9), (5, 5), (12), (7, 7), (15), \ldots\}.$$

□

2. Asymptotics of counting function

The main task of the theory of partitions is to study the number of partitions satisfying some conditions. But it appears to be really difficult to find the exact formula for the counting functions, therefore one attempts to find some estimates on the growth. These estimates are called the asymptotics of the counting function.

Recall that $p(n, S)$ is the counting function of the set of partitions S and $p_A(n)$ is the counting function of the partitions with the parts in the set of positive integers A.

2.1 Partitions without restriction

First question that comes in mind is how many partitions positive integer n has, that means we have no restrictions on the partitions. G. H. Hardy and S. Ramanujan [17] found the answer in 1918.

Theorem 2.1.1 (Hardy, Ramanujan) The asymptotic formula for the number $p(n)$ of all partitions of n is

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Proofs of this famous theorem are of two types, analytical and combinatorial. Whilst the original one of Hardy and Ramanujan used complex analysis, the elementary one published by Erdős [13] (without evaluating the constant $1/(4\sqrt{3})$) in 1942 and improved by Newman [26] in 1951 is based on the recurrent formula

$$np(n) = \sum_{\substack{kv \le n \\ k,v \ge 1}} vp(n - kv),$$

whose proof is not difficult. But "elementary" here means that the proof does not use complex analysis, because the proof itself is fairly technical. We will not prove this theorem, we present only some estimates on the counting function $p(n)$, as given in [22, Chapter 12].

Theorem 2.1.2 For sufficiently large n the counting function $p(n)$ satisfies the estimate

$$e^{1.99\sqrt{n}} < p(n) < e^{\pi\sqrt{\frac{2}{3}n}} = e^{2.56\sqrt{n}}.$$

Proof. First we prove the upper bound. Dividing the result of Theorem 1.2.9 by x^n we obtain for every $x \in (0, 1)$ the inequality

$$p(n) \le \frac{1}{x^n} \sum_{k=0}^{\infty} p(k)x^k = \frac{1}{x^n} \prod_{k=1}^{\infty} \frac{1}{1 - x^k}.$$

We want to determine the value x in order to minimize the right side. Applying logarithm on both sides gives

$$\log p(n) \le -n \log x - \sum_{k=1}^{\infty} \log(1 - x^k).$$

Taylor formula for $\log(1-y)$, $y \in (0,1)$, and some algebraic manipulation imply

$$\begin{aligned}
-\sum_{k=1}^{\infty} \log\left(1-x^k\right) &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{x^{kj}}{j} = \sum_{j=1}^{\infty} \frac{1}{j} \sum_{k=1}^{\infty} x^{kj} = \sum_{j=1}^{\infty} \frac{1}{j} \frac{x^j}{1-x^j} \\
&= \sum_{j=1}^{\infty} \frac{1}{j} \frac{x^j}{(1-x)(1+x+\cdots+x^{j-1})} \\
&\leq \sum_{j=1}^{\infty} \frac{1}{j} \frac{x^j}{(1-x)jx^{j-1}} = \frac{x}{1-x} \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} \frac{x}{1-x},
\end{aligned}$$

therefore

$$\log p(n) \leq -n \log x + \frac{\pi^2}{6} \frac{x}{1-x}.$$

Substituting $u = x/(1-x)$ we transform the interval $x \in (0,1)$ to the interval $u \in (0,\infty)$. Variable x is expressed by the formula $x = u/(1+u)$, so we obtain

$$\log p(n) \leq n \log\left(1+\frac{1}{u}\right) + \frac{\pi^2}{6} u < \frac{n}{u} + \frac{\pi^2}{6} u.$$

Simple calculation finds the minimum of the right side in $u = \sqrt{6n}/\pi$, so the final and required estimate is

$$\log p(n) < \pi \sqrt{\frac{2n}{3}}.$$

The lower bound is the result of combinatorial interpretation and is not so sharp. Denote $p^*(n,k)$ the number of partitions of n with k parts, where the order of parts is important (e.g. $(2,1,1)$ and $(1,1,2)$ are different partitions). Then simple argument gives $p^*(n,k) = \binom{n-1}{k-1}$, because we distribute $k-1$ breakpoints into $n-1$ positions. But this process counts every partition of $\mathcal{P}$ at most $k!$ times, therefore

$$p(n) \geq \frac{\binom{n-1}{k-1}}{k!} \geq \frac{(n-1)(n-2)\ldots(n-k+1)}{(k!)^2}$$

for any $k \in [n]$. Simple reasoning inspire us to take $k = \lfloor \sqrt{n} \rfloor$, because we want to find k such that $(k+1)^2 \geq n-k$ and $k^2 < n-k+1$, which is roughly $\sqrt{n}$. Since we have the estimates $k^2 \leq n$ and $(1-1/k)^{k-1} > e^{-1}$, from the monotonicity of $(1+1/m)^m \to e$ we have

$$\begin{aligned}
(n-1)(n-2)\cdots(n-k+1) \geq (n-k)^{k-1} &= n^{k-1}\left(1-\frac{k}{n}\right)^{k-1} \\
&\geq n^{k-1}\left(1-\frac{1}{k}\right)^{k-1} \geq \frac{n^k}{en}.
\end{aligned}$$

And the basic approximation $k! \leq ek(k/e)^k$ concludes the computing

$$p(n) \geq \frac{n^k}{en} \frac{e^{2k-2}}{k^{2k+2}} = \left(\frac{n}{k^2}\right)^k \frac{e^{2k-3}}{nk^2} \geq \frac{e^{2k-3}}{n^2} \geq \frac{1}{e^5 n^2} e^{2\sqrt{n}},$$

which for sufficiently large n implies the desired result. □

2.2 Partitions with parts from a finite set A

Another famous theorem which counts partitions came from I. J. Schur, 1926. In his paper [32] he found asymptotic formula for partitions with parts from a finite set A of relatively prime positive integers

Let $A = \{a_1, a_2, \ldots, a_k\}$ such that $\gcd(a_1, a_2, \ldots, a_k) = 1$. Using the famous claim stating that there are integers c_i, $i \in [k]$, for which $\sum_{i=1}^{k} c_i a_i = 1$ gives us that the equation $n = \sum_{i=1}^{k} d_i a_i$ has solution in non-negative integers d_i, $i \in [k]$ for every sufficiently large positive integer n. Therefore the counting function $p_A(n)$ is non-zero for sufficiently large n.

In the case $A = \{1, 2, 3, \ldots, k\}$ it is known the asymptotics

$$p_A(n) \sim \frac{n^{k-1}}{k!(k-1)!}.$$

We generalize this result to the theorem called Schur's asymptotic theorem. Although Schur's proof was analytical, we present here a combinatorial proof from M. B. Nathanson [25, Chapter 15].

Theorem 2.2.1 (Schur) Let A be a nonempty finite set of relatively prime positive integers, such that $|A| = k$. Then

$$p_A(n) = \frac{n^{k-1}}{(k-1)!}\left(\frac{1}{\prod_{a\in A} a}\right) + O(n^{k-2}).$$

Proof. We use induction on the cardinality of $|A| = k$. When $k = 1$, then $A = \{1\}$ due to the condition on greatest common divisor of the elements of A. That means $p_A(n) = 1$, because the only partition of n in this case is the sum of 1's.

Let $k \geq 2$. Assume the theorem holds for $k-1$. Set $A = \{a_1, a_2, \ldots, a_k\}$. The idea of the induction step is simple. According to the $\gcd(a_1, \ldots, a_{k-1})$ we find the possible multiplicities u' of the last partition a_k and due to the assumption we count $p_{A^*}(n - u'a_k)$, where $A^* = A - \{a_k\}$. Then we sum it over all possible multiplicities u' and several estimates conclude the proof.

Denote $d = \gcd(a_1, \ldots, a_{k-1})$. Because $a_1, \ldots, a_k$ are relatively prime, we have $\gcd(d, a_k) = 1$. For $i \in [k-1]$ we set

$$a'_k = \frac{a_k}{d}.$$

Then $\gcd\left(a'_1, \ldots, a'_{k-1}\right) = 1$ and we can use the induction assumption on the set $A' = \{a'_1, \ldots, a'_{k-1}\}$ and get for all positive integers n the formula

$$p_{A'}(n) = \frac{n^{k-2}}{(k-2)!}\left(\frac{1}{\prod_{a'\in A'} a'}\right) + O(n^{k-3}). \tag{2.1}$$

Let $n \geq (d-1)a_k$. Because d and a_k are relatively prime, there is a unique integer $u \in [d-1]_0$ that satisfies the congruence

$$n \equiv ua_k \pmod{d}.$$

Let

$$m = \frac{n - ua_k}{d}.$$

Then m is non-negative, because $n \geq (d-1)a_k$. Trivially $m \leq n$. Now u is one of the possible multiplicities for a_k. If we want to find all of them, we solve for non-negative integer v the congruence

$$n \equiv va_k \pmod{d}. \tag{2.2}$$

Hence modulo d we have $va_k \equiv ua_k$, which implies $u \equiv v$. Because u was the lowest non-negative integer satisfying $n \equiv ua_k \pmod{d}$, that means there exists a non-negative integer l such that $v = u + ld$. We want $n - va_k = n - (u + ld)a_k$ to be non-negative, so we have the estimate

$$0 \leq l \leq \left\lfloor \frac{n - ua_k}{da_k} \right\rfloor = \left\lfloor \frac{m}{a_k} \right\rfloor =: r \leq m.$$

So there are r solutions v of the congruence (2.2) satisfying $n - va_k \geq 0$.

Let λ be a partition of n with parts in A. If exactly v parts of λ are equal to a_k, then the rest $n - va_k$ is partitioned by parts from $\{a_1, \dots a_{k-1}\}$, therefore $d \mid n - va_k$ and v is the solution of the congruence (2.2), hence $v = u + ld$ for some $l \in [r]_0$.

So every partition λ of n with the parts belonging to A has the part a_k exactly v times, where $v = u + ld$ for some $0 \leq l \leq r$. It suffices to find the number of partitions of $n - va_k$ with parts belonging to $\{a_1, \dots a_{k-1}\}$, or equivalently to find the number of partitions of

$$\frac{n - va_k}{d} = \frac{n - (u + ld)a_k}{d} = m - la_k$$

with parts in the set $A' = \{a'_1, \dots, a'_{k-1}\}$. Hence we use the induction hypothesis (2.2) and get

$$\begin{aligned} p_A(n) &= \sum_{l=0}^{r} p_{A'}(m - la_k) \\ &= \left(\frac{1}{\prod_{i=1}^{k-1} a'_i} \right) \sum_{l=0}^{r} \left(\frac{(m - la_k)^{k-2}}{(k-2)!} + O(m^{k-3}) \right) \\ &= \left(\frac{d^{k-1}}{\prod_{i=1}^{k-1} a_i} \right) \sum_{l=0}^{r} \frac{(m - la_k)^{k-2}}{(k-2)!} + O(n^{k-2}). \end{aligned}$$

Now we sum up three claims, whose consequence gives the proof. Since we know the estimate

$$\sum_{l=0}^{r} l^j = \frac{r^{j+1}}{j+1} + O(r^j),$$

can modify the expression

$$\binom{k-2}{j} \frac{1}{(k-2)!(j+1)} = \frac{1}{(k-2-j)!(j+1)!} = \binom{k-1}{j+1} \frac{1}{(k-1)!},$$

and binomial theorem gives

$$\sum_{j=0}^{k-2} (-1)^j \binom{k-1}{j+1} = -\sum_{j=1}^{k-1} (-1)^j \binom{k-1}{j} = -(1-1)^{k-1} + 1 = 1,$$

hence we have

$$\begin{aligned}
\sum_{l=0}^{r} \frac{(m-la_k)^{k-2}}{(k-2)!} &= \frac{1}{(k-2)!} \sum_{l=0}^{r} \sum_{j=0}^{k-2} \binom{k-2}{j} m^{k-2-j} (-la_k)^j \\
&= \frac{1}{(k-2)!} \sum_{j=0}^{k-2} \binom{k-2}{j} m^{k-2-j} (-a_k)^j \sum_{l=0}^{r} l^j \\
&= \frac{1}{(k-2)!} \sum_{j=0}^{k-2} \binom{k-2}{j} m^{k-2-j} (-a_k)^j \left(\frac{r^{j+1}}{j+1} + O(r^j) \right) \\
&= \frac{1}{(k-2)!} \sum_{j=0}^{k-2} \binom{k-2}{j} m^{k-2-j} (-a_k)^j \left(\frac{m^{j+1}}{a_k^{j+1}(j+1)} + O(m^j) \right) \\
&= \frac{m^{k-1}}{a_k} \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{(-1)^j}{(k-2)!(j+1)} + O(m^{k-2}) \\
&= \frac{m^{k-1}}{a_k (k-1)!} \sum_{j=0}^{k-2} (-1)^j \binom{k-1}{j+1} + O(m^{k-2}) \\
&= \frac{m^{k-1}}{a_k (k-1)!} + O(m^{k-2}).
\end{aligned}$$

Applying to the formula for $p_A(n)$ gives

$$\begin{aligned}
p_A(n) &= \left(\frac{d^{k-1}}{\prod_{i=1}^{k-1} a_i} \right) \sum_{l=0}^{r} \frac{(m-la_k)^{k-2}}{(k-2)!} + O(n^{k-2}) \\
&= \left(\frac{d^{k-1}}{\prod_{i=1}^{k-1} a_i} \right) \left(\frac{m^{k-1}}{a_k (k-1)!} + O(m^{k-2}) \right) + O(n^{k-2}) \\
&= \left(\frac{1}{\prod_{i=1}^{k} a_i} \right) \frac{(n-ua_k)^{k-1}}{(k-1)!} + O(n^{k-2}) \\
&= \left(\frac{1}{\prod_{i=1}^{k} a_i} \right) \frac{n^{k-1}}{(k-1)!} + O(n^{k-2}),
\end{aligned}$$

which completes the proof. □

Corollary 2.2.2 Let $p_k(n)$ be the number of partitions of a positive integer n into at most k parts. Then

$$p_k(n) = \frac{n^{k-1}}{k!(k-1)!} + O(n^{k-2}).$$

Proof. The proof of Theorem 1.2.5 maps the partitions with at most k parts bijectively to the partitions with parts in the set $A_k = \{1, 2, \ldots, k\}$. Applying Schur's Theorem 2.2.1 to the set A_k concludes the proof. □

Corollary 2.2.3 Let A be an infinite set of positive integers such that $\gcd(A) = 1$. Then

$$\lim_{n \to \infty} \frac{\log p_A(n)}{\log n} = \infty.$$

In particular, $p_A(n) > 0$ for sufficiently large n.

Proof. For any positive integer k, the condition $\gcd(A) = 1$ gives existence of a set $A_k \subset A$ with cardinality k and $\gcd(A_k) = 1$. By Schur's Theorem 2.2.1

$$p_A(n) \geq p_{A_k}(n) = \frac{n^{k-1}}{(k-1)!}\left(\frac{1}{\prod_{a\in A_k} a}\right) + O(n^{k-2}),$$

so there is a constant c_k such that for all sufficiently large positive integers n we have

$$p_A(n) \geq c_k n^{k-1}.$$

Hence

$$\frac{\log p_A(n)}{\log n} \geq \frac{(k-1)\log n + \log c_k}{\log n} > k-1.$$

Because we can make k sufficiently large, applying limes inferior on the both sides gives us the desired result. □

2.3 Asymptotics for other sets of partitions

In 1954 G. Meinardus published two papers [23] and [24], where he described the asymptotics for the wide range of restricted partition functions. As a corollary we present here the asymptotic formula for the partitions with all parts congruent to a modulo k, which we use in the Chapter 3. It can be found in [3]

Theorem 2.3.1 (Meinardus) Let $H_{a,k}$ be the set of all partitions with parts congruent to a modulo k, $a \in [k]$. Then we have asymptotic formula

$$p(n, H_{a,k}) \sim Cn^{\kappa} \exp\left(\pi\sqrt{\frac{2n}{3k}}\right),$$

where

$$C = C(a,k) = \Gamma\left(\frac{a}{k}\right)\pi^{(a/k)-1}2^{-(3/2)-(a/2k)}3^{-(a/2k)}k^{-(1/2)+(a/2k)}$$

depends on the Gamma function Γ and

$$\kappa = -\frac{1}{2}\left(1 + \frac{a}{k}\right).$$

There are many asymptotic formulas for various sets of partitions. Here are examples of three asymptotics. The first is for partitions consisting of primes, the second is for partitions with parts equal to some square number and third is for partitions with parts equal to powers of 2, all can be found in [15, Chapter VIII]. Note that the second result is also the corollary of general Meinardus theorem [25].

Theorem 2.3.2 If $\mathbb{P}$ is the set of all prime numbers Then

$$\log p_{\mathbb{P}}(n) \sim 2\pi\sqrt{\frac{n}{3\log n}}.$$

Theorem 2.3.3 Let $\mathbb{S}$ be the set of all squares, that is $\mathbb{S} = \{1, 4, 9, 16, \dots\}$, then

$$p_{\mathbb{S}}(n) \sim \frac{C}{n^{7/6}} \exp\left(K \sqrt[3]{n}\right),$$

where C and K are some constants.

Theorem 2.3.4 Let $\mathbb{T}$ be the set of all powers of 2, that is $\mathbb{T} = \{1, 2, 4, 8, \dots\}$, then

$$\log p_{\mathbb{T}}(2n) \sim \frac{(\log n)^2}{2 \log 2}.$$

The asymptotic formulas for the partitions with parts k-th power of primes or squares are also known and can be found in [15], but this is not the aim of this work.

2.4 Explicit formula for $p(n)$

The work of Hardy and Ramanujan [17] on asymptotic formula for $p(n)$ was extended nearly twenty years later by Rademacher, who improved the asymptotical expansion of Hardy and Ramanujan to the convergent series. This was the first explicit formula for the counting function $p(n)$.

Theorem 2.4.1 Let $p(n)$ be the number of partitions of n. Then

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{\frac{1}{2}} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}(x - \frac{1}{24})}\right)}{\sqrt{x - \frac{1}{24}}} \right]_{x=n},$$

where

$$A_k(n) = \sum_{\substack{h \in [k-1]_0 \\ \gcd(h,k)=1}} \exp\left(-2\pi i \frac{nh}{k} + \pi i s(h,k)\right)$$

and

$$s(h,k) = \sum_{m=1}^{k-1} \left(\frac{m}{k} - \left\lfloor \frac{m}{k} \right\rfloor - \frac{1}{2}\right)\left(\frac{hm}{k} - \left\lfloor \frac{hm}{k} \right\rfloor - \frac{1}{2}\right).$$

Another man who studied the paper of Hardy and Ramanujan [17] was A. Selberg. He independently proved explicit formula 2.4.1 (not published) and Nathanson [25] mentions his funny quotation: "I am inclined to believe that Rademacher and I were the only ones to have studied this paper thoroughly since the time it was written."

Recently J. H. Bruiner and K. Ono discovered a new explicit algebraic formula for $p(n)$ using modular equations. In their preprint [11] they express $p(n)$ as a finite sum of algebraic numbers. They also show an example how to count $p(1) = 1$, but it take almost one page to evaluate it.

3. Oscillations of counting function

This chapter is devoted to studying fastest possible growth of counting functions of partition ideals, which are infinitely many times zero. All obtained results are original and are based on several combinatorial and fundamental analytical methods like principle of inclusion and exclusion or Taylor series.

First we introduce several definitions, then we make lower bound for the sought counting function. Afterwards we find asymptotic formula for fastest counting function of special family of partition ideals. This chapter is concluded by the upper bound with little characterization of partition ideals with fastest counting funtions.

3.1 Partition ideals

Definition 3.1.1 *We say that a set of partitions $X \subset P$ is a partition ideal, if $\lambda \prec \mu$ and $\mu \in X$ always implies $\lambda \in X$.*

Definition 3.1.2 *Let A be a (finite) set of distinct positive integers and p be a positive integer. Denote by $X_p(A)$ the set of all partitions λ, whose parts lie in A and their multiplicity in λ is at most p. In addition, let $X_\infty(A)$ be the set all partitions with parts in A and unbounded multiplicity. We say that A is the set of allowable parts of partition ideal $X_p(A)$, $p \in \overline{\mathbb{N}}$.*

Clearly, $X_p(A)$ is a partition ideal for $p \in \overline{\mathbb{N}}$. Because the mapping $X_p : A \mapsto X_p(A)$ is injective, the definition of allowable parts is correct.

Definition 3.1.3 *The sum $X + Y$ of two partition ideals X and Y is the set of all partitions λ of the form $\lambda = \mu + \nu$, where $\mu \in X$ and $\nu \in Y$.*

Similarly we define the finite sum $X_1 + X_2 + \ldots + X_k$ of partition ideals X_1, $X_2, \ldots, X_k$ as a set of all partitions λ of the form $\lambda = \mu^1 + \mu^2 + \cdots + \mu^k$, where $\mu^i \in X_i$, $i \in [k]$.

Finally, if $X_1, X_2, \ldots$ are partition ideals, denote the infinite sum $X_1 + X_2 + \cdots$ as a set of all partitions λ of the form $\lambda = \mu^1 + \mu^2 + \cdots + \mu^k$ such that $\mu^i \in X_{g(i)}$, $i \in [k]$, $g(i) \in \mathbb{N}$ and g is strictly increasing function.

In particular, we have $X \subset X + Y$ and $Y \subset X + Y$

Definition 3.1.4 *We say that the partition ideal X is proper partition ideal, if there are disjoint sets of positive integers $A_1, A_2, \ldots, A_\infty$ such that*

$$X = X_\infty(A_\infty) + \sum_{L=1}^{\infty} X_L(A_L).$$

That means X is a proper partition ideal if in any partition from X every part from A_k has multiplicity at most k and every part of A_∞ has unrestricted multiplicity.

These definitions allow us to present the main theorems of this thesis.

3.2 Lower bound for the greatest counting function satisfying (P)

Let X be a partition ideal. Suppose that X satisfies the condition (P), which states that there is an infinite set of integers S such that $\mathcal{P}_n \cap X = \varnothing$ for all $n \in S$, that is

$$p(n, X) = 0 \quad \text{for all} \quad n \in S. \tag{P}$$

Our aim is to study maximal growth of the counting function $p(n, X)$. Discovering this we get the oscillation of the counting function. We call it oscillation at the bottom, because of the condition (P).

Little trying gives basic example with very fast growth $p(n/2)$.

Example 3.2.1. Considering $E = \{2, 4, 6, 8, \dots\}$ the set of all even numbers, we get for even numbers n the asymptotic formula

$$p(n, X_\infty(E)) = p(n/2) \sim \frac{1}{2n\sqrt{3}} \exp\left(\pi\sqrt{\frac{n}{3}}\right).$$

and for odd n we have $p(n, X_\infty(E)) = 0$.

Now we can ask, if there is some partition ideal X with the counting function growing faster.

The answer is "yes". Moreover, for every $\beta \in (0, 1)$ we will construct a partition ideal X^β, whose counting function is infinitely many times zero and grows faster than $p(\beta n)$. Exact formulation gives Corollary 3.2.6.

Before that we make several computations in the next lemma.

Definition 3.2.2 *Let X be a partition ideal and $Q = \{q_1, q_2, \dots, q_m\}$ a finite set of positive integers. Denote*

$$p_{-Q}(n, X) = |\{\lambda \in X : q_i \not\prec \lambda \text{ for all } i \in [m]\}|$$

the number of partitions from X with no part from Q.

Lemma 3.2.3 Let X be a partition ideal such that $X = X_\infty(A)$ for some set A of positive integers. Let $p(n, X)$ be the counting function of X with asymptotic growth

$$p(n, X) \sim Cn^\kappa \exp\left(K\sqrt{n}\right),$$

for some nonzero constants C, K and κ. Suppose we have a finite set $Q \subset A$. Then

$$p_{-Q}(n, X) \sim C\left(\frac{K}{2\sqrt{n}}\right)^m \frac{\prod_{i=1}^m q_i}{n^{-\kappa}} \exp\left(K\sqrt{n}\right).$$

Proof. First let $m = 1$ and $Q_1 = \{q\}$. If we take some partition $\lambda \in X$ of n, which contains part q, and remove part q from λ, we get partition of $n - q$. Conversely to any partition of $n - q$ we can add another part q to get partition of n from X, because $q \in A$. Therefore

$$p_{-Q_1}(n, X) = p(n, X) - p(n - q, X).$$

Using assumption on asymptotic of $p(n,X)$ with some basic computing gives

$$\begin{aligned}
p_{-Q_1}(n,X) &= C\left(e^{K\sqrt{n}+\kappa\log n} - e^{K\sqrt{n-q}+\kappa\log(n-q)}\right)\\
&= Ce^{K\sqrt{n}+\kappa\log n}\left(1 - e^{K(\sqrt{n-q}-\sqrt{n})+\kappa(\log(n-q)-\log n)}\right)\\
&= Cn^{\kappa}e^{K\sqrt{n}}\left(1 - e^{K\left(-\frac{q}{2\sqrt{n}}+O(n^{-3/2})\right)+\kappa\left(-\frac{q}{n}+O(n^{-2})\right)}\right)\\
&= Cn^{\kappa}e^{K\sqrt{n}}\left(1 - e^{-\frac{Kq}{2\sqrt{n}}+O(n^{-1})}\right)\\
&= Cn^{\kappa}e^{K\sqrt{n}}\left(\frac{Kq}{2\sqrt{n}} + O(n^{-1})\right)\\
&\sim C\left(\frac{K}{2\sqrt{n}}\right)\frac{q}{n^{-\kappa}}\,\exp\left(K\sqrt{n}\right).
\end{aligned}$$

Hence we have the required asymptotics for $p_{-Q_1}(n,X)$, which is counting function of partition ideal $X_\infty(A\backslash Q_1)$. Applying the preceding procedure m-times for $q_1, q_2, \ldots, q_m$, we get the desired result

$$p_{-Q}(n,X) \sim C\left(\frac{K}{2\sqrt{n}}\right)^m \frac{\prod_{i=1}^m q_i}{n^{-\kappa}}\,\exp\left(K\sqrt{n}\right).$$

□

Theorem 3.2.4 There is a non-decreasing function $f : \mathbb{N} \to [0,1)$ such that $f(n) \to 1$ for $n \to \infty$ and a partition ideal X with the counting function $p(n,X)$ satisfying $p(n,X) = 0$ for infinitely many positive integers n and $p(n,X) > p(nf(n))$ for infinitely many positive integers n.

Proof. The idea of construction is simple. We consider integer intervals[1] I_t, $t \in \mathbb{N}$, ordered such that $i < j$ for all $i \in I_k$, $j \in I_l$, $k < l$. Desired partition ideal will be of the type

$$X = \bigcup_{t=3}^{\infty} X_t(I_t).$$

Property P will be ensured by the suitable length of the sets between I_t and I_{t+1} and the asymptotic growth will be guaranteed by the length of sets I_t.

Now we try to be more specific. Denote

$$A_k = \{k, k+1, k+2, \ldots\}, \quad A_k^* = \{1, 2, \ldots, k-1\},$$

the set of positive integers greater than $k-1$ (resp. less than k). Let L be a positive integer, $k = Lm$ multiple of L and $s \in [L-1]$. Denote $B(s)$ the set of positive integers, which are congruent to s modulo L. Moreover, denote $B_k(s) = B(s) \cap A_k$ the set of positive integers greater than $k-1$, which are congruent to s modulo L. Let

$$B_k = \bigcup_{s=1}^{L-1} B_k(s)$$

be the set of all positive integers greater or equal to k, which are not congruent to 0 modulo L. Then Theorem of Cohen and Remmel 1.4.3 gives

$$p(n, X_{L-1}(A_k)) = p(n, X_\infty(B_k)), \tag{3.1}$$

[1] Integer interval is a real interval intersected with the set of positive integers.

because assumptions for this theorem are satisfied for sets

$$\Lambda = \{(1), (2), (3), \ldots, (k-1), (k^L), ((k+1)^L), \ldots\}$$
$$\Gamma = \{(1), (2), (3), \ldots, (k-1), (Lk), (L(k+1)), \ldots\}.$$

According to Theorem 2.3.1 we have asymptotic formula for $p(n, X_\infty(B(s)))$, which states

$$p(n, X_\infty(B(s))) \sim C_s n^{-\frac{1}{2}(1+\frac{s}{L})} \exp\left(\pi\sqrt{\frac{2n}{3L}}\right)$$

and Lemma 3.2.3 used with

$$X = X_\infty(B(s)), \quad C = C_s, \quad K = \pi\sqrt{\frac{2}{3L}}, \quad \kappa = -\frac{1}{2}\left(1+\frac{s}{L}\right),$$
$$Q = \{s, s+L, s+2L, \ldots, s+(m-1)L\}$$

implies

$$\begin{aligned} p(n, X_\infty(B_k(s))) &\sim C_s \left(\pi\sqrt{\frac{1}{6Ln}}\right)^m \frac{\prod_{i=0}^{m-1}(s+iL)}{n^{\frac{1}{2}(1+\frac{s}{L})}} \exp\left(\pi\sqrt{\frac{2n}{3L}}\right) \\ &= C_{s,k}\ n^{-\frac{1}{2}(m+1+\frac{s}{L})} \exp\left(\pi\sqrt{\frac{2n}{3L}}\right), \end{aligned} \tag{3.2}$$

where

$$C_{s,k} = C_s \left(\pi\sqrt{\frac{1}{6L}}\right)^m \prod_{i=0}^{m-1}(s+iL).$$

We can write every partition in $X_\infty(B_k)$ of some positive integer n as a sum of partitions of positive integers $k_1, \ldots, k_{L-1}$ in partition ideals $X_\infty(B_k(1))$, ..., $X_\infty(B_k(L-1))$, with the property $k_1 + \cdots + k_{L-1} = n$. Combining this with (3.1) and (3.2) we get

$$\begin{aligned} p(n, X_{L-1}(A_k)) &= p(n, X_\infty(B_k)) \\ &= \sum_{k_1+\cdots+k_{L-1}=n} \prod_{s=1}^{L-1} p(k_s, X_\infty(B_k(s))) \\ &\geq \prod_{s=1}^{L-1} p(n/(L-1), X_\infty(B_k(s))) \\ &\sim \frac{C_{k,1}C_{k,2}\cdots C_{k,L-1}}{\left(\frac{n}{L-1}\right)^{\frac{1}{2}\sum_{s=1}^{L-1}(1+m+\frac{s}{L})}} \exp\left(\pi(L-1)\sqrt{\frac{2n}{3L(L-1)}}\right) \\ &= Cn^{-(m+2)(L-1)/2} \exp\left(\pi\sqrt{\frac{2n}{3}}\sqrt{\frac{L-1}{L}}\right), \end{aligned}$$

where $C = C_{k,1}C_{k,2}\cdots C_{k,L-1}$ is a constant depending only on L and k. Hence if we choose k, L arbitrary positive integers such that $L > 1$, then from preceding asymptotic we have existence of positive integer $n_L(k)$ such that for all $n \geq n_L(k)$ we have

$$p(n, X_{L-1}(A_k)) \geq \exp\left(\pi\sqrt{\frac{2n}{3}}\sqrt{\frac{L-2}{L}}\right). \tag{3.3}$$

We construct sequences $\{k_t\}_{t=3}^\infty$ and $\{n_t\}_{t=3}^\infty$ inductively in following two steps.

- Take $k_3 = 1$. From (3.3) there is positive integer n_3 such that for all $n \geq n_3$ we have
$$p(n, X_3(A_{k_3})) \geq \exp\left(\pi\sqrt{\frac{2n}{3}}\sqrt{\frac{1}{3}}\right).$$
- Assume we have k_i and n_i for $3 \leq i \leq t-1$. Define
$$k_t = (t-1)\frac{n_{t-1}(n_{t-1}+1)}{2} + 2.$$
From (3.3) there exist n_t such that for all $n \geq n_t$ we have
$$p(n, X_t(A_{k_t})) \geq \exp\left(\pi\sqrt{\frac{2n}{3}}\sqrt{\frac{t-2}{t}}\right).$$

Denote the partition ideal by
$$X = \sum_{t=3}^{\infty} X_t(A_{k_t} \cap A^*_{n_t})$$
and the function f by the formula
$$f(n) = \frac{t-2}{t} \quad \text{for} \quad n_t \leq n < n_{t+1}. \tag{3.4}$$
We show that X and f have the required properties.

Clearly $f \to 1$ for $n \to \infty$. Furthermore, for all positive integers t we have
$$p(k_t - 1, X) = 0,$$
because the sum of all allowable parts of partitions of X, which are less than k_t, is less than $k_t - 2$, due to definition of k_t. Finally
$$p(n_t, X) \geq p(n_t, X_t(A_{k_t})) \geq \exp\left(\pi\sqrt{\frac{2n_t}{3}}\sqrt{\frac{t-2}{t}}\right) > p(n_t f(n_t)),$$
which completes the proof. □

Note 3.2.5. Assuming some conditions, we made claim that there is some $f : \mathbb{N} \to [0,1)$ such that $f(n) \to 1$ for $n \to \infty$. One can ask how fast this function goes to 1. According the formula (3.4) the growth of f depends on the growth of n_t for $t \to \infty$, which partly depends on size of k and partly on the strength of estimates which we made.

Corollary 3.2.6 Let $\beta \in (0,1)$. Then there exists partition ideal X^β such that its counting function $p(n, X^\beta)$ is infinitely many time equal to zero and infinitely many times greater than $p(\beta n)$.

Proof. Theorem 3.2.4 ensures the existence of a non-decreasing function $f : \mathbb{N} \to [0,1)$ with the property that $f(n) \to 1$ for $n \to \infty$ and a partition ideal X_f, whose counting function is infinitely many times zero and for infinitely many positive integers $p(n, X) > p(nf(n))$ holds. Because $f(n) > \beta$ for sufficiently large n, we have also $p(n, X) > p(\beta n)$ for sufficiently large n. □

3.3 Upper bound for a special case

We proved that there is a partition ideal with fast growing function. What kind of assumptions we have to add in order to receive the growth of the counting function from Example 3.2.1 as best as possible? Natural condition is to demand the partition ideal to be proper. But Theorem 3.2.4 shows that this is not enough. So we add the condition about the partitions of some numbers, as the following Theorem 3.3.2 shows.

Definition 3.3.1 *Let X be partition ideal with counting function $p(n, X)$. Define*

$$F(n, X) = \max\{p(m, X) : m \in [n]\}.$$

Theorem 3.3.2 Let X be a proper partition ideal such that two following conditions are true:

(i) There is a positive integer r, such that for almost all[2] positive integers k the number kr has in X at least one partition with parts in the set $R = \{r, 2r, 3r, \dots\}$ of multiples of r.

(ii) The counting function $p(n, X) = 0$ for infinitely many positive integers n.

Then the maximal asymptotic growth of $F(n, X)$ is $p(n/2)$.

Proof. Let

$$X = X_\infty(A_\infty) + \sum_{L=1}^{\infty} X_L(A_L).$$

Let $\mathcal{R}$ be the set of all positive integers satisfying (i) and r be the smallest of them. If $A_\infty \not\subset R$, say $a \in A_\infty \backslash R$, then $\gcd(a, r) = r_0 < r$. Because $r_0 \in \mathcal{R}$, we have a contradiction with the choice of r. Therefore $A_\infty \subset R$.

Denote $A = \cup_{i=1}^{\infty} A_i$. The condition (ii) gives $r > 1$. For $r = 2$, there cannot be any odd number in A, because then condition (ii) is false. So we have $F(n, X) \le F(n, X_\infty(A)) \le p(n/2)$.

Let $r > 2$. If $A \subset R$, then $X \subset X_\infty(R)$ and

$$F(n, X) < F(n, X_\infty(R)) = p(n/r) \ll p(n/2),$$

so we assume that the set $A\backslash R$ is nonempty.

Suppose that $A\backslash R$ is finite. Then there is a positive integer i_0 such that $A_L\backslash R = \varnothing$ for all $L > i_0$. Therefore the partition ideal $X^* := \sum_{L=1}^{i_0} X_L(A_L\backslash R)$ is finite and we can bound the number of partitions in this partition ideal by

$$|X^*| \le \prod_{L=1}^{i_0} L^{|A_L|}.$$

Because every partition of n contains some parts from $A\backslash R$ and the remaining parts from R, we have the estimate

$$p(n, X) \le \left(\prod_{L=1}^{i_0} L^{|A_L|}\right) p(n, X_\infty(R)) \le \left(\prod_{L=1}^{i_0} L^{|A_L|}\right) p(n/r) \ll p(n/2),$$

[2] in the sense at most finitely many positive integers does not have this property

because $r > 2$. Therefore $F(n, X) \ll p(n/2)$.

If the set $A \backslash R$ is infinite, we distribute the numbers of $A \backslash R$ to the residue classes $M_1, M_2, \ldots, M_{r-1}$, so that $x \in M_a$ iff $x \equiv a \pmod r$. Define numbers $a_1 < a_2 < \cdots < a_m$ with condition that the sets $M_{a_1}, M_{a_2}, \ldots, M_{a_m}$ are infinite and m is maximal possible.

If $\gcd(r, a_1, a_2, \ldots, a_m) = 1$, then there is a positive integer n_0, such that for every $n > n_0$ the statement $p(n, X) > 0$ holds, which is in contradiction with the condition (ii). So we have $\gcd(r, a_1, a_2, \ldots, a_m) = s_0 > 1$. Define $S_0 = \{s_0, 2s_0, 3s_0, \ldots\}$ the set of multiples of s_0. If $s_0 = 2$, then because of condition (ii), there cannot be any odd number in A, therefore

$$p(n, X) \le p(n, X_\infty(S_0)) = p(n/2),$$

which implies $F(n, X) \le p(n/2)$.

In the case $s_0 > 2$, we have the set $A - S_0$ finite and using the same argument like above we get desired result. We find positive integer i_1 such that $A_L \backslash R = \varnothing$ for all $L > i_1$. Then

$$p(n, X) \le \left(\prod_{L=1}^{i_1} L^{|A_L|} \right) p(n, X_\infty(S_0)) \le \left(\prod_{L=1}^{i_1} L^{|A_L|} \right) p(n/s_0) \ll p(n/2).$$

Hence $F(n, X) \ll p(n/2)$.

Note that we have $F(n, X) = p(n/2)$ only for $X = X_\infty(E)$, where E is the set of all even numbers. □

3.4 Counting function at the top

We have shown in Theorem 3.2.4 and Corollary 3.2.6 several examples of fast growing counting functions with property (P). One could conjecture that we can make partition ideal with counting function growing arbitrarily fast but slower than the function $p(n)$.

This conjecture is false because we can show that for every real constants κ and C and every partition ideal with property (P) we have

$$p(n, X) < C n^{-\kappa} \exp\left(\pi \sqrt{\frac{2n}{3}} \right).$$

In this section we set the constants c and K as

$$c = \frac{1}{4\sqrt{3}}, \quad K = \pi \sqrt{\frac{2}{3}}.$$

Definition 3.4.1 *Let Z be a set of partitions that forms an antichain in the sense, that there are no two partitions in Z such that one is the subpartition of the other. Denote $F(Z)$ set of all partitions $\lambda \in \mathcal{P}$ such that for any $\mu \in Z$ we have $\lambda \not> \mu$. We call the set Z the forbidden set of partition ideal $F(Z)$.*

Observe that the set $F(Z)$ is partition ideal, because $\lambda \not> \mu$ and $\nu < \lambda$ implies $\nu \not> \mu$.

Remark that if Z is a set of positive integers, then Z is also the set of partitions, and we have double notation $p(n, F(Z))$ and $p_{-Z}(n) = p_{-Z}(n, \mathcal{P})$ for the number of partitions of n that do not contain any part from Z.

Definition 3.4.2 *We say that two partitions $\lambda, \mu \in \mathcal{P}$ are independent, if the parts of λ and μ are pairwise distinct. We say that a set S of partitions is independent, if any two partitions in S are independent.*

Theorem 3.4.3 Let X be a partition ideal with counting function $p(n, X)$ and forbidden set of partitions Z. Let Z contains an infinite independent subset. Let κ, C be any positive real constants. Then

$$p(n, X) < Cn^{-\kappa} e^{K\sqrt{n}}$$

for sufficiently large n.

Proof. Let Z' be infinite independent subset of Z. Denote

$$Z' = \{\lambda^1, \lambda^2, \dots\} \quad \text{and} \quad Z'' = \{|\lambda^1|, |\lambda^2|, \dots\}.$$

Because Z' is independent, the conditions of Cohen–Remmel Theorem 1.4.3 are satisfied for the sets

$$\begin{aligned} \Lambda &= \{\lambda^1, \lambda^2, \lambda^3, \dots\} \\ \Gamma &= \{|\lambda^1|, |\lambda^2|, |\lambda^3|, \dots\}, \end{aligned}$$

and we have

$$p(n, F(Z')) = p(n, F(Z'')).$$

Now fix C and κ positive real numbers independent on n. Denote

$$Z''_m = \{|\lambda^1|, |\lambda^2|, \dots, |\lambda^m|\},$$

where $m > 2(\kappa - 1)$ is a positive integer. Using Lemma 3.2.3 with

$$X = \mathcal{P}, \quad C = c, \quad \kappa = -1, \quad Q = Z'',$$

we get

$$\begin{aligned} p(n, F(Z''_m)) &\sim c \left(\frac{K}{2\sqrt{n}}\right)^m \frac{\prod_{i=1}^m |\lambda^i|}{n} e^{K\sqrt{n}} \\ &= \frac{cK^m \prod_{i=1}^m |\lambda^i|}{2^m} n^{-\frac{m}{2}-1} e^{K\sqrt{n}}. \end{aligned}$$

Because $m > 2(\kappa - 1)$, we have $-m/2 - 1 < -\kappa$. Hence

$$p(n, F(Z''_m)) < Cn^{-\kappa} e^{K\sqrt{n}}$$

for sufficiently large n. In total

$$\begin{aligned} p(n, X) &\le p(n, F(Z)) \le p(n, F(Z')) \\ &= p(n, F(Z'')) < p(n, F(Z''_m)) \\ &< Cn^{-\kappa} e^{K\sqrt{n}}, \end{aligned}$$

which yields the theorem. □

Theorem 3.4.4 Let X be a partition ideal satisfying (P). Then following is true.

(1) There is a partition ideal X and a sequence $\{n_k\}_{k=1}^{\infty}$ such that $p(n_k, X) > p(n_k f(n_k))$ for some $f : \mathbb{N} \to [0, 1)$ with condition $f(n) \to 1$ for $n \to 1$.

(2) For any κ and C positive reals we have $p(n, X) < Cn^{-\kappa} e^{K\sqrt{n}}$ for sufficiently large n.

Proof. The first part of the statement is a result of Theorem 3.2.4, the second part we prove using Theorem 3.4.3.

Denote S the set of positive integers, such that $p(n, X) = 0$ for $n \in S$. We can regard the set S also as a set of partitions with only one part. Because S is infinite, Theorem 3.4.3 implies

$$p(n, X) < p(n, F(S)) < Cn^{-\kappa} e^{K\sqrt{n}}$$

for sufficiently large n, which is what we wanted to prove. □

From the Lemma 3.2.3 we have the asymptotics for counting functions of partition ideals, which cannot use numbers from some finite set Y, and we call them $p(n, F(Y))$. Are there other counting functions of some partition ideals, which grows faster than $p(n, F(Y))$? The answer is partially hidden in the following theorem.

Theorem 3.4.5 Let $X = F(Z)$ be the partition ideal with the finite nonempty forbidden set Z. Than the asymptotics for the counting function $p(n, F(Z))$ is of the form

$$p(n, F(Z)) \sim Cn^{\kappa} e^{K\sqrt{n}},$$

where C is a constant and $\kappa = -m/2$ for some $m \in \mathbb{N} \setminus \{1, 2\}$.

Proof. First we show that the counting function $p(n, X)$ has asymptotics greater than $K_1 n^{\kappa_1} e^{K\sqrt{n}}$ for some constants $K_1 = K_1(Z)$ and $\kappa_1 = \kappa_1(Z)$. We define Z^* the set of all distinct parts, which are contained in some partition of Z. Because Z is finite, so is Z^*. Let $Z^* = \{a_1, a_2, \ldots, a_m\}$. From $F(Z^*) \subset F(Z) = X$ we have

$$p(n, X) \geq p(n, F(Z^*)).$$

But from Lemma 3.2.3 we know the asymptotic for $p(n, F(Z^*))$, which is

$$p(n, F(Z^*)) \sim c \left(\frac{K}{2\sqrt{n}} \right)^m \frac{\prod_{i=1}^m a_i}{n} \exp\left(K\sqrt{n}\right).$$

Hence

$$K_1 = c \left(\frac{K}{2} \right)^m \prod_{i=1}^m a_i, \quad \kappa_1 = -\frac{m}{2} - 1.$$

Second observation counts the number of partitions of n which contain partition λ as a subpartition. We show that $p(n - |\lambda|)$ is wanted number, because there is a one-to-one mapping between partitions of n which contain λ and all partitions of $n - |\lambda|$. If μ is a partition of $n - |\lambda|$, then partition $\mu + \lambda$ is a partition of n which contains λ. In the other way, from any partition μ of n which contains λ we can remove λ, so we obtain partition $\mu - \lambda$ of $n - |\lambda|$.

Now we focus on the proof of theorem. Because Z is finite, we denote $k = |Z|$ the number of partitions in Z. We use the principle of inclusion and exclusion (PIE) to count all partitions in $F(Z)$ in the following way.

First allow all partitions from $\mathcal{P}$, their number is $p(n)$. Then subtract all partitions which contain any partition from the forbidden set Z[3], whose number

[3] That means if any partition contains more partitions from Z, we subtract them more times.

is the sum $\sum_{\lambda_1 \in Z} p(n - |\lambda_1|))$. Now we have subtracted several partitions more times, so we add all partitions, which are the union of two partitions of Z. Their number is $\sum_{\{\lambda_1,\lambda_2\}\in\binom{Z}{2}} p(n-|\lambda_1 \cup \lambda_2|))$. And continue according to the principle of inclusion and exclusion until we add $(-1)^k \sum_{\{\lambda_1,\ldots,\lambda_k\}\in\binom{Z}{k}} p(n - |\cup_{j=1}^k \lambda_j|))$. We get

$$\begin{aligned} p(n,X) &= p(n) - \sum_{\lambda_1 \in Z} p(n|\lambda_1|)) + \sum_{\{\lambda_1,\lambda_2\}\in\binom{Z}{2}} p(n - |\lambda_1 \cup \lambda_2|)) + \ldots \\ &\quad + (-1)^k \sum_{\{\lambda_1,\ldots,\lambda_k\}\in\binom{Z}{k}} p(n - |\cup_{j=1}^k \lambda_j|)) \\ &= \sum_{i=0}^{k} (-1)^i \sum_{\{\lambda_1,\ldots,\lambda_i\}\in\binom{Z}{i}} p(n - |\cup_{j=1}^i \lambda_j|)). \end{aligned}$$

Together with Theorem 2.1.1 we get

$$p(n,X) \sim \sum_{i=0}^{k} (-1)^i \sum_{\lambda_1,\ldots,\lambda_i\in\binom{Z}{i}} \frac{c}{n - |\cup_{j=1}^i \lambda_j|}\, e^{K\sqrt{n-|\cup_{j=1}^i \lambda_j|}},$$

hence denoting $q = |\cup_{j=1}^i \lambda_j|$ we get

$$\begin{aligned} \frac{p(n,X)}{p(n)} &\sim \sum_{i=0}^{k} (-1)^i \sum_{\lambda_1,\ldots,\lambda_i\in\binom{Z}{i}} \exp\left(K\sqrt{n-q} - K\sqrt{n} - \log\frac{n-q}{n}\right) \\ &= -\sum_{i=0}^{k} (-1)^i \sum_{\lambda_1,\ldots,\lambda_i\in\binom{Z}{i}} \underbrace{\left(1 - \exp\left(K\sqrt{n-q} - K\sqrt{n} - \log\frac{n-q}{n}\right)\right)}_{Q(n,q)}, \end{aligned}$$

because the ones sum up to 0 due to the binomial theorem. Let us modify the expression $Q(n,q)$ by expanding $\sqrt{n-q} - \sqrt{n}$ and $\log(1-q/n)$ for $n \to \infty$ and $1 - e^x$ for $x \to 0$. We have

$$\begin{aligned} Q(n,q) &= 1 - \exp\left(K\sqrt{n-q} - K\sqrt{n} - \log\frac{n-q}{n}\right) \\ &= 1 - \exp\left(K\sum_{i=1}^{\infty}(-1)^i\binom{1/2}{i}\sqrt{n}\left(\frac{q}{n}\right)^i + \sum_{i=1}^{\infty}\frac{1}{i}\left(\frac{q}{n}\right)^i\right) \\ &= \sum_{i=1}^{\infty} C_{i,q} n^{-i/2}, \end{aligned}$$

where $C_{i,q}$ are some constants depending on i and q. Therefore the fraction

$$\frac{p(n,X)}{p(n)} \sim \sum_{i=1}^{\infty} C_i n^{-i/2},$$

where C_i are some constants, must be of same type. Hence

$$p(n,X) \sim p(n) \sum_{i=1}^{\infty} C_i n^{-i/2} = c\left(\frac{C_1}{n^{3/2}} + \frac{C_2}{n^2} + \frac{C_3}{n^{5/2}} + \ldots\right) e^{K\sqrt{n}}.$$

From the first part of the proof we have lower bound $K_1 n^{\kappa_1} e^{K\sqrt{n}}$ for the asymptotic of $p(n,X)$, therefore there is i_0 such that $C_{i_0} \neq 0$. We take the smallest i_0 with such a condition. Then

$$p(n,X) \sim c\, C_{i_0} n^{-i_0/2-1} e^{K\sqrt{n}},$$

so defining $C = cC_{i_0}$ and $\kappa = -i_0/2 - 1$ we concludes the proof.

□

Example 3.4.6. The preceding proof gives us exact way how to count the counting functions of partition ideals with finite forbidden sets. We shall show one example, how to count them. We analyse the counting function $p(n, X)$ of the partition ideal $X = F(Z)$, where $Z = \{(2,3),(2,4),(3,4)\}$.

According to the PIE we have

$$p(n,X) = p(n) - p(n-5) - p(n-6) - p(n-7) + 2p(n-9).$$

Hence

$$\begin{aligned}
\frac{p(n,X)}{p(n)} &= \sum_{q\in\{5,6,7\}} \left(1 - \exp\left(-K(\sqrt{n} - \sqrt{n-q}) - \log\frac{n-q}{n}\right)\right) + \\
&\quad 2\exp\left(-K(\sqrt{n} - \sqrt{n-9}) - \log\frac{n-9}{n}\right) - 2 \\
&= \sum_{q\in\{5,6,7\}} \left(1 - \exp\left(-\frac{Kq}{2\sqrt{n}} + O(n^{-3/2}) + \frac{q}{n} + O(n^{-2})\right)\right) + \\
&\quad 2\exp\left(-\frac{9K}{2\sqrt{n}} + O(n^{-3/2}) + \frac{9}{n} + O(n^{-2})\right) - 2 \\
&= \frac{K}{2\sqrt{n}}(5+6+7-2\cdot 9) - \frac{1}{n}(5+6+7-2\cdot 9) \\
&\quad - \frac{K^2}{8n}(5^2+6^2+7^2-2\cdot 9^2) + O(n^{-3/2}) \\
&= \frac{13K^2}{2n} + O(n^{-3/2}).
\end{aligned}$$

Therefore

$$p(n,X) \sim \frac{13K^2}{8\sqrt{3}n^2} e^{K\sqrt{n}}.$$

Example 3.4.7. If $Z = \{(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)\}$ and $X = F(Z)$, then

$$p(n,X) \sim \frac{77K^3}{16\sqrt{3}n^{5/2}} e^{K\sqrt{n}}.$$

4. Combinatorial enumeration

Up to now we were studying partitions, partition ideals and growth of their counting functions. In order to make idea what claims can be made in partitions, we introduce similar structures and sum up couple of theorems that are proved for them.

So the first aim of this chapter is to generalize this theory to more combinatorial structures and then present several results for special cases like graphs, permutations and words. We put emphasize on asymptotics and oscillations of "counting functions" of these structures.

This chapter is inspired by surveys of Bollobás [4] and Klazar [20].

4.1 General definitions

Definition 4.1.1 *We say that a set U is a universe, if it is a set of some combinatorial structure. We introduce a partial ordering $\prec$ on U, that means $\prec$ is a substructure relation U, and a size function $s : U \to \mathbb{N}_0$. We denote this fact by a triple $(U, \prec, s)$. Let $P \subset U$, denote*

$$p(n, P) = \#\{A \in P : s(A) = n\}$$

the labeled counting function of the set P.

We say that a set $X \subset U$ is an ideal, if $A \prec B$ and $B \in X$ implies $A \in X$.

Most of the time we consider the size function s strictly increasing. That means for every $A, B \in X$ such that $A \prec B$ holds $s(A) < s(B)$.

Example 4.1.2. In the case of partitions, the universe U is the set of all partitions $\mathcal{P}$, the partial ordering is the subpartition relation and the size function is the absolute value of the partition. The ideals are called partition ideals and the labeled counting function of the set has the same notion when we drop labeled.

Definition 4.1.3 *Let $Z \subset U$. Define*

$$F(Z) = \{A \in U : A \not\succ B \text{ for every } B \in Z\}.$$

the set of all elements of U that do not contain (in the sense of $\prec$) any element of Z.

Moreover we request that the set Z is an antichain, which means that no two elements of Z are comparable by relation $\prec$. Then the mapping $F : 2^A \to 2^A$ defines a one-to-one correspondence between antichains of U and the ideals of U.

Definition 4.1.4 *Let $(U, \prec, s)$ be the universe with relation $\prec$ and size function s. In addition, suppose we have on U an equivalence relation $\sim$, hence we get $(U, \prec, s, \sim)$. Then the function*

$$q(n, P) = \#(\{A \in P : s(A) = n\}/\sim)$$

is called the unlabeled counting function.

Example 4.1.5. Now suppose, that $\mathcal{P}^*$ is the set of all partitions, where the order of parts is important. That means partitions $(2,1,1)$ and $(1,2,1)$ are different partitions, both belong to $\mathcal{P}^*$. On $\mathcal{P}^*$ we introduce the equivalence $\sim$, such that $\lambda \sim \mu$ if and only if they differ only in the order of the parts. That means $\mathcal{P}^*/_\sim = \mathcal{P}$.

Taking $U = \mathcal{P}^*$, $\prec$ to be the subpartition relation and s to be the absolute value, we have two counting functions, labeled and unlabeled. While unlabeled counting function counts the number of partitions in $\mathcal{P}$ with certain property, labeled counting function counts the number of partitions in $\mathcal{P}^*$ with the same property.

4.2 Properties of the counting function of graphs

This section is devoted to several combinatorial properties of finite graphs. Our terminology is standard, $V(G)$ denotes the vertex set of a graph G and $E(G)$ its edge set. The order or the number of vertices of graph G is denoted by $|V(G)| = |G|$ and the size or the number of edges is denoted by $|E(G)|$. For a graph G and its vertex $x \in V(G)$ we define graph $G - x$, which is obtained from G by deleting vertex x and all edges incident with it. Graph K_r denotes the complete graph on r vertices, graph C_r denotes the cycle on r vertices.

Definition 4.2.1 *Let $\mathcal{G}_n$ be the set of graphs with the vertex set $[n]$. Denote $\mathcal{G} = \cup_{i=1}^{\infty}\mathcal{G}_i$ set of all finite graphs.*

Now we apply definitions from the general theory to the graph theory. The universe U is the set of all graphs $\mathcal{G}$. The ordering $\prec$ here means the relation to be induced subgraph and the size function denotes the order of graph G. If $P \subset \mathcal{G}$, we have the labeled counting function $|P_n|_p$, which we mostly denote by $|P_n|$.

If we introduce a (isomorphic) relation $\sim$ on $\mathcal{G}$ such that $G_1 \sim G_2$ whenever G_1 is isomorphic to G_2, that means the labeling of the vertices is not important, we get structure $(\mathcal{G}, \prec, |\cdot|, \sim)$. Then for a set $P \subset \mathcal{G}$ we have the unlabeled counting function $|P_n|_q$. To prevent misunderstanding, we will always write the lower index q in this notation.

Clearly, $|P_n| \geq |P_n|_q$ for all $P \subset \mathcal{G}$, and moreover $|P_n| \leq n!|P_n|_q$, because there are at most $n!$ graphs with the vertices in $[n]$, which are mutually isomorphic. Therefore we have

$$|P_n|_q \leq |P_n| \leq n!|P_n|_q.$$

Example 4.2.2. Let K^* be a star graph with infinitely many terminal vertices and $P = P(K^*)$ be a set of all graphs which are induced subgraphs of K^*. Then

$$|P_n| = n + 1 \quad \text{and} \quad |P_n|_q = 2.$$

Definition 4.2.3 *A set $\mathcal{L} \subset \mathcal{G}$ of graphs is called hereditary, if it is closed under taking induced subgraphs. Equivalently, $\mathcal{L}$ is hereditary, if $G - x \in \mathcal{L}$ for all $G \in \mathcal{L}$ and $x \in V(G)$. Further, defining $\mathcal{L}_n = \mathcal{L} \cap \mathcal{G}_n$ we can write $\mathcal{L} = \cup_{i=1}^{\infty}\mathcal{L}_i$.*

Finally, we say that property $\mathcal{L}$ is monotone, if it is closed under taking any subgraph.

Obviously, if the set $\mathcal{L}$ of graphs is monotone, then it is hereditary.

Hereditary or monotone sets of graphs are analogical to partition ideals. When we have some properties about hereditary or monotone sets of graphs, we can try to carry them to partition ideals. Therefore we are interested in the labeled counting functions, their asymptotic formulas and their oscillations.

The next theorem summarizes the results about the growth of labeled counting function $p(n, P)$ for P hereditary set of graphs. We will not prove it, but make only several observations. The proofs can be found in papers [6], [8], [9] and [30].

Theorem 4.2.4 Let $\mathcal{L}$ be the hereditary property of graphs. Then one of the following cases holds for sufficiently large n.

1. $|\mathcal{L}_n|$ is identically zero, one or two.
2. There is an integer $k > 0$ such that $|\mathcal{L}_n|$ is a polynomial degree k in n.
3. There is an integer $k > 1$ such that $|\mathcal{L}_n|$ has exponential order of the form $\sum_{i=1}^{k} p_i(x)i^n$, where $p_i(n)$ is a polynomial in n and $p_k(n)$ is non-zero.
4. There is an integer $r > 1$ such that $|\mathcal{L}_n| = n^{(1-1/r)n+o(n)}$.
5. One has $B(n) \leq |\mathcal{L}_n| \leq 2^{o(n^2)}$, where $B(n)$ is the n-th Bell number, which denotes the number of partitions[1] of a set S with $|S| = n$.
6. There is an integer $r > 1$ such that $|\mathcal{L}_n| = 2^{(1-1/r)n^2/2+o(n^2)}$.

The jump from 1 to 2 is surprisingly easy to justify. As observed in [30], if $|\mathcal{L}_n| > 2$, then $\mathcal{L}_n$ contains a graph G_1 which is neither empty nor complete. Such a graph must contain a vertex x such that $1 \leq \deg(x) \leq n-2$. So there must be at least $\binom{n-1}{\deg(x)}$ labellings of G_1, thus $|\mathcal{L}_n| > n-1$.

If we restrict boundaries for $\deg(x)$ in the sense $2 \leq \deg(x) \leq n-3$, we get $|\mathcal{L}_n| > n^2/2 - n/2$. With a little more effort one can show, as shown in [6], that the polynomial range in case 2. is divided into separated subranges with minimum and maximum, which implies case 2.

For the jump between cases 2 and 3, the structure of the case 3 and the jump between cases 3 and 4 the statement is not so easy, there is important the cardinality of the set $\mathcal{L}$. The proof can be found in [6].

The jump from 4 to 5 is associated with the set S consisting of all graphs, whose components are complete graphs. Clearly S is a hereditary set. To find the cardinality of S is not difficult. Suffices to partition the set of vertices $[n]$ to the sets, which can be made exactly $B(n)$ times. So every hereditary set R of graphs such that $S \subset R$ has the counting function at least $B(n)$. The other case is described in [8].

Till this time we could expect that the function $|\mathcal{L}_n|$ is well-behaved. But as the following Theorem 4.2.5, which can be found in [7], shows, growth of the counting functions in the case 5 is really confusing. Although the statement proves only existence of monotone set with big oscillations, it is conjectured that it holds for hereditary sets.

[1] This is not integer partition, but partition of the set. That means we want to divide set to non-overlapping and non-empty sets.

Theorem 4.2.5 Let $1 < c < c'$ and $\varepsilon > 1/c$. Let $f(n)$ be a function such that $n^{c'n} < f(n) < 2^{n^{2-\varepsilon}}$ for all n. Then there are integer sequences $\{r_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=1}^{\infty}$ and a monotone set $\mathcal{L}$ of graphs such that

(1) $|\mathcal{L}_n| = n^{cn+o(n)}$ for $n = r_i$,

(2) $n^{cn+o(n)} \leq |\mathcal{L}_n| \leq f(n)$ for all n,

(3) $|\mathcal{L}_n| > f(n) - n!$ for $n = s_i$.

Notice that the constant ε depends on c. It is conjectured that preceding Theorem 4.2.5 holds for c, ε independent.

Let us sum up gained properties of hereditary sets of graphs. Theorem 4.2.5 shows, that the counting function $|\mathcal{L}_n|$ is well-behaved at the bottom. In particular, if we have $|\mathcal{L}_n| = 0$ for some n, then $|\mathcal{L}_n| = 0$ for all n. Therefore entirely at the bottom oscillations are not possible. Going higher we have exactly polynomial or exponential growth, so there could be oscillations, but very small. Only case, where the oscillations can be very large, is case 5. in Theorem 4.2.5.

4.3 Permutations

Definition 4.3.1 *A permutation π of $[n]$ is a injective mapping $\pi : [n] \to [n]$. We mark π as a sequence $\pi(1)\pi(2)\ldots\pi(n)$. Number n denotes the length or the absolute value of π. We denote the set of all permutations of $[n]$ as $\mathcal{S}_n$ and the set of all finite permutations as $\mathcal{S}$.*

On the set $\mathcal{S}$ we introduce a relation $\prec$ in the following way. Let $\pi = a_1 a_2 \ldots a_n$ be a permutation on $[n]$ and $\rho = b_1 b_2 \ldots b_m$ be a permutation on $[m]$. We say, that $\pi \prec \rho$ if $n \leq m$ and π can be obtained from ρ by deleting $m - n$ terms of ρ and reducing the remaining into $[n]$ while keeping their order. More precisely there are indices $1 \leq m_1 < m_2 < \cdots < m_n$ such that $\rho(m_i) < \rho(m_j)$ if and only if $\pi(i) < \pi(j)$, for $i, j \in [n]$. Instead of $\pi \prec \rho$ we sometimes say that ρ contains π. When this is not true, we say that ρ avoids π.

Let us make an example. Permutation $\rho = 14752638$ contains permutation $\pi = 14325$, because numbers $1, 7, 6, 3$ and 8 has the same order in ρ as numbers $1, 4, 3, 2, 5$ in π.

At this time we have structure $(\mathcal{S}, \prec, |\cdot|)$. We introduce an ideal sets.

Definition 4.3.2 *Let $X \subset \mathcal{S}$ be a set of permutations. We say, that X is hereditary, if X is closed under containment. Denote the counting function of X as*

$$p(n, X) = |X \cap \mathcal{S}|.$$

Definition 4.3.3 *Let $Z \subset \mathcal{S}$ be a set of permutations. Denote $F(Z)$ the set of all permutations that do not contain any permutation of Z. We say, that set Z is the forbidden set for the hereditary set $F(Z)$.*

One can easily check, that F maps sets of permutations on the hereditary sets of permutation. If Z is one-element set, $Z = \{\rho\}$, we write $F(\rho)$ instead of $F(\{\rho\})$. Now we can focus on the growth of the counting functions.

Obviously, $p(n, \mathcal{S}) = n! = n^{n+o(n)}$. But what speed has the counting function of a non-trivial hereditary set? In particular, we want to find the counting function of $F(\rho)$ for some permutation ρ. Stanley–Wilf conjecture (first mentioned by Bóna [10]) states that for every permutation $\rho \in \mathcal{S}$ there is a constant c such that $p(n, F(\rho)) \leq c^n$. Although this statement is rather strong, because avoiding only one permutation ρ the growth of the counting function falls from $n!$ to the exponential growth, it was proved by Marcus and Tardos [21] in 2004 with a contribution by Klazar [19].

Theorem 4.3.4 Let X be the hereditary set of permutations which is not equal to the set of all permutations. Then there is a constant c such that we have the estimate

$$p(n, X) < c^n$$

for every positive integer n.

Second theorem about the growth of counting functions was published Kaiser and Klazar [18] in 2002. Before that we introduce generalized Fibonacci numbers as follows: $F_{n,k} = 0$ for $n \leq 0$, $F_{n,1} = 1$ and

$$F_{n,k} = F_{n-1,k} + F_{n-2,k} + \cdots + F_{n-k,k}$$

for $n > 0$. Thus the function $F_{n,k}$ grows as α_k, as a function of n, where α_k is the largest positive root of $x^k - x^{k-1} - x^{k-2} - \ldots - 1$. And now promised theorem.

Theorem 4.3.5 Let X be a hereditary set of permutations. Then exactly one of the following possibilities holds.

1. Counting function $p(n, X)$ is bounded.
2. There are integers $k, l \geq 1$ such that for every n

$$F_{n,k} \leq p(n, X) \leq n^l F_{n,k}.$$

3. We have the estimate $p(n, X) \geq 2^{n-1}$ for every n.

These two Theorems 4.3.4 and 4.3.5 are sufficient to make a observation, that oscillations of permutations at the bottom are not large.

4.4 Words

Let A be a finite set and n positive integer. We say that ω is a word of length n over the alphabet A, or n-word, if a is a sequence $a = a_1 a_2 a_3 \ldots a_n$, where $a_i \in A$ for $i \in [n]$. A word a is finite, if it is n-word for some n.

An n-block of a word $a = a_1 a_2 a_3 \ldots a_n$ is a sequence $a_{j+1} a_{j+2} \ldots a_{j+n}$ for some $j \in \mathbb{N}$. Let $n \geq k$. Observe that n-word has exactly $n - k + 1$ blocks of length k.

Definition 4.4.1 *Set $\mathcal{L}$ of words is said to be hereditary, if any block of a word from $\mathcal{L}$ is in $\mathcal{L}$. Denote all n-words of $\mathcal{L}$ as $\mathcal{L}_n$. We have $\mathcal{L} = \cup_{i=1}^{\infty} \mathcal{L}_n$. We say that function $n \mapsto |\mathcal{L}_n|$ is a counting function of the set $\mathcal{L}$.*

Let W be the set of (finite) words. Denote $\mathcal{L}(W)$ the set of all words which forms a block of words from W. In particular, $\mathcal{L}(a)$ is a set of blocks of the word a. We mostly write $\mathcal{L}(a)$ instead of $\mathcal{L}(\{a\})$.

Example 4.4.2. Let $A = \{0, 1\}$ and $a = 10101010\ldots$. Then $|\mathcal{L}_4(a)| = 2$, because $\mathcal{L}_4(a) = \{1010, 0101\}$. Actually $|\mathcal{L}_n(a)| = 2$ for all $n \geq 1$.

Using the theory of word graphs, Balogh and Bollobás [5] proved in 2005 theorem about the growth of counting function of hereditary sets of words. This paper contains the note about possible oscillations, therefore it is important for us and we sum it to the next theorem and consecutive note.

Theorem 4.4.3 Let $\mathcal{L}$ be hereditary set of finite words over an alphabet A. Then the counting function $\mathcal{L}_n$ is either bounded or at least $n + 1$ for every n.

Although $|\mathcal{L}_n|$ is in the first case bounded, Balogh and Bollobás [5] proved, that for every k there is a hereditary set $\mathcal{L}$ of the words such that

$$\limsup_{n\to\infty} |\mathcal{L}_n| = k^2 \quad \text{and} \quad \liminf_{n\to\infty} |\mathcal{L}_n| = 2k - 1,$$

or similarly

$$\limsup_{n\to\infty} |\mathcal{L}_n| = k(k+1) \quad \text{and} \quad \liminf_{n\to\infty} |\mathcal{L}_n| = 2k.$$

However, if $|\mathcal{L}_n| \leq n$ for some n, then the counting function is not just bounded, but cannot even be larger than examples above.

Therefore possible oscillations at the bottom in the case of finite words over alphabet A are bounded independently on n.

Conclusion

We made upper bound and lower bound in Theorem 3.4.4 for the fastest counting function of the partition ideal of integer partitions, which satisfies condition (P). Especially the lower bound grows very fast, which means there is an enormous oscillation of the counting function of some partition ideals with the condition (P).

When we consider partition ideals satisfying both condition (P) and condition on partitions of some positive integers, we deduced in Theorem 3.3.2 that the counting function cannot have greater asymptotics than $p(n/2)$.

Chapter 4 shows that in similar combinatorial structures like graphs, permutations or words there are no such big oscillations, therefore our paper signifies that this results cannot be generalized to any theory, which contains integer partitions.

Open problems

1. It would be interesting to find the exact asymptotics for the fastest growing function of partition ideals with the property P.

2. Theorem 3.4.5 shows that asymptotic formula of $p(n, F(Z))$ for finite set of partitions Z is of the form $Cn^{\kappa}e^{K\sqrt{n}}$, where $C = C(Z)$ is constant and $\kappa = -m/2$ for some $m \in \mathbb{N}\setminus\{1,2\}$. Are there any antichains Z such that $p(n, F(Z)) \sim Cn^{\kappa}e^{K\sqrt{n}}$ for some constant $C \in \mathbb{R}$ and κ, which is not of the form $\kappa = -m/2$ for some $m \in \mathbb{N}$?

Notation

Notation	Meaning
$\varnothing$	empty set
$\mathbb{N}$	$\{1, 2, 3, 4, \dots\}$
$\mathbb{N}_0$	$\{0, 1, 2, 3, 4, \dots\}$
$\overline{\mathbb{N}}$	$\mathbb{N} \cup \infty$
$[k]$	$\{1, 2, 3, \dots, k\}$
$[k]_0$	$\{0, 1, 2, \dots, k\}$
$\binom{A}{k}$	set of all k-element subsets of A
$\gcd(a_1, \dots, a_k)$	greatest common divisor of numbers $a_1, \dots, a_k$
$\gcd(A)$	greatest common divisor of all numbers from A
$A \backslash B$	$\{a \in A : a \notin B\}$
$\lvert A \rvert$ or $\#A$	number of elements (cardinality) of A
2^A	the power set of A
$f(n) \ll g(n)$	$f(n) = o(g(n))$

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