Weak and Measure-valued Solutions to Evolutionary PDEs

J. MÁLEK

Charles University, Prague,
Czech Republic

J. NEČAS

Charles University, Prague,
Czech Republic,
and Northern Illinois University,
DeKalb, USA

M. ROKYTA

Charles University, Prague,
Czech Republic

and

M. RŮŽIČKA

University of Bonn,
Germany

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Preface

This book deals with evolution partial differential equations of both hyperbolic and parabolic type with particular emphasis on problems that arise in nonlinear fluid mechanics. If an alternative title were to be given to the book, it could be 'on the passage to the limit within nonlinearities'. Fortunately enough, the preface is usually longer than one sentence, which gives us the opportunity to describe briefly the contents of the book.

After presenting some preliminary results, we devote the second chapter to the study of scalar hyperbolic equations of first order (or scalar hyperbolic conservation laws) in arbitrary spatial dimensions. In the first part we treat the usual Cauchy problem, following the presentation of Godlewski and Raviart [1991]. The second part focuses on recent results of Otto [1992] concerning the solvability of a scalar hyperbolic conservation law in a bounded smooth domain. In both cases, we prove the existence and uniqueness of the entropy weak solution via the method of parabolic perturbation. This method, together with the Galerkin method, are the basic means for constructing convenient approximations of the original problems.

In the third chapter we introduce the concept of the Young measure. This is a very effective tool to describe the behaviour of weakly convergent sequences under superpositions of nonlinearities. As an application, we prove again the existence of an entropy weak solution to a scalar hyperbolic conservation law in one space dimension exploiting the reduction of the support of a corresponding Young measure.

The last two chapters deal with problems where nonlinearities depend on gradients of the solution. In the fourth chapter we study the nonlinear scalar hyperbolic equation of the second order. Chapter 5 is devoted to a class of non-Newtonian fluids, sometimes called fluids with shear-dependent viscosity or generalized Newtonian fluids. Both compressible and incompressible models are studied here.

Using the fundamental theorem on Young measures, we prove the global-in-time existence of measure-valued solutions to the above problems. Although the measure-valued solution can be subject to further investigation we want to emphasize that more attention is paid to the questions of existence, uniqueness and regularity of weak solutions. We have addressed these questions for incompressible fluids with shear dependent viscosity, studied in Sections 5.3 and 5.4. For the nonlinear hyperbolic equation of second order (studied in Chapter 4) as well as for the compressible fluid with shear dependent viscosity (studied in Section 5.5), the existence of weak solutions is still open. Nevertheless, the question of existence of a weak solution to approximating equations, shown here, is an interesting problem on its own.

This monograph is one of the few attempts to carry out a detailed analysis for a class of evolution equations for non-linear fluids (essentially in Section 1.1 and Chapter 5). Although we have tried to provide a systematic investigation, the text should be considered as an introduction to the topic, since many problems remain to be studied and a lot of interesting questions are still unanswered. We feel that the reader can easily find interesting issues for further investigation, here.

In order to make the book self-contained, we give in Section 1.2 an overview of the definitions and basic properties of the function spaces needed. The Appendix contains some useful assertions concerning the linear theory. For the benefit of the reader we have included some references that are not cited in the main text but are related to the subject of the book.

For readers interested in particular problems, we indicate the main topics together with the sections where they are discussed.

- *Non-Newtonian fluids:* Sections 1.1.1, 1.1.4–1.1.5, Section 4.2, Chapter 5.
- *Hyperbolic conservation laws:* Sections 1.1.1–1.1.2, Chapter 2, Sections 3.3–3.4.
- *Young measures:* Sections 3.1–3.2 and Sections 4.1–4.2 with applications in Sections 3.3–3.4, 4.3, 5.2 and 5.5.
- *Hyperbolic equations of second order:* Chapter 4, see also Section 1.1.2.

We are thankful to many people for their help, advice and time spent in discussions, as well as for their support and interest. First of all, we would like to thank Professor G.P. Galdi and Professor

K.R. Rajagopal. Since there are several authors to this book, the number of reasons for expressing gratitude is rather extensive. Nevertheless, G.P. Galdi's essential support to organize periodically the Winter School on Mathematical Theory in Fluid Mechanics as well as K.R. Rajagopal's permanent effort to provide us with new views on continuum mechanics are common for all of the authors.

A large part of this monograph has been written at the Department of Applied Analysis, University of Bonn, headed by Professor Jens Frehse. We wish to thank him for his permanent support as well as for the pleasant atmosphere that we found there.

We are also very obliged to Felix Otto, who agreed that we use results of his thesis (see OTTO [1992, 1993]) on scalar hyperbolic conservation laws in bounded domains and provided us with a preliminary version of the text.

Our thanks go as well to numerous colleagues who have pointed out misprints and imprecise statements. Of these, special thanks go to Mária Lukáčová, Antonín Novotný, Luboš Pick, Ondřej Pokluda and Jan Seidler, and especially to Milan Pokorný and Mark Steinhauer for reading a large portion of the manuscript and for valuable comments. Very special thanks for many valuable remarks go to Endre Süli, University of Oxford, who read the final draft of the whole manuscript.

We also want to express our thanks to Oldřich Ulrych for solving our TEX problems, to Michal Kubeček for helping us with graphics and to Silke Thiedemann for correcting the English in a part of the manuscript.

Writing of the book has some unpleasant side-effects. These include evenings and weekends devoted to the preparation of the manuscript instead of spending that time with wives and children. Very special thanks for their enduring support and understanding are, therefore, expressed to Jana and Markéta.

This work was partially supported by the grant GAUK-353/1993 of Charles University Prague (Czech Republic) and SFB 256, Universität Bonn (Germany).

Josef Málek
Jindřich Nečas
Mirko Rokyta
Michael Růžička

Prague
June 1995