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Vlastnosti zobrazení s konečnou distorzí

Katedra matematické analýzy

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Abstrakt: Zkoumáme spojitost zobrazení s konečnou distorzí, funkce které mají sloužit jako model elastických deformací při nelineární elasticitě. Zaměřujeme se na podmínky pro spojitost na vnitřní distorzi a navíc ukážeme, že jistý odhad modulu spojitosti je ostrý, t.j. nemůže být vylepšen. Uvedeme důkaz spojitosti pro zobrazení s konečnou distorzí za zjednodušených předpokladů na distorzi.
Klíčová slova: Zobrazení s konečnou distorzí, spojitost, modulus spojitosti.

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Abstract: We study the continuity of mappings of finite distortion, a set of mappings intended to model elastic deformations in non-linear elasticity. We focus on continuity criteria for the inner-distortion function and prove that certain modulus of continuity estimates are sharp, i.e. cannot be improved. We also give a proof of the continuity of mappings of finite distortion under simplified conditions on the integrability of the distortion function.
Key words: Mapping of finite distortion, continuity, modulus of continuity.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ is an open and non-empty set. It is well-known that functions and mappings from the Sobolev space $W^{1,p}(\Omega)$ are continuous if $p > n$ and may be discontinuous for $p \le n$. Mappings in the corresponding spaces $W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n)$ (see Preliminaries for the definition) are often considered as model deformations in nonlinear elasticity (see [2] for reference and motivation) and therefore continuity is an important property, which is of interest to us. To be specific, if we imagine that such a mapping represents an elastic deformation, then continuity guarantees that the material does not break.

Let us first introduce mappings of finite outer distortion (often referred to as mappings of finite distortion), which are usually considered in this context. A mapping $f : \Omega \to \mathbb{R}^n$ is said to have finite outer distortion if the determinant of the Jacobi matrix $J(x, f) = J_f(x)$ belongs to $L^1_{\text{loc}}(\Omega)$ and there exists some positive measurable function $K_O(x)$ finite almost everywhere in Ω such that the following condition is satisfied

$$|Df(x)|^n \le K_O(x) J_f(x) \ \text{ for almost all } x \in \Omega.$$

If we replace this condition with the existence of a function $K_I(x)$ such that

$$|D^\sharp f(x)|^n \le K_I(x) J_f^{n-1}(x) \ \text{ for almost all } x \in \Omega$$

then we arrive at the class of mappings of finite inner distortion. See also Definitions 2.16 and 2.18. To be more precise we examine continuity conditions and results for these two classes of mappings. It can be proved by linear algebra that the following inequality holds between the two distortion functions, $K_I^{\frac{1}{n-1}} \le K_O \le K_I^{n-1}$. More detailed information can be found in [2] on pages 108-112.

Theorems 1.1 and 1.2 are known and their proof can be found in [2] and [4]. Their results are actually more general and we have extracted these proofs in the two most important cases where some technical details can be simplified making them easier to understand.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be open and let $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ be a mapping of finite outer distortion. Then f is continuous.*

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be open and let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite outer distortion. Suppose that there is $\lambda > 0$ such that $\exp(\lambda K_O) \in L^1_{\text{loc}}(\Omega)$. Then f is continuous.*

To prove these theorems it is necessary to build a theory of weakly monotonous mappings and the distributional Jacobian. We do this in sections 3 and 4 and our proofs are self-contained.

It would be desirable to know the optimal integrability of inner distortion function that guarantees continuity. From the inequality $K_O \leq K_I^{n-1}$ and Theorem 1.2 it is easy to deduce that the condition $\exp(K_I^{n-1}) \in L^1$ implies continuity. Surprisingly it was shown by Onninen [9] that a weaker condition, given in Theorem 1.3 suffices .

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite outer distortion. Suppose that there is $\lambda > (n-1)^2 - 1$ such that $\exp(\lambda K_I^{\frac{1}{n-1}}) \in L^1_{\text{loc}}(\Omega)$. Then f is continuous.*

The condition $\lambda > (n-1)^2 - 1$ seems to be unnatural and actually comes from some technical estimates in the proof. Let us recall that all the classical counter-examples of discontinuity in this theory are radial and have the form

$$f(x) = \frac{x}{|x|}\varphi(|x|)$$

where the differentiable function φ satisfies $\lim_{t\to 0} \varphi(t) > 0$ (see Section 5). Therefore the usual counterexamples form a cavity at the origin and map a ball to some annulus. We show that there are no such counterexamples of discontinuity even for $\lambda \leq (n-1)^2 - 1$. This new result gives a strong indication that the condition $\lambda > (n-1)^2 - 1$ in the previous theorem is superfluous.

Theorem 1.4. *Let $f \in W^{1,1}(B(0,1), \mathbb{R}^n)$ have finite outer distortion. Further let f be a homeomorphism $B(0,1) \setminus \{0\}$ onto $B(0,2) \setminus L$, where $L \subset B(0,1)$ is a closed, pathwise-connected, whose projection onto some hyperplane ρ has positive $(n-1)$ dimensional measure. Let also $0 \in X$, then it holds that*

$$\exp(\lambda K_I^{\frac{1}{n-1}}) \notin L^1(B(0,1)) \textit{ for all } \lambda > 0.$$

Let us note that it has been proved [5] that the condition $\exp(\lambda K_O) \in L^1$ in Theorem 1.2 has some degree of sharpness in the sense of Orlicz spaces, namely that for all δ there exists a mapping of finite distortion such that $\exp(K_O^{1-\delta}) \in L^1$, which is not continuous (see Example 5.5). We even show that $\exp(K_I^{(1-\delta)/(n-1)}) \in L^1$ for this map and we give further examples in Section 5 showing that other conditions like $J_f \in L^1_{\text{loc}}$ or $J_f(x) = 0 \Rightarrow |Df(x)| = 0$ cannot be omitted.

Another interesting question is to find the sharp modulus of continuity for the mappings of finite outer distortion which satisfy $\exp(\lambda K_O) \in L^1$. Let us first recall the history of such estimates. First it was shown in [4] that mappings in this class satisfy

$$|f(x)-f(y)| \le \frac{C}{\log^{1/n}\log(e^e+1/|x-y|)}$$

and this was later improved using the isoperimetric inequality in [7] to

$$|f(x)-f(y)| \le \frac{C}{\log^{\lambda/n-\varepsilon}(1/|x-y|)}.$$

Finally using very delicate arguments it has been shown in [10] that

$$|f(x)-f(y)| \le \frac{C}{\log^{\lambda/n}(1/|x-y|)}. \tag{1.1}$$

Extremal mappings for continuity of mappings of finite distortion are usually radial maps and therefore the natural candidate for the extremal map is

$$f_0(x) = \frac{x}{|x|}\frac{1}{\log^{\lambda/n}(1/|x|)}.$$

Standard computation will give us

$$K(x) = \frac{n}{\lambda}\log\frac{1}{|x|}$$

and hence

$$\int_{B(0,\frac{1}{2})} \exp(\lambda K(x))\, dx = \int_{B(0,\frac{1}{2})} \frac{1}{|x|^n}\, dx = \infty.$$

This elementary computation suggests that there is some room for the improvement in the estimate (1.1) and maybe we can add some supplementary factor like $\log\log 1/|x-y|$ to some negative power to our estimate. We show that surprisingly this is not the case and the modulus of continuity (1.1) is already sharp. Let us point out that this new fact was not only surprising for us but also for all three authors of [7] and [10] who actually expected a better estimate to hold.

Theorem 1.5. *For every $\lambda > 0$ there is a ball $B := B(0,r)$, some $C > 0$ and a mapping of finite distortion $f : B \to \mathbb{R}^n$ such that*

$$\int_B \exp\big(\lambda K_f(x)\big)\, dx < \infty$$

and

$$|f(x)-f(0)| \ge C\frac{1}{\log^{\lambda/n}(1/|x|)}. \tag{1.2}$$

Let us consider the mapping

$$f(x) = \frac{x}{|x|} \frac{(\log 1/|x|)^{\frac{a}{\log 1/|x|}}}{\log^{\lambda/n}(1/|x|)}$$

where $a > 0$. The additional factor clearly satisfies

$$\lim_{|x| \to 0} (\log 1/|x|)^{\frac{a}{\log 1/|x|}} = 1$$

and thus the modulus of continuity of our f is exactly as required in (1.1). On the other hand this alteration slightly affects the distortion and the standard computation (see Section 7) will give us

$$K(x) \sim \frac{n}{\lambda} \log \frac{1}{|x|} - \frac{n^2 a}{\lambda^2} \log\log \frac{1}{|x|}$$

and hence

$$\int_{B(0,\frac{1}{2})} \exp(\lambda K(x))\, dx < \infty$$

for sufficiently large a.

2 Preliminaris

2.1 Basic definitions

Let us begin this section with some basic and well known but necessary definitions.

Definition 2.1. *We define the open ball $B(x,r) \subset \mathbb{R}^n$ for $x \in \mathbb{R}^n$ and $r \in (0,\infty)$ as*

$$B(x,r) = \{y \in \mathbb{R}^n \mid |x-y| < r\}.$$

To simplify notation we will write

$$\lambda B = \lambda B(x,r) = B(x,\lambda r).$$

Definition 2.2. *We define the sphere $S(x,r) \subset \mathbb{R}^n$ for $x \in \mathbb{R}^n$ and $r \in (0,\infty)$ as*

$$S(x,r) = \{y \in \mathbb{R}^n \mid |x-y| = r\}.$$

Definition 2.3. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be open and $f : \Omega_1 \to \Omega_2$. Let f be a continuous mapping, whose inverse f^{-1} is defined everywhere on Ω_2 and is also continuous. Then we call f a homeomorphism.*

We will use the letter c to represent some positive constant which is dependent only on certain parameters like the dimension of $\mathbb{R}^n$. The specific value of c, however, may change during the process of a proof, even during a single string of estimates. We will classify what c is dependent on (respectively independent of), whenever necessary or unclear.

As we will be dealing with mappings, which may be differentiated in different senses it is necessary to define the symbols we will use to denote this.

Definition 2.4. *Let f be a function on some open set $\Omega \subset \mathbb{R}^n$. We define*

$$\partial_\gamma f(x) = \lim_{t\to 0} \frac{f(x+t\gamma) - f(x)}{t} \qquad \text{for all } x \in \Omega,\ \gamma \in \mathbb{R}^n \setminus \{0\},$$

if the limit exists. If $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{R}^m$ then we define

$$\partial_\gamma f(x) := (\partial_\gamma f_1(x), \ldots, \partial_\gamma f_m(x)).$$

Definition 2.5. *Let f be a function mapping some open set $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^m$. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. Further let the partial derivatives $\partial_{e_i} f(x)$ exist for all $i \in \{1, \ldots, n\}$. Then we define*

$$\partial_i f(x) := \partial_{e_i} f(x)$$

and

$$\nabla f(x) := (\partial_1 f(x), \dots, \partial_n f(x)).$$

Definition 2.6. *Let f be a function mapping some open set $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^m$ and let $\alpha = (\alpha(1), \dots, \alpha(n))$ be a multi-index, whose height is $|\alpha|$. We define $\partial_\alpha f$ as follows*

$$\partial_\alpha f(x) := \frac{\partial^{|\alpha|}}{\prod_{k=1}^n \partial x_k^{\alpha(k)}} f(x),$$

given that all of the partial derivatives, which come into consideration exist and are continuous on some open neighborhood of x, irrespective of what order we derive in and that they are all continuous.

Definition 2.7. *Let f be a function from some open set $\Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and let $k \in \mathbb{N}$. If $\partial_\alpha f$ is a continuous mapping on Ω for all α multi-indices, $|\alpha| \leq k$, then we say $f \in C^k(\Omega, \mathbb{R}^m)$. If $f \in C^k$ for all $k \in \mathbb{N}$, then we say that $f \in C^\infty$.*

Definition 2.8. *Let f be a function on some open set $\Omega \subset \mathbb{R}^n$. We define*

$$\operatorname{supp} f := \overline{\{x \in \Omega \mid f(x) \neq 0\}}$$

where we take the closure in $\mathbb{R}^n$.

Definition 2.9. *For $k \in \mathbb{N}_0$ we define $C_c^k(\Omega)$ as the set of all functions such that $f \in C^k(\Omega)$ and $\operatorname{supp} f \subset \Omega$. Further we denote $C_c^\infty(\Omega)$ as $\mathcal{D}(\Omega)$.*

Definition 2.10. *Let $f \in L^1_{\text{loc}}(\Omega)$ for some open set $\Omega \subset \mathbb{R}^n$. Let $i \in \{1, \dots, n\}$ and let there exist some $g \in L^1_{\text{loc}}(\Omega)$ such that*

$$\int_\Omega g(x)\varphi(x)dx = -\int_\Omega f(x)\partial_i \varphi(x)dx$$

for all $\varphi \in \mathcal{D}(\Omega)$. Then we define the weak derivative of f in the direction x_i (we write $D_i f$) as g.

For the above defined weak derivative we may also use the term distributional derivative. All relevant distributional derivatives of all functions we consider in our results can be expressed as some L^1_{loc} function.

Definition 2.11. *Let $f \in L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$ for some open set $\Omega \subset \mathbb{R}^n$ be such that $D_i f$ exist. We define*

$$Df := \begin{pmatrix} D_1 f_1, & D_2 f_1, & \dots & D_n f_1 \\ D_1 f_2, & D_2 f_2, & \dots & D_n f_2 \\ \vdots & \vdots & \dots & \vdots \\ D_1 f_m, & D_2 f_m, & \dots & D_n f_m \end{pmatrix}$$

Definition 2.12. *Let $f \in L^1_{\text{loc}}(\Omega, \mathbb{R}^n)$ for some open set $\Omega \subset \mathbb{R}^n$ be such that Df exists. We define the Jacobian as $J_f := \det Df$.*

2.2 Sobolev spaces and mappings of finite distortion

Definition 2.13. *Let $p \in [1, \infty]$ and $\Omega \subset \mathbb{R}^n$ be an open set. We define the symbol $W^{1,p}(\Omega)$, as the set of all elements $f \in L^p(\Omega)$, whose weak derivatives $|D_i f| \in L^p(\Omega)$ for $i \in \{1, \dots, n\}$. Further for all $f \in W^{1,p}(\Omega)$ we define*

$$||u||_{W^{1,p}(\Omega)} := \left(||f||_p^p + \sum_{i=1}^{n} ||D_i f||_p^p \right)^{\frac{1}{p}}$$

for $p < \infty$ and

$$||u||_{W^{1,\infty}(\Omega)} := \max\{||f||_\infty, ||D_1 f||_\infty, \dots ||D_n f||_\infty\}.$$

We use the shortened notation $||u||_{W^{1,p}(\Omega)} = || \dots ||_{1,p}$, when there is no danger of confusion to the reader. It is well known that $\left(W^{1,p}(\Omega), || \dots ||_{W^{1,p}(\Omega)}\right)$ is a Banach space. We say that a Sobolev function, that is a function belonging to $W^{1,p}(\Omega)$, is continuous when it has a continuous representative.

Definition 2.14. *Let $p \in [1, \infty]$ and $\Omega \subset \mathbb{R}^n$ be an open set. We define the symbol $W^{1,p}_0(\Omega)$ as the closure of $C^\infty_c(\Omega)$ in the space $W^{1,p}(\Omega)$.*

Definition 2.15. *Let $p \in [1, \infty]$ and $\Omega \subset \mathbb{R}^n$ be an open set. We define the symbol $W^{1,p}_{\text{loc}}(\Omega)$ as the set of elements of L^p_{loc}, for whom $D_i f \in L^p_{\text{loc}}(\Omega)$, $i \in \{1, \dots, n\}$.*

We define the spaces $W^{1,p}(\Omega, \mathbb{R}^n)$ as a set of functions whose coordinate functions belong to $W^{1,p}(\Omega)$ and analogously we define $W^{1,p}_0(\Omega, \mathbb{R}^n)$ and $W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n)$.

We may now define the class of mappings, which will be the focus of our study.

Definition 2.16. *We say that a mapping $f : \Omega \to \mathbb{R}^n$ on the open set $\Omega \subset \mathbb{R}^n$, has finite outer distortion if $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$, $J_f \geq 0$ a.e., $J_f \in L^1_{\text{loc}}(\Omega)$ and there exists some function $K_O : \Omega \to [1, \infty]$, $K_O(x) < \infty$ almost everywhere such that*

$$|Df(x)|^n \leq K_O(x) J_f(x) \qquad \text{for almost all } x \in \Omega.$$

For mappings of finite outer distortion we can define $K_O(x) = 1$ on the set $\{J_f = 0\}$ and

$$K_O(x) := \frac{|Df(x)|^n}{J_f(x)} \qquad \text{for all } x \in \{J_f > 0\}.$$

Definition 2.17. *Let A be a real $n \times n$ matrix. Define $\tilde{A}_{k,m}$ for all $k, m \in \{1, \dots, n\}$ as the $(n-1) \times (n-1)$ matrix given by the omission of the k-th column and the m-th row from A and put $A_{k,m} := \det \tilde{A}_{k,m}$. Then we may define the adjoint matrix of A as*

$$A^\sharp := ((-1)^{i+j} A_{i,j})_{j=1,\dots,n}^{i=1,\dots,n}.$$

Definition 2.18. *We say that a mapping $f : \Omega \to \mathbb{R}^n$ on the open set $\Omega \subset \mathbb{R}^n$, has finite inner distortion if $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$, $J_f \geq 0$ a.e., $J_f \in L^1_{\text{loc}}(\Omega)$ and there exists some function $K_I(x) : \Omega \to [1, \infty]$, $K_I(x) < \infty$ almost everywhere such that*

$$|D^\sharp f(x)|^n \leq K_I(x) J_f^{n-1}(x) \qquad \text{for almost all } x \in \Omega.$$

For mappings of finite outer distortion we can define $K_I = 1$ on the set $\{J_f = 0\}$ and

$$K_I(x) := \frac{|D^\sharp f(x)|^n}{J_f^{n-1}(x)} \qquad \text{for all } x \in \{J_f > 0\}.$$

Hereby K_I and K_O are defined almost everywhere in Ω. Where $J_f(x) < 0$ or is not defined we do not define the distortion functions.

2.3 Properties of Sobolev functions

Now we cite a special case of the first part of the Sobolev embedding theorem.

Theorem 2.19. *Let Ω be an open non-empty set with a Lipschitz boundary in $\mathbb{R}^n$. Further let $p \in [1, n)$ and $f \in W^{1,p}(\Omega)$. Then $f \in L^{\frac{np}{n-p}}(\Omega)$ and moreover*

$$||f||_{L^{\frac{np}{n-p}}(\Omega)} \leq c||f||_{W^{1,p}(\Omega)}.$$

Definition 2.20. *Let $f \in L^1(E)$ for some measurable set $E \subset \mathbb{R}^n$. Then we define*

$$f_E := \fint_E f := \frac{1}{|E|} \int_E f.$$

Theorem 2.21 (Poincaré)**.** *Let $f \in W^{1,p}(B(0,r))$, $p \in [1, \infty)$. Then there exists some c, which depends only on n and p such that*

$$\int_B |f - f_B|^p \leq c r^p \int_B |Df|^p.$$

Definition 2.22. *Let f be a mapping of some set X into $\mathbb{R}^n$. We define*

$$\operatorname{osc}_X f := \operatorname{diam}(f(X)) = \sup\{|f(x_1) - f(x_2)| \mid x_1, x_2 \in X\}.$$

Lemma 2.23 (Sobolev imbedding theorem on spheres)**.** *Let $p > n - 1$ and $u \in W^{1,p}(B(0,R))$. Then there is a representative of u such that for almost every $t \in (0,R)$ we have*

$$\operatorname{osc}_{\partial B(0,t)} u \leq ct \left(\fint_{\partial B(0,t)} |Du|^p \right)^{\frac{1}{p}}.$$

The following lemma is a special version of the area formula, proved in [8], for spheres. It is easy to note that the dimension of the sphere is $n - 1$ and that the area formula applies to Sobolev functions on the sphere as long as the integrability is higher than that of the dimension, here $n - 1$.

Lemma 2.24. *Let $p > n - 1$ and let $f \in W^{1,p}(B(0,R), \mathbb{R}^n)$ be a mapping such that $|f(B(0,R))| < \infty$. Then for almost all $t \in (0,R)$ we have*

$$\int_{S(0,t)} |D^\sharp f| \geq c\mathcal{H}^{n-1}(f(S(0,t))).$$

The following isoperimetric inequality can be found in [1, Theorem 3.2.43].

Lemma 2.25 (Isoperimetric inequality)**.** *Let $X \subset \mathbb{R}^n$ be a measurable set of finite measure and let us denote by ∂X its boundary. Then*

$$\mathcal{H}^{n-1}(\partial X) \geq c|X|^{\frac{n-1}{n}}.$$

Definition 2.26. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We define the Hardy-Littlewood maximal function*

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)| dy.$$

Theorem 2.27. *Let f be in $L^1(\mathbb{R}^n)$. Then there exists some c dependent only on n such that*

$$|\{x \in \mathbb{R}^n \mid Mf(x) > \alpha\}| \leq \frac{c}{\alpha} \int_{\{|f| \geq \frac{\alpha}{2}\}} |f|.$$

We will need the following Kirszbraun theorem, which can be found in [6].

Theorem 2.28. *Let $K \subset \mathbb{R}^n$ be a compact. If f is a Lipschitz function (with the constant c_f) defined on K then there exists some F Lipschitz function defined on $\mathbb{R}^n$ such that $f(x) = F(x)$ for all $x \in K$ and whose Lipschitz constant satisfies $c_F = c_f$.*

We will need to use functions in $C^\infty(\Omega)$ to approximate elements of $W^{1,p}(\Omega)$. For this we use so-called mollification. Generally let $\Phi \in C_0^\infty(\mathbb{R}^n)$ fulfill the following

$$\begin{aligned} &\operatorname{supp}\Phi \subset \overline{B(0,1)} \\ &\Phi(x) = \Phi(y) \qquad\qquad \text{for } |x| = |y| \\ &\Phi(x) \geq 0 \\ &\int_{B(0,1)} \Phi = 1. \end{aligned} \tag{2.1}$$

For a given Φ we define the family $\{\Phi_j\}_{j\in\mathbb{N}}$ on $\mathbb{R}^n$ as follows

$$\Phi_j(x) := j^n \Phi(jx).$$

As n is fixed for all proofs there is no danger of misunderstanding. Hereby Φ_j satisfy (2.1) for all $j \in \mathbb{N}$ and $\operatorname{supp}\Phi_j \subset \overline{B(0, \frac{1}{j})}$ for all j.

For simplicity and to be precise we will work with

$$\Phi^*(x) = c \exp\left(\frac{1}{|x|^2 - 1}\right),$$

where c is chosen such as to fulfill (2.1). We may define the mollification $\{f_j\}_{j\in\mathbb{N}}$ of all elements $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ as the convolution of f with Φ_j that is,

$$f_j(x) := \int_{\mathbb{R}^n} f(y)\Phi_j(x-y)dy. \tag{2.2}$$

Should f not be defined on the whole of $\mathbb{R}^n$ we may still define $f_j(x)$ if f is defined on some open neighborhood U of x. Then there exists some j_0 such that $B(x, j_0^{-1}) \subset U$ and we redefine f as zero outside U for the purpose of integration in (2.2).

If $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ we define the mollification of $f = (f_1, \ldots, f_m)$ as the mollification of its component functions $f_1, \ldots, f_m$. The following theorem on the mollifications of a function can be found in [2] on page 58.

Theorem 2.29. *Let $f \in L^1_{\text{loc}}(\Omega)$ for Ω some open set in $\mathbb{R}^n$. Then*

1. $\lim_{j\to\infty} f_j(x) = f(x)$ *for almost all $x \in \Omega$.*

2. *If f is continuous then the convergence in 1. is locally uniform.*

3. *If $f \in W^{1,p}(\Omega)$ for $p \in [1,\infty]$, $i \in \{1,\ldots,n\}$ and if $X \subset\subset \Omega$ is a compact set such that* $\mathrm{dist}(X,\partial\Omega) > j^{-1}$ *we have that*

$$||\partial_i f_j||_{L^p(X)} \leq ||D_i f||_{L^p(\Omega)}.$$

4. *If in 3. we assume moreover that $p < \infty$ then*

$$\lim_{j\to\infty} ||D_i f_j - D_i f||_{L^p(X)} = 0.$$

5. *If $\Omega = \mathbb{R}^n$ then 3. and 4. hold for $X = \mathbb{R}^n$.*

3 The distributional Jacobian.

The proofs of Theorem 1.2 and Theorem 1.1 use a concept called weak monotonicity. In order to show that a mapping of finite distortion is weakly monotonous we need to use Theorem 3.3 on the so-called distributional Jacobian.

Definition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be open. For functions $f \in W^{1,\frac{n^2}{n+1}}(\Omega, \mathbb{R}^n)$ we define the distributional Jacobian of f, we write $\mathcal{J}_f$, as follows*

$$\mathcal{J}_f(\varphi) = -\int_\Omega f_1 J(\varphi, f_2, ..., f_n) \qquad \text{for all } \varphi \in \mathcal{D}(\Omega),$$

where $J(\varphi, f_2, ..., f_n)$ is the classical Jacobian defined as the determinant of the Jacobi matrix of $(\varphi, f_2, \dots, f_n)$.

We claim that the integral on the right hand side in the definition is finite for all $\varphi \in \mathcal{D}(\Omega)$ and for all $f \in W^{1,\frac{n^2}{n+1}}(\Omega)$. Any product of the form

$$\prod_{j=2}^{n} D_{\gamma_j} f_j \quad \gamma_j \in \{1, \dots, n\}$$

lie in the space $L^{\frac{n^2}{n^2-1}}$. Because $\varphi \in W^{1,\infty}(\Omega)$ it follows that

$$J(\varphi, f_2, ..., f_n) \in L^{\frac{n^2}{n^2-1}}(\Omega)$$

for $n \geq 2$. By Theorem 2.19 we have that

$$f \in L^{n^2}_{\text{loc}}(\Omega).$$

Now we can prove that the integral is finite by using the Hölder inequality on f_1 and $J(\varphi, f_2, ..., f_n)$ because

$$\frac{1}{n^2} + \frac{n^2-1}{n^2} = 1.$$

Therefore the given integral is finite.

Definition 3.2. *Let f be a measurable function on some open set $\Omega \subset \mathbb{R}^n$. We say that $f \in L^n \log^{-1} L(\Omega)$ whenever,*

$$\int_\Omega \frac{|f|^n}{\log(e + |f|)} < \infty.$$

We can note the following. Firstly $f \in (L^n \log^{-1} L)_{\mathrm{loc}}(\Omega)$ implies that $f \in L^p_{\mathrm{loc}}(\Omega)$ for all $p \in [1, n)$. The Sobolev embedding theorem 2.19 implies, that $f \in L^p_{\mathrm{loc}}(\Omega)$ for all $p \in [1, \infty)$, given that $f \in W^{1,1}(\Omega)$, and $|Df| \in (L^n \log^{-1} L)_{\mathrm{loc}}(\Omega)$. Therefore if f has finite distortion and $|Df| \in (L^n \log^{-1} L)_{\mathrm{loc}}(\Omega)$ then the distributional determinant is well defined as $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for all $p < n$.

The following theorem was published in [3].

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$. Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and let J_f be non-negative almost everywhere. Further let $|Df| \in L^n \log^{-1} L(\Omega)$. Then we have*

$$J_f \in L^1_{\mathrm{loc}}(\Omega)$$

and

$$\mathcal{J}(\varphi) = \int_\Omega \varphi J_f \qquad \textit{for all } \varphi \in \mathcal{D}(\Omega).$$

In order to prove this we will need the following lemmata.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $n - 1 < p < n$ and $B := B(x, r)$ be a ball such that $B \subset\subset \Omega$. Let $f \in W^{1,1}(\Omega)$ and $|Df| \in L^n \log^{-1} L(\Omega)$ with $f_1 \in W^{1,\infty}(\Omega) \cap W^{1,p}_0(B)$. Then*

$$\int_B J_f = 0.$$

Proof. Firstly let

$$f \in C^2(\Omega, \mathbb{R}^n) \quad \text{and} \quad f_1 \in C_0^\infty(B). \tag{3.1}$$

Let for the purpose of this proof S_m signify the set of all permutations on $\{1, \dots m\}$. Take any $\pi \in S_n$ and any $j \in \{2, \dots, n\}$ and define

$$A_{\pi,j} := \operatorname{sgn} \pi f_1 \partial_{\pi(j)\pi(1)} f_j \prod_{i \neq 1,j} \partial_{\pi(i)} f_i.$$

Let the mapping $\omega : S_n \times \{2, \dots, n\} \to S_n$ be defined as,

$$\omega(\pi, j) = \begin{cases} \pi(1) & i = j \\ \pi(i) & i \neq 1, j \\ \pi(j) & i = 1. \end{cases}$$

Hereby we see that

$$\operatorname{sgn} \omega(\pi, j) = -\operatorname{sgn} \pi \quad \text{for all } j \in \{2, \dots, n\} \tag{3.2}$$

and ω is surjective and the pre-image of a permutation in ω is always a set with $n-1$ elements in $S_n \times \{2,\dots,n\}$. Moreover $(\pi_1, j_1), (\pi_2, j_2) \in \omega^{-1}(\pi)$ implies that either the two elements are identical or $\pi_1 \neq \pi_2$ and $j_1 \neq j_2$.

Choose any $\pi \in S_n$. Let χ be the orthogonal projection of $\mathbb{R}^n$ onto $(\mathrm{span}\{e_1\})^\perp$ and $B' := \chi(B)$. Further for every $b \in B'$ define

$$\begin{aligned}
\vec{b} &:= \{b + \lambda e_1 \mid \lambda \in \mathbb{R}\} \cap B \\
\lambda_1 &:= \inf\{\lambda \in \mathbb{R} \mid b + \lambda e_1 \in \vec{b}\} \\
\lambda_2 &:= \sup\{\lambda \in \mathbb{R} \mid b + \lambda e_1 \in \vec{b}\} \\
b_1 &:= b + \lambda_1 e_1 \\
b_2 &:= b + \lambda_2 e_1.
\end{aligned}$$

Then by the Fubini theorem and by integration by parts we get that

$$\begin{aligned}
\int_B \operatorname{sgn}\pi \prod_{i=1}^n \partial_{\pi(i)} f_i = \operatorname{sgn}\pi \int_{B'} \int_{\vec{b}} \partial_{\pi(1)} f_1 \prod_{i\neq 1} \partial_{\pi(i)} f_i = \\
= -\operatorname{sgn}\pi \int_{B'} \int_{\vec{b}} f_1 \partial_{\pi(1)} (\prod_{i \neq 1} \partial_{\pi(i)} f_i) = -\int_B A_{\pi,j}
\end{aligned}$$

where we have used the fact that $f_1 \in C_0(B)$ and thus the boundary terms are zero.

Now we take the sum over all $\pi \in S_n$ and $j = 2, \dots, n$ to get

$$(n-1)\int_B J_f = \int_B \left(\sum_{j\in\{2,\dots n\}} \sum_{\pi \in S_n} \operatorname{sgn}\pi \prod_{i=1}^n \partial_{\pi(i)} f_i \right) = -\int_B \left(\sum_{\pi\in S_n} \sum_{j \in \{2,\dots n\}} A_{\pi,j} \right).$$

Similarly however, thanks to the fact that ω is $(n-1)$-tuply surijective, we have that

$$(n-1)\int_B J_f = -\int_B \left(\sum_{\pi\in S_n} \sum_{j \in \{2,\dots n\}} A_{\omega(\pi,j)} \right).$$

Here we can interchange the second derivatives because $f \in C^2$. Further by (3.2) we have

$$A_{\omega(\pi,j)} = -A_{\pi,j},$$

which finally gives that

$$2(n-1)\int_B J_f = -\int_B \left(\sum_{\pi\in S_n} \sum_{j \in \{2,\dots n\}} A_{\omega(\pi,j)} + A_{\pi,j} \right) = 0.$$

Let us assume we have f given satisfying the assumptions of the Lemma and chose $\varepsilon \in (0,1)$. We use Theorem 2.29 for the mollification of components $f_2, \ldots, f_n$ individually to find a sequence of functions $f_2^m, \ldots, f_n^m$, which converge to $f_2, \ldots, f_n$ in the space $W^{1,n-\varepsilon}(B)$. Further, thanks to the fact that $C_c^\infty(B)$ is dense in $W_0^{1,s}(B)$ for all $s < \infty$ we may assume that there exists some sequence of functions $\{g_m\}_{m=1}^\infty$, which converge to f_1 in the space $W^{1,\frac{n-\varepsilon}{1-\varepsilon}}(B)$ and also belong to the space $C_c^\infty(B)$. Note that for the exponents chosen

$$\frac{1-\varepsilon}{n-\varepsilon} + \sum_{i=1}^{n-1} \frac{1}{n-\varepsilon} = 1. \tag{3.3}$$

We will now, for the purpose of this proof, simplify notation by denoting $f_1^m := g_m$ instead of the mollification of f_1 and $f^m := (f_1^m, \ldots, f_n^m)$. Using the triangle inequality we have that,

$$\begin{aligned}
|\prod_{i=1}^n D_{\pi(i)} f_i - \prod_{i=1}^n D_{\pi(i)} f_i^m| &\leq |D_{\pi(1)} f_1 \prod_{i=2}^n D_{\pi(i)} f_i - D_{\pi(1)} f_1^m \prod_{i=2}^n D_{\pi(i)} f_i| \\
&\quad + |D_{\pi(1)} f_1^m \prod_{i=2}^n D_{\pi(i)} f_i - D_{\pi(1)} f_1^m \prod_{i=2}^n D_{\pi(i)} f_i^m| \\
&= |D_{\pi(1)} f_1 - D_{\pi(1)} f_1^m| |\prod_{i=2}^n D_{\pi(i)} f_i| \\
&\quad + |D_{\pi(1)} f_1^m| |\prod_{i=2}^n D_{\pi(i)} f_i^m - \prod_{i=2}^n D_{\pi(i)} f_i|.
\end{aligned}$$

The integral over B of the first of these terms tends to zero for m tending to ∞ thanks to the fact that $f_1^m \to f_1$ in $W^{1,\frac{n-\varepsilon}{1-\varepsilon}}(B)$ and the Hölder inequality, where the exponents are as in (3.3). We now iterate this operation until we get,

$$\begin{aligned}
|\prod_{i=1}^n D_{\pi(i)} f_i - \prod_{i=1}^n D_{\pi(i)} f_i^m| \leq \sum_{j=1}^n |\prod_{i<j} D_{\pi(i)} f_i^m| \; |D_{\pi(j)} f_j^m - D_{\pi(j)} f_j| \\
\times |\prod_{j<k\leq n} D_{\pi(k)} f_k|
\end{aligned}$$

where for simplification we use the definition that $\prod_\emptyset = 1$. But for $j > 1$ we have that the second factor tends to zero in $L^{n-\varepsilon}(\Omega)$ in all terms and the product of the first and third factors is in $L^{\frac{n-\varepsilon}{n-1-\varepsilon}}(\Omega)$. Therefore by the Hölder inequality, using $\frac{n-1-\varepsilon}{n-\varepsilon} + \frac{1}{n-\varepsilon} = 1$, the sum tends to zero as m tends to ∞.

Because the set S_n is finite, the same follows for the sum of all $\pi \in S_n$, which means nothing different than

$$\lim_{m\to\infty}\int_B |J_f - J_{f^m}| = 0.$$

But as

$$\int_B J_{f^m} = 0 \qquad \text{for all } m\in\mathbb{N}$$

we get

$$\int_B J_f = \lim_{m\to\infty}\int_B J_{f^m} = 0.$$

□

Lemma 3.5. *Let $B = B(x_0, r_0)$ be a ball in $\mathbb{R}^n$ and let $u \in W^{1,1}(4B)$. For all $\lambda > 0$ we define*

$$F_\lambda = \{x\in B : M|Du(x)| < \lambda\} \cap \{x \in B \mid x \text{ is a Lebesgue point of } u\}. \tag{3.4}$$

Then there is some constant $c > 0$ such that,

$$|u(x) - u(y)| \le c\lambda|x-y| \text{ for all } x, y \in F_\lambda.$$

Proof. Let us choose $\lambda > 0$ and $x, y \in F_\lambda$. We define

$$\begin{aligned} B_j &:= B(x, 2^{-j}|x-y|) \qquad && j\in\mathbb{N}_0 \\ B_j &:= B(y, 2^{j+1}|x-y|) && j \in -\mathbb{N}. \end{aligned}$$

Since x and y are Lebesgue points we obtain

$$|u(x) - u(y)| \le \sum_{j\in\mathbb{Z}} |u_{B_j} - u_{B_{j+1}}|$$

but because for such u and for any of the balls B' we have

$$|u_{B'} - u_{2B'}| = \left|\frac{1}{|B'|}\int_{B'}(u(t) - u_{2B'})dt\right| \le \frac{|2B'|}{|B'|}\fint_{2B'}|u(t) - u_{2B'}|dt \tag{3.5}$$

we see that

$$\sum_{j\in\mathbb{Z}\setminus\{-1\}} |u_{B_j} - u_{B_{j+1}}| \le c\sum_{j=-\infty}^{-2}\fint_{B_{j+1}}|u(t) - u_{B_{j+1}}|dt + c\sum_{j=0}^{\infty}\fint_{B_j}|u(t) - u_{B_j}|dt$$

Now applying the Poincaré inequality (Lemma 2.21) we get

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \fint_{B_j} |u(t)-u_{B_j}|dt &\le c\sum_{j\in\mathbb{Z}} r_j \fint_{B_j} |Du| \\ &\le c\left(M|Du|(x)+M|Du|(y)\right)\sum_{j\in\mathbb{Z}} r_j \\ &\le \lambda c|x-y|.\end{aligned}$$

Let us consider $|u_{B_0}-u_{B_{-1}}|$. Clearly

$$|u_{B_0}-u_{B_{-1}}| \le |u_{B_0}-u_{2B_{-1}}| + |u_{2B_{-1}}-u_{B_{-1}}|.$$

Using exactly the same reasoning as above, the second of these two can be estimated in the same way and is therefore less than $c\lambda|x-y|$. The first can be estimated as follows,

$$\frac{1}{|B_0|}\left|\int_{B_0} u(t)-u_{2B_{-1}}dt\right| \le \frac{|2B_{-1}|}{|B_0|}\fint_{2B_{-1}} |u(t)-u_{2B_{-1}}|.$$

Applying the Poincaré inequality gives the required estimate. Note that c was at no point dependent on λ and therefore we have the proposition. □

Lemma 3.6. *Let $p>1$, $\lambda>0$ and $v\in L^p$. Then there exists some $c>0$ such that*

$$\int_{\{Mv>\lambda\}} (Mv)^p \le c\int_{\{v>\frac{\lambda}{2}\}} |v|^p.$$

Proof. Using the Fubini theorem and the maximal estimate in Theorem 2.27 we see that

$$\begin{aligned}\int_{\{Mv>\lambda\}} (Mv)^p &= \int_{\{Mv>\lambda\}}\int_0^{Mv(x)} pt^{p-1}dtdx = \\ &= p\int_\lambda^\infty t^{p-1}|\{Mv>t\}|dt + \lambda^p|\{Mv>\lambda\}| \\ &\le c\int_\lambda^\infty t^{p-2}\int_{\{|v|>\frac{t}{2}\}} |v(x)|dxdt + c\lambda^{p-1}\int_{\{|v|>\frac{\lambda}{2}\}} |v(x)|dx \\ &\le c\int_\lambda^\infty \int_{\{|v|>\frac{t}{2}\}} |v(x)|^{p-1}dxdt + c\int_{\{|v|>\frac{\lambda}{2}\}} |v(x)|^p dx \\ &= c\int_{\{|v|>\frac{\lambda}{2}\}} |v(x)|^{p-1}\int_\lambda^{2|v(x)|} dtdx + c\int_{\{|v|>\frac{\lambda}{2}\}} |v(x)|^p dx \\ &\le c\int_{\{|v|>\frac{\lambda}{2}\}} |v(x)|^p dx.\end{aligned}$$

□

Lemma 3.7. *Let $B := B(a, r_0)$ be a ball in $\mathbb{R}^n$ with $2B \subset \Omega$, let $f \in W^{1,1}(B)$ and let $\lambda > 0$. Let us also asume that*

$$\int_B f = 0.$$

There exists some $C > 0$, which depends only on n and R such that for all $x \in B$, which are Lebesgue points of f and Df, we have that,

$$Mf(x) > \lambda \Rightarrow M|Df|(x) > C\lambda.$$

Proof. Let $x \in B$ be a Lebesgue point of f and $|Df|$ and let $Mf(x) > \lambda$. It follows that there exists some $r > 0$ such that

$$\fint_{B(x,r)} |f(y)|dy > \lambda.$$

Define $x_0 := a$. Take the smallest $K \in \mathbb{N}$ such that $2^{-K} r_0 \leq r$ and for $1 \geq j \leq K$ define

$$r_j := r_0 \left(\frac{r}{r_0}\right)^{\frac{j}{K}}.$$

$$x_j := \begin{cases} \frac{x - x_{j-1}}{|x - x_{j-1}|}(r_{j-1} - r_j) & \big(x \neq x_{j-1}\big) \wedge \big(r_{j-1} - r_j \leq |x - x_{j-1}|\big) \\ x & \text{else.} \end{cases}$$

Now define $B_j := B(x_j, r_j)$ for $j \in \{0, \dots K\}$. Note that

$$\sum_{j=1}^{K} r_j < 4r_0. \tag{3.6}$$

It is clear that $(f)_{B_K} > \lambda$. It follows hereby that

$$\sum_{j=1}^{K} |(f)_{B_j} - (f)_{B_{j-1}}| > \lambda \tag{3.7}$$

as $f_{B_0} = 0$. We can estimate all of the terms above as follows,

$$\begin{aligned} |f_{B_i} - f_{B_{i-1}}| = \left| \frac{1}{|B_i|} \int_{B_i} (f(t) - f_{B_{i-1}})dt \right| &\leq \frac{|B_{i-1}|}{|B_i|} \fint_{B_{i-1}} |f(t) - f_{B_{i-1}}|dt \\ &\leq c \fint_{B_{i-1}} |f(t) - f_{B_{i-1}}|dt, \end{aligned} \tag{3.8}$$

where c does not depend on i or x. Now combining (3.7) with (3.8) and applying the Poincaré inequality (Lemma 2.21) we get

$$\begin{aligned}\lambda < c\sum_{j=1}^{k} \fint_{B_{j-1}} |f(t) - f_{B_{j-1}}|dt &\leq c\sum_{j=1}^{k} r_j \fint_{B_j} |Df(t)|dt \\ &\leq cM|Df|(x)\sum_{j\in\mathbb{Z}} r_j \\ &\leq cM|Df|(x).\end{aligned}$$

But c is derived from the Poincaré Lemma, (3.6) and (3.8) and therefore depends and only on n and r_0. □

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^n$. Let $f \in W^{1,1}_{loc}(\Omega)$, $J_f \geq 0$ and $|Df| \in L^n \log^{-1} L(\Omega)$. Let $B = B(a,R)$ be a ball such that $2B \subset\subset \Omega$ and $\varphi \in C_c^\infty(B)$. Define*

$$u := \varphi(x)\,(f_1(x) - (f_1)_B)$$

and let F_λ corresponding to u be as in (3.4)*. Then*

$$\operatorname{liminf}_{\lambda\to\infty} \lambda \int_{B\setminus F_\lambda} |Df|^{n-1} = 0.$$

Proof. It follows from per-partes integration that

$$|Du(x)| \leq |Df_1(x)||\varphi(x)| + |D\varphi(x)||f_1(x) - (f_1)_{2B}|$$

Let us take some point $y \in B\setminus F_\lambda$, and integrate the previous inequality over $B(y,r)$ with $r \in (0, R-|y-a|)$

$$\begin{aligned}\fint_{B(y,r)} |Du(x)|dx &\leq \fint_{B(y,r)} |Df_1(x)||\varphi(x)|dx + \fint_{B(y,r)} |D\varphi(x)||f_1(x) - (f_1)_B|dx. \\ &\leq cM|Df_1|(y) + cM|f_1 - (f_1)_B|(y).\end{aligned}$$

Now taking the supremum over r we get

$$\lambda < M|Du|(y) \leq cM|Df_1|(y) + cM|f_1 - (f_1)_B|(y).$$

This implies that at least one of the right hand side terms is greater than $\frac{\lambda}{2c}$. But this by Lemma 3.7, where we put $f := f_1 - (f_1)_B$, implies that there exists some $C > 0$ depending only on n such that

$$CM|Df_1| > \lambda.$$

Hereby we see that,

$$B \setminus F_\lambda \subset \{M|Df| > c\lambda\} \cup Z$$

where $|Z| = 0$. This however shows, in combination with Lemma 3.6, that for any $\delta \in (0,1)$,

$$\begin{aligned} \lambda \int_{B\setminus F_\lambda} |Df|^{n-1} \leq & \lambda \int_{\{M|Df|>c\lambda\}} |Df|^{n-1} \\ & \leq c\lambda^{1-\delta} \int_{\{M|Df|>c\lambda\}} (M|Df|)^{n-1+\delta} \\ & \leq c\lambda^{1-\delta} \int_{\{|Df|>\frac{c\lambda}{2}\}} |Df|^{n-1+\delta}. \end{aligned} \tag{3.9}$$

We now show that the latter of these is small. Define

$$h(t) := \frac{1}{t\log(e+t)}.$$

Let us note that there exists some $c > 0$ such that for all $a > 1$ we have

$$\int_1^a h(t)t^{1-\delta} = \int_1^a \frac{1}{t^\delta \log(e+t)} \leq c\frac{a^{1-\delta}}{\log(e+a)},$$

which can easily be seen by deriving the right hand side with respect to a. Therefore using the Fubini theorem we get,

$$\int_1^\infty h(t)t^{1-\delta} \left(\int_{\{|v|>t\}} |v(x)|^{n-1+\delta} dx \right) dt =$$

$$= \int_{\{|v|>1\}} |v(x)|^{n-1+\delta} \left(\int_1^{|v(x)|} h(t)t^{1-\delta} dt \right) dx \leq c \int_{\{|v|>1\}} \frac{|v(x)|^n}{\log(e+|v(x)|)} dx.$$

This however is finite for all $v \in L^n \log^{-1} L(\Omega)$.

Let $t > 0$. We consider the following,

$$\int_1^\infty h(t) = \infty$$

$$h(t) > 0$$

$$t^{1-\delta} \left(\int_{\{|v|>t\}} |v|^{n-1+\delta}(x) dx \right) \geq 0.$$

These, combined with the fact that

$$\int_1^\infty h(t)t^{1-\delta} \left(\int_{\{|v|>t\}} |v(x)|^{n-1+\delta} dx \right) dt < \infty$$

for all $v \in L^n \log^{-1} L(\Omega)$ gives that

$$\text{liminf}_{t\to\infty} t^{1-\delta} \left(\int_{\{|v|>t\}} |v|^{n-1+\delta}(x)dx \right) = 0,$$

Now substitute in (3.9) Df for v and use $|Df| \in L^n \log^{-1} L(\Omega)$ to get the required result. $\square$

Proof of Theorem 3.3. Let B be a ball in Ω such that $2B \subset\subset \Omega$. We choose some $\varphi \in C_0^\infty(B)$ such that $\varphi \geq 0$ and $\varphi = 1$ everywhere on B/2. We now define $u(x) = (f_1(x) - (f_1)_B)\varphi(x)$. Clearly $u \in W_0^{1,1}(B)$. We may therefore, for this first stage of our proof, redefine u as zero everywhere outside B. We define

$$F_\lambda = \{x \in \mathbb{R}^n : M|Du(x)| < \lambda\} \cap \{x \in \mathbb{R}^n \mid x \text{ is a Lebesgue point of } u\}$$

using our new redefined function u. We choose some $\lambda > 0$ and by Lemma 3.5 in B (outside of B it is obvious) we have that

$$|u(x) - u(y)| \leq c\lambda|x - y| \qquad \text{for all } x, y \in F_\lambda.$$

We define

$$\tilde{u}_\lambda(x) := u(x) \qquad x \in F_\lambda.$$

Evidently the Lipschitz function $\tilde{u}_\lambda$ can be extended (uniquely) onto $\overline{F_\lambda}$ while maintaining the Lipschitz quality of $\tilde{u}_\lambda$. We extend $\tilde{u}_\lambda$ using Theorem 2.28 wherever not yet defined (a subset of $2B$) keeping the $c\lambda$ Lipschitz quality of the function. Here take K as $\overline{F_\lambda \cap 2B}$.

Note that by Lemma 3.4 it holds that

$$\int_B J(\tilde{u}_\lambda, f_2, ..., f_n) = 0.$$

This in conjunction with $|D\tilde{u}_\lambda| \leq c\lambda$, gives that

$$\left| \int_{F_\lambda} J(\tilde{u}_\lambda, f_2, \ldots, f_n) \right| = \left| \int_{B\setminus F_\lambda} J(\tilde{u}_\lambda, f_2, \ldots, f_n) \right| \tag{3.10}$$

$$\leq c\lambda \int_{B\setminus F_\lambda} |Df|^{n-1}. \tag{3.11}$$

We use the definition of $\tilde{u}_\lambda$ and the derivation of products rule to get

$$\begin{aligned} J(\tilde{u}_\lambda, f_2, \ldots, f_n) =& \varphi J(f_1, f_2, \ldots, f_n) + \\ &+ (f_1 - (f_1)_B) J(\varphi, f_2, \ldots, f_n) \end{aligned} \tag{3.12}$$

almost everywhere on F_λ.

Thus we have, using $J_f \geq 0$, that

$$\int_{F_\lambda \cap \frac{B}{2}} J_f \leq \int_{F_\lambda \cap B} \varphi J_f$$
$$\leq \left| \int_{F_\lambda \cap B} J(\tilde{u}_\lambda, f_2, \dots, f_n) \right| + \left| \int_{F_\lambda \cap B} (f_1 - (f_1)_B) J(\varphi, f_2, \dots, f_n) \right|.$$

Now using (3.10), (3.11) and the fact that $|Df| \in L^n \log^{-1} L(\Omega)$ we deduce that

$$\int_{F_\lambda \cap \frac{B}{2}} J_f \leq c\lambda \int_{B \setminus F_\lambda} |Df|^{n-1} + \int_B |(f_1 - (f_1)_B) J(\varphi, f_2, \dots, f_n)| < C$$

for all $\lambda > 0$, where by choosing some sequence of λ_i, which tend to infinity but for which the first term tends to zero by Lemma 3.8, we have that constant C does not depend on λ.

Therefore we get that

$$\int_{\frac{B}{2}} J_f(x) dx = \int_{\frac{B}{2}} |J_f(x)| dx < \infty$$

for all B such that $2B \subset\subset \Omega$. This however means that $J_f \in L^1_{\text{loc}}(\Omega)$.

Notice that at the beginning of the proof we could have taken $\varphi \in \mathcal{D}(B)$ without any other restriction, defined $u^\varphi(x) := (f_1(x) - (f_1)_B)\varphi(x)$, and then repeated the extension process as described above to get $\tilde{u}^\varphi_\lambda \in W^{1,1}_0(B)$. We use the definition of $\tilde{u}^\varphi_\lambda$ and the derivation of products rule to get

$$\begin{aligned} \varphi J(f_1, f_2, \dots, f_n) =& J(\tilde{u}^\varphi_\lambda, f_2, \dots, f_n) - \\ & - (f_1 - (f_1)_B) J(\varphi, f_2, \dots, f_n) \end{aligned} \tag{3.13}$$

almost everywhere on F_λ (taking the set F_λ, which corresponds to our new choice of φ).

Thanks to (3.13) we have

$$\begin{aligned} \int_B \varphi J_f &= \int_{B \setminus F_\lambda} \varphi J_f + \int_{F_\lambda} \varphi J_f \\ &= \int_{B \setminus F_\lambda} \varphi J_f + \int_{F_\lambda} J(\tilde{u}^\varphi_\lambda, f_2, \dots, f_n) + \\ &+ \int_{F_\lambda} (f_1)_B J(\varphi, f_2, \dots, f_n) - \int_{F_\lambda} f_1 J(\varphi, f_2, \dots, f_n). \end{aligned} \tag{3.14}$$

By Lemma 3.4 and the Lebesgue theorem we have

$$\lim_{\lambda \to \infty} \int_{F_\lambda} (f_1)_B J(\varphi, f_2, \dots, f_n) = \int_B (f_1)_B J(\varphi, f_2, \dots, f_n) = 0.$$

Notice that φJ_f is not dependent on λ and thanks to the continuity of the Lebesgue integral, the first term tends to zero for λ tending to ∞. Let us now prove that the second term tends to zero as λ tends to infinity. Firstly we note that thanks to Lemma 3.4

$$\int_B J(\tilde{u}^\varphi_\lambda, f_2, \dots, f_n) = 0.$$

This however also gives us that

$$\int_{F_\lambda} J(\tilde{u}^\varphi_\lambda, f_2, \dots, f_n) = -\int_{B \setminus F_\lambda} J(\tilde{u}^\varphi_\lambda, f_2, \dots, f_n).$$

The right hand side can be estimated as follows

$$\left| \int_{B \setminus F_\lambda} J(\tilde{u}^\varphi_\lambda, f_2, \dots, f_n) \right| \leq c\lambda \int_{B \setminus F_\lambda} |Df|^{n-1}.$$

Now applying Lemma 3.8 we get

$$\operatorname{liminf}_{\lambda \to \infty} \left| \int_{F_\lambda} J(\tilde{u}^\varphi_\lambda, f_2, \dots, f_n) \right| \leq \operatorname{liminf}_{\lambda \to \infty} \lambda \int_{B \setminus F_\lambda} |Df|^{n-1} = 0. \quad (3.15)$$

From (3.15) we chose a sequence of numbers $\{\lambda_i\}_{i=1}^\infty$ tending to infinity such that

$$\lim_{i \to \infty} \left| \int_{F_{\lambda_i}} J(\tilde{u}^\varphi_{\lambda_i}, f_2, \dots, f_n) \right| = 0$$

and consider the equation (3.14) with respect to this sequence, getting

$$\begin{aligned} \int_B \varphi J_f &= -\lim_{i \to \infty} \int_{F_{\lambda_i}} f_1 J(\varphi, f_2, \dots, f_n) \\ &= -\int_B f_1 J(\varphi, f_2, \dots, f_n) \end{aligned}$$

for all $\varphi \in \mathcal{D}(B)$. Hereby we prove that

$$\mathcal{J}_f(\varphi) = -\int_B \varphi J(f_1, f_2, \dots, f_n) \qquad \text{for all } \varphi \in \mathcal{D}(B).$$

Let us now take $\varphi \in \mathcal{D}(\Omega)$ without any other restriction. Let us take some finite covering of $\operatorname{supp} \varphi$ by balls and a division of unity. The equality holds on each ball and hereby on $\operatorname{supp} \varphi$. □

4 Weakly monotonous functions and continuity

4.1 Weakly monotonous functions

We now use the result of Theorem 3.3 to show weak monotonicity of a certain class of functions. We start with the following definition.

Definition 4.1. *Let $f \in W^{1,p}(\Omega)$, $p \in [1,\infty)$. Then f is p-weakly monotonous if the following holds: For all balls $B \subset\subset \Omega$ and for all $m, M \in \mathbb{R}, m < M$ both of the following implications are satisfied*

$$\begin{aligned} (m-f)^+ \in W_0^{1,p}(B) &\Rightarrow \quad f \geq m \text{ a.e. in } B, \\ (f-M)^+ \in W_0^{1,p}(B) &\Rightarrow \quad f \leq M \text{ a.e. in } B. \end{aligned}$$

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $f = (f_1, \dots, f_n)$ have finite distortion and $|Df| \in L^n \log^{-1} L(\Omega)$. Then $f_1, \dots, f_n$ are p-weakly monotonous for all $p < n$.*

Proof. We shall conduct the proof for f_1 the proofs for other component functions are analogous. Let B be a ball whose closure lies in Ω. Let us make the following definitions,

$$\begin{aligned} &v := (f_1 - M)^+ \chi_B \\ &g := (v, f_2, \dots, f_n) \\ &\varphi \in C_c^\infty(\Omega): \quad \varphi \geq 0, \quad \varphi(x) = 1 \quad \text{for } \ x \in B \end{aligned}$$

and pose the hypothesis that $v \in W_0^{1,p}(B)$. If we make the following definition

$$E := \{x \in B \ : \ f_1(x) > M\}$$

then we have

$$J_g = \begin{cases} 0 & \text{a.e. in } B \setminus E \\ J_f & \text{a.e. in } E. \end{cases}$$

Here the equation holds for such points that are Lebesgue points of the derivative and density points of the respective sets. We use in turn that $J_g \geq 0$, Theorem 3.3, the fact that $v(x) = 0$ for $x \notin B$ and $\nabla\varphi(x) = 0$ for $x \in B$ to show that

$$\begin{aligned} \int_B J_g \leq \int_\Omega \varphi J_g = \\ = -\int_\Omega v J(\varphi, f_2, \dots, f_n) = 0. \end{aligned}$$

Since $J_g \geq 0$ it follows that $J_g = 0$ almost everywhere in B, giving $J_f = 0$ almost everywhere in E. Nevertheless f has finite distortion, which implies that $|Df| = 0$ on E. Therefore $Df_1 = 0$ on $\{f_1 > M\}$ yielding $Dv = 0$ a.e. in B. Because $v \in W_0^{1,p}(B)$ we have that $v = 0$ a.e. in B.

We prove the second implication with $(m - f_1)^+$ analogously. □

4.2 Oscillation estimates

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $u \in W^{1,p}(\Omega)$ be p-weakly monotonous in $B(a,R) \subset\subset \Omega$ and $r < R$. Let u_j, $j \in \mathbb{N}$ be the convolution approximations of u as defined in* (2.2). *For any two Lebesgue points of u, x_0, $y_0 \in B(a,r)$ and for any $\delta > 0$ there exists some $N \in \mathbb{N}$ such that for all $j > N$ and for all $t \in (r,R)$ we have*

$$|u_j(x_0) - u_j(y_0)| \leq \operatorname{osc}(u_j, S(a,t)) + 2\delta.$$

Proof. It suffices to show that

$$u_j(x_0),\ u_j(y_0) \in (\min_{x \in S(a,t)} u_j(x) - \delta, \max_{x \in S(a,t)} u_j(x) + \delta)$$

for all $j \in \mathbb{N}$ greater than some N. We prove only $u_j(x_0) < \max_{x \in S(a,t)} u_j(x) + \delta$ for all $j > N$ as the other inequalities are similar. We prove this by contradiction. Let there exist some sequence of natural numbers $\{j_k\}_{k=1}^{\infty}$ and radii $\{t_k\}_{k=1}^{\infty} \in [r,R]$ such that

$$u_{j_k}(x_0) \geq \max_{x \in S(a,t_k)} u_{j_k}(x) + \delta.$$

Without loss of generality we may assume that t_k converges to some t. We now make the definition

$$v_{j_k}(x) := u_{j_k}(x) - u_{j_k}(x_0) + \delta \text{ for all } x \in B(a,t_k).$$

But $v_{j_k}(x) < 0$ for all $x \in S(a,t_k)$, which shows that $(v_{j_k})^+ \in W_0^{1,p}(B(a,t_k))$. Since $t_k \to t$ there exists an N_0 such that $|t_k - t| < \frac{t}{9}$ for all $k > N_0$. If $t_k \leq t$ let us define $\eta_k(s) = s$ on $\mathbb{R}$. If $t_k > t$ we define η_k on $\mathbb{R}$ as follows

$$\eta_k(s) = \begin{cases} s & s \in (-\infty, \frac{3t - t_k}{2}] \\ 3s + t_k - 3t & s \in [\frac{3t - t_k}{2}, \infty). \end{cases} \tag{4.1}$$

We can now make the following definition,

$$\tilde{v}_{j_k}(x) := \begin{cases} v_{j_k}(a) & x = a \\ v_{j_k}(\eta_k(|x-a|)\frac{x-a}{|x-a|} + a) & |x-a| \in (0,t]. \end{cases}$$

Clearly η_k maintains the absolute continuity of v_{j_k} on almost all radial lines and without reducing its integrability, giving us $\tilde{v}_{j_k} \in W^{1,p}(B(a,t))$. Because $v_{j_k}(x) < 0$ for all $x \in S(a,t_k)$ we have that $\tilde{v}_{j_k}^+ \in W_0^{1,p}(B(a,t))$ for all $k > N_0$.

We want to prove that $||v_{j_k} - \tilde{v}_{j_k}||_{W^{1,p}(B(a,t))} \to 0$ for $k \to \infty$. This is clear for those k, for which $t_k \leq t$. We now consider the case where $t_k > t$. Here $\tilde{v}_{j_k}(x) = v_{j_k}(x)$, whenever $|x-a| < \frac{3t-t_k}{2}$. Define

$$P_k := B(a,t_k) \setminus B\left(a, \frac{3t-t_k}{2}\right).$$

We have that $||v_{j_k}||_{L^p(P_k)} \leq ||u||_{L^p(P_k)}$ (see Theorem 2.29), which tends to zero because $|P_k| \to 0$. By considering the integration over radial line segments in P_k we see that $||\tilde{v}_{j_k}||_{L^p(P_k)} \leq ||v_{j_k}||_{L^p(P_k)}$ yielding

$$||v_{j_k} - \tilde{v}_{j_k}||_{L^p(P_k)} \leq 2||v_{j_k}||_{L^p(P_k)} \leq 2||u||_{L^p(P_k)} \to 0.$$

It is easy to observe that thanks to by combining (4.1) with the absolute continuity property on almost all radials and using per-partes integration and thanks to the fact that $\frac{t_k}{t} \leq 3$ combined with Lemma 5.2 and absolute continuity on almost all circles concentric with the sphere, we have

$$||D\tilde{v}_{j_k}||_{L^p(P_k)} \leq 3||Dv_{j_k}||_{L^p(P_k)}.$$

By using the triangle inequality we get,

$$||v_{j_k} - \tilde{v}_{j_k}||_{W^{1,p}(P_k)} \leq c||v_{j_k}||_{W^{1,p}(P_k)} \leq c||u||_{W^{1,p}(P_k)} \to 0.$$

This gives that $||v_{j_k} - \tilde{v}_{j_k}||_{W^{1,p}(B(a,t))} \to 0$ and therefore $\tilde{v}_{j_k} \to u - u(x_0) + \delta$ in $W^{1,p}(B(a,t))$ because x_0 is a Lebesgue point of u. This implies however that $(u - u(x_0) + \delta)^+ \in W_0^{1,p}(B(a,t))$.

Thanks to the weak monotonicity of u we now have,

$$u(x) \leq u(x_0) - \delta$$

for almost all $x \in B(a,t)$. This however cannot be as x_0 is a Lebesgue point of u. □

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $n-1 < p < n$, let $u \in W^{1,p}(B(a,R))$ be p-weakly monotonous in $B(a,R) \subset\subset \Omega$ and let us define $B_r := B(a,r)$. Then for almost every $t \in (r,R)$ we have*

$$\operatorname{diam}\big(u(B_r)\big) \leq ct\left(\fint_{S(a,t)} |Du|^p\right)^{\frac{1}{p}}.$$

Proof. Let $r \in (0, R)$ and $x_0, y_0 \in B(a, r)$ be Lebesgue points of u. Using Lemma 4.3 and then Lemma 2.23 for sufficiently large j we have

$$|u_j(x_0) - u_j(y_0)| \leq ct \left(\fint_{S(a,t)} |Du_j|^p \right)^{\frac{1}{p}} + \delta_j \text{ for almost all } t \in (r, R), \quad (4.2)$$

where $\delta_j \to 0$ for $j \to \infty$. Because, by Theorem 2.29, the convolution approximations converge to u in $W^{1,p}(B(a, R))$ it holds that

$$\int_{B(a,r)} |Du_j - Du|^p \to 0.$$

From this it follows that

$$\int_0^r \left(\int_{S(a,t)} |Du_j - Du|^p \right) dt \to 0 \quad \text{for } j \to \infty.$$

This implies the existence of a subsequence u_{j_k}, for which for almost all $t \in (0, r)$ holds

$$\int_{S(a,t)} |Du_{j_k} - Du|^p \to 0.$$

Let us remember that x_0 and y_0 are Lebesgue points of u and therefore $u_j(x_0) \to u(x_0)$ and $u_j(y_0) \to u(y_0)$. Using this and taking any such t for which the above covergence holds we take the limit over k in (4.2) getting

$$|u(x_0) - u(y_0)| \leq ct \left(\fint_{S(a,t)} |Du|^p \right)^{\frac{1}{p}}$$

where c is derived from Lemma 2.23 and is independent on r. □

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ with $|Df| \in L^n \log^{-1} L(\Omega)$ be p-weakly monotonous for some $p \in (n-1, n)$. Then f is continuous.*

Proof. Clearly there exists an increasing convex function $\Phi \in C^\infty(0, \infty)$ and an $M > 0$, such that for the $p \in (n-1, n)$ given,

$$\lim_{s \to 0_+} \frac{\Phi(s)}{s} = 0$$

and for all $s > M$ it holds that

$$\Phi(s) = \frac{s^n}{\log(s)}.$$

We can moreover suppose that,

$$\varphi(t) := \Phi(\sqrt[p]{t}),$$

is convex. Using Lemma 4.4 and then the Jensen inequality we see that

$$\operatorname{diam} f(B_r) \leq ct\left(\varphi^{-1}\circ\varphi\Big(\fint_{S(a,t)} |Df|^p\Big)\right)^{\frac{1}{p}} \leq ct\Big(\varphi^{-1}\Big(\fint_{S(a,t)} \Phi(|Df|)\Big)\Big)^{\frac{1}{p}}.$$

Divide by ct, take the power p and apply φ to get

$$\varphi\Big(\frac{\operatorname{diam}^p f(B_r)}{c^p t^p}\Big) = \Phi\left(\frac{\operatorname{diam} f(B_r)}{ct}\right) \leq \fint_{S(a,t)} \Phi(|Df|).$$

By multiplying by $\omega_n t^{n-1}$, where ω_n is the $n-1$ dimensional measure of the unit sphere, and then integrating over t from r to R we find

$$\begin{aligned} \omega_{n-1}\int_r^R \Phi\left(\frac{\operatorname{diam} f(B_r)}{ct}\right) t^{n-1}dt &\leq \int_{B(a,R)\setminus B(a,r)} \Phi(|Df|) \\ &\leq \int_{B(a,R)} \Phi(|Df|) < \infty \end{aligned} \tag{4.3}$$

because $|Df| \in L^n \log^{-1} L(\Omega)$. The above however holds for all $r > 0$. This implies that $\lim_{r\to 0+} \operatorname{diam} f(B_r) = 0$. To show this let us assume the converse is true i.e. that

$$\limsup_{r\to 0+} \operatorname{diam} f(B_r) =: z > 0.$$

Then however, because $\operatorname{diam} f(B_r)$ is non-decreasing in r, we have a $\delta > 0$ such that

$$\begin{aligned} \lim_{r\to 0_+} \omega_{n-1}\int_r^R \Phi\left(\frac{\operatorname{diam} f(B_r)}{ct}\right) t^{n-1}dt &\geq \lim_{r\to 0_+} \omega_{n-1}\int_r^R \Phi\left(\frac{z}{ct}\right) t^{n-1}dt \\ &\geq C\int_0^\delta \frac{1}{t\log t^{-1}}dt = \infty \end{aligned}$$

which is in contradiction with (4.3). □

4.3 Proof of Theorem 1.2 and Theorem 1.1

Proof of Theorem 1.1. Easily follows from Theorem 4.2 and Theorem 4.5. □

Lemma 4.6. *Let $a \geq 1$, $b \geq 0$ then for all $\lambda > 0$ it holds that*

$$ab \leq \exp(\lambda a) + \frac{2b}{\lambda}\log\big(e + \frac{b}{\lambda}\big).$$

Proof. Either

$$ab \le \exp(\lambda a)$$

and the proof is finished or

$$ab \ge \exp(\lambda a).$$

But in the second case the fact that $\exp(x) > x^2$ for all $x \ge 0$ implies that

$$ab > \lambda^2 a^2.$$

This implies

$$\frac{b}{\lambda^2} > a$$

and therefore

$$\exp(\lambda a) \le ab < \frac{b^2}{\lambda^2}.$$

Hereby we get

$$a < \frac{2}{\lambda} \log\big(e + \frac{b}{\lambda}\big)$$

and thus

$$ab < \frac{2b}{\lambda} \log\big(e + \frac{b}{\lambda}\big).$$

$\square$

Lemma 4.7. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ have finite outer distortion and suppose there is some $\lambda > 0$ such that $\exp(\lambda K_O) \in L^1_{\mathrm{loc}}(\Omega)$. Then $|Df| \in L^n \log^{-1} L_{\mathrm{loc}}(\Omega)$*

Proof. Easily

$$\log(e + |Df|^n) \le \log\left((e + |Df|)^n\right) = n \log(e + |Df|).$$

Therefore

$$\frac{1}{n}|Df|^n \log^{-1}(e + |Df|) \le |Df|^n \log^{-1}(e + |Df|^n) \le \frac{K_O J_f}{\log(e + K_O J_f)}.$$

As $K_O \ge 1$ and $J_f \ge 0$ a.e. we see that

$$0 < \frac{K_O(x) J_f(x)}{\log(e + K_O(x) J_f(x))} \le \frac{K_O(x) J_f(x)}{\log(e + J_f(x))} \qquad \text{for almost all } x \in \Omega.$$

Fix $x \in \Omega$ and find some ball $B = B(x, r)$ such that

$$\int_B \exp(\lambda K_O) < \infty \quad \text{and} \quad \int_B J_f < \infty.$$

We use Lemma 4.6 where $a := K_O$ and $b := \frac{J_f}{\log(e+J_f)}$ to obtain

$$\begin{aligned}\int_B |Df|^n \log^{-1}(e+|Df|) &\leq n \int_B \frac{K_O J_f}{\log(e+J_f)} \\ &\leq n \int_B \exp(\lambda K_O) + \frac{2n}{\lambda} \int_B \frac{J_f}{\log(e+J_f)} \log\left(e + \frac{J_f}{\lambda \log(e+J_f)}\right).\end{aligned}$$

The first of these two terms is finite by the hypothesis. We separate the second integral into two over the sets $A_1 := \{x \in B \mid \lambda \log(e+J_f) \leq 1\}$ and $A_2 := B \setminus A_1$. The integrand is bounded on A_1 and on A_2 is dominated by J_f, which is integrable. $\square$

Proof of Theorem 1.2. Let $x \in \Omega$. For x find some neighborhood U_x of x such that $J_f \in L^1(U_x)$ and $\exp(\lambda K_O) \in L^1(U_x)$. By Theorem 4.2, Lemma 4.7 and Theorem 4.5 we have f continuous on U_x. $\square$

5 Counter-examples of continuity

Definition 5.1. *We shall say that* $f \in W^{1,1}(B(0,r),\mathbb{R}^n)$, $r > 0$ *is radially homogenous if there exists some positive function* $\varphi \in C^1((0,r))$ *such that*

$$f(x) = \frac{x}{|x|}\varphi(|x|).$$

Lemma 5.2. *Let* $f \in W^{1,1}(B(0,1),\mathbb{R}^n)$ *be radially homogeonous. Then for all* $x \in B(0,1)$, $x \neq 0$ *there exists some positively oriented orthogonal basis* $\alpha(x) = (\frac{x}{|x|}, y_2, \ldots, y_n)$ *of* $\mathbb{R}^n$ *such that*

$$D^{\alpha(x)}f(x) := A(x)Df(x)A(x)^{-1} = \begin{pmatrix} \varphi'(|x|) & 0 & \ldots & 0 \\ 0 & \frac{\varphi(|x|)}{|x|} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{\varphi(|x|)}{|x|} \end{pmatrix},$$

where $A(x)$ *is the transitional matrix between the canonic basis and* $\alpha(x)$. *Moreover*

$$|Df(x)| = \max\left\{|\varphi'(x)|, \frac{\varphi(|x|)}{|x|}\right\}$$

$$\text{and } J_f = \left|\varphi'(x)\frac{\varphi^{n-1}(|x|)}{|x|^{n-1}}\right|.$$

Proof. Given the regularity of φ we have that $f \in C^1(B(0,1)\setminus\{0\})$. This and $f \in W^{1,1}(B(0,1),\mathbb{R}^n)$ implies equality almost everywhere between ∇f and Df. For the existence of the basis $\alpha(x)$ use $y_1 := \frac{x}{|x|}$ and Gramm-Schmidt orthogonalisation process. Then

$$\begin{aligned}\partial_{\frac{x}{|x|}} f(x) &= \lim_{t\to 0}\Big(\frac{x + t\frac{x}{|x|}}{|x|+t}\varphi(|x|+t) - \frac{x}{|x|}\varphi(|x|)\Big)t^{-1} \\ &= \lim_{t\to 0}\frac{x}{|x|}\frac{\varphi(|x|+t)-\varphi(|x|)}{t} = \frac{x}{|x|}\varphi'(|x|)\end{aligned}$$

because

$$\frac{x + t\frac{x}{|x|}}{(|x|+t)} = \frac{x}{|x|}.$$

Chose some $i \in \{2, \dots n\}$ then almost everywhere

$$\begin{aligned} D_{y_i} f(x) = \partial_{y_i} f(x) &= \lim_{t\to 0} \frac{f(x+ty_i) - f(x)}{t} \\ &= \lim_{t\to 0} \frac{f(x+ty_i) - f\left(\frac{|x|}{|x+ty_i|}(x+ty_i)\right)}{t} + \\ &\quad + \lim_{t\to 0} \frac{f\left(\frac{|x|}{|x+ty_i|}(x+ty_i)\right) - f(x)}{t} \\ &= \lim_{t\to 0} \frac{\frac{x+ty_i}{|x+ty_i|}\varphi(|x+ty_i|) - \frac{x+ty_i}{|x+ty_i|}\varphi(|x|)}{t} + \\ &\quad + \lim_{t\to 0} \frac{\frac{x+ty_i}{|x+ty_i|}\varphi(|x|) - \frac{x}{|x|}\varphi(|x|)}{t}. \end{aligned} \tag{5.1}$$

The second limit in (5.1) is

$$\lim_{t\to 0} \frac{x|x| - x|x+ty_i| + t|x|y_i}{t|x||x+ty_i|}\varphi(|x|) = \frac{y_i}{|x|}\varphi(|x|)$$

because $y_i \perp x$ and because

$$\frac{\partial}{\partial t}(|x+ty_i|) = \lim_{t\to 0} \frac{\sqrt{|x|^2+t^2} - |x|}{t} = \lim_{t\to 0} \frac{|x|^2+t^2-|x|^2}{t(\sqrt{|x|^2+t^2}+|x|)} = 0.$$

We use the derivation of compound functions on the first of these two limits to get

$$\lim_{t\to 0} \frac{x+ty_i}{|x+ty_i|} \frac{\varphi(|x+ty_i|) - \varphi(|x|)}{t} = \frac{x}{|x|}\varphi'(|x|)\frac{\partial}{\partial t}(|x+ty_i|) = 0.$$

□

In Theorem 1.1 we require that f is a mapping of finite outer distortion. Let us now show that finite outer distortion cannot be replaced with finite inner distortion. Evidently a.e. finite outer distortion implies a.e. finite inner distortion (the Jacobian is integrable and therefore almost everywhere finite) but the reverse is clearly not true. In fact in this example we have $K_I \in L^\infty$, which therefore shows that the condition of finite outer distortion in Theorem 1.3 and Theorem 1.2 cannot be replaced with finite inner distortion. For any meaningful results concerning continuity we must therefore require that f has finite outer distortion.

Example 5.3. *There exists a function $f \in W^{1,n}(B(0,\frac{1}{2}),\mathbb{R}^n)$, with $J_f \geq 0$ such that f is not continuous at 0 but satisfies the condition*

$$J_f(x) = 0 \Rightarrow |D^\sharp f(x)| = 0$$

almost everywhere in $B(0,1)$.

Proof. Define $\Omega := B(0, \frac{1}{2}) \setminus \{0\}$ and

$$f := e_1 \log\left(\log\left(\frac{1}{|x|}\right)\right) \qquad \text{for all } \ x \in \Omega,$$

where e_1 is the first unit vector of the canonical basis in $\mathbb{R}^n$. Using the properties of log we find

$$\int_\Omega |f|^n dx < C \int_\Omega \log\left(\frac{1}{|x|}\right) dx < \infty.$$

Note that $f \in C^\infty(\Omega)$, which gives that $f \in W^{1,1}_{\text{loc}}(\Omega)$ and,

$$D_\alpha f(x) = \partial_\alpha f(x) \qquad \text{for almost all } x \in \Omega. \tag{5.2}$$

We need $|\nabla f| \in L^n(B(0, \frac{1}{2}))$. We find that

$$\partial_{-\frac{x}{|x|}} f_1(x) = \frac{1}{|x| \log\left(\frac{1}{|x|}\right)} \text{ and } \partial_y f_1(x) = 0$$

for all $y \in S(0,1)$ perpendicular to x. We have that

$$|\nabla f(x)| = \frac{1}{|x| \log\left(\frac{1}{|x|}\right)}.$$

We use the formula on change of variables, polar coordinates and the fact that $n > 1$ to prove that that the integral of $|\nabla f|^n$ over $B(0, \frac{1}{2})$ is finite.

$$\begin{aligned} \int_{B(0,\frac{1}{2})} \Big(\frac{1}{|x| \log\left(\frac{1}{|x|}\right)}\Big)^n dx &= c \int_0^{\frac{1}{2}} \frac{s^{n-1}}{s^n \log^n \frac{1}{|x|}} ds \\ &= c \int_0^{\frac{1}{2}} \frac{1}{s \log^n \frac{1}{|x|}} ds \quad < \infty. \end{aligned}$$

Since the range of f is one dimensional and $n \geq 3$, we have that $|Df|$ has $n-1$ zero rows yielding that

$$|D^\sharp f(x)| = J_f(x) = 0 \qquad \text{for all } x \in B(0, 1/2) \setminus \{0\}.$$

Hereby the function K_I exists and is almost everywhere equal to 1. Let us also note that we can consider f to be a mapping of finite inner distortion on $B(0, \frac{1}{2})$ but f is not continuous on this set because $\lim_{x\to 0} |f(x)| = \infty$. □

Further it is worth noting that no restriction on distortion can guarantee continuity if we omit the assumption that J_f is locally integrable.

Example 5.4. *Let $n \geq 3$. There exists a function $f \in W^{1,1}(B(0,1), \mathbb{R}^n)$ such that*

$$J_f(x) > 0,\ J_f(x) = |Df(x)|^n,\ |D^\sharp f|^n = J_f^{n-1} \text{ for all } x \neq 0$$

but f is not continuous at zero.

Proof. Let us define

$$g(x) := \frac{x}{|x|^2} \qquad \text{for all } x \in B(0,1) \setminus \{0\}$$

Then because g is radially homogeneous we use Lemma 5.2 to get the positively oriented orthogonal basis $\alpha(x)$ and

$$D^{\alpha(x)}g = \begin{pmatrix} -\frac{1}{|x|^2} & 0 & \dots & 0 \\ 0 & \frac{1}{|x|^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{|x|^2} \end{pmatrix}$$

Note that $|\nabla g| = |\partial_y g|$ for all $y \in S(0,1)$ and $\partial_y g \perp \partial_\gamma g$ for all $y \perp \gamma$, $y, \gamma \in S(0,1)$. We also have that $-J_g(x) = |Df(x)|^n = \frac{1}{|x|^{2n}}$. Because $n \geq 3$, we use the substitution theorem with polar coordinates to find that

$$\int_{B(0,1)} g < \infty.$$

We also have, however, that

$$\int_{B(0,1)} |\nabla g| = \int_{B(0,1)} \frac{1}{|x|^2} dx < \infty.$$

We define

$$f_1(x) = -g_1(x)$$
$$f_i(x) = g_i(x) \qquad i \neq 1$$

giving $f \in W^{1,1}(B(0,1), \mathbb{R}^n)$, $J_f(x) > 0$, for all $x \neq 0$, $|D^\sharp f|^n = |Df|^{n(n-1)} = J_f^{n-1}$ and f is not continuous at 0. □

Example 5.5. *Let $\delta > 0$ and $n \geq 3$. There exists an $f \in W^{1,1}(B(0,1/2), \mathbb{R}^n)$ with finite outer distortion such that $\exp(\lambda K_O^{1-\delta}) \in L^1(B(0,1/2))$ for all $\lambda > 0$ but f is not continuous at the origin. It also holds that $\exp(\lambda K_I^{\frac{1-\delta}{n-1}}) \in L^1(B(0,1/2))$.*

Proof. Let us have $\delta > 0$ given. Without loss of generality we may assume that $\frac{1}{2} > \delta > 0$. Put $\varepsilon := (1 - \frac{\delta}{2})^{-1} - 1$ and define

$$f(x) := \frac{x}{|x|} \left(1 + \log^{-\varepsilon}\left(\frac{1}{|x|}\right)\right).$$

Using Lemma 5.2 we get

$$D^{\alpha(x)} f(x) = \begin{pmatrix} \varepsilon \frac{\log^{-1-\varepsilon}\left(\frac{1}{|x|}\right)}{|x|}, & 0, & \dots, & 0 \\ 0 & \frac{1+\log^{-\varepsilon}\left(\frac{1}{|x|}\right)}{|x|}, & \dots, & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0, & 0, & \dots, & \frac{1+\log^{-\varepsilon}\left(\frac{1}{|x|}\right)}{|x|} \end{pmatrix}, \tag{5.3}$$

and

$$|Df| = \frac{1 + \log^{-\varepsilon}\left(\frac{1}{|x|}\right)}{|x|} \text{ and}$$

$$J_f = \varepsilon \frac{\log^{-1-\varepsilon}\left(\frac{1}{|x|}\right)}{|x|} \frac{\left(1 + \log^{-\varepsilon}\left(\frac{1}{|x|}\right)\right)^{n-1}}{|x|^{n-1}}.$$

Thus we have

$$K_O = \varepsilon^{-1} \log^{\frac{1}{1-\delta/2}}\left(\frac{1}{|x|}\right) + \varepsilon^{-1} \log\left(\frac{1}{|x|}\right).$$

This easily gives that

$$\exp((\varepsilon K_O)^{1-\delta/2}) \leq \frac{c}{|x|},$$

which is in $L^1(B(0,1))$. It now follows that $\exp(\lambda K_O^{1-\delta}) \in L^1(B(0,1/2))$ for all $\lambda > 0$. As can be seen from (5.3), we have

$$|D^\sharp f(x)| = |Df(x)|^{n-1}$$

and therefore

$$K_I = K_O^{n-1}.$$

This gives that $\exp\left((\lambda K_I)^{\frac{1-\delta}{n-1}}\right) \in L^1(B(0,1))$ but $\exp(\lambda K_I^{\frac{1}{n-1}}) \notin L^1(B(0,1))$ for all $\lambda > 0$. $\square$

6 Discontinuity and inner distortion

In order to prove Theorem 1.4 we will need a couple of lemmata.

Lemma 6.1. *Let* $a, \alpha, \lambda > 0$ *and* $n \in \mathbb{N}$, $n \geq 2$ *then*

$$a\alpha \leq t^{n-1}\exp(\lambda a) + \frac{2\alpha}{\lambda}\log\left(e + \frac{\alpha}{\lambda t^{n-1}}\right)$$

for all $t > 0$.

Proof. Either

$$a\alpha \leq t^{n-1}\exp(\lambda a)$$

and the proof is finished or

$$a\alpha > t^{n-1}\exp(\lambda a) > t^{n-1}a^2\lambda^2.$$

This, however, implies that

$$a < \frac{\alpha}{\lambda^2 t^{n-1}}.$$

Hereby

$$t^{n-1}\exp(\lambda a) < a\alpha < \frac{\alpha^2}{\lambda^2 t^{n-1}}.$$

From here, however, we have that

$$\lambda a < 2\log\left(e + \frac{\alpha}{\lambda t^{n-1}}\right)$$

and therefore

$$a\alpha < \frac{2\alpha}{\lambda}\log\left(e + \frac{\alpha}{\lambda t^{n-1}}\right).$$

□

Lemma 6.2. *Let* $b : (0,1) \to [0,\infty)$ *be a measurable function with the following properties,*

$$\int_0^1 \frac{1}{tb(t)}dt < \infty.$$

Then

$$\int_0^1 t^{n-1}\exp(\lambda b(t))dt = \infty$$

for all $\lambda > 0$.

Proof. Without loss of generality we may assume that $b(t) \geq 1$ almost everywhere because if this was not true we define $b_2 := b + 1$ giving that

$$\int_0^1 \frac{1}{tb_2(t)} dt < \infty$$

and

$$\exp(\lambda b_2(t)) = \exp\big(\lambda(b(t)+1)\big) = \exp(\lambda)\exp(\lambda b(t)).$$

Clearly

$$\infty = \int_0^1 \frac{1}{t\log(\frac{2}{t})} dt = \int_0^1 \frac{b(t)}{t\log(\frac{2}{t})b(t)} dt. \tag{6.1}$$

Put $a := b(t)$, $\alpha := \frac{1}{t\log(\frac{2}{t})b(t)}$ and let $\lambda > 0$ be arbitrary. Use Lemma 6.1 and (6.1) to get that

$$\infty = \int_0^1 t^{n-1}\exp(\lambda b(t))dt + \frac{2}{\lambda}\int_0^1 \frac{\log\left(e + \frac{1}{\lambda t^n \log(\frac{2}{t})b(t)}\right)}{t\log\left(\frac{2}{t}\right)b(t)} dt. \tag{6.2}$$

We now show that the second term is finite to give the required result. For $t^n < \min\{e^{-1}\lambda^{-1}, 2e^{-1}\} =: q$ we get,

$$\frac{2}{\lambda}\int_0^q \frac{\log\left(e + \frac{1}{\lambda t^n \log(\frac{2}{t})b(t)}\right)}{t\log\left(\frac{2}{t}\right)b(t)} dt \leq \frac{2n}{\lambda}\int_0^q \frac{\log\left(2\lambda^{-1}t^{-n}\right)}{t\log(2^n t^{-n})b(t)} dt < \infty$$

because $b(t) \geq 1$ and $\log\left(2\lambda^{-1}t^{-n}\right) < c\log(2^n t^{-n})$ for $t \in (0,q)$. For $t > q$ we have,

$$\frac{2}{\lambda}\int_q^1 \frac{\log\left(e + \frac{1}{\lambda t^n \log(\frac{2}{t})b(t)}\right)}{t\log\left(\frac{2}{t}\right)b(t)} dt \leq \frac{2}{\lambda}\int_q^1 \frac{\log\left(e + \frac{1}{\lambda q^n \log(\frac{2}{q})}\right)}{q\log\left(\frac{2}{q}\right)} dt < \infty,$$

which by (6.2) gives the result. □

Lemma 6.3. *Let f be a homeomorphism from $B(0,1)\backslash\{0\}$ onto $B(0,2)\backslash L =: G$, where $L \subset B(0,1)$ is closed in $\mathbb{R}^n$ and path-wise connected such that there exists some hyperplane ρ such that the projection of L onto ρ has positive $(n-1)$ dimensional measure. Then for all $t \in (0,1)$ and for all lines segments given as $P_\gamma(l) := \{r\gamma + l \mid l \in L\ \gamma \in S(0,1),\ r \in [0,4]\}$ it holds that*

$$f(S(0,t)) \cap P_\gamma(l) \neq \emptyset. \tag{6.3}$$

Proof. For contradiction let there exist some $l \in L$ and some $\gamma \in S(0,1)$ such that $f(S(0,t)) \cap P_\gamma(l) = \emptyset$. Clearly $\partial G \cap f(S(0,t)) = \emptyset$. Because $f(S(0,t))$ and $(P_\gamma(l) \cup \partial G)$ are closed and disjoint there exists some $\delta > 0$ such that

$$\big((P_\gamma(l) \cup \partial G) + B(0,\delta)\big) \cap f(S(0,t)) = \emptyset.$$

This gives that the pre-images $f^{-1}(y_m)$ of all sequences y_m, which converge to some $y \in \partial G$, either converge to zero or have all accumulation points on the unit sphere. If this were not so we could find two sequences w_m and z_m both of which converge to elements on the boundary $w_m \to w$ and $z_m \to z$, with z, $w \in \partial G$ and for whom $f^{-1}(w_m) \to 0$ and $f^{-1}(z_m) \to z^* \in S(0,1)$.

Therefore for some $m_0 \in \mathbb{N}$, w_m, $z_m \in \big((P_\gamma(l) \cup \partial G) + B(0,\delta)\big) =: Q$ for all $m \geq m_0$. Notice further that Q is open and pathwise connected. For any $m, k \geq m_0$ we may join any pair (w_m, z_k) with a continuous curve $\alpha_m^k : [0,1] \to G$, whose image is disjoint with $f(S(0,t))$.

We find some m and k such that $|f^{-1}(w_m)| < t$ and $|f^{-1}(z_k)| > t$. Because α_m^k is continuous and f^{-1} is continuous, $|f^{-1} \circ \alpha_m^k(s)|$ is a continuous function of s. But $|f^{-1} \circ \alpha_m^k(0)| < t$, $|f^{-1} \circ \alpha_m^k(1)| > t$ and $|f^{-1} \circ \alpha_m^k(s)| \neq t$ for all $s \in [0,1]$, which is in contradiction with the intermediate value property of continuous functions.

There are two possibilities either $|f^{-1}(y_m)| \to 0$ for all sequences y_m in G with $y_m \to y \in \partial G$ or $|f^{-1}(y_m)| \to 1$ for all sequences y_m in G with $y_m \to y \in \partial G$. Let us assume the second case. Hereby we have that for all sequences x_m in $B(0,1) \setminus \{0\}$, $x_m \to 0$ there exists some $m_0 \in \mathbb{N}$ such that for all $m > m_0$ we have that

$$f(x_m) \in \overline{G \setminus Q}.$$

But this sequence must have at least one accumulation point, which lies in G and which we will denote as z. Note that $z \in G$ implies that there exists some x such that $f(x) = z$ and $x_n \to 0 \neq x$. Nevertheless $f(x_m)$ converges to $f(x)$, which is in contradiction with the continuity of f^{-1}.

In the first case we get analogously for all sequences x_m in $B(0,1)$, $x_m \to x \in S(0,1)$, that $f(x_m)$ is a sequence in G with all accumulation points (of which there must be at least one) at a distance of at least δ from the boundary. Similarly as before this is in contradiction with the continuity of f^{-1}. □

Proof of Theorem 1.4. Choose $\lambda > 0$. Use Lemma 6.3 to get that

$$f(S(0,t)) \cap P_\gamma(l) \neq \emptyset$$

for all $l \in L, \gamma \in S(0,1)$ and for all $t \in (0,1)$ where

$$P_\gamma(l) = \{s\gamma + l \,|l \in L \; s \in [0,4]\}.$$

From this however we see that the projection of $f(S(0,t))$ onto the hyperplane ρ is a superset of the projection of L and therefore has $(n-1)$ dimensional measure greater than some positive c fixed and independent on t. Without loss of generality we may assume that $K_O \in L^p$ for all $p < \infty$. Otherwise $K_O \le K_I^{n-1}$ easily implies $\exp(\lambda K_I^{\frac{1}{n-1}}) \notin L^1$ and thereby the result. Therefore using the definition of K_O and the Hölder inequality we get

$$\begin{aligned}\int_\Omega |Df|^{n-\frac{1}{2}} &\le \int_\Omega (K_O^{\frac{2n-1}{2n}} J_f^{\frac{2n-1}{2n}}) \\ &\le \Big(\int_\Omega K_O^{2n-1}\Big)^{\frac{1}{2n}} \Big(\int_\Omega J_f\Big)^{\frac{2n-1}{2n}}.\end{aligned}$$

Therefore as $J_f \in L^1(B(0,1))$, $K_O \in L^{2n-1}(B(0,1))$ we have that $|Df| \in L^{n-\frac{1}{2}}(\Omega)$. Therefore by Lemma 2.24 on f and $S(0,t)$ and using the fact that projections do not increase Hausdorf measures we have

$$\int_{S(0,t)} |D^\sharp f| \ge c\mathcal{H}^{n-1}\big(f(S(0,t))\big) \ge c\mathcal{H}^{n-1}\big(\rho\big(f(S(0,t))\big)\big) \ge c$$

for almost all $t \in (0,1)$ where $c > 0$ is fixed independent of t. This gives that

$$c \le \int_{S(0,t)} |D^\sharp f| = \int_{S(0,t)} \big(\frac{\lambda^{n-1}}{\lambda^{n-1}} K_I\big)^{\frac{1}{n}} J_f^{\frac{n-1}{n}} \text{for almost all } t \in (0,1). \tag{6.4}$$

Now using the Hölder inequality we get

$$c \le \lambda^{-\frac{n-1}{n}} \Big(\int_{S(0,t)} J_f\Big)^{\frac{n-1}{n}} \Big(\int_{S(0,t)} \lambda^{n-1} K_I\Big)^{\frac{1}{n}}. \tag{6.5}$$

Let ω_n be the $n-1$ dimensional measure of the unit sphere in $\mathbb{R}^n$. Let us note that there exists some $S = S(n)$ for which $\exp(s^{\frac{1}{n-1}})$ is convex for all $s > S$. Further let us note that there exists some smooth convex function $\Phi : [0,\infty) \to [0,\infty)$ such that $\Phi(s) = \exp(s^{\frac{1}{n-1}})$ for all $s > S$. Dividing the second integral in the equation above by the measure $|S(0,t)| = \omega_n t^{n-1}$ we get,

$$c \le \Big(\int_{S(0,t)} J_f\Big)^{\frac{n-1}{n}} \omega_n^{\frac{1}{n}} t^{\frac{n-1}{n}} \Big[\Phi^{-1} \circ \Phi\Big(\fint_{S(0,t)} \lambda^{n-1} K_I\Big)\Big]^{\frac{1}{n}}.$$

We may now use the Jensen inequality to find that

$$c \le \Big(\int_{S(0,t)} J_f\Big)^{\frac{n-1}{n}} t^{\frac{n-1}{n}} \Big[\Phi^{-1}\Big(\fint_{S(0,t)} \Phi(\lambda^{n-1} K_I)\Big)\Big]^{\frac{1}{n}}.$$

Thus we get

$$\omega_n t^{n-1}\Phi\left(\frac{c}{t^{n-1}\left(\int_{S(0,t)} J_f\right)^{n-1}}\right) \le \int_{S(0,t)} \Phi(\lambda^{n-1}K_I). \tag{6.6}$$

Put

$$b(t) := \begin{cases} \frac{c}{t\int_{S(0,t)} J_f} & \int_{S(0,t)} J_f > 0 \\ \infty & \int_{S(0,t)} J_f = 0. \end{cases}$$

But as it can be seen from (6.5), the set $\{t \in (0,1) \mid \int_{S(0,t)} J_f = 0\}$ has zero measure. Therefore b is finite almost everywhere. Then we have

$$\int_0^1 \frac{1}{tb(t)} = c\int_{B(0,1)} J_f < \infty$$

and therefore by Lemma 6.2 we have that

$$\int_{\{b(t)>S\}} t^{n-1}\exp(b(t))dt = \int_{\{b(t)>S\}} t^{n-1}\Phi(b^{n-1}(t)) = \infty.$$

In connection with (6.6) we get

$$\int_0^1\int_{S(0,t)} \Phi(\lambda^{n-1}K_I) \ge c\int_0^1 t^{n-1}\Phi(b^{n-1}(t)) = \infty.$$

But this means precisely that

$$\int_{B(0,1)} \Phi(\lambda^{n-1}K_I) = \infty.$$

This however is equivalent (as $|B(0,1)| < \infty$) with

$$\int_{B(0,1)} \exp(\lambda K_I^{\frac{1}{n-1}}) = \infty.$$

□

Remark 6.4. *Let us note that a special case of what we have proven is where* $L := \overline{B(0,\frac{1}{2})}$. *This shows that no radial mapping forming a cavity at the origin could have the required integrability of the distortion function. Our above result applies to a much more general class of mappings.*

7 Sharp modulus of continuity

We will actually prove a theorem much more general than Theorem 1.5. We will study the class of mappings with exponentially integrable distortion in a more general setting (see e.g. [5]). We require that

$$\int_B \exp\big(\mathcal{A}(K(x))\big)\, dx < \infty \tag{7.1}$$

for some Orlicz function $\mathcal{A}$ and the case in Theorem 1.5 corresponds to the case $\mathcal{A}(t) = \lambda t$. We call an infinitely differentiable and strictly increasing function $\mathcal{A} : [0,\infty) \to [0,\infty)$ with $\mathcal{A}(0) = 0$ and $\lim_{t\to\infty} \mathcal{A}(t) = \infty$ an Orlicz function. As usual we impose the additional condition

$$\int_1^\infty \frac{\mathcal{A}'(s)}{s} = \infty. \tag{7.2}$$

It is easy to see that the critical functions for this condition are

$$\begin{aligned} &\mathcal{A}_1(t) = \lambda t,\ \mathcal{A}_2(t) = \lambda \frac{t}{\log(e+t)}, \\ &\mathcal{A}_3(t) = \lambda \frac{t}{\log(e+t)\log(e+\log(e+t))} \quad \text{and so on.} \end{aligned} \tag{7.3}$$

We will also require that

$$\begin{aligned} &(i)\ \exists t_0 > 0\ \forall t > t_0 \ \text{ such that } \mathcal{A}(t) > n t^{\frac{2}{3}} \\ &(ii)\ \mathcal{A}'(t) \text{ is non-increasing} \\ &(iii)\ b'(t) \text{ is non-increasing for } b(t) := \frac{t}{\mathcal{A}(t)} \\ &(iv)\ b(0) := \lim_{t\to 0} b(t) \text{ is finite and positive.} \end{aligned} \tag{7.4}$$

Let us note that the critical functions from (7.3) satisfy these conditions and therefore these assumptions are not restrictive. It has been shown in [5] that a mapping f is continuous under the assumptions (7.1) and (7.2) and that the assumption (7.2) is sharp. We show that the modulus of continuity estimate for these mappings from [10, Theorem 2] is sharp:

Theorem 7.1. *Suppose that an Orlicz function $\mathcal{A}$ satisfies* (7.2) *and* (7.4)*. Then there is a ball $B := B(0,r)$ and a mapping of finite distortion $f : B \to \mathbb{R}^n$ such that*

$$\int_B \exp\big(\mathcal{A}(K_f(x))\big)\, dx < \infty$$

and there exists some $C > 0$ such that

$$|f(x) - f(0)| \geq C \exp\Big(-\int_{|x|}^{1/2} \frac{dt}{t\mathcal{A}^{-1}(\log 1/t^n)}\Big) \text{ for all } x \in B. \tag{7.5}$$

Let us at this point note that it was proved in [10] that for certain restrictions on the size of R it holds that

$$|f(x) - f(y)| \leq C \exp\Big(-\int_{|x-y|}^{R} \frac{dt}{t\mathcal{A}^{-1}(\log C/t^n)}\Big).$$

Our result shows the sharpness of this estimate. Note further that if we put $\mathcal{A}_1(t) = \lambda t$ we arrive at the modulus given in (1.1).

Proof of Theorem 7.1. Let us put $B := B(0, \min\{\exp(-t_0), e^{-e^4}\})$ and choose $\alpha > b(0)^{-1}$. Without loss of generality we can assume that t_0 is big enough such that

$$t^{\frac{3}{2}} < \frac{1}{\alpha(\alpha+1)} \frac{t^2}{\log t} \text{ for all } t > t_0. \tag{7.6}$$

We define the function f as,

$$f(x) := \frac{x}{|x|} \exp\Big(-\int_{|x|}^{\frac{1}{2}} \frac{1}{t\mathcal{A}^{-1}(n \log \frac{1}{t})} dt\Big) (\log |x|^{-1})^{\frac{\alpha+2}{\log |x|^{-1}}}$$

Note that

$$\lim_{t\to\infty} (\log t)^{\frac{\alpha+2}{\log t}} = \lim_{t\to\infty} \exp\Big(\frac{(\alpha+2)\log\log t}{\log t}\Big) = 1,$$

which easily gives that f satisfies the condition given in (7.5).

Using Lemma 5.2 we find that,

$$|Df(x)| = \frac{|f(x)|}{|x|} \max\Big\{1, \Big(\frac{1}{\mathcal{A}^{-1}(n \log |x|^{-1})} + (\alpha+2)\frac{\log\log|x|^{-1} - 1}{\log^2 |x|^{-1}}\Big)\Big\}.$$

Clearly

$$\lim_{x\to 0} \Big(\frac{1}{\mathcal{A}^{-1}(n \log |x|^{-1})} + (\alpha+2)\frac{\log\log|x|^{-1} - 1}{\log^2 |x|^{-1}}\Big) = 0$$

and therefore the greater element is the first. From (7.4) (i) and (7.6) we have

$$\mathcal{A}^{-1}(nt) < t^{\frac{3}{2}} < \frac{1}{\alpha(\alpha+1)} \frac{t^2}{\log t} \text{ for all } \; t > t_0.$$

This however implies that

$$\alpha(\alpha+1)\frac{\mathcal{A}^{-1}(nt)\log t}{t^2} < 1.$$

Now by multiplying on both sides by $\frac{\mathcal{A}^{-1}(nt)\log t}{t^2}$ and by substituting $t = \log|x|^{-1}$ we get that

$$\mathcal{A}^{-1}(n\log|x|^{-1})\frac{\log\log|x|^{-1}}{\log^2|x|^{-1}} > \alpha(\alpha+1)\Big(\mathcal{A}^{-1}(n\log|x|^{-1})\frac{\log\log|x|^{-1}}{\log^2|x|^{-1}}\Big)^2.$$

Using this fact and because $\log\log|x|^{-1} > 4$ for all $x \in B$, we deduce that

$$\begin{aligned}
K_O(x) &= \frac{1}{\Big(\frac{1}{\mathcal{A}^{-1}(n\log|x|^{-1})} + (\alpha+2)\frac{\log\log|x|^{-1}-1}{\log^2|x|^{-1}}\Big)} \\
&\le \frac{\mathcal{A}^{-1}(n\log|x|^{-1})}{1+(\alpha+1)\mathcal{A}^{-1}(n\log|x|^{-1})\frac{\log\log|x|^{-1}}{\log^2|x|^{-1}}} \\
&\le \mathcal{A}^{-1}(n\log|x|^{-1})\Big(1-\alpha\mathcal{A}^{-1}(n\log|x|^{-1})\frac{\log\log|x|^{-1}}{\log^2|x|^{-1}}\Big) =: \tilde{K}(x).
\end{aligned}$$

Note that,

$$\mathcal{A}^{-1}(n\log|x|^{-1}) - \tilde{K}(x) = \alpha n^2\Big(\frac{\mathcal{A}^{-1}(n\log|x|^{-1})}{n\log|x|^{-1}}\Big)^2\log\log|x|^{-1}. \tag{7.7}$$

By (7.4) (iii) we obtain that

$$b(s) - b(0) = b'(\xi)s \ge b'(s)s$$

for all $s > 0$ and therefore

$$\mathcal{A}'(s)\Big(\frac{s}{\mathcal{A}(s)}\Big)^2 = \frac{b(s)-sb'(s)}{b^2(s)}b^2(s) \ge b(0). \tag{7.8}$$

From (7.4) (ii) we know that $\mathcal{A}'(t)$ is a non-increasing function and therefore

$$\mathcal{A}(a-d) = \mathcal{A}(a) - \mathcal{A}'(\xi)d \le \mathcal{A}(a) - \mathcal{A}'(a)d \tag{7.9}$$

for some $\xi \in (a-d, a)$. We now use (7.9) putting

$$a := \mathcal{A}^{-1}\Big(n\log\frac{1}{|x|}\Big),\ d := \mathcal{A}^{-1}\Big(n\log\frac{1}{|x|}\Big) - \tilde{K}(x)$$

using (7.7) and then (7.8) (where we put $s := \mathcal{A}^{-1}(n\log|x|^{-1})$) to get that

$$\begin{aligned}
\mathcal{A}(K(x)) &\le \mathcal{A}(\tilde{K}(x)) \\
&\le \mathcal{A}\big(\mathcal{A}^{-1}(n\log|x|^{-1})\big) - \\
&\quad - \alpha n^2\mathcal{A}'(\mathcal{A}^{-1}(n\log|x|^{-1}))\Big(\frac{\mathcal{A}^{-1}(n\log|x|^{-1})}{n\log|x|^{-1}}\Big)^2\log\log|x|^{-1} \\
&\le n\log\frac{1}{|x|} - b(0)\alpha n^2\log\log\frac{1}{|x|}.
\end{aligned}$$

This however implies that

$$\mathcal{A}(K(x)) \le n \log |x|^{-1} - b(0)\alpha n^2 \log\log |x|^{-1} \text{ for almost all } \ x \in B.$$

But this, for $\alpha > b(0)^{-1}$, yields

$$\begin{aligned}\int_B \exp(\mathcal{A}(K(x)))dx &\le \int_B \exp\left(n \log |x|^{-1} - b(0)\alpha n^2 \log\log |x|^{-1}\right) dx \\ &\le \int_B \frac{1}{|x|^n \log^{b(0)\alpha n^2} |x|^{-1}} dx < \infty.\end{aligned}$$

□

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