CHAPTER 5

Mathematical theory for a class of non-Newtonian fluids

5.1 Introduction

In this chapter we will study solutions (global in time) to systems describing the motion of both incompressible liquids and compressible isothermal gases in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$. The equations in question have already been formulated in Section 1.1.4 and Section 1.1.5 in Chapter 1. It is worth recalling the assumptions put on the stress $\boldsymbol{\tau}^E$ for both compressible and incompressible models. As before, $\mathbb{R}^{d^2}_{\mathrm{sym}}$ represents the set of all symmetric $d \times d$ matrices, i.e.,

$$\mathbb{R}^{d^2}_{\mathrm{sym}} \equiv \left\{ \mathbf{M} \in \mathbb{R}^{d^2} ; M_{ij} = M_{ji},\, i,j = 1,\ldots,d \right\}.$$

For $\mathbf{u} : (0,T) \times \Omega \longrightarrow \mathbb{R}^d$ we denote, as usual, by $\mathbf{e} = \mathbf{e}(\mathbf{u}) : \mathbb{R}^d \longrightarrow \mathbb{R}^{d^2}_{\mathrm{sym}}$ the symmetric part of the gradient of $\mathbf{u}$. That is, the components of $\mathbf{e}$ are defined by

$$2\, e_{ij} \equiv \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\,, \quad i,j = 1,\ldots,d\,.$$

Further, let $\boldsymbol{\tau}, \boldsymbol{\sigma} : \mathbb{R}^{d^2}_{\mathrm{sym}} \longrightarrow \mathbb{R}^{d^2}_{\mathrm{sym}}$ be given continuous functions.

Considering a flow of a **compressible** fluid described by (1.50)–(1.51) in Chapter 1, when $\boldsymbol{\tau}^E = \boldsymbol{\tau}(\mathbf{e})$, we assume that for a certain $p > 1$ and $q \in [p-1, p)$ there exist α, $\beta > 0$ such that for all $\boldsymbol{\eta} \in \mathbb{R}^{d^2}_{\mathrm{sym}}$,

$$\boldsymbol{\tau}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq \alpha |\boldsymbol{\eta}|^p\,, \tag{1.1}$$

$$|\boldsymbol{\tau}(\boldsymbol{\eta})| \leq \beta \big(1 + |\boldsymbol{\eta}|\big)^q\,. \tag{1.2}$$

Considering a flow of an **incompressible** fluid described by system (1.55)–(1.56) in Chapter 1, when $\boldsymbol{\tau}^E = \boldsymbol{\tau}(\mathbf{e}) + \boldsymbol{\sigma}(\mathbf{e})$, we assume that there exist $p > 1$, $q \in [p-1, p)$ and α, $\beta > 0$ (usually different from α, β above) such that for all $\boldsymbol{\eta} \in \mathbb{R}^{d^2}_{\mathrm{sym}}$,

$$\boldsymbol{\tau}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq \alpha |\boldsymbol{\eta}|^p \,, \tag{1.3}$$

$$\boldsymbol{\sigma}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq 0, \tag{1.4}$$

$$|\boldsymbol{\tau}(\boldsymbol{\eta}) + \boldsymbol{\sigma}(\boldsymbol{\eta})| \leq \beta\big(1 + |\boldsymbol{\eta}|\big)^q \,. \tag{1.5}$$

Notice that if $\boldsymbol{\tau}$, $\boldsymbol{\sigma}$ fulfil (1.3)–(1.5), $\boldsymbol{\tau} + \boldsymbol{\sigma}$ satisfies (1.1)–(1.2). Unless additional assumptions are imposed on $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, the decomposition of $\boldsymbol{\tau}^E$ into $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ is not useful. In such cases we will assume that $\boldsymbol{\sigma} \equiv \mathbf{0}$ and $\boldsymbol{\tau}$ satisfies (1.1)–(1.2).

Our aim is to find global solutions to equations of motion for models of both incompressible and compressible fluids. In this chapter we will try to answer the following questions.

1. *For which p do measure-valued solutions exist?*

Let us emphasize that measure-valued functions are expected to appear only as a tool to describe the behaviour of a sequence $\boldsymbol{\tau} \circ \big(\mathbf{e}(\mathbf{u}^k)\big)$, where $\boldsymbol{\tau}$ is as above and $\mathbf{u}^k$ are some approximate solutions to our problem.† In other words, the limits of the other nonlinearities appearing in the equations will be characterized by integrable functions and will be obtained in the classical weak sense. The restriction on p comes in fact from these limits and not from the limit of $\boldsymbol{\tau} \circ \big(\mathbf{e}(\mathbf{u}^k)\big)$.

Assumption (1.1) ensures uniform boundedness of the sequence $\mathbf{e}(\mathbf{u}^k)$ in $L^p(Q_T)^{d^2}$. Using the Korn inequality (see Theorem 1.10 below), a uniform bound on $\nabla \mathbf{u}^k$ in $L^p(Q_T)^{d^2}$ can be found. Because $\boldsymbol{\tau}$ satisfies (1.2), we can apply Theorem 2.1 and Corollary 2.10 from Chapter 4 directly in order to obtain the existence of a measure-valued function characterizing the limit of $\boldsymbol{\tau} \circ \big(\mathbf{e}(\mathbf{u}^k)\big)$, cf. Chapter 4.

The existence of measure-valued solutions will be proved in Section 5.2 (incompressible case) and in Section 5.5 (compressible case).

2. *For which p are measure-valued solutions Dirac ones?*—or equivalently—*For which p do weak solutions exist?*
3. *For which p, if any, do weak solutions have some qualitative properties such as higher regularity, uniqueness, etc.?*

† Let us remark that in this chapter $\mathbf{u}^k$ will denote Galerkin approximations, in contrast to Chapter 4, where $\mathbf{u}^k$ were solutions to perturbed problems.

Satisfactory answers to these questions will be given only in the case of models for incompressible fluids, see Sections 5.3 and 5.4. A complete overview of known results is presented in Section 5.2.3. As will be seen, the most interesting results require some additional assumptions on the tensor function $\boldsymbol{\tau}$. More precisely, for the incompressible model we will assume the existence of a scalar function $U \in C^2(\mathbb{R}^{d^2})$, called **potential of $\boldsymbol{\tau}$**, such that for some $p \in (1,\infty)$, C_1, $C_2 > 0$ we have for all $\boldsymbol{\eta}$, $\boldsymbol{\xi} \in \mathbb{R}^{d^2}_{\mathrm{sym}}$ and i, j, k, $\ell = 1,\dots,d$,

$$\frac{\partial U(\boldsymbol{\eta})}{\partial \eta_{ij}} = \tau_{ij}(\boldsymbol{\eta})\,, \tag{1.6}$$

$$U(\mathbf{0}) = \frac{\partial U(\mathbf{0})}{\partial \eta_{ij}} = 0\,, \tag{1.7}$$

$$\frac{\partial^2 U(\boldsymbol{\eta})}{\partial \eta_{mn}\partial \eta_{rs}}\xi_{mn}\xi_{rs} \ge C_1 \begin{cases} |\boldsymbol{\eta}|^{p-2}|\boldsymbol{\xi}|^2\,, \\ (1+|\boldsymbol{\eta}|)^{p-2}|\boldsymbol{\xi}|^2\,, \end{cases} \tag{1.8}$$

$$\Big|\frac{\partial^2 U(\boldsymbol{\eta})}{\partial \eta_{ij}\partial \eta_{k\ell}}\Big| \le C_2(1+|\boldsymbol{\eta}|)^{p-2}\,. \tag{1.9}$$

Let us recall that examples of functions $\boldsymbol{\tau}$ for which a potential U can be found were presented in Chapter 1, Example 1.69.

Notice that $(1.8)_2$ implies $(1.8)_1$ if $p \ge 2$. We understand (1.8) (and the cases in it) in the sense that only one of the two conditions is considered. Let us remark that the typical example of the tensor $\boldsymbol{\tau}$ whose potential satisfies the condition $(1.8)_1$ is

$$\boldsymbol{\tau}\big(\mathbf{e}(\mathbf{u})\big) = |\mathbf{e}(\mathbf{u})|^{p-2}\mathbf{e}(\mathbf{u})\,,$$

while the tensors

$$\begin{aligned} \boldsymbol{\tau}\big(\mathbf{e}(\mathbf{u})\big) &= \big(1+|\mathbf{e}(\mathbf{u})|\big)^{p-2}\mathbf{e}(\mathbf{u})\,, \\ \boldsymbol{\tau}\big(\mathbf{e}(\mathbf{u})\big) &= \big(1+|\mathbf{e}(\mathbf{u})^2|\big)^{\frac{p-2}{2}}\mathbf{e}(\mathbf{u})\,, \end{aligned}$$

are standard examples of $\boldsymbol{\tau}$s whose potentials satisfy $(1.8)_2$. Compare with Example 1.73 in Chapter 1.

The properties of $\boldsymbol{\tau}$ which follow from (1.6)–(1.9) are derived at the end of this section. In particular, it will be shown that (1.6)–$(1.8)_1$ imply (1.3), see Lemma 1.19 below.

Finally, let us note that to the knowledge of the authors nothing is known about the existence of weak solutions to the compressible isothermal model even for very large p. This remains, similarly to the case of the nonlinear hyperbolic equation of second order, an interesting open problem (for $d \geq 2$).

Before finishing this section, we first prove the Korn inequality which turns out to be an important mathematical tool in the sequel, and then we will study the properties of $\boldsymbol{\tau}$ which follow from (1.6)–(1.9).

5.1.1 Korn's inequality

Theorem 1.10 *Let $1 < p < \infty$. Then there exists a constant $K_p = K_p(\Omega)$ such that the inequality*

$$K_p \|\mathbf{v}\|_{1,p} \leq \|\mathbf{e}(\mathbf{v})\|_p \tag{1.11}$$

is fulfilled for all $\mathbf{v}$ satisfying either

- $\mathbf{v} \in W_0^{1,p}(\Omega)^d$, *where $\Omega \subset \mathbb{R}^d$ is open and bounded, $\partial\Omega \in C^{0,1}$, or*[‡]
- $\mathbf{v} \in W_{\mathrm{per}}^{1,p}(\Omega)^d$, *where $\Omega = (0, L)^d$, $L > 0$.*

Remark 1.12 There are several proofs of this theorem if $p = 2$, see for example Hlaváček and Nečas [1970], Nitsche [1981], Kondratiev and Oleinik [1989], covering more general boundary conditions. For general $p \in (1, \infty)$ see Gobert [1962, 1971], Nečas [1966], Temam [1985] or Fuchs [1994]. The case of $p = 1$ must be excluded because of the counterexample due to Ornstein [1962]. For $p = 1$ it holds only

$$K_1 \|\mathbf{v}\|_{\frac{d}{d-1}} \leq \|\mathbf{e}(\mathbf{v})\|_1 \,. \tag{1.13}$$

For the proof of (1.13) see Anzelotti and Giaquinta [1980].

In order to prove Theorem 1.10 we need one general result concerning distributions with derivatives in $\left(W_0^{1,p}(\Omega)\right)^*$, $\left(W_{\mathrm{per}}^{1,p}(\Omega)\right)^*$, respectively. We restrict ourselves to the first case of Theorem 1.10; the second case is easier.

Theorem 1.14 *Let Ω be an open bounded subset of $\mathbb{R}^d$ with $\partial\Omega \in C^{0,1}$. Let $T \in \mathcal{D}'(\Omega)$. If T, $\frac{\partial T}{\partial x_i} \in \left(W_0^{1,q}(\Omega)\right)^*$ for some $q \in$*

[‡] By $W_{\mathrm{per}}^{1,p}(\Omega)^d$ for $\Omega = (0, L)^d$, $L > 0$, we understand the space of space-periodic functions from $W^{1,p}(\Omega)^d$ with mean value zero (see also Section 1.2.4 in Chapter 1).

$(1,\infty)$, and all $i=1,\dots,d$, then there exists a function $u \in L^{q'}(\Omega)$, $q' = \frac{q}{q-1}$, such that

$$\langle T, \varphi\rangle = \int_\Omega u\varphi \, dx \qquad \forall\, \varphi \in \mathcal{D}(\Omega)\,.$$

Moreover, there exists $C>0$ such that

$$\|u\|_{q'}^{q'} \le C\Big(\|T\|_{-1,q'}^{q'} + \sum_{i=1}^{d} \|\frac{\partial T}{\partial x_i}\|_{-1,q'}^{q'}\Big)\,. \tag{1.15}$$

PROOF : See NEČAS [1966], where a more general statement is proved. In fact, if $m \in \mathbb{Z}$, $T \in W^{m-1,q'}(\Omega)$ and $\frac{\partial T}{\partial x_i} \in W^{m-1,q'}(\Omega)$ then $T \in W^{m,q'}(\Omega)$ and an inequality corresponding to (1.15) holds. ■

PROOF (of Theorem 1.10): Let us define the space

$$E(\Omega)^d \equiv \big\{\mathbf{u} \in L^p(\Omega)^d; \mathbf{e}(\mathbf{u}) \in L^p(\Omega)^{d^2}\big\}$$

with $\|\mathbf{u}\|_{E(\Omega)^d} \equiv \|\mathbf{u}\|_p + \|\mathbf{e}(\mathbf{u})\|_p$. Then $E(\Omega)^d$ is a Banach space. Let

$$I : W^{1,p}(\Omega)^d \longrightarrow E(\Omega)^d$$

be the identity mapping. Clearly I is a continuous map. We want to prove the surjectivity of I. Taking $\mathbf{v} \in E(\Omega)^d$ we have, in the sense of distributions, for all $i,\, j,\, k = 1,\dots,d$,

$$\frac{\partial^2 v_i}{\partial x_j \partial x_k} = \frac{\partial e_{ik}(\mathbf{v})}{\partial x_j} + \frac{\partial e_{ij}(\mathbf{v})}{\partial x_k} - \frac{\partial e_{jk}(\mathbf{v})}{\partial x_i}\,. \tag{1.16}$$

Since $\mathbf{e}(\mathbf{v}) \in L^p(\Omega)^{d^2}$, (1.16) implies $\frac{\partial^2 v_i}{\partial x_j \partial x_k} \in \big(W_0^{1,p'}(\Omega)\big)^*$, where p' is the dual exponent to p. Further, we obtain from $\mathbf{v} \in L^p(\Omega)^d$ that $\frac{\partial v_i}{\partial x_j} \in \big(W_0^{1,p'}(\Omega)\big)^*$. Due to Theorem 1.14 we obtain $\frac{\partial v_i}{\partial x_j} \in L^p(\Omega)$ for $i,j=1,\dots,d$, therefore $\mathbf{v} \in W^{1,p}(\Omega)^d$ and I is surjective. Hence $W^{1,p}(\Omega)^d$ coincides with $E(\Omega)^d$ and the Open Mapping Theorem (see YOSIDA [1965]) gives the inequality

$$\|\mathbf{v}\|_{1,p} \le c(p,\Omega)\,\big(\|\mathbf{v}\|_p + \|\mathbf{e}(\mathbf{v})\|_p\big)\,. \tag{1.17}$$

It remains to show that for all $\mathbf{v} \in W_0^{1,p}(\Omega)^d$,

$$\|\mathbf{v}\|_p \le \widetilde{c}(p,\Omega)\,\|\mathbf{e}(\mathbf{v})\|_p\,. \tag{1.18}$$

To obtain a contradiction, we will assume that there is a sequence $\{\mathbf{v}^n\}_{n=1}^\infty \subset W_0^{1,p}(\Omega)^d$ such that $\|\mathbf{v}^n\|_p = 1$ and $n\,\|\mathbf{e}(\mathbf{v}^n)\|_p < 1$.

Thus $\mathbf{e}(\mathbf{v}^n) \to \mathbf{0}$ in $L^p(\Omega)^{d^2}$ for $n \to \infty$. Using (1.17), we can choose a subsequence of $\{\mathbf{v}^n\}$ still labelled $\{\mathbf{v}^n\}$ such that $\mathbf{v}^n \rightharpoonup \mathbf{v}$ weakly in $W^{1,p}(\Omega)^d$ and strongly in $L^p(\Omega)^d$. Then $\|\mathbf{v}\|_p = 1$, $\mathbf{v}|_{\partial\Omega} = 0$ and $\mathbf{e}(\mathbf{v}) = \mathbf{0}$. A vector field $\mathbf{v}$ satisfying $\mathbf{e}(\mathbf{v}) = \mathbf{0}$ has an equivalent form $\mathbf{v} = \mathbf{a} + \mathbf{b} \times \mathbf{x}$, see NEČAS AND HLAVÁČEK [1981]. Due to the homogeneous boundary conditions $\mathbf{v}$ has to be identically zero, which contradicts $\|\mathbf{v}\|_p = 1$.

Therefore, inequality (1.18) holds and the proof of Theorem 1.10 is complete. ∎

5.1.2 Two algebraic lemmas

In this section we will derive some inequalities which are consequences of assumptions (1.6)–(1.9).

The cases in (1.20) and (1.25) below are understood to be in agreement with (1.8). This means that $(1.8)_1$ implies $(1.20)_1$ and $(1.25)_1$, and $(1.8)_2$ implies $(1.20)_2$ and $(1.25)_2$.

Lemma 1.19 *Let $p > 1$ and $\boldsymbol{\tau} : \mathbb{R}^{d^2}_{\mathrm{sym}} \longrightarrow \mathbb{R}^{d^2}_{\mathrm{sym}}$, $U : \mathbb{R}^{d^2}_{\mathrm{sym}} \longrightarrow \mathbb{R}$ satisfy (1.6)–(1.9). Then there exist positive constants C_3, C_4, C_5 such that for all $\mathbf{e} \in \mathbb{R}^{d^2}_{\mathrm{sym}}$,*

$$\tau_{ij}(\mathbf{e})e_{ij} \geq C_3 \begin{cases} |\mathbf{e}|^p\,, \\ |\mathbf{e}|(|\mathbf{e}|^{p-1} - 1) \geq C_4(|\mathbf{e}|^p - 1)\,, \end{cases} \tag{1.20}$$

$$|\tau_{ij}(\mathbf{e})| \leq C_5\,(1 + |\mathbf{e}|)^{p-1}\ , \tag{1.21}$$

$i, j = 1, \ldots, d$, and for all $\mathbf{e}$, $\widehat{\mathbf{e}} \in \mathbb{R}^{d^2}_{\mathrm{sym}}$,

$$(\tau_{ij}(\mathbf{e}) - \tau_{ij}(\widehat{\mathbf{e}}))(e_{ij} - \widehat{e}_{ij}) \geq 0\,. \tag{1.22}$$

Further, the inequality $(1.20)_2$ can be replaced by

$$\tau_{ij}(\mathbf{e})e_{ij} \geq C_3\, \min(|\mathbf{e}|^2, |\mathbf{e}|^p)\,. \tag{1.23}$$

Moreover, if $p \geq 2$ then

$$\tau_{ij}(\mathbf{e})e_{ij} \geq C_3(1 + |\mathbf{e}|^{p-2})|\mathbf{e}|^2\,, \tag{1.24}$$

and there also exists C_6 such that

$$(\tau_{ij}(\mathbf{e}) - \tau_{ij}(\widehat{\mathbf{e}}))(e_{ij} - \widehat{e}_{ij}) \geq C_6 \begin{cases} |\mathbf{e} - \widehat{\mathbf{e}}|^p\,, \\ \left(|\mathbf{e} - \widehat{\mathbf{e}}|^2 + |\mathbf{e} - \widehat{\mathbf{e}}|^p\right). \end{cases} \tag{1.25}$$

If $p \in (1, 2)$ and $|\mathbf{e}|$, $|\widehat{\mathbf{e}}| \leq R$ then there exists $C_7 = C_7(R)$ such that

$$(\tau_{ij}(\mathbf{e}) - \tau_{ij}(\widehat{\mathbf{e}}))(e_{ij} - \widehat{e}_{ij}) \geq C_7|\mathbf{e} - \widehat{\mathbf{e}}|^2\,. \tag{1.26}$$

PROOF : Due to (1.6) and (1.7), we have

$$\begin{aligned}\tau_{ij}(\mathbf{e}) = \frac{\partial U(\mathbf{e})}{\partial e_{ij}} - \frac{\partial U(\mathbf{0})}{\partial e_{ij}} &= \int_0^1 \frac{d}{ds}\frac{\partial U(s\mathbf{e})}{\partial e_{ij}}\, ds \\ &= \int_0^1 \frac{\partial^2 U(s\mathbf{e})}{\partial e_{ij}\partial e_{k\ell}} e_{k\ell}\, ds\,.\end{aligned} \tag{1.27}$$

Using (1.9) we obtain

$$\begin{aligned}|\tau_{ij}(\mathbf{e})| &\le C_2 d^2 \int_0^1 (1+s|\mathbf{e}|)^{p-2}|\mathbf{e}|\, ds = \frac{C_2 d^2}{p-1}\left[(1+s|\mathbf{e}|)^{p-1}\right]_0^1 \\ &\le \frac{C_2 d^2}{p-1}(1+|\mathbf{e}|)^{p-1}\,,\end{aligned}$$

which is (1.21) with $C_5 = \frac{C_2 d^2}{p-1}$. From (1.27) and (1.8) we also have

$$\tau_{ij}(\mathbf{e}) e_{ij} \ge C_1 \begin{cases} \int_0^1 s^{p-2}\, ds\ |\mathbf{e}|^p = \frac{1}{p-1}|\mathbf{e}|^p\,, \\ \int_0^1 (1+s|\mathbf{e}|)^{p-2}\, ds\ |\mathbf{e}|^2\,. \end{cases} \tag{1.28}$$

Since $(1.28)_1$ proves $(1.20)_1$, we henceforth pay attention to an estimate of $(1.28)_2$ from below.

Let $\alpha > 0$ be arbitrary. Then

$$\max_{x\in[0,\infty)} \frac{1+x^\alpha}{(1+x)^\alpha} < 2\,.$$

This implies

$$\begin{aligned}(1+s|\mathbf{e}|)^{p-2} &\ge \tfrac{1}{2}(1+(s|\mathbf{e}|)^{p-2})\,, &&\text{if } p\ge 2\,, \\ (1+|\mathbf{e}|)^{p-1} &\ge \tfrac{1}{2}(1+|\mathbf{e}|^{p-1})\,, &&\text{if } p>1\,.\end{aligned} \tag{1.29}$$

Therefore, for $p \ge 2$,

$$\tau_{ij}(\mathbf{e}) e_{ij} \ge \frac{C_1}{2}\int_0^1 (1+(s|\mathbf{e}|)^{p-2})\, ds\, |\mathbf{e}|^2 \ge \frac{C_1}{2(p-1)}(1+|\mathbf{e}|^{p-2})\, |\mathbf{e}|^2\,,$$

which proves (1.24) with $C_3 = \frac{C_1}{2(p-1)}$ and also trivially (1.23) for $p \ge 2$.

If $p > 1$, then it follows from $(1.28)_2$ that

$$\begin{aligned}\tau_{ij}(\mathbf{e}) e_{ij} &\ge \frac{C_1}{p-1}|\mathbf{e}|\left[(1+s|\mathbf{e}|)^{p-1}\right]_0^1 \\ &= \frac{C_1}{p-1}|\mathbf{e}|\big((1+|\mathbf{e}|)^{p-1} - 1\big)\,.\end{aligned} \tag{1.30}$$

Using $(1.29)_2$ we obtain

$$\tau_{ij}(\mathbf{e})e_{ij} \geq \frac{C_1}{2}\frac{|\mathbf{e}|}{p-1}(1+|\mathbf{e}|^{p-1}-2) = \frac{C_1|\mathbf{e}|}{2(p-1)}(|\mathbf{e}|^{p-1}-1)\,,$$

which is the first inequality in $(1.20)_2$. The second inequality in $(1.20)_2$ then follows from the fact that

$$|\mathbf{e}| = 1|\mathbf{e}| \leq \frac{1}{2}|\mathbf{e}|^p + \hat{C}_3 1^{p'}\,.$$

In order to prove (1.23) for $p \in (1,2)$ we start with (1.30) and we will show that

$$\begin{aligned} &(1+|\mathbf{e}|)^{p-1} - 1 \geq c|\mathbf{e}| && \text{for } 0 \leq |\mathbf{e}| \leq 1\,,\\ &(1+|\mathbf{e}|)^{p-1} - 1 \geq c|\mathbf{e}|^{p-1} && \text{for } 1 \leq |\mathbf{e}| < \infty\,. \end{aligned} \tag{1.31}$$

Looking for the minimum of the function $f(y) = \frac{(1+y)^{p-1}-1}{y}$ on the interval $[0,1]$, we find $f(1) = 2^{p-1}-1 > 0$, $f(0) = p-1 > 0$ and $f(y) > 0$ on $(0,1)$. Thus $(1.31)_1$ is proved. Seeking the minimum of the function $g(y) = \frac{(1+y)^{p-1}-1}{y^{p-1}}$ on the interval $[1,\infty)$, we see that

$$\begin{aligned} \min_{y\in[1,\infty)} g(y) &= \min_{y\in[1,\infty)} \left(1+\frac{1}{y}\right)^{p-1} - \left(\frac{1}{y}\right)^{p-1}\\ &= \min_{z\in(0,1]} (1+z)^{p-1} - z^{p-1} \equiv \min_{z\in(0,1]} h(z)\,. \end{aligned}$$

We find $h(0) = 1$, $h(1) = 2^{p-1}-1$ and $h'(z) < 0$ on $[0,1]$. Hence we have $\min g(y) = g(1) > 0$, which shows $(1.31)_2$ and therefore also (1.23).

Next,

$$\begin{aligned} &(\tau_{ij}(\mathbf{e}) - \tau_{ij}(\widehat{\mathbf{e}}))(e_{ij} - \widehat{e}_{ij})\\ &\quad\overset{(1.6)}{=} \int_0^1 \frac{d}{ds}\Big(\frac{\partial U\big(\widehat{\mathbf{e}} + s(\mathbf{e}-\widehat{\mathbf{e}})\big)}{\partial e_{ij}}\Big)\,ds\,(e_{ij} - \widehat{e}_{ij})\\ &\quad= \int_0^1 \frac{\partial^2 U\big(\widehat{\mathbf{e}} + s(\mathbf{e}-\widehat{\mathbf{e}})\big)}{\partial e_{ij}\partial e_{k\ell}}(e_{ij} - \widehat{e}_{ij})(e_{k\ell} - \widehat{e}_{k\ell})\,ds\\ &\quad\overset{(1.8)}{\geq} C_1|\mathbf{e}-\widehat{\mathbf{e}}|^2 \begin{cases} \int_0^1 |\widehat{\mathbf{e}} + s(\mathbf{e}-\widehat{\mathbf{e}})|^{p-2}\,ds\,,\\ \int_0^1 \big(1+|\widehat{\mathbf{e}} + s(\mathbf{e}-\widehat{\mathbf{e}})|\big)^{p-2}\,ds\,, \end{cases} \end{aligned} \tag{1.32}$$

which gives (1.22) for general $p > 1$. If $p \in (1,2)$ and $|\mathbf{e}| \leq R$ and

$|\widehat{\mathbf{e}}| \le R$, then

$$\int_0^1 |s\mathbf{e} + (1-s)\widehat{\mathbf{e}}|^{p-2}\, ds \ge \frac{1}{R^{2-p}}\,,$$

and

$$\int_0^1 \big(1 + |s\mathbf{e} + (1-s)\widehat{\mathbf{e}}|\big)^{p-2}\, ds \ge \frac{1}{(1+R)^{2-p}}\,,$$

and the strong monotonicity condition (1.26) holds.

If $p \ge 2$, then according to $(1.29)_1$,

$$\begin{aligned}\int_0^1 &\big(1 + |\widehat{\mathbf{e}} + s(\mathbf{e} - \widehat{\mathbf{e}})|\big)^{p-2}\, ds \\ &\ge \frac{1}{2}\Big(1 + \int_0^1 |\widehat{\mathbf{e}} + s(\mathbf{e} - \widehat{\mathbf{e}})|^{p-2}\, ds\Big)\,.\end{aligned} \tag{1.33}$$

If we show the existence of a constant C such that

$$\int_0^1 |\widehat{\mathbf{e}} + s(\mathbf{e} - \widehat{\mathbf{e}})|^{p-2}\, ds \ge C\,|\mathbf{e} - \widehat{\mathbf{e}}|^{p-2}\,, \tag{1.34}$$

then assertion (1.25) follows from (1.32)–(1.34).

In order to prove (1.34), let us consider two cases. If $|\widehat{\mathbf{e}}| \ge |\mathbf{e} - \widehat{\mathbf{e}}|$ then $|\widehat{\mathbf{e}} + s(\mathbf{e} - \widehat{\mathbf{e}})| \ge \big||\widehat{\mathbf{e}}| - s|\mathbf{e} - \widehat{\mathbf{e}}|\big| \ge (1-s)|\mathbf{e} - \widehat{\mathbf{e}}|$ and (1.34) holds.

If $|\widehat{\mathbf{e}}| < |\mathbf{e} - \widehat{\mathbf{e}}|$ then

$$\begin{aligned}\int_0^1 |\widehat{\mathbf{e}} + s(\mathbf{e} - \widehat{\mathbf{e}})|^{p-2}\, ds &= \int_0^1 \frac{\big(|\widehat{\mathbf{e}} + s(\mathbf{e} - \widehat{\mathbf{e}})|^2\big)^{\frac{p}{2}}}{|\widehat{\mathbf{e}} + s(\mathbf{e} - \widehat{\mathbf{e}})|^2}\, ds \\ &\ge \frac{1}{2|\mathbf{e} - \widehat{\mathbf{e}}|^2}\Big(\int_0^1 |s\mathbf{e} + (1-s)\widehat{\mathbf{e}}|^2\, ds\Big)^{\frac{p}{2}} \\ &= \frac{1}{2|\mathbf{e} - \widehat{\mathbf{e}}|^2}\,\frac{1}{3^{\frac{p}{2}}}\big(|\mathbf{e}|^2 + (\mathbf{e}, \widehat{\mathbf{e}}) + |\widehat{\mathbf{e}}|^2\big)^{\frac{p}{2}} \\ &\ge \frac{1}{2|\mathbf{e} - \widehat{\mathbf{e}}|^2}\,\frac{1}{6^{\frac{p}{2}}}\big(|\mathbf{e}|^2 + |\widehat{\mathbf{e}}|^2\big)^{\frac{p}{2}}\,.\end{aligned}$$

However, $|\mathbf{e} - \widehat{\mathbf{e}}|^2 \le 2\big(|\mathbf{e}|^2 + |\widehat{\mathbf{e}}|^2\big)$ and (1.34) follows. The proof of Lemma 1.19 is complete. ∎

Lemma 1.35 *Let $p > 1$ and the assumptions (1.6)–(1.9) be satisfied. Then there exist C_8, C_9, C_{10} such that*

$$C_9(1 + |\mathbf{e}|)^p \ge U(\mathbf{e}) \ge C_8 \begin{cases} |\mathbf{e}|^p\,, \\ |\mathbf{e}|(|\mathbf{e}|^{p-1} - p)\,. \end{cases} \tag{1.36}$$

If $p \geq 2$ then

$$C_{10}(1+|\mathbf{e}|^{p-2})|\mathbf{e}|^2 \leq U(\mathbf{e})\,. \tag{1.37}$$

PROOF : Starting from

$$U(\mathbf{e}) \overset{(1.7)}{=} \int_0^1 \frac{d}{ds}U(s\mathbf{e})\,ds = \int_0^1 \frac{\partial U(s\mathbf{e})}{\partial e_{ij}} e_{ij}\,ds = \int_0^1 \frac{1}{s}\tau_{ij}(s\mathbf{e})\,se_{ij}\,ds\,,$$

we use the previous lemma directly and obtain (1.36), (1.37) with $C_8 = C_3/p$, $C_9 = d^2 C_5/p$ and $C_{10} = C_3/p$. ∎

5.2 Incompressible non-Newtonian fluids and measure-valued solutions

The introductory part of this section contains the formulation of the basic problem called $(\mathrm{NS})_{\mathrm{p}}$. We have striven for having Sections 5.2–5.4 consistent in the sense that the existence of measure-valued solutions, the existence of weak solutions and further properties of solutions are studied basically for the same problem under the same assumptions. Later on, the possibilities for how to extend the validity of the results to more general situations are discussed in remarks and theorems and some of the assertions are proved under weaker assumptions or in more general settings in comparison with the basic version. Section 5.2.3 of this section provides an exposition of all the results related to the subject, at least to the knowledge of the authors.

Before presenting this summary, the existence of a measure-valued solution to the problem $(\mathrm{NS})_{\mathrm{p}}$ is demonstrated with the aim of illustrating some basic techniques and methods used in the evolution theory of incompressible fluids, see Section 5.2.2.

5.2.1 Formulation of the problem

In the following three sections we will study the initial-boundary value problem, called shortly the **problem $(\mathbf{NS})_{\mathrm{p}}$**.

Let $\Omega = (0,L)^d$, $L \in (0,\infty)$, be a cube in $\mathbb{R}^d$. Let us denote $\Gamma_j = \partial\Omega \cap \{x_j = 0\}$ and $\Gamma_{j+d} = \partial\Omega \cap \{x_j = L\}$ for $j = 1,\ldots,d$. For $T \in (0,\infty)$, we denote by Q_T the time-space cylinder $I \times \Omega$, where $I = (0,T)$ is a time interval. We assume that $d \geq 2$.

Let $\mathbf{f} : Q_T \to \mathbb{R}^d$ and $\mathbf{u}_0 : \Omega \to \mathbb{R}^d$ be given and assume that some tensor function $\boldsymbol{\tau} : \mathbb{R}^{d^2}_{\mathrm{sym}} \to \mathbb{R}^{d^2}_{\mathrm{sym}}$ satisfies for some $p \in (1, \infty)$ the assumptions (1.6), (1.7), $(1.8)_2$ and (1.9). We seek $\mathbf{u} = (u_1, \ldots, u_d) : Q_T \to \mathbb{R}^d$ and $\pi : Q_T \to \mathbb{R}$ solving the system of $d+1$ equations in Q_T,

$$\operatorname{div} \mathbf{u} = 0\,, \tag{2.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + u_k \frac{\partial \mathbf{u}}{\partial x_k} = -\nabla \pi + \operatorname{div} \boldsymbol{\tau}(\mathbf{e}(\mathbf{u})) + \mathbf{f}\,, \tag{2.2}$$

satisfying the initial condition

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0 \quad \text{ in } \Omega \tag{2.3}$$

and the space-periodicity requirements

$$\begin{aligned} \mathbf{u}|_{\Gamma_j} = \mathbf{u}|_{\Gamma_{j+d}}, \qquad \nabla \mathbf{u}|_{\Gamma_j} &= \nabla \mathbf{u}|_{\Gamma_{j+d}}\,, \\ \pi|_{\Gamma_j} = \pi|_{\Gamma_{j+d}}\,, & \end{aligned} \tag{2.4}$$

for $j = 1, \ldots, d$.

5.2.2 Measure-valued solutions

The goal of this section is to define the notion of a measure-valued solution to the problem $(\mathrm{NS})_{\mathrm{p}}$ and to prove its existence. For that purpose, the reader will easily find that (except for the definition of function spaces) no changes in the proof are needed if:

1. we replace the space-periodicity condition (2.4) by the Dirichlet condition

$$\mathbf{u} = \mathbf{0} \qquad \text{on } I \times \partial\Omega\,; \tag{2.5}$$

2. instead of assuming the existence of the potential U for $\boldsymbol{\tau}$, see (1.6), we use the inequalities

$$\boldsymbol{\tau}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq C_4(|\boldsymbol{\eta}|^p - 1)\,, \tag{2.6}$$

$$|\tau_{ij}(\boldsymbol{\eta})| \leq C_5(1 + |\boldsymbol{\eta}|)^{p-1}\,, \tag{2.7}$$

which are (due to Lemma 1.19) consequences of (1.6), (1.7), $(1.8)_2$ and (1.9).

Due to the constraint (2.1) we introduce spaces of **divergence-free** functions. Let us denote for $q \in (1, \infty)$[§]

$$\begin{aligned}
&\mathcal{V} \equiv \{\boldsymbol{\phi} \in C^{\infty}_{\text{per}}(\Omega)^d ; \operatorname{div} \boldsymbol{\phi} = 0 \,, \int_\Omega \boldsymbol{\phi}\, dx = \mathbf{0}\}; \\
&H \equiv \text{closure of } \mathcal{V} \text{ in the } L^2(\Omega)^d\text{-norm} ; \\
&V_q \equiv \text{closure of } \mathcal{V} \text{ in the } L^q(\Omega)^{d^2}\text{-norm of gradients.}
\end{aligned} \tag{2.8}$$

The dual space of V_q is denoted by V_q^* and $\langle \cdot, \cdot \rangle_q$ denotes the duality between V_q and V_q^*.

The use of spaces of divergence-free functions has some special consequences; these are summarized in the following lemma.

Lemma 2.9 *Let $\pi : \Omega \to \mathbb{R}$ and $\boldsymbol{\varphi}, \mathbf{u}, \mathbf{v} : \Omega \to \mathbb{R}^d$ be periodic functions. If $\boldsymbol{\varphi}, \mathbf{u}$ are divergence-free, then*

$$\int_\Omega \frac{\partial \pi}{\partial x_i} \varphi_i \, dx = 0 \,, \tag{2.10}$$

$$\int_\Omega u_j \frac{\partial u_i}{\partial x_j} \varphi_i \, dx = - \int_\Omega u_j u_i \frac{\partial \varphi_i}{\partial x_j} \, dx \,, \tag{2.11}$$

$$\int_\Omega u_j \frac{\partial v_i}{\partial x_j} v_i \, dx = 0 \,, \tag{2.12}$$

whenever the integrals in (2.10)–(2.12) are well defined.

PROOF : The proof is based on the use of Green's Theorem 2.20 in Chapter 1 and the fact that $\mathbf{u}, \boldsymbol{\varphi}$ are divergence-free. Thus (the boundary terms vanish due to periodicity),

$$\begin{aligned}
\int_\Omega \frac{\partial \pi}{\partial x_i} \varphi_i \, dx &= - \int_\Omega \pi \operatorname{div} \boldsymbol{\varphi} \, dx = 0 \,, \\
\int_\Omega u_j \frac{\partial v_i}{\partial x_j} v_i \, dx &= \frac{1}{2} \int_\Omega u_j \frac{\partial |\mathbf{v}|^2}{\partial x_j} \, dx \\
&= - \frac{1}{2} \int_\Omega (\operatorname{div} \mathbf{u}) |\mathbf{v}|^2 \, dx = 0 \,,
\end{aligned}$$

[§] For the definition of $C^{\infty}_{\text{per}}(\Omega)$ see Chapter 1.

and

$$\int_\Omega u_j \frac{\partial u_i}{\partial x_j} \varphi_i \, dx = -\int_\Omega u_j u_i \frac{\partial \varphi_i}{\partial x_j} \, dx - \int_\Omega \operatorname{div} \mathbf{u} \, u_i \varphi_i \, dx$$
$$= -\int_\Omega u_j u_i \frac{\partial \varphi_i}{\partial x_j} \, dx \,,$$

■

From identity (2.10) it follows that the pressure π disappears from the weak formulation of (2.1), (2.2) if divergence-free test functions are used. Consequently, we end up with a formulation of (2.1), (2.2) where the only unknown is the velocity $\mathbf{u}$. In this way, we will consider only the function $\mathbf{u}$ to be the solution of our problem and will not be interested in the pressure π. However, having solved the problem for $\mathbf{u}$, it is a standard matter to determine the pressure π from the velocity field $\mathbf{u}$ and the weak formulation of (2.1), (2.2) at least in the sense of distributions. For more details we refer to Temam [1977], Galdi [1994a, 1994b] and Girault and Raviart [1986]. We will not discuss the regularity of the pressure π in this book.

Definition 2.13 *Let $p > 1$, $\mathbf{u}_0 \in H$, $\mathbf{f} \in L^{p'}(I; (W^{1,p}_{\mathrm{per}}(\Omega)^d)^*)$, where $p' = \frac{p}{p-1}$. Then a couple $(\mathbf{u}, \nu)$ is called a* **measure-valued solution** *to the problem (NS)$_\mathrm{p}$ if*

$$\begin{aligned} &\mathbf{u} \in L^\infty(I; H) \cap L^p(I; V_p) \,, \\ &\nu \in L^\infty_\omega\big(Q_T; \operatorname{Prob}(\mathbb{R}^{d^2})\big) \,, \end{aligned} \tag{2.14}$$

and if $(\mathbf{u}, \nu)$ satisfy

$$\begin{aligned} \int_{Q_T} \Big[-u_i \frac{\partial \varphi_i}{\partial t} - u_j u_i \frac{\partial \varphi_i}{\partial x_j} & \\ + e_{ij}(\boldsymbol{\varphi}) \int_{\mathbb{R}^{d^2}} \tau_{ij}\big(\tfrac{\boldsymbol{\lambda}+\boldsymbol{\lambda}^T}{2}\big) \, d\nu_{t,x}(\boldsymbol{\lambda}) \Big] \, dx \, dt & \\ - \int_{Q_T} f_i \varphi_i \, dx \, dt = \int_\Omega u_{0i} \varphi_i \, dx & \end{aligned} \tag{2.15}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}\big((-\infty, T); \mathcal{V}\big)$, and

$$\nabla \mathbf{u}(t,x) = \int_{\mathbb{R}^{d^2}} \boldsymbol{\lambda} \, d\nu_{t,x}(\boldsymbol{\lambda}) \qquad \text{a.e. in } Q_T \,. \tag{2.16}$$

Theorem 2.17 *Let* $\mathbf{u}_0 \in H$ *and* $\mathbf{f} \in L^{p'}(I; (W^{1,p}_{\mathrm{per}}(\Omega)^d)^*)$. *Then there exists a measure-valued solution to the problem* $(NS)_{\mathrm{p}}$ *provided that*

$$p > \frac{2d}{d+2} \,. \tag{2.18}$$

PROOF : The proof is split into several smaller parts.

- *The choice of basis*

We will show the existence of a measure-valued solution to the system (2.1)–(2.4) via Galerkin approximations. For this purpose we can take the set $\{\widehat{\boldsymbol{\omega}}^r\}_{r=1}^{\infty}$ formed by the eigenvectors $\widehat{\boldsymbol{\omega}}^r$, $r = 1, 2, \dots,$ of the Stokes operator. However, as shown in Lemma 4.26 in the Appendix, the projectors $P^N\mathbf{u} \equiv \sum_{r=1}^{N}(\mathbf{u}, \widehat{\boldsymbol{\omega}}^r)\,\widehat{\boldsymbol{\omega}}^r$ are uniformly continuous at most in the norm of $W^{2,2}(\Omega)^d$. Due to imbedding, it follows that, e.g. in three dimensions, we have an upper bound for p, $p < 6$. On the other hand, the use of $\{\widehat{\boldsymbol{\omega}}^r\}$ allows us to exploit the Stokes operator as a test function in an easy way in order to get the so-called second energy estimate. This will be needed in Sections 5.3 and 5.4.

Nevertheless, the second energy estimate is not needed for proving the existence of measure valued solutions. Therefore, a more convenient set $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ is chosen here. For

$$s > \frac{d}{2} + 1, \tag{2.19}$$

we denote by ($\mathcal{V}$ is defined in (2.8))

$$V^s \equiv \text{ the closure of } \mathcal{V} \text{ with respect to the } W^{s,2}(\Omega)^d\text{-norm.}$$

Let us denote the scalar product in V^s by $((\cdot,\cdot))_s$ and let $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ be the set of eigenvectors to the problem

$$((\boldsymbol{\omega}^r, \boldsymbol{\varphi}))_s = \lambda_r(\boldsymbol{\omega}^r, \boldsymbol{\varphi}) \qquad \forall\, \boldsymbol{\varphi} \in V^s \,, \tag{2.20}$$

which are orthonormal in H and orthogonal in V^s, see Theorem 4.11 in the Appendix. The reason why we choose s satisfying (2.19) is the following: if $\mathbf{v} \in W^{s,2}(\Omega)^d$ then $\nabla\mathbf{v} \in W^{s-1,2}(\Omega)^{d^2}$ and

$$W^{s-1,2}(\Omega) \hookrightarrow L^{\infty}(\Omega) \quad \text{ if } \quad \frac{1}{2} - \frac{s-1}{d} < 0\,,$$

which is just (2.19). Consequently, for all $p > 1$ we have $\nabla\mathbf{v} \in L^p(\Omega)^{d^2}$ and $V^s \hookrightarrow V_p$.

• *Galerkin system and a priori estimates*

Let us define $\mathbf{u}^N(t,x) \equiv \sum_{r=1}^N c_r^N(t)\boldsymbol{\omega}^r(x)$, where the coefficients $c_r^N(t)$ solve the so-called **Galerkin system**

$$\begin{aligned}\Big(\frac{d}{dt}\mathbf{u}^N, \boldsymbol{\omega}^r\Big) + \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(\boldsymbol{\omega}^r)\,dx + \int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}\omega_i^r\,dx \\ = (\mathbf{f}, \boldsymbol{\omega}^r)\,, \qquad 1 \le r \le N\,, \qquad (2.21)\\ \mathbf{u}^N(0) = P^N\mathbf{u}_0\,.\end{aligned}$$

Here, P^N is the orthogonal continuous projector of H (resp. V^s) onto the linear hull of the first N eigenvectors $\boldsymbol{\omega}^r$, $r = 1,\dots,N$, see (4.12)–(4.13) in the Appendix. Due to the orthonormality of $\{\boldsymbol{\omega}^r\}$ in H, the system (2.21) can be rewritten as

$$\begin{aligned}\frac{d}{dt}c_r^N &= \mathcal{F}_r(c_1^N,\dots,c_N^N,t)\,,\\ c_r^N(0) &= (\mathbf{u}_0, \boldsymbol{\omega}^r)\,, \qquad (2.22)\end{aligned}$$

where

$$\begin{aligned}\mathcal{F}_r(c_1^N,\dots,c_N^N,t) \equiv (\mathbf{f}, \boldsymbol{\omega}^r) - c_\ell^N c_s^N \int_\Omega \omega_j^\ell \frac{\partial \omega_i^s}{\partial x_j}\omega_i^r\,dx \\ - \int_\Omega \tau_{ij}(\mathbf{e}(c_k^N\boldsymbol{\omega}^k))e_{ij}(\boldsymbol{\omega}^r)\,dx\,,\end{aligned}$$

$r = 1,\dots,N$. Before discussing the solvability (local and global) of (2.21), let us derive the following *a priori* estimates:

There is a constant C depending on $T, \Omega, \|\mathbf{f}\|_{L^{p'}(I;(W^{1,p}_{\mathrm{per}}(\Omega)^d)^*)}$ and $\|\mathbf{u}_0\|_2$ such that for all $N = 1, 2, \dots$,

$$\|\mathbf{u}^N\|_{L^\infty(I;H)} \le C\,, \qquad (2.23)$$

$$\|\mathbf{u}^N\|_{L^p(I;V_p)} \le C\,, \qquad (2.24)$$

$$\Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_{L^{p'}(I;(V^s)^*)} \le C\,. \qquad (2.25)$$

In order to prove (2.23), (2.24) we multiply the rth equation of the Galerkin system (2.21) by $c_r^N(t)$ and add the equations. The result can be written in the form

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^N\|_2^2 + \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(\mathbf{u}^N)\,dx = (\mathbf{f}, \mathbf{u}^N)\,, \qquad (2.26)$$

because of $\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j} u_i^N\,dx = 0$ (see (2.12)).

Due to assumption $(1.8)_2$ and its consequence $(1.20)_2$,

$$\int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(\mathbf{u}^N)\,dx \geq C_4\big(\|\mathbf{e}(\mathbf{u}^N)\|_p^p - |\Omega|\big)\,.$$

Using Korn's inequality (1.11) we obtain from (2.26),

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^N\|_2^2 + K_p^p C_4\|\nabla\mathbf{u}^N\|_p^p & \\
&\leq |(\mathbf{f},\mathbf{u}^N)| + C_4|\Omega| \\
&\leq \|\mathbf{f}\|_{-1,p'}\|\nabla\mathbf{u}^N\|_p + C_4|\Omega| \\
&\underset{\text{ineq.}}{\overset{\text{Young}}{\leq}} \frac{K_p^p C_4}{2}\|\nabla\mathbf{u}^N\|_p^p + c\,\|\mathbf{f}\|_{-1,p'}^{p'} + C_4|\Omega|
\end{aligned}$$

or finally,

$$\frac{d}{dt}\|\mathbf{u}^N\|_2^2 + K_p^p C_4\|\nabla\mathbf{u}^N\|_p^p \leq c\|\mathbf{f}\|_{-1,p'}^{p'} + C_4|\Omega|\,. \tag{2.27}$$

Integrating (2.27) between 0 and t, $t \in I$, we get

$$\begin{aligned}
\|\mathbf{u}^N(t)\|_2^2 + K_p^p C_4 \int_0^t \|\nabla\mathbf{u}^N(\tau)\|_p^p\,d\tau & \\
\leq c\int_0^T \|\mathbf{f}(\tau)\|_{-1,p'}^{p'}\,d\tau + \|\mathbf{u}^N(0)\|_2^2 + C_4 T\,|\Omega| & \\
\leq C\,. &
\end{aligned} \tag{2.28}$$

Thus (2.23) and (2.24) are proved.

The *a priori* estimate (2.23) implies

$$|\mathbf{c}^N(t)|^2 \leq C \qquad \text{for all } t \in I\,. \tag{2.29}$$

Since $\mathcal{F}_r$, $r = 1,\dots,N$, in (2.22) satisfy the Carathéodory conditions (see Appendix, Definition 3.2), we obtain by Theorem 3.4 in the Appendix the local existence of a continuous function $\mathbf{c}^N : (0,t^*) \longrightarrow \mathbb{R}^N$ solving (2.22), having $\frac{d\mathbf{c}^N}{dt}$ defined almost everywhere. But due to the continuity of $\mathbf{c}^N$ on $[0,t^*)$ and the uniform boundedness (2.29) we can shift t^* to T. See Zeidler [1990b, Chapter 30] for more details.

To prove (2.25), let us take $\boldsymbol{\varphi} \in L^p(I;V^s)$, $\|\boldsymbol{\varphi}\|_{L^p(I;V^s)} \le 1$. Then

$$\Big(\frac{d\mathbf{u}^N}{dt},\boldsymbol{\varphi}\Big) = \Big(\frac{d\mathbf{u}^N}{dt},P^N\boldsymbol{\varphi}\Big) \overset{(2.21)}{=} -\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j} P^N\varphi_i \,dx - \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(P^N\boldsymbol{\varphi})\,dx + (\mathbf{f},P^N\boldsymbol{\varphi})\,.$$

By (4.12) in the Appendix, $\|P^N\boldsymbol{\varphi}\|_{s,2} \le \|\boldsymbol{\varphi}\|_{s,2}$. Thus,

$$\begin{aligned}
\Big|\int_0^T\int_\Omega u_j^N\frac{\partial u_i^N}{\partial x_j}P^N\varphi_i\,dx\,dt\Big|
&= \Big|\int_0^T\int_\Omega u_i^N u_j^N\frac{\partial(P^N\varphi_i)}{\partial x_j}\,dx\,dt\Big| \\
&\le \int_0^T\int_\Omega |\mathbf{u}^N|^2|\nabla(P^N\boldsymbol{\varphi})|\,dx\,dt \\
&\le \int_0^T \|\nabla(P^N\boldsymbol{\varphi})\|_\infty\|\mathbf{u}^N\|_2^2\,dt \\
&\overset{(2.19)}{\le} c\int_0^T \|P^N\boldsymbol{\varphi}\|_{s,2}\|\mathbf{u}^N\|_2^2\,dt \overset{(2.23)}{\le} C\,,
\end{aligned}$$

$$\begin{aligned}
&\Big|\int_0^T\int_\Omega \tau_{ij}(e(\mathbf{u}^N))e_{ij}(P^N\boldsymbol{\varphi})\,dx\,dt\Big| \\
&\quad\overset{(1.21)}{\le} C_5 d^2\int_0^T\int_\Omega (1+|\mathbf{e}(\mathbf{u}^N)|)^{p-1}|\mathbf{e}(P^N\boldsymbol{\varphi})|\,dx\,dt \\
&\quad\underset{\text{ineq.}}{\overset{\text{Hölder}}{\le}} c\int_0^T \|\mathbf{e}(P^N\boldsymbol{\varphi})\|_p\|(1+|\mathbf{e}(\mathbf{u}^N)|)\|_p^{p-1}\,dt \\
&\quad\le c\int_0^T \|\mathbf{e}(P^N\boldsymbol{\varphi})\|_p(1+\|\nabla\mathbf{u}^N\|_p)^{p-1}\,dt \\
&\quad\overset{(2.19)}{\le} c\int_0^T \|P^N\boldsymbol{\varphi}\|_{s,2}(1+\|\nabla\mathbf{u}^N\|_p)^{p-1}\,dt \\
&\quad\underset{\text{ineq.}}{\overset{\text{Hölder}}{\le}} c\Big(\int_0^T \|\boldsymbol{\varphi}\|_{s,2}^p\,dt\Big)^{1/p}\Big(\int_0^T (1+\|\nabla\mathbf{u}^N\|_p)^p\,dt\Big)^{\frac{p-1}{p}} \\
&\quad\overset{(2.24)}{\le} C\,,
\end{aligned}$$

$$\int_0^T |(\mathbf{f}, P^N \boldsymbol{\varphi})|\, dt \le \|\mathbf{f}\|_{L^{p'}(I;(W^{1,p}_{\mathrm{per}}(\Omega)^d)^*)} \|P^N \boldsymbol{\varphi}\|_{L^p(I;V_p)} \le C\,.$$

Since

$$\Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_{L^{p'}(I;(V^s)^*)} = \sup_{\substack{\boldsymbol{\varphi}\in L^p(I;V^s)\\ \|\boldsymbol{\varphi}\|_{L^p(I;V^s)}\le 1}} \Big| \int_0^T \Big(\frac{\partial \mathbf{u}^N}{\partial t}, \boldsymbol{\varphi}\Big)\, dt \Big|\,,$$

the uniform estimate (2.25) is proved.

• *Limiting process*

From (2.23), (2.24) follows the existence of $\mathbf{u}$ satisfying for all $r > 1$ that

$$\mathbf{u}^N \rightharpoonup \mathbf{u} \qquad \text{weakly in } L^r(I;H) \cap L^p(I;V_p) \tag{2.30}$$

at least for a subsequence. In order to prove

$$\begin{aligned}\int_0^T \int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j} \varphi_i \, dx\, dt &\overset{(2.11)}{=} -\int_0^T \int_\Omega u_j^N u_i^N \frac{\partial \varphi_i}{\partial x_j}\, dx\, dt \\ &\to -\int_0^T \int_\Omega u_j u_i \frac{\partial \varphi_i}{\partial x_j}\, dx\, dt\end{aligned} \tag{2.31}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$, we need the strong convergence

$$\mathbf{u}^N \to \mathbf{u} \qquad \text{in } L^p(I;H)\,. \tag{2.32}$$

Let us first show (2.32) and then we will prove (2.31). By virtue of the compact imbedding $W^{1,p}(\Omega)^d \hookrightarrow\hookrightarrow L^2(\Omega)^d$ if $p > \frac{2d}{d+2}$ (here the lower bound (2.18) for the parameter p appears!), we can apply the Aubin-Lions Lemma 1.2.48 with $X_0 = V_p, X = H, X_1 = (V^s)^*$, $\alpha = p$ and $\beta = p'$. As a conclusion we obtain (2.32) at least for some subsequence still denoted by $\mathbf{u}^N$.¶

¶ Notice that due to (2.23) and (2.32), $\mathbf{u}^N \to \mathbf{u}$ in $L^r(I;H)$ for arbitrary $r \in (1,\infty)$. Indeed, for the interesting case $r > p$ we have

$$\begin{aligned}\int_0^T \|\mathbf{u}^N - \mathbf{u}\|_2^r\, dt &= \int_0^T \|\mathbf{u}^N - \mathbf{u}\|_2^p \|\mathbf{u}^N - \mathbf{u}\|_2^{r-p}\, dt \\ &\overset{(2.23)}{\le} 2C \int_0^T \|\mathbf{u}^N - \mathbf{u}\|_2^p\, dt\,,\end{aligned}$$

and the last integral vanishes due to (2.32).

Let us now prove (2.31). First, for all $\boldsymbol{\omega}^r$, $r \in \mathbb{N}$,

$$\begin{aligned}\int_0^T \int_\Omega (u_j^N u_i^N - u_j u_i) \frac{\partial \omega_i^r}{\partial x_j}\, dx\, dt &= \int_0^T \int_\Omega (u_j^N - u_j) u_i^N \frac{\partial \omega_i^r}{\partial x_j}\, dx\, dt \\ &+ \int_0^T \int_\Omega u_j (u_i^N - u_i) \frac{\partial \omega_i^r}{\partial x_j}\, dx\, dt \\ &\equiv I_1 + I_2\,.\end{aligned}$$

Then,

$$\begin{aligned}|I_1| &\le \|\nabla \boldsymbol{\omega}^r\|_\infty \int_0^T \|\mathbf{u}^N - \mathbf{u}\|_2 \|\mathbf{u}^N\|_2\, dt \\ &\overset{(2.23)}{\le} C \|\nabla \boldsymbol{\omega}^r\|_\infty \int_0^T \|\mathbf{u}^N - \mathbf{u}\|_2\, dt\,,\end{aligned}$$

which tends to zero due to (2.32). On the other hand the weak convergence (2.30) implies $|I_2| \to 0$ when $N \to \infty$. Since the functions of the type $g(t)\boldsymbol{\omega}^r(x)$, $g \in \mathcal{D}(-\infty, T)$, are dense in $\mathcal{D}(-\infty, T; \mathcal{V})$, we obtain

$$\int_0^T \int_\Omega u_j^N u_i^N \frac{\partial \psi_i}{\partial x_j}\, dx\, dt \to \int_0^T \int_\Omega u_j u_i \frac{\partial \psi_i}{\partial x_j}\, dx\, dt$$

for all $\boldsymbol{\psi} \in \mathcal{D}(-\infty, T; \mathcal{V})$. Next,

$$\begin{aligned}\int_0^T \Big(\frac{\partial \mathbf{u}^N(t)}{\partial t}, \boldsymbol{\omega}^r\Big) g(t)\, dt &= -\int_0^T (\mathbf{u}^N(t), \boldsymbol{\omega}^r) \frac{d}{dt} g(t)\, dt \\ &- (\mathbf{u}^N(0), \boldsymbol{\omega}^r) g(0)\end{aligned}$$

for $g \in \mathcal{D}(-\infty, T)$, $r = 1, 2, \ldots$. Due to the weak convergence (2.30) and $\mathbf{u}^N(0) = P^N \mathbf{u}_0 \to \mathbf{u}_0$ in H, we obtain

$$\begin{aligned}\int_0^T \Big(\frac{\partial \mathbf{u}^N(t)}{\partial t}, \boldsymbol{\varphi}(t)\Big)\, dt \to &-\int_0^T \Big(\mathbf{u}(t), \frac{\partial \boldsymbol{\varphi}(t)}{\partial t}\Big)\, dt \\ &- (\mathbf{u}_0, \boldsymbol{\varphi}(0)) \qquad (2.33)\end{aligned}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$.

• *Limiting process in* $\boldsymbol{\tau}$

It remains to find the limit of the nonlinear term given by $\boldsymbol{\tau}$. According to (2.24) the sequence $\{\nabla \mathbf{u}^N\}$ is bounded in $L^p(Q_T)^{d^2}$. Because the components of the nonlinear continuous function $\boldsymbol{\tau}$ have the $(p-1)$-growth due to (1.9) and (1.21), we can use Corollary

2.10 from Chapter 4 with $\mathbf{z}^j = \nabla \mathbf{u}^j$, $\tau = \tau_{ij}$, $q = p-1$, $s = d^2$ and $Q = Q_T$. We obtain

$$\tau_{ij}\big(\tfrac{\nabla \mathbf{u}^N + (\nabla \mathbf{u}^N)^T}{2}\big) \rightharpoonup \overline{\tau}_{ij} \qquad \text{in } L^{\frac{p}{p-1}}(Q_T)\,, \tag{2.34}$$

where

$$\overline{\tau}_{ij} \overset{\text{a.e.}}{=} \int_{\mathbb{R}^{d^2}} \tau_{ij}\big(\tfrac{\boldsymbol{\lambda}+\boldsymbol{\lambda}^T}{2}\big)\, d\nu_{t,x}(\boldsymbol{\lambda})\,.$$

This means that

$$\begin{aligned} &\int_{Q_T} \tau_{ij}(\mathbf{e}(\mathbf{u}^N)) e_{ij}(\boldsymbol{\varphi})\, dx\, dt \\ &\qquad \to \int_{Q_T} e_{ij}(\boldsymbol{\varphi}) \int_{\mathbb{R}^{d^2}} \tau_{ij}\big(\tfrac{\boldsymbol{\lambda}+\boldsymbol{\lambda}^T}{2}\big) d\nu_{t,x}(\boldsymbol{\lambda})\, dx\, dt \end{aligned} \tag{2.35}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$. Taking into account (2.31), (2.33) and (2.35), equation (2.15) is derived.

Using Corollary 2.10 from Chapter 4 with $\tau = \mathrm{Id}$, $r = p$ and $q = 1$ (the other parameters are as above) we obtain for all $\psi \in L^{p'}(Q_T)$, $i, j = 1, \dots, d$,

$$\int_{Q_T} \frac{\partial u_i^N}{\partial x_j} \psi\, dx\, dt \to \int_{Q_T} \psi \int_{\mathbb{R}^{d^2}} \lambda_{ij}\, d\nu_{t,x}(\boldsymbol{\lambda})\, dx\, dt\,.$$

Since $\nabla \mathbf{u}^N \rightharpoonup \nabla \mathbf{u}$ weakly in $L^p(Q_T)$,

$$\frac{\partial u_i}{\partial x_j}(t,x) \overset{\text{a.e.}}{=} \int_{\mathbb{R}^{d^2}} \lambda_{ij}\, d\nu_{t,x}(\boldsymbol{\lambda})\,,$$

which is (2.16) and the proof of the theorem is complete. ■

Remark 2.36 The measure-valued function ν satisfies besides (2.16) more restrictive conditions. It can be found in Kinderlehrer and Pedregal [1992b] that for arbitrary quasiconvex and bounded below φ such that $\lim_{|\mathbf{A}|\to\infty} \frac{\varphi(\mathbf{A})}{1+|\mathbf{A}|^p}$ exists, we have

$$\varphi(\langle \nu_x, \mathbf{Id}\rangle) \le \langle \nu_x, \varphi\rangle \qquad \text{for a.a. } x \in \Omega\,,$$

whenever ν is generated by gradients $\nabla \mathbf{u}_j$ of a sequence $\mathbf{u}_j$ bounded in $W^{1,p}(\Omega)^d$, $p > 1$. Let us recall a function $\varphi : \mathbb{R}^{d^2} \to \mathbb{R}$ is quasiconvex provided that for all $\boldsymbol{\xi} \in \mathbb{R}^{d^2}$ and $\mathbf{u} \in W_0^{1,\infty}(\Omega)^d$,

$$\varphi(\boldsymbol{\xi}) \le \frac{1}{|\Omega|} \int_\Omega \varphi(\boldsymbol{\xi} + \nabla \mathbf{u})\, dx\,.$$

Remark 2.37 (to the proof of Theorem 2.17) As can be seen from the proof of Theorem 2.17, we can weaken some of the assumptions immediately. We formulate this in the following two theorems.

Theorem 2.38 *Let $p > \frac{2d}{d+2}$, $\mathbf{u}_0 \in H$, $\mathbf{f} \in L^{p'}\big(I; (W^{1,p}_{\mathrm{per}}(\Omega)^d)^*\big)$, $p' = \frac{p}{p-1}$. Assume that for certain positive constants C_4, C_5 and $q \in [p-1, p)$ we have for all $\boldsymbol{\eta} \in \mathbb{R}^{d^2}_{\mathrm{sym}}$,*

$$\boldsymbol{\tau}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq C_4(|\boldsymbol{\eta}|^p - 1)\,, \tag{2.39}$$

$$|\tau_{ij}(\boldsymbol{\eta})| \leq C_5(1 + |\boldsymbol{\eta}|)^q\,. \tag{2.40}$$

Then there exists a measure-valued solution to the problem (2.1)–(2.4).

Proof : If $q = p-1$ then the proof coincides with the proof of Theorem 2.17. For $q \in (p-1, p)$ two insignificant changes take place. Firstly, instead of (2.25), we obtain

$$\Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_{L^{\frac{p}{q}}(I;(V^s)^*)} \leq C\,.$$

Secondly, (2.34) is replaced by

$$\tau_{ij}\Big(\tfrac{\nabla \mathbf{u}^N + (\nabla \mathbf{u}^N)^T}{2}\Big) \rightharpoonup \overline{\tau}_{ij} \qquad \text{in } L^{\frac{p}{q}}(Q_T)\,.$$

The rest of the proof remains as before. ∎

Let us now consider the problem (2.1)–(2.3), where $\boldsymbol{\tau}$ fulfills (2.39)–(2.40). Let us study the Dirichlet boundary problem. It means that $\Omega \subset \mathbb{R}^d$ is a bounded domain with $\partial\Omega \in C^{0,1}$, and (2.4) is replaced by (2.5). Then we modify the definition of the space $\mathcal{V}$ in (2.8) to

$$\mathcal{V} \equiv \big\{\boldsymbol{\phi} \in \mathcal{D}(\Omega)^d; \operatorname{div} \boldsymbol{\phi} = 0\big\}\,.$$

Following the proofs of previous theorems, we get

Theorem 2.41 *Let $p > \frac{2d}{d+2}$, $\mathbf{u}_0 \in H$, $\mathbf{f} \in L^{p'}\big(I; (W^{1,p}(\Omega)^d)^*\big)$, $p' = \frac{p}{p-1}$. Then there exists a measure-valued solution to the problem (2.1)–(2.3), (2.5), (2.39)–(2.40).*

5.2.3 Survey of known results related to the problem (NS)$_{\mathrm{p}}$

We will now summarize the results connected with the problem $(\mathrm{NS})_{\mathrm{p}}$ from the points of view of

1. existence theory,
2. uniqueness and its consequences,
3. extensions to other problems.

This survey will be completed by some bibliographical remarks.

1. *The existence theory* includes the discussion on the values of p for which one obtains

- the existence of a measure-valued solution,
- the existence of a weak solution,
- the existence of a strong solution,
- the existence of a strong solution for small data,
- the local existence of a strong solution for arbitrary data.

See Figures 5.1 and 5.2 for the graphic presentation of the results in three and two dimensions, respectively.

The existence mentioned in the first four items above is understood globally in time, i.e., for arbitrary $T \in (0, +\infty)$ the solution of a given type is constructed. A measure-valued solution has been defined in Definition 2.13. A function $\mathbf{u} \in L^\infty(I; H) \cap L^p(I; V_p)$ will be called a **weak solution** if, besides the limiting processes (2.31) and (2.33), the limit

$$\int_0^T \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(\boldsymbol{\varphi})\, dx\, dt \to \int_0^T \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}))e_{ij}(\boldsymbol{\varphi})\, dx\, dt \tag{2.42}$$

can be justified also for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$. By a **strong solution** (or **semiregular** one), we mean a function

$$\mathbf{u} \in L^2\big(I; W^{2,2}_{\text{per}}(\Omega)^d\big) \cap L^\infty(I; V_p \cap V_2) \cap L^p(I; V_{3p}),$$
$$\frac{\partial \mathbf{u}}{\partial t} \in L^2(Q_T),$$

satisfying the weak formulation (3.7) at least for all $\boldsymbol{\varphi} \in L^p(I; V_p)$.

• *The existence of a measure-valued solution* has been proved both for the space-periodic problem $(\mathrm{NS})_\mathrm{p}$ and for the Dirichlet problem in Theorems 2.17, 2.38 and 2.41 for

$$p > \frac{2d}{d+2}.$$

The nonlinear tensor function $\boldsymbol{\tau}$ is assumed to satisfy the p-coercivity condition (2.39) and the polynomial growth condition of order q, $q < p$, see (2.40). This result was first obtained in MÁLEK, NEČAS AND NOVOTNÝ [1992] and NEČAS [1991], where $\mathbf{u}^N$ were not Galerkin approximations but solutions of the so-called **bipolar model problem**. Let us recall that the bound $p > \frac{2d}{d+2}$ is due to the strong convergence $\mathbf{u}^N \to \mathbf{u}$ in $L^p(I, H)$ needed to pass to the limit in the convective term, cf. (2.31).

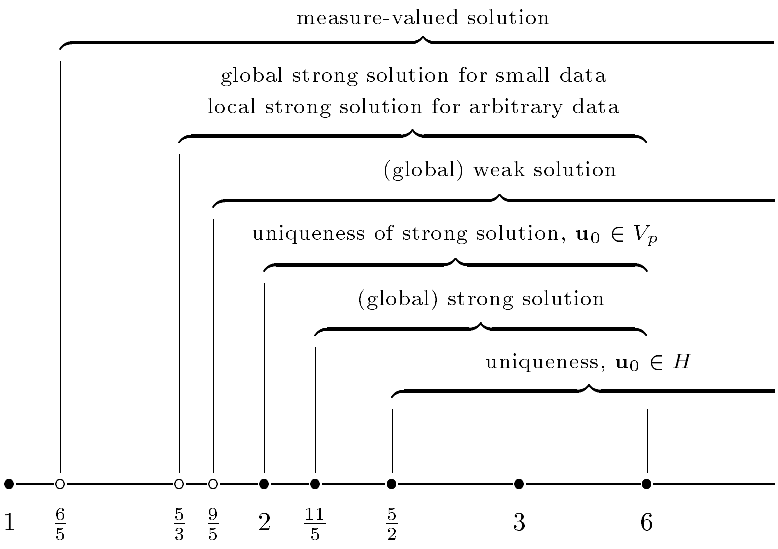

Figure 5.1 *Existence, semiregularity and uniqueness results for problem (NS)$_{\mathrm{p}}$ in three dimensions.*

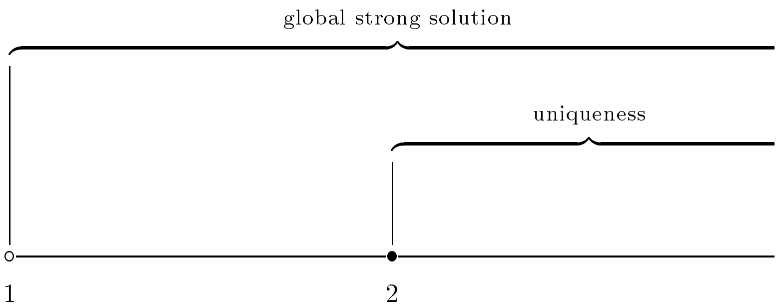

Figure 5.2 *Existence, semiregularity and uniqueness results for problem (NS)$_{\mathrm{p}}$ in two dimensions.*

• *The existence of a weak solution* requires in addition (2.42). The first results were obtained at the end of sixties by LADYZHENSKAYA [1969, 1970a, 1970b] and LIONS [1969], and in a different setting by KANIEL [1970]. While Ladyzhenskaya derived the nonlinear $\boldsymbol{\tau} \sim \mathbf{e}$ dependence by kinetic theory argument, Lions used the nonlinear p-Laplace operator.[‖] Combining the methods of monotone operators and compactness, they both showed the existence of a weak solution for

$$p \geq 1 + \frac{2d}{d+2} = \frac{3d+2}{d+2}\,.$$

This result holds both for space-periodic and Dirichlet problems, provided that the p-coercivity, the $(p-1)$-growth condition and the monotonicity of nonlinear operator hold.

For the particular case $d = 3$, the results of Ladyzhenskaya and Lions are valid for $p \geq \frac{11}{5}$. Thus, the special subcase of $p = 2$, the Navier-Stokes system (1.64)–(1.65) in Chapter 1, is not covered by these results, although the existence of a weak solution to the Navier-Stokes system is well-known due to LERAY [1934] and many others. Moreover, even in two dimensions, no existence proof follows from the results of Ladyzhenskaya and Lions if $p < 2$, but this case in particular has a lot of applications as shown, e.g., in Examples 1.78, 1.80 and 1.83 in Chapter 1.

This fact has motivated an effort to extend the results of Ladyzhenskaya and Lions. Thus, first the notion of a measure-valued solution to $(\mathrm{NS})_{\mathrm{p}}$ was defined and after proving its existence, attention was paid to the question of when the measure-valued function reduces to a Dirac one. Note that this is equivalent to (2.42) or to the construction of a weak solution.

For space-periodic problem $(\mathrm{NS})_{\mathrm{p}}$ BELLOUT, BLOOM AND NEČAS [1994] and MÁLEK, NEČAS AND RŮŽIČKA [1993] have proved the existence of a weak solution whenever

$$p > \frac{3d}{d+2}\,.$$

[‖] However, then the stress

$$\boldsymbol{\tau}^E = |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \tag{2.43}$$

does not satisfy the principle of frame indifference. Nevertheless, a modification of (2.43), namely,

$$\boldsymbol{\tau}^E = |\mathbf{e}(\mathbf{u})|^{p-2}\mathbf{e}(\mathbf{u})$$

brings no deeper change into Lions' mathematical approach.

Thus the particular case '$p = 2$ if $d = 3$' is captured. This existence proof is presented in detail in Section 5.3.

Regarding the Dirichlet boundary value problem, the first attempt in Málek, Nečas and Růžička [1996] gives a proof of the existence of a weak solution for $p \geq 2$ in three space dimensions.

• *The existence of a strong solution* to the problem $(\mathrm{NS})_{\mathrm{p}}$ is demonstrated also in Bellout, Bloom and Nečas [1994] as well as in Málek, Nečas and Růžička [1993], whenever

$$p \geq 1 + \frac{2d}{d+2}\,.$$

One of the key points in getting the above-mentioned results on weak and strong solutions lies in estimating the term

$$\mathcal{K} \equiv \Big| \sum_{i,j,k} \int_0^T \int_\Omega \frac{\partial u_j^N}{\partial x_k} \frac{\partial u_i^N}{\partial x_j} \frac{\partial u_i^N}{\partial x_k}\, dx\, dt \Big|\,.$$

In two dimensions, however, the term $\mathcal{K}$ vanishes due to the condition $\operatorname{div} \mathbf{u}^N = 0$ (or, $\frac{\partial u_1^N}{\partial x_1} = -\frac{\partial u_2^N}{\partial x_2}$). This almost immediately implies the existence of a strong solution for $p > 1$ (if $d = 2$).

• *The existence of a strong solution for small data* was first proved in Málek, Rajagopal and Růžička [1995]. It is shown (see also Section 5.4.2) that whenever

$$p > \frac{3d-4}{d} \qquad \Big(\text{and } p < 1 + \frac{2d}{d+2}\Big)\,,$$

strong solutions exist provided that the $W^{1,2}$-norm of the initial value $\mathbf{u}_0$ is small enough.

The global existence of a strong solution for small data belonging to $C(I; W^{2,q}(\Omega)^d)$, $q > d$, can be found in Amann [1994] under the sole assumption that the stress $\boldsymbol{\tau}$ does not differ too much from that one of the Newtonian fluid (given by (1.1.63)) for small values of $\mathbf{e}$.

All of the results mentioned above are global in time, i.e., they remain valid for arbitrarily large finite time interval $(0, T)$.

• As regards *the local existence of a strong solution*, the results are similar to the previous item. Whenever

$$p > \frac{3d-4}{d}\,,$$

we can find for arbitrary data a t^* such that a strong solution exists on time interval $(0, t^*)$. See Section 5.4.2 for details.

2. *Uniqueness and its consequences*

In the works of LADYZHENSKAYA [1969] and LIONS [1969], the existence of the unique weak solution is proved if

$$p \geq \frac{d+2}{2} \quad \text{and} \quad \mathbf{u}_0 \in H\,,$$

see also Theorem 4.29.

Let $\mathbf{u}$ and $\mathbf{v}$ be a weak and a strong solution, respectively, corresponding to the same data $\mathbf{u}_0 \in V_2$ and $\mathbf{f}$. Then they must coincide if $p \geq 2$. Since the strong solution exists globally for

$$p \geq 1 + \frac{2d}{d+2} \quad \text{and} \quad \mathbf{u}_0 \in V_2\,,$$

we also get 'global' uniqueness for these ps, see Theorem 4.37. For $p \in [2, 1 + \frac{2d}{d+2})$ the uniqueness result holds only locally or for sufficiently small data. Note that if $d = 2$ all limit bounds coincide.

As a consequence of uniqueness, we see that for $p \geq \frac{d+2}{2}$ the operators

$$S_t : \mathbf{u}_0 \in H \mapsto \mathbf{u}(t) \in H$$

form a semigroup. This allows us, following the approach developed by LADYZHENSKAYA [1972], CONSTANTIN AND FOIAS [1985], TEMAM [1988] and others, to study the existence of a global attractor and to investigate its properties (e.g. finite Hausdorff or fractal dimensions). See MÁLEK AND NEČAS [1994] and MÁLEK, RŮŽIČKA AND THÄTER [1994] for results in this direction.

Considering asymptotic behaviour, there are also results concerning the stability of the rest state. In MÁLEK, RAJAGOPAL AND RŮŽIČKA [1995], the exponential stability is proved for

$$p > \frac{2d}{d+2}\,.$$

Thus, in three space dimensions, for $p \in (\frac{2d}{d+2}, \frac{3d-4}{d}]$, the rest state is exponentially stable without any hypothesis about the local existence of the solution.

The exponential stability of the rest state is also studied in AMANN [1994] within the framework of his setting mentioned earlier.

3. *Extensions of the results to other problems*

Until now, we have discussed mostly the results related to the space-periodic problem $(\mathrm{NS})_\mathrm{p}$. These results can be extended in many directions.

Firstly, it is possible to weaken the assumptions on $\mathbf{u}_0$ and to perturb conveniently the potential tensor function $\boldsymbol{\tau}$ by $\boldsymbol{\sigma}$ in the sense of decomposition (1.58) in Chapter 1. Then existence results also hold, and they are discussed in Section 5.3.3. Compare with Málek, Rajagopal and Růžička [1995].

Secondly, we can modify the domain. As already mentioned, some (but not all) results have been extended from the space-periodic case to the Dirichlet problem. Except for the paper of Pokorný [1996] ($\Omega = \mathbb{R}^d$), no research has been done for the case of unbounded domains.

Thirdly, the presented approach can be applied to other systems. Thus, for example, Málek, Růžička and Thäter [1994] studied the properties of solutions of the modified Boussinesq approximation (1.88)–(1.90) in Chapter 1. Since the temperature equation obeys a sort of maximum principle, many of the results are valid for the same range of p as in the problem $(\mathrm{NS})_\mathrm{p}$. An analogous procedure can also be used for simple turbulence models, compare with Parés [1992].

As for fluids of rate type (such as Oldroyd-B fluid, described by (1.81)–(1.82) in Chapter 1) or n-grade fluids with shear-dependent viscosity, see for example (1.79) in Chapter 1, rigorous mathematical results are rare. We are aware only of the paper of Man [1992], where a one-dimensional channel flow for a second grade fluid with power-law viscosity is studied, and of the paper of Hakim [1994] considering the system of type (1.81) in Chapter 1 in two dimensions. We hope, however, that the theory developed here and in the above-mentioned papers could play a positive role in studying questions of existence, uniqueness and regularity of solutions for more complicated problems.

The following lemma shows that the obtained bounds

$$\frac{3d}{d+2} \qquad \text{and} \qquad 1+\frac{2d}{d+2}$$

are natural and come from the convective term.

Lemma 2.44

1. *Let* $\mathbf{u}, \mathbf{v} \in V_p$ *be arbitrary. Then*
$$\int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \, dx$$
is finite if $p \geq \frac{3d}{d+2}$.
2. *Let* $\mathbf{u} \in L^p(I;V_p) \cap L^\infty(I;H)$ *and* $\mathbf{v} \in V_p$ *be arbitrary. Then*
$$\int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \, dx \, dt$$
is finite if $p \geq \max(\frac{d+\sqrt{3d^2+4d}}{d+2}, \frac{3d}{d+2})$.
3. *Let* $\mathbf{u} \in L^p(I;V_p) \cap L^\infty(I;H)$ *be arbitrary. Then*
$$\int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \, dx \, dt$$
is finite for all $\mathbf{v} \in L^p(I;V_p)$ *if* $p \geq 1 + \frac{2d}{d+2}$.

Remark 2.45 From the third assertion of the previous lemma it follows that $u_j \frac{\partial u_i}{\partial x_j} \in L^{p'}(Q_T)$ just for $p \geq 1 + \frac{2d}{d+2}$, which implies $\frac{\partial \mathbf{u}}{\partial t} \in L^{p'}(I, L^{p'}(\Omega)^d)$. This is a crucial point in applying the theory of monotone operators to the problem $(\mathrm{NS})_\mathrm{p}$, see Lions [1969].

Proof (of Lemma 2.44):

Ad 1. Due to the imbedding $W^{1,p}_{\mathrm{per}}(\Omega)^d \hookrightarrow L^{\frac{dp}{d-p}}_{\mathrm{per}}(\Omega)$, we get by Hölder's inequality
$$\Big| \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \, dx \Big| \leq \|\mathbf{u}\|_{\frac{dp}{d-p}} \|\nabla \mathbf{u}\|_p \|\mathbf{v}\|_{\frac{dp}{d-p}} \, ,$$
provided that
$$\frac{1}{p} + \frac{2(d-p)}{dp} \leq 1 \, .$$
However, the last inequality is equivalent to $p \geq \frac{3d}{d+2}$.

Ad 2. Again by Hölder's inequality
$$\Big| \int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \, dx \, dt \Big| \leq \|\mathbf{v}\|_{\frac{dp}{d-p}} \int_0^T \|\nabla \mathbf{u}\|_p \|\mathbf{u}\|_q \, dt \, , \qquad (2.46)$$
where
$$\frac{1}{q} = 1 - \frac{1}{p} - \frac{d-p}{dp} = \frac{(d+1)p - 2d}{dp} \, .$$

If $p \geq \frac{4d}{d+2}$ then $q \leq 2$ and the assertion follows immediately from the assumptions on $\mathbf{u}$. Whenever $p \in \left[\frac{3d}{d+2}, \frac{4d}{d+2}\right)$ we can use the interpolation inequality

$$\|\mathbf{v}\|_q \leq \|\mathbf{v}\|_2^{1-\alpha} \|\mathbf{v}\|_{\frac{dp}{d-p}}^{\alpha} \quad \text{with } \alpha = \frac{4d-(d+2)p}{(d+2)p-2d} \tag{2.47}$$

and together with (2.46) we obtain

$$\begin{aligned}\Big|\int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \, dx \, dt\Big| &\leq c \, \|\nabla \mathbf{v}\|_p \int_0^T \|\mathbf{u}\|_2^{1-\alpha} \|\nabla \mathbf{u}\|_p^{1+\alpha} \, dt \\ &\leq c \, \|\nabla \mathbf{v}\|_p \|\mathbf{u}\|_{L^\infty(I,H)}^{1-\alpha} \int_0^T \|\nabla \mathbf{u}\|_p^{1+\alpha} \, dt \, .\end{aligned}$$

However,

$$1 + \alpha = \frac{2d}{(d+2)p - 2d} \leq p$$

whenever

$$p \geq \frac{d + \sqrt{3d^2 + 4d}}{d+2} \qquad \Big(> \frac{3d}{d+2} \text{ for } d < 4 \Big) \, .$$

Ad 3. Similarly to before we obtain

$$\begin{aligned}\Big|\int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \, dx \, dt\Big| &\underset{\text{ineq.}}{\overset{\text{Hölder}}{\leq}} \int_0^T \|\mathbf{v}\|_{\frac{dp}{d-p}} \|\nabla \mathbf{u}\|_p \|\mathbf{u}\|_q \, dt \\ &\overset{(2.47)}{\leq} c \int_0^T \|\nabla \mathbf{v}\|_p \|\mathbf{u}\|_2^{1-\alpha} \|\nabla \mathbf{u}\|_p^{1+\alpha} \, dt \\ &\leq c \, \|\mathbf{u}\|_{L^\infty(I;H)}^{1-\alpha} \int_0^T \|\nabla \mathbf{v}\|_p \|\nabla \mathbf{u}\|_p^{1+\alpha} \, dt \\ &\underset{\text{ineq.}}{\overset{\text{Hölder}}{\leq}} c \, \|\mathbf{u}\|_{L^\infty(I;H)}^{1-\alpha} \|\mathbf{v}\|_{L^p(I;V_p)} \Big(\int_0^T \|\nabla \mathbf{u}\|_p^{(1+\alpha)\frac{p}{p-1}} \, dt \Big)^{\frac{p-1}{p}} \, .\end{aligned}$$

However,

$$(1+\alpha) \frac{p}{p-1} \leq p \qquad \text{if} \quad p \geq 1 + \frac{2d}{d+2} \, .$$

∎

5.3 Incompressible non-Newtonian fluids and weak solutions

5.3.1 Basic theorem and idea of the proof

In the previous section we have introduced the notion of measure-valued solutions in order to describe

$$\lim_{N\to\infty}\int_0^T\int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big)e_{ij}(\boldsymbol{\varphi})\,dx\,dt\,. \tag{3.1}$$

Here, $\boldsymbol{\varphi}$ is a smooth function and $\mathbf{u}^N$ are Galerkin approximations.

As was shown, (3.1) can be described by means of a measure-valued function if $p > \frac{2d}{d+2}$. In this section, we show that whenever $p > \frac{3d}{d+2}$, the limit of (3.1) can be characterized classically, i.e., for some $\mathbf{u} \in L^p(I, V_p)$ the limiting process (for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$)

$$\int_0^T\int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big)e_{ij}(\boldsymbol{\varphi})\,dx\,dt \longrightarrow \int_0^T\int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big)e_{ij}(\boldsymbol{\varphi})\,dx\,dt \tag{3.2}$$

can be justified and we consequently prove the existence of a weak solution.

In order to focus in the proof of the first theorem only on verifying (3.2), we will use the definition of a weak solution which coincides with the definition of a measure-valued one, of course, except for the nonlinear term given by $\boldsymbol{\tau}$. We will show the existence of such weak solutions in Theorem 3.4.

Let us now assume that

$$\mathbf{u}_0 \in V_2 \quad \text{and} \quad \begin{array}{ll} \mathbf{f} \in L^2(Q_T)^d & \text{if } p \geq 2\,, \\ \mathbf{f} \in L^{p'}(Q_T)^d & \text{if } p < 2\,. \end{array} \tag{3.3}$$

The aim of this section is to prove the following theorem.

Theorem 3.4 *Let* $\mathbf{u}_0$, $\mathbf{f}$ *satisfy (3.3). Let*

$$\begin{array}{ll} \dfrac{3d}{d+2} < p < \dfrac{2d}{d-2} & \text{if } d \leq 4\,, \\ \dfrac{3d-4}{d} < p < \dfrac{2d}{d-2} & \text{if } 5 \leq d \leq 9\,. \end{array} \tag{3.5}$$

Then[††] *there exists a* **weak solution** $\mathbf{u} : Q_T \longrightarrow \mathbb{R}^d$ *to the problem*

[††] Note that for $d > 9$ the interval $\left(\frac{3d-4}{d}, \frac{2d}{d-2}\right)$ is an empty set.

$(NS)_p$. This means

$$\mathbf{u} \in L^\infty(I;H) \cap L^p(I;V_p) \tag{3.6}$$

and the integral identity

$$\int_0^T \int_\Omega \Big(-u_i \frac{\partial \varphi_i}{\partial t} - u_j u_i \frac{\partial \varphi_i}{\partial x_j} + \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big) e_{ij}(\boldsymbol{\varphi}) - f_i \varphi_i \Big)\, dx\, dt \\ = (\mathbf{u}_0, \boldsymbol{\varphi}) \tag{3.7}$$

holds for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$.

Remark 3.8 In Remark 3.66 we show that weak solutions in fact satisfy stronger versions of weak formulations. In Section 5.3.3 we will also extend the existence proof to initial values $\mathbf{u}_0 \in H$ and to broader classes of stress tensors.

Section 5.3.2 will be devoted to the proof of Theorem 3.4, but an idea of the proof together with some preliminaries are given immediately. As pointed out already, the key point is to justify the limiting process in the nonlinearity $\boldsymbol{\tau}$, see (3.2). The basic scheme of our approach is the following:

1. We define for some set $\{\boldsymbol{\omega}^r\}_{r=1}^\infty \subset V_p$ the Galerkin approximations $\mathbf{u}^N(t,x) = \sum_{r=1}^N c_r^N(t)\boldsymbol{\omega}^r(x)$. Because of the independence of the first *a priori* estimates on the choice of $\{\boldsymbol{\omega}^r\}_{r=1}^\infty$, we have (compare with (2.23), (2.24))

$$\|\mathbf{u}^N\|_{L^\infty(I;H)} \le C\,, \tag{3.9}$$
$$\|\mathbf{u}^N\|_{L^p(I;V_p)} \le C\,. \tag{3.10}$$

2. Assume that for some $\tilde{p} \ge 1$, in general less than p,

$$\nabla \mathbf{u}^N \to \nabla \mathbf{u} \qquad \text{strongly in } L^{\tilde{p}}(Q_T)^{d^2}\,. \tag{3.11}$$

   Then $\nabla \mathbf{u}^N \to \nabla \mathbf{u}$ almost everywhere in Q_T and also (since $\tau_{ij} \in C^1(\mathbb{R}^{d^2}_{\mathrm{sym}})$)

$$\tau_{ij}(\mathbf{e}(\mathbf{u}^N)) \to \tau_{ij}(\mathbf{e}(\mathbf{u})) \qquad \text{a.e. in } Q_T\,. \tag{3.12}$$

   As the growth parameter is less than the coercivity parameter

$(p-1<p)$, we obtain for all measurable $M \subset Q_T$,

$$
\begin{aligned}
\int_M & \big|\tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big)\big|\,dx\,dt \\
&\overset{(1.21)}{\leq} C_5 \int_M \big(1+|\mathbf{e}(\mathbf{u}^N)|\big)^{p-1}\,dx\,dt \\
&\leq C_5 \Big(\int_0^T \int_\Omega \big(1+|\mathbf{e}(\mathbf{u}^N)|\big)^p\,dx\,dt\Big)^{\frac{p-1}{p}} |M|^{\frac{1}{p}} \\
&\overset{(3.10)}{\leq} c\,|M|^{\frac{1}{p}}\,.
\end{aligned}
\tag{3.13}
$$

Taking into account Lemma 2.11 from Chapter 1, we see that (3.12) and (3.13) imply that

$$
\int_0^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big)e_{ij}(\boldsymbol{\varphi})\,dx\,dt \longrightarrow \int_0^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big)e_{ij}(\boldsymbol{\varphi})\,dx\,dt
\tag{3.14}
$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty,T;\mathcal{V})$, which is just (3.2).

3. In order to verify (3.11), it is sufficient to find uniform bounds of $\{\nabla\mathbf{u}^N\}$ in some fractional Sobolev space, namely, for some σ positive

$$
\|\mathbf{u}^N\|_{L^{\tilde p}(I;W^{1+\sigma,\tilde p}_{\mathrm{per}}(\Omega)^d)} \leq c\,, \qquad \tilde p < p\,.
\tag{3.15}
$$

Provided that for $\gamma \geq 1$,

$$
\Big\|\frac{\partial\mathbf{u}^N}{\partial t}\Big\|_{L^\gamma(I;(W^{2,2}_{\mathrm{per}}(\Omega)^d\cap V_p)^*)} \leq c\,,
\tag{3.16}
$$

we can use Lemma 2.48 from Chapter 1 with $X_1 = (W^{2,2}_{\mathrm{per}}(\Omega)^d \cap V_p)^*$, $X = W^{1,\tilde p}_{\mathrm{per}}(\Omega)^d$, $X_0 = W^{1+\sigma,\tilde p}_{\mathrm{per}}(\Omega)^d$, $\alpha = \tilde p$ and $\beta = \gamma$, and we obtain (3.11) at least for some subsequence of $\mathbf{u}^N$.

Notice that (3.16), (3.9), (3.10) are sufficient for the limiting process in the terms not containing τ_{ij} (similarly to the existence proof for measure-valued solutions). Moreover, (3.15) and (3.16) imply (3.11) which is enough for the limiting process in (3.14). Thus the existence of weak solutions is demonstrated as soon as the assumptions (3.15), (3.16) are fulfilled.

For the proof of (3.16) see Lemma 3.30 below.

4. To get (3.15), we will use the second energy estimate. For this purpose we choose a special set of $\{\boldsymbol{\omega}^r\}$ consisting of eigenvectors of the Stokes operator denoted by A. Define $P^N\mathbf{u} \equiv \sum_{r=1}^N(\mathbf{u},\boldsymbol{\omega}^r)\,\boldsymbol{\omega}^r$. By Lemma 4.26 in the Appendix, we know that P^N are continuous uniformly with respect to the norm of

$W^{2,2}(\Omega)^d$. Because of the imbedding $W^{2,2}(\Omega)^d \hookrightarrow W^{1,p}(\Omega)^d$, valid for $p \leq \frac{2d}{d-2}$, we consider henceforth the problem $(\mathrm{NS})_{\mathrm{p}}$ only for such p.

If we use the set $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ in the Galerkin system (2.21), we can multiply the rth equation by $\lambda_r c_r^N(t)$, where λ_r are the corresponding eigenvalues. Using

$$\lambda_r(\boldsymbol{\omega}^r, \mathbf{u}^N) = (A\boldsymbol{\omega}^r, \mathbf{u}^N) = (\nabla\boldsymbol{\omega}^r, \nabla\mathbf{u}^N)\,,$$

(see (4.22) in the Appendix) and summing over $r = 1, \dots, N$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}^N\|_2^2 &+ \int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}(A\mathbf{u}^N)_i\,dx \\ +\int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big)&e_{ij}(A\mathbf{u}^N)\,dx = (\mathbf{f}, A\mathbf{u}^N).\end{aligned} \tag{3.17}$$

Because of the periodicity of $\boldsymbol{\omega}^r$,

$$A\mathbf{u}^N = -\Delta\mathbf{u}^N\,.$$

Hence, using $\int_\Omega u_j^N \frac{\partial^2 u_i^N}{\partial x_j \partial x_k}\frac{\partial u_i^N}{\partial x_k} = 0$, we have

$$\begin{aligned}&\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}(A\mathbf{u}^N)_i\,dx = \int_\Omega \frac{\partial u_j^N}{\partial x_k}\frac{\partial u_i^N}{\partial x_j}\frac{\partial u_i^N}{\partial x_k}\,dx\,,\\ &\int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(A\mathbf{u}^N)\,dx \\ &\qquad \overset{(1.6)}{=} \int_\Omega \frac{\partial^2 U(\mathbf{e}(\mathbf{u}^N))}{\partial e_{k\ell}\partial e_{ij}}\frac{\partial e_{k\ell}(\mathbf{u}^N)}{\partial x_s}\frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_s}\,dx\,.\end{aligned} \tag{3.18}$$

Using (3.18) together with $(1.8)_2$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}^N\|_2^2 + C_1\int_\Omega \big(1+|\mathbf{e}(\mathbf{u}^N)|\big)^{p-2}\frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_k}\frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_k}\,dx \\ \leq \|\nabla\mathbf{u}^N\|_3^3 + |(\mathbf{f}, \Delta\mathbf{u}^N)|\,.\end{aligned} \tag{3.19}$$

It may be worth commenting, at this moment, on the nice behaviour of the 'convective' term $(3.18)_1$ in the two-dimensional space-periodic setting. Namely, if $d = 2$ then

$$\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}(A\mathbf{u}^N)_i\,dx = 0\,. \tag{3.20}$$

Let us verify this. As above, we have

$$\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}(A\mathbf{u}^N)_i\,dx = -\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}\Delta u_i^N\,dx$$
$$= \int_\Omega \frac{\partial u_j^N}{\partial x_k}\frac{\partial u_i^N}{\partial x_j}\frac{\partial u_i^N}{\partial x_k}\,dx\,,$$

where we sum over all indices from 1 to 2. By a simple comparison, we see that the relation $\frac{\partial u_1^N}{\partial x_1} = -\frac{\partial u_2^N}{\partial x_2}$ implies

$$\sum_{i,j,k=1}^{2}\int_\Omega \frac{\partial u_j^N}{\partial x_k}\frac{\partial u_i^N}{\partial x_j}\frac{\partial u_i^N}{\partial x_k} = 0\,. \tag{3.21}$$

Consequently, the term $\|\nabla\mathbf{u}^N\|_3^3$ does not enter into (3.19). This almost straightforwardly implies the existence of a strong solution for $p > 1$ in two dimensions with space-periodic boundary conditions, as shown in Theorem 4.21 below.

However, mainly for the historical reasons, in the proof of Theorem 3.4 we will not use the fact that the convective term in $(3.18)_1$ vanishes for $d = 2$.

Let us denote by

$$\mathcal{I}_p(\mathbf{u}) \equiv \int_\Omega \big(1 + |\mathbf{e}(\mathbf{u})|\big)^{p-2}\frac{\partial e_{ij}(\mathbf{u})}{\partial x_k}\frac{\partial e_{ij}(\mathbf{u})}{\partial x_k}\,dx\,. \tag{3.22}$$

The main part of the proof of Theorem 3.4, see Section 5.3.2, will be devoted to the following implication: if $p > \frac{3d-4}{d}$ then it is possible to find a $\lambda > 0$ such that

$$\int_0^T \frac{\mathcal{I}_p(\mathbf{u}^N)}{\big(1 + \|\nabla\mathbf{u}^N\|_2^2\big)^\lambda}\,dt \le C\,. \tag{3.23}$$

Thus, roughly speaking, (3.23) provides a uniform estimate for the time integral of the ratio of $W^{2,p}$- and $W^{1,2}$-norms of $\mathbf{u}^N$. Having (3.23), we will prove for $p > \frac{3d}{d+2}$ that some L^s-norm[‡‡] of the second derivatives of $\mathbf{u}^N$ is uniformly summable with some fractional exponent (which vanishes for p approaching $\frac{3d}{d+2}+$). Nevertheless, it reveals that this fractional information is sufficient for proving (3.15).

In order to handle the right-hand side of (3.19), we need estimates

[‡‡] Namely, $s = 2$ if $p \ge 2$ and $s = p$ if $p < 2$.

from below of the integral $\mathcal{I}_p(\mathbf{u})$ by means of some norms of the first and second derivatives. This is provided by the following result.

Lemma 3.24 *Let $\mathbf{u} \in C^2_{\mathrm{per}}(\Omega)^d$. Then there exists a constant c depending only on Ω, p and d such that*

$$\|D^2\mathbf{u}\|_p \le c\big(\mathcal{I}_p(\mathbf{u})\big)^{\frac{1}{2}}\big(1+\|\nabla\mathbf{u}\|_p\big)^{\frac{2-p}{2}} \quad \text{if } p \in (1,2)\,, \tag{3.25}$$

$$\|D^2\mathbf{u}\|_2 \le c\big(\mathcal{I}_p(\mathbf{u})\big)^{\frac{1}{2}} \quad \text{if } p \ge 2\,. \tag{3.26}$$

For $1 \le q \le 2$, $q \neq d$,

$$\|\nabla\mathbf{u}\|_{\frac{dp}{d-q}} \le c\big(\mathcal{I}_p(\mathbf{u})\big)^{\frac{q}{2p}}\big(1+\|\nabla\mathbf{u}\|_p\big)^{\frac{2-q}{2}} \quad \text{if } p > 1\,. \tag{3.27}$$

PROOF : Simple calculations imply

$$\begin{aligned}
\|\nabla\mathbf{e}(\mathbf{u})\|_p^p &= \int_\Omega \left(\frac{\partial e_{ij}(\mathbf{u})}{\partial x_k}\frac{\partial e_{ij}(\mathbf{u})}{\partial x_k}\right)^{\frac{p}{2}} dx \\
&= c\int_\Omega \left\{(1+|\mathbf{e}(\mathbf{u})|)^{p-2}\frac{\partial e_{ij}(\mathbf{u})}{\partial x_k}\frac{\partial e_{ij}(\mathbf{u})}{\partial x_k}\right\}^{\frac{p}{2}} \\
&\qquad \times (1+|\mathbf{e}(\mathbf{u})|)^{-\frac{p}{2}(p-2)}\, dx \\
&\underset{\text{ineq.}}{\overset{\text{Hölder}}{\le}} c\,\big(\mathcal{I}_p(\mathbf{u})\big)^{\frac{p}{2}}\Big(\int_\Omega (1+|\mathbf{e}(\mathbf{u})|)^p\, dx\Big)^{\frac{2-p}{2}} \\
&\le c\,\big(\mathcal{I}_p(\mathbf{u})\big)^{\frac{p}{2}}(1+\|\nabla\mathbf{u}\|_p)^{p\frac{2-p}{2}}\,.
\end{aligned}$$

Assertion (3.25) then follows from (1.16).

If $p \ge 2$ then $\big(1+|\mathbf{e}(\mathbf{u})|\big)^{p-2} \ge 1$, and (3.26) is a direct consequence of Korn's inequality (1.11) and the definition of $\mathcal{I}_p(\mathbf{u})$.

Let $1 \le q \le 2$. From the equality

$$\frac{\partial}{\partial x_s}\big(1+|\mathbf{e}(\mathbf{u})|\big)^{\frac{p}{q}} = \frac{p}{q}\big(1+|\mathbf{e}(\mathbf{u})|\big)^{\frac{p-q}{q}}\frac{e_{ij}(\mathbf{u})\frac{\partial e_{ij}(\mathbf{u})}{\partial x_s}}{|\mathbf{e}(\mathbf{u})|}$$

it follows that

$$\begin{aligned}
&\|\nabla\big(1+|\mathbf{e}(\mathbf{u})|\big)^{\frac{p}{q}}\|_q^q \\
&\quad\le c\left(\frac{p}{q}\right)^q \int_\Omega \big(1+|\mathbf{e}(\mathbf{u})|\big)^{p-q}|\mathbf{e}(\nabla\mathbf{u})|^q\, dx \\
&\quad\le c\int_\Omega \big(1+|\mathbf{e}(\mathbf{u})|\big)^{\frac{p-2}{2}q}|\mathbf{e}(\nabla\mathbf{u})|^q\big(1+|\mathbf{e}(\mathbf{u})|\big)^{\frac{2-q}{2}p}\, dx \\
&\quad\underset{\text{ineq.}}{\overset{\text{Hölder}}{\le}} c\,\big(\mathcal{I}_p(\mathbf{u})\big)^{\frac{q}{2}}\Big(\int_\Omega \big(1+|\mathbf{e}(\mathbf{u})|\big)^p\, dx\Big)^{\frac{2-q}{2}}\,.
\end{aligned}$$

Hence

$$\|\nabla\big(1+|\mathbf{e}(\mathbf{u})|\big)^{\frac{p}{q}}\|_q \le c\,\big(\mathcal{I}_p(\mathbf{u})\big)^{\frac{1}{2}}\big(1+\|\nabla\mathbf{u}\|_p\big)^{\frac{2-q}{2q}p}. \tag{3.28}$$

Since $W^{1,q}(\Omega)\hookrightarrow L^{\frac{dq}{d-q}}(\Omega)$ if $q<d$ (this implies that $q<2$ for $d=2$), we obtain from (3.28) that

$$\|\big(1+|\mathbf{e}(\mathbf{u})|\big)^{\frac{p}{q}}\|_{\frac{dq}{d-q}} \le c\,\left(\mathcal{I}_p(\mathbf{u})\right)^{\frac{1}{2}}\left(1+\|\nabla\mathbf{u}\|_p\right)^{\frac{2-q}{2q}p}.$$

On the other hand,

$$\|\big(1+|\mathbf{e}(\mathbf{u})|\big)^{\frac{p}{q}}\|_{\frac{dq}{d-q}} \ge \||\mathbf{e}(\mathbf{u})|^{\frac{p}{q}}\|_{\frac{dq}{d-q}} = \|\mathbf{e}(\mathbf{u})\|^{\frac{p}{q}}_{\frac{dp}{d-q}}.$$

Using again Korn's inequality (1.11) we obtain

$$\|\nabla\mathbf{u}\|_{\frac{dp}{d-q}} \le c\,\left(\mathcal{I}_p(\mathbf{u})\right)^{\frac{q}{2p}}\left(1+\|\nabla\mathbf{u}\|_p\right)^{\frac{2-q}{2}},$$

which is (3.27). ∎

Remark 3.29 Let us use Lemma 3.24 in order to estimate the term $\int_0^T|(\mathbf{f},\Delta\mathbf{u}^N)|\,dt$, which appears on the right-hand side of (3.19) after integrating it with respect to time.
If $p<2$,

$$\begin{aligned}
&\int_0^T|(\mathbf{f},\Delta\mathbf{u}^N)|\,dt\\
&\quad\le\int_0^T\|\mathbf{f}\|_{p'}\|D^2\mathbf{u}^N\|_p\,dt\\
&\quad\overset{(3.25)}{\le} c\int_0^T\|\mathbf{f}\|_{p'}\big(1+\|\nabla\mathbf{u}^N\|_p\big)^{\frac{2-p}{2}}\left(\mathcal{I}_p(\mathbf{u}^N)\right)^{\frac{1}{2}}\,dt\\
&\quad\le\frac{C_1}{4}\int_0^T\mathcal{I}_p(\mathbf{u}^N)\,dt\\
&\qquad\qquad+c\int_0^T\|\mathbf{f}\|^2_{p'}\big(1+\|\nabla\mathbf{u}^N\|_p\big)^{2-p}\,dt\\
&\quad\le\frac{C_1}{4}\int_0^T\mathcal{I}_p(\mathbf{u}^N)\,dt\\
&\qquad\qquad+c\Big(\int_0^T\|\mathbf{f}\|^{p'}_{p'}\,dt\Big)^{\frac{2}{p'}}\Big(\int_0^T\big(1+\|\nabla\mathbf{u}^N\|_p\big)^p\,dt\Big)^{\frac{2-p}{p}}.
\end{aligned}$$

If $p \geq 2$,

$$
\begin{aligned}
&\int_0^T |(\mathbf{f}, \Delta \mathbf{u}^N)| \, dt \\
&\qquad \leq \int_0^T \|\mathbf{f}\|_2 \|D^2 \mathbf{u}^N\|_2 \, dt \\
&\qquad \overset{(3.26)}{\leq} c \int_0^T \|\mathbf{f}\|_2 \left(\mathcal{I}_p(\mathbf{u}^N)\right)^{\frac{1}{2}} \, dt \\
&\qquad \leq c\|\mathbf{f}\|^2_{L^2(Q_T)} + \frac{C_1}{4} \int_0^T \mathcal{I}_p(\mathbf{u}^N) \, dt \, .
\end{aligned}
$$

In both cases the term $\frac{C_1}{4} \int_0^T \mathcal{I}_p(\mathbf{u}^N) \, dt$ can be moved to the left-hand side of (3.19) integrated with respect to time. The remaining terms coming from the estimate of $\int_0^T |(\mathbf{f}, \Delta \mathbf{u}^N)| \, dt$ are uniformly bounded due to (3.10) and (3.3).

Since in this section we work with a different basis in V_p than before, the estimate of $\frac{\partial \mathbf{u}^N}{\partial t}$ is slightly different. In fact, the following lemma holds (compare with (2.25)).

Lemma 3.30 *Let $\mathbf{u}^N(t,x) = \sum_{r=1}^N c_r^N(t) \boldsymbol{\omega}^r(x)$ be solutions to the Galerkin systems (2.21), where $\boldsymbol{\omega}^r$ are eigenvectors of the Stokes operator. Let us denote*

$$
Y \equiv \left\{ \mathbf{u} \in W^{2,2}_{\mathrm{per}}(\Omega)^d ; \operatorname{div} \mathbf{u} = 0, \int_\Omega \mathbf{u} \, dx = \mathbf{0} \right\} .
$$

Then for arbitrary $p \in (\frac{3d-2}{d+2}, \frac{2d}{d-2})$ there exist a constant C and a parameter $\gamma \in (1, \infty)$ such that

$$
\left\| \frac{\partial \mathbf{u}^N}{\partial t} \right\|_{L^\gamma(I; Y^*)} \leq C \, . \tag{3.31}
$$

Proof : We proceed similarly to the derivation of (2.25). We look for an appropriate $\gamma \in (1, \infty)$ such that the estimate

$$
\sup_{\substack{\boldsymbol{\varphi} \in L^{\gamma'}(I;Y) \\ \|\boldsymbol{\varphi}\|_{L^{\gamma'}(I;Y)} \leq 1}} \int_0^T \left| \left(\frac{\partial \mathbf{u}^N}{\partial t}, \boldsymbol{\varphi} \right) \right| dt \leq C \, , \qquad \gamma' = \frac{\gamma}{\gamma - 1}
$$

holds. Now,

$$\begin{aligned}\Big|\Big(\frac{\partial \mathbf{u}^N}{\partial t},\boldsymbol{\varphi}\Big)\Big| &= \Big|\Big(\frac{\partial \mathbf{u}^N}{\partial t},P^N\boldsymbol{\varphi}\Big)\Big| \\ &\le \Big|\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}\big(P^N\boldsymbol{\varphi}\big)_i\,dx\Big| \\ &\quad + \Big|\int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big)e_{ij}\big(P^N\boldsymbol{\varphi}\big)\,dx\Big| \\ &\quad + \big|\big(\mathbf{f},P^N\boldsymbol{\varphi}\big)\big| \equiv I_1 + I_2 + I_3\,.\end{aligned}$$

Due to our choice of basis, compare with (4.27) in the Appendix,

$$\|P^N\boldsymbol{\varphi}\|_{2,2} \le C\|\boldsymbol{\varphi}\|_{2,2} \qquad \forall\,\boldsymbol{\varphi}\in Y\,, \tag{3.32}$$

where C is independent of N.

Let us estimate I_1, I_2, I_3 separately starting with I_2. It holds that

$$\begin{aligned}\int_0^T I_2\,dt &\overset{(1.21)}{\le} C\int_0^T\int_\Omega \big(1+|\mathbf{e}(\mathbf{u}^N)|\big)^{p-1}|\nabla P^N\boldsymbol{\varphi}|\,dx\,dt \\ &\underset{\text{ineq.}}{\overset{\text{Hölder}}{\le}} C\int_0^T \big(1+\|\nabla\mathbf{u}^N\|_p\big)^{p-1}\|\nabla P^N\boldsymbol{\varphi}\|_p\,dt\,.\end{aligned}$$

Since $W^{2,2}(\Omega)^d \hookrightarrow W^{1,p}(\Omega)^d$ if $p < \frac{2d}{d-2}$, we get not only the upper bound on p but also

$$\begin{aligned}\int_0^T I_2\,dt &\le C\int_0^T \big(1+\|\nabla\mathbf{u}^N\|_p\big)^{p-1}\|P^N\boldsymbol{\varphi}\|_{2,2}\,dt \\ &\overset{(3.32)}{\le} C\int_0^T \big(1+\|\nabla\mathbf{u}^N\|_p\big)^{p-1}\|\boldsymbol{\varphi}\|_{2,2}\,dt \\ &\underset{\text{ineq.}}{\overset{\text{Hölder}}{\le}} C\,\|\boldsymbol{\varphi}\|_{L^p(I;Y)}\big(1+\|\nabla\mathbf{u}^N\|^{p-1}_{L^p(I;V_p)}\big) \overset{(3.10)}{\le} C\,.\end{aligned}$$

Since the estimate of I_3 is trivial, we can focus on estimating I_1. We will distinguish four cases:

- $d > 2$ and $p < d$,
- $d > 2$ and $p \ge d$,
- $d = 2$ and $p \ge 2$,
- $d = 2$ and $p < 2$.

• The case $d > 2$ and $p < d$: Having

$$\begin{aligned}\int_0^T I_1\,dt &= \int_0^T \Big|\int_\Omega u_k^N \frac{\partial u_i^N}{\partial x_k}(P^N\boldsymbol{\varphi})_i\,dx\Big|\,dt \\ &= \int_0^T \Big|\int_\Omega u_k^N u_i^N \frac{\partial (P^N\boldsymbol{\varphi})_i}{\partial x_k}\,dx\Big|\,dt \\ &\underset{\text{ineq.}}{\overset{\text{Hölder}}{\leq}} \int_0^T \|\nabla P^N\boldsymbol{\varphi}\|_{\frac{2d}{d-2}}\|\mathbf{u}^N\|^2_{\frac{4d}{d+2}}\,dt\,,\end{aligned} \tag{3.33}$$

we use the interpolation inequality

$$\|\mathbf{v}\|_{\frac{4d}{d+2}} \leq \|\mathbf{v}\|_2^{1-\lambda}\,\|\mathbf{v}\|^\lambda_{\frac{dp}{d-p}} \qquad \text{with} \quad \lambda = \frac{(d-2)p}{2(dp-2d+2p)}\,,$$

and we obtain

$$\begin{aligned}\int_0^T I_1\,dt &\leq C\int_0^T \|\mathbf{u}^N\|_2^{2(1-\lambda)}\|\nabla\mathbf{u}^N\|_p^{\frac{(d-2)p}{dp-2d+2p}}\|P^N\boldsymbol{\varphi}\|_{2,2}\,dt \\ &\leq C\,\|\mathbf{u}^N\|^{2(1-\lambda)}_{L^\infty(I;H)}\int_0^T \Big(\|\nabla\mathbf{u}^N\|_p^p\Big)^{\frac{d-2}{dp-2d+2p}}\|\boldsymbol{\varphi}\|_{2,2}\,dt \\ &\underset{\text{ineq.}}{\overset{\text{Hölder}}{\leq}} C\,\|\mathbf{u}^N\|^{2(1-\lambda)}_{L^\infty(I;H)}\|\mathbf{u}^N\|^{\frac{(d-2)p}{dp-2d+2p}}_{L^p(I;V_p)}\Big(\int_0^T\|\boldsymbol{\varphi}\|^{\delta'}_{2,2}\,dt\Big)^{\frac{1}{\delta'}} \\ &\underset{(3.10)}{\overset{(3.9)}{\leq}} C\,,\end{aligned}$$

where $\delta' = \frac{\delta}{\delta-1}$ and $\delta = \frac{dp-2d+2p}{d-2}$. Note that $\delta > 1$ if $p > \frac{3d-2}{d+2}$.

• The case $d > 2$ and $p \geq d$: With help of (3.33) and since $W^{1,d}(\Omega)^d \hookrightarrow L^q(\Omega)^d$, for all $q < \infty$, we get

$$\begin{aligned}\int_0^T I_1\,dt &\leq \int_0^T \|\mathbf{u}^N\|^2_{\frac{4d}{d+2}}\|P^N\boldsymbol{\varphi}\|_{2,2}\,dt \\ &\leq C\int_0^T \|\mathbf{u}^N\|^2_{1,d}\|\boldsymbol{\varphi}\|_{2,2}\,dt \\ &\underset{\text{ineq.}}{\overset{\text{Hölder}}{\leq}} C\,\|\mathbf{u}^N\|^2_{L^p(I;V_p)}\|\boldsymbol{\varphi}\|_{L^{\delta'}(I;Y)} \overset{(3.10)}{\leq} C\,,\end{aligned}$$

now with $\delta = \frac{p}{2}$.

- The case $d = 2$ and $p \geq 2$: We estimate I_1 as follows:

$$\int_0^T I_1\,dt \leq \int_0^T \int_\Omega |\mathbf{u}^N|\,|\nabla\mathbf{u}^N|\,|P^N\boldsymbol{\varphi}|\,dx\,dt$$
$$\underset{\text{ineq.}}{\overset{\text{Hölder}}{\leq}} \int_0^T \|P^N\boldsymbol{\varphi}\|_\infty \|\nabla\mathbf{u}^N\|_2 \|\mathbf{u}^N\|_2\,dt$$
$$\overset{(3.9)}{\leq} C \int_0^T \|\nabla\mathbf{u}^N\|_p \|\boldsymbol{\varphi}\|_{2,2}\,dt$$
$$\underset{\text{ineq.}}{\overset{\text{Hölder}}{\leq}} C\,\|\mathbf{u}^N\|_{L^p(I;V_p)} \|\boldsymbol{\varphi}\|_{L^{p'}(I;Y)} \overset{(3.10)}{\leq} C\,.$$

Put $\delta = p$ in this case.

- The case $d = 2$ and $p < 2$: Similarly to the previous case we obtain

$$\int_0^T I_1\,dt \leq \int_0^T \|P^N\boldsymbol{\varphi}\|_\infty \|\nabla\mathbf{u}^N\|_p \|\mathbf{u}^N\|_{\frac{p}{p-1}}\,dt.$$

Due to the interpolation inequality

$$\|\mathbf{v}\|_{\frac{p}{p-1}} \leq \|\mathbf{v}\|_2^{\frac{3p-4}{2(p-1)}}\,\|\mathbf{v}\|_{\frac{2p}{2-p}}^{\frac{2-p}{2(p-1)}}\,,$$

and (3.9), we have

$$\int_0^T I_1\,dt \leq C \int_0^T \|\boldsymbol{\varphi}\|_{2,2} \|\nabla\mathbf{u}^N\|_p^{1+\frac{2-p}{2(p-1)}}\,dt$$
$$\underset{\text{ineq.}}{\overset{\text{Hölder}}{\leq}} C\,\|\mathbf{u}^N\|_{L^p(I;V_p)}^{\frac{1}{2(p-1)}} \|\boldsymbol{\varphi}\|_{L^{\delta'}(I;Y)} \overset{(3.10)}{\leq} C\,,$$

where $\delta = \frac{\delta'}{\delta'-1} = 2(p-1)$.

Therefore, $\int_0^T I_1\,dt \leq C$ in all cases. Putting

$$\gamma = \min(p, \delta)\,,$$

we obtain from previous calculations the required estimate (3.31). The proof of Lemma 3.30 is finished. ∎

5.3.2 Proof of the basic theorem

PROOF (of Theorem 3.4): First, let us summarize the situation: taking the basis $\{\boldsymbol{\omega}^r\}_{r=1}^\infty$ consisting of eigenvectors of the Stokes operator, we have derived not only the first *a priori* estimates (3.9), (3.10) and (3.31), but also the inequality (3.19), from which we want to find the estimate of the type (3.15): to establish $\sigma > 0$ and $C > 0$ such that

$$\|\mathbf{u}^N\|_{L^{\tilde{p}}(I;W^{1+\sigma,\tilde{p}}_{\mathrm{per}}(\Omega)^d)} \le C \qquad \text{for some } \tilde{p} < p\,. \tag{3.34}$$

Let us recall that the estimate (3.34) is the only one which remains to be proved in order to get the existence of a weak solution to the problem $(\mathrm{NS})_{\mathrm{p}}$.

Finally, it has been also shown in Remark 3.29 that the term $|(\mathbf{f}, \Delta\mathbf{u}^N)|$ can be handled easily. Therefore, we put $\mathbf{f} \equiv \mathbf{0}$ to simplify the following calculations.

Thus, our starting inequality (3.19) has the form of

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}^N\|_2^2 + C_1\mathcal{I}_p(\mathbf{u}^N) \le \|\nabla\mathbf{u}^N\|_3^3\,, \tag{3.35}$$

(see (3.22) for the definition of $\mathcal{I}_p(\mathbf{u})$).

From now on we keep $d \ge 2$ and $p > 1$ arbitrary but fixed. Of course, $p < \frac{2d}{d-2}$.

Let us consider two cases:

1. $p \ge 3$,
2. $p < 3$.

Ad 1. The first case is trivial. Indeed, after integrating (3.35) with respect to time between 0 and t, $t \le T$, we obtain

$$\begin{aligned}\frac{1}{2}\|\nabla\mathbf{u}^N(t)\|_2^2 + C_1\int_0^t \mathcal{I}_p(\mathbf{u}^N)\,dt &\le \frac{1}{2}\|\nabla\mathbf{u}_0\|_2^2 + \|\nabla\mathbf{u}^N\|^3_{L^3(Q_T)} \\ &\le \frac{1}{2}\|\nabla\mathbf{u}_0\|_2^2 + c\|\nabla\mathbf{u}^N\|^3_{L^p(Q_T)} \\ &\le C\,,\end{aligned}$$

due to (3.3) and (3.10). Particularly,

$$\int_0^T \mathcal{I}_p(\mathbf{u}^N)\,dt \le C$$

and we use (3.26) to obtain immediately (3.34) with $\widetilde{p} = 2$ and $\sigma = 1$.

Ad 2. Let us consider $p < 3$ in the sequel. In order to estimate the right-hand side of (3.35), we will have to include into the consideration all possible norms of $\nabla\mathbf{u}^N$; it means

- the L^2-norm, appearing in the first term of (3.35),

- the L^p-norm, for which *a priori* estimates of $\nabla \mathbf{u}^N$ are available, see (3.10),
- the $L^{\frac{dp}{d-q}}$-norm, estimating from below the elliptic term $\mathcal{I}_p(\mathbf{u}^N)$, see (3.27).

To combine all norms mentioned above we use the following two interpolation inequalities which are correct for $q \geq \frac{d(3-p)}{3}$:

$$\|\mathbf{v}\|_3 \leq \|\mathbf{v}\|_2^{\frac{2(dp+3q-3d)}{3(dp+2q-2d)}} \|\mathbf{v}\|_{\frac{dp}{d-q}}^{\frac{dp}{3(dp+2q-2d)}}, \tag{3.36}$$

$$\|\mathbf{v}\|_3 \leq \|\mathbf{v}\|_p^{\frac{dp+3q-3d}{3q}} \|\mathbf{v}\|_{\frac{dp}{d-q}}^{\frac{d(3-p)}{3q}}. \tag{3.37}$$

Since for $\alpha \in (0,1)$ we have

$$\|\nabla \mathbf{u}^N\|_3^3 = \|\nabla \mathbf{u}^N\|_3^{3(1-\alpha)+3\alpha}, \tag{3.38}$$

we obtain from (3.36)–(3.38)

$$\begin{aligned} \|\nabla \mathbf{u}^N\|_3^3 \leq & \|\nabla \mathbf{u}^N\|_2^{2(1-\alpha)\frac{dp+3q-3d}{dp+2q-2d}} \|\nabla \mathbf{u}^N\|_p^{\alpha\frac{dp+3q-3d}{q}} \\ & \times \|\nabla \mathbf{u}^N\|_{\frac{dp}{d-q}}^{\alpha\frac{d(3-p)}{q}+\frac{(1-\alpha)dp}{dp+2q-2d}}. \end{aligned} \tag{3.39}$$

Set

$$\begin{aligned} Q_1 &\equiv (1-\alpha)\frac{dp+3q-3d}{dp+2q-2d}, \\ Q_2 &\equiv \alpha\frac{dp+3q-3d}{q}, \\ Q_3 &\equiv \alpha\frac{d(3-p)}{q} + \frac{(1-\alpha)dp}{dp+2q-2d}. \end{aligned}$$

Then, combining (3.35) and (3.39), we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}^N\|_2^2 + C_1\mathcal{I}_p(\mathbf{u}^N) \\ & \quad \leq \left(\|\nabla \mathbf{u}^N\|_2^2\right)^{Q_1}\left(1+\|\nabla \mathbf{u}^N\|_p\right)^{Q_2}\|\nabla \mathbf{u}^N\|_{\frac{dp}{d-q}}^{Q_3}. \end{aligned} \tag{3.40}$$

By the inequality (3.27) we obtain from (3.40) that

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}^N\|_2^2 + C_1\mathcal{I}_p(\mathbf{u}^N) \\ & \quad \leq C\left(\|\nabla \mathbf{u}^N\|_2^2\right)^{Q_1}\left(1+\|\nabla \mathbf{u}^N\|_p\right)^{Q_2+\frac{2-q}{2}Q_3}\left(\mathcal{I}_p(\mathbf{u}^N)\right)^{\frac{q}{2p}Q_3}. \end{aligned} \tag{3.41}$$

By Young's inequality we obtain

$$\begin{aligned}&\frac{d}{dt}\|\nabla\mathbf{u}^N\|_2^2 + C_1\mathcal{I}_p(\mathbf{u}^N)\\&\quad\le C\left(\|\nabla\mathbf{u}^N\|_2^2\right)^{Q_1\cdot\delta'}\left(1+\|\nabla\mathbf{u}^N\|_p\right)^{\left(Q_2+\frac{2-q}{2}Q_3\right)\cdot\delta'},\end{aligned}\tag{3.42}$$

provided that

$$\frac{1}{\delta}+\frac{1}{\delta'}=1 \qquad \text{and} \qquad \frac{q}{2p}\,Q_3\,\delta = 1\,, \quad \delta,\,\delta' > 1\,. \tag{3.43}$$

Requiring

$$\Big(Q_2+\frac{2-q}{2}Q_3\Big)\delta' = p\,, \tag{3.44}$$

we can easily compute α, δ and δ'. Indeed, we have from (3.43) and (3.44),

$$1=\frac{1}{\delta}+\frac{1}{\delta'}=\frac{Q_2}{p}+\frac{Q_3}{p}=\frac{3\alpha}{p}+\frac{(1-\alpha)d}{dp+2q-2d}\,.$$

Therefore,

$$\begin{aligned}\alpha &= \frac{p(dp+2q-3d)}{2(dp+3q-3d)}\,,\\ 1-\alpha &= \frac{(3-p)(dp+2q-2d)}{2(dp+3q-3d)}\,,\end{aligned}$$

and

$$\delta' = \frac{4}{dp-3d+4}\,.$$

Notice that the necessary condition for α to be from $(0,1)$ and δ' to be defined, reads

$$p > \frac{3d-4}{d}\,; \tag{3.45}$$

compare with (3.5). Notice also that δ' does not depend on the choice of q.

Inserting α, $1-\alpha$ and δ' into the inequality (3.42), we obtain

$$\frac{d}{dt}\|\nabla\mathbf{u}^N\|_2^2 + C_1\mathcal{I}_p(\mathbf{u}^N) \le C\left(\|\nabla\mathbf{u}^N\|_2^2\right)^{\frac{2(3-p)}{dp-3d+4}}\left(1+\|\nabla\mathbf{u}^N\|_p\right)^p$$

or

$$\frac{d}{dt}\left(1+\|\nabla\mathbf{u}^N\|_2^2\right)+C_1\mathcal{I}_p(\mathbf{u}^N) \le C\left(1+\|\nabla\mathbf{u}^N\|_2^2\right)^{\lambda}\left(1+\|\nabla\mathbf{u}^N\|_p\right)^p\,, \tag{3.46}$$

where

$$\lambda = \frac{2(3-p)}{dp-3d+4}\,. \tag{3.47}$$

Dividing (3.46) by $\big(1+\|\nabla \mathbf{u}^N\|_2^2\big)^{\lambda}$, we obtain for $\lambda \neq 1$

$$\begin{aligned}\frac{1}{1-\lambda}\frac{d}{dt}\big(1+\|\nabla \mathbf{u}^N\|_2^2\big)^{1-\lambda} + C_1\big(1+\|\nabla \mathbf{u}^N\|_2^2\big)^{-\lambda}\mathcal{I}_p(\mathbf{u}^N)&\\ \leq C\left(1+\|\nabla \mathbf{u}^N\|_p\right)^p.&\end{aligned} \tag{3.48}$$

If $\lambda = 1$ then the first term in (3.46) is replaced by

$$\frac{1}{2}\frac{d}{dt}\ln\left(1+\|\nabla \mathbf{u}^N\|_2^2\right). \tag{3.49}$$

Let us keep in mind that one should always replace the term with power $1-\lambda$ by the logarithm term whenever $\lambda = 1$.

Integrating (3.48) between 0 and t, $t \in (0,T]$, we get

$$\begin{aligned}\frac{1}{(1-\lambda)}\big(1+\|\nabla \mathbf{u}^N(t)\|_2^2\big)^{1-\lambda}&\\ + C_1\int_0^t \mathcal{I}_p(\mathbf{u}^N(\tau))\big(1+\|\nabla \mathbf{u}^N(\tau)\|_2^2\big)^{-\lambda}\,d\tau&\\ \underset{(3.10)}{\overset{(3.3)}{\leq}} C = C(\mathbf{u}_0)\,.&\end{aligned} \tag{3.50}$$

We are interested in getting 'global' estimates. Here, the word 'global' means 'independent of the size of initial values (in an appropriate space) and of the length of the time interval'. Because the first term in (3.50) gives this 'global' information only when $\lambda \leq 1$ and since

$$\lambda \leq 1 \qquad \text{if and only if} \qquad p \geq \frac{3d+2}{d+2} = 1 + \frac{2d}{d+2}\,,$$

we will consider two possibilities:

a) $p \geq 1 + \frac{2d}{d+2}$, i.e., $\lambda \leq 1$,
b) $p < 1 + \frac{2d}{d+2}$, i.e., $\lambda > 1$.

Ad a), $p \geq 1 + \frac{2d}{d+2}$. Taking the supremum over all $t \in (0,T]$ (in (3.50)), we obtain the estimate

$$\|\nabla \mathbf{u}^N\|_{L^\infty(I;L^2_{\mathrm{per}}(\Omega))} \leq C\,. \tag{3.51}$$

Consequently,

$$\int_0^T \mathcal{I}_p(\mathbf{u}^N(t))\,dt \le C\,. \tag{3.52}$$

Let us remark that $1+\frac{2d}{d+2} \ge 2$ for all $d \ge 2$. Using (3.26) we finally have

$$\int_0^T \|D^2\mathbf{u}^N\|_2^2\,dt \le C \tag{3.53}$$

and we are in the situation of (3.34) with $\widetilde{p}=2$ and $\sigma=1$. Case a) is finished.

Ad b), $p < 1+\frac{2d}{d+2}$. Because $\lambda > 1$, the first term in (3.50) is negative. However, it can be moved across to the right-hand side and estimated by $\frac{1}{\lambda-1}$. Therefore, for $\lambda > 1$ we can dispose of

$$\int_0^T \mathcal{I}_p(\mathbf{u}^N(\tau))\big(1+\|\nabla\mathbf{u}^N(\tau)\|_2^2\big)^{-\lambda}\,d\tau \le C\,. \tag{3.54}$$

Due to different types of the lower estimates of $\mathcal{I}_p(\mathbf{u}^N)$ by means of some norms of the second derivatives of $\mathbf{u}^N$, compare (3.25) with (3.26), we further divide the proof into two subcases:

(i) $p \ge 2$,
(ii) $p < 2$.

In what follows, we expect to find the lower bound for p.

Ad (i). This case is nontrivial only for $d \ge 3$. Let us first show that (3.54) implies

$$\int_0^T \|D^2\mathbf{u}^N\|_2^{2\beta}\,dt \le C \quad \text{with } \beta = \frac{dp-3d+4}{(d-2)p-3d+10}\,. \tag{3.55}$$

Indeed,

$$\begin{aligned}
\int_0^T \|D^2\mathbf{u}^N\|_2^{2\beta}\,dt &\overset{(3.26)}{\le} C\int_0^T \big\{\mathcal{I}_p(\mathbf{u}^N)(1+\|\nabla\mathbf{u}^N\|_2^2)^{-\lambda}\big\}^{\beta} \\
&\qquad \times\quad (1+\|\nabla\mathbf{u}^N\|_2^2)^{\lambda\beta}\,dt \\
&\underset{\text{ineq.}}{\overset{\text{Hölder}}{\le}} C\Big(\int_0^T \mathcal{I}_p(\mathbf{u}^N)(1+\|\nabla\mathbf{u}^N\|_2^2)^{-\lambda}\,dt\Big)^{\beta} \\
&\qquad \times\quad \Big(\int_0^T (1+\|\nabla\mathbf{u}^N\|_2^2)^{\frac{\lambda\beta}{1-\beta}}\,dt\Big)^{1-\beta} \underset{(3.10)}{\overset{(3.54)}{\le}} C\,,
\end{aligned}$$

if

$$\frac{\lambda\beta}{1-\beta} \overset{(3.47)}{=} \frac{2(3-p)}{dp-3d+4}\,\frac{\beta}{1-\beta} = 1 \qquad \Longleftrightarrow$$

$$\beta = \frac{dp-3d+4}{(d-2)p-3d+10} = 1 - \frac{2(3-p)}{(d-2)p-3d+10}\,.$$

Notice that

$$\begin{aligned} &\beta \in \big[\tfrac{1}{3}, \tfrac{1}{2}\big) && \text{for } p \in \big[2, \tfrac{11}{5}\big)\,, && d = 3\,, \\ &\beta \in \big(0, \tfrac{3}{4}\big) && \text{for } p \in \big(\tfrac{3d-4}{d}, 1 + \tfrac{2d}{d+2}\big)\,, && d \geq 4\,. \end{aligned}$$

Thus, we have obtained the lower bound for p if $d \geq 4$.

Having (3.55), we use

$$W^{2,2}(\Omega)^d \hookrightarrow W^{1+s,p}(\Omega)^d \quad \text{with } s = \frac{2d-(d-2)p}{2p} \tag{3.56}$$

and the interpolation inequality (see Lemma 2.18 in Chapter 1)

$$\|\mathbf{v}\|_{1+\sigma,p} \leq c\|\mathbf{v}\|_{1,p}^{1-\frac{\sigma}{s}}\|\mathbf{v}\|_{1+s,p}^{\frac{\sigma}{s}} \tag{3.57}$$

with $0 < \sigma < s$, in order to show that for chosen $r \in (1,p)$,

$$\int_0^T \|\mathbf{u}^N\|_{1+\sigma,p}^r\,dt \leq C \qquad \text{with} \quad \sigma = s\,\frac{2\beta(p-r)}{r(p-2\beta)}\,. \tag{3.58}$$

Once we have (3.58), the estimate of type (3.34), with $\widetilde{p} = r$, follows immediately. Let us therefore verify (3.58). We have that

$$\begin{aligned} \int_0^T \|\mathbf{u}^N\|_{1+\sigma,p}^r\,dt &\underset{(3.56)}{\overset{(3.57)}{\leq}} C\int_0^T \|\mathbf{u}^N\|_{1,p}^{r(1-\frac{\sigma}{s})}\|\mathbf{u}^N\|_{2,2}^{\frac{\sigma r}{s}}\,dt \\ &\underset{\text{ineq.}}{\overset{\text{Hölder}}{\leq}} C\Big(\int_0^T \|\mathbf{u}^N\|_{1,p}^p\Big)^{1/\delta'}\Big(\int_0^T \|\mathbf{u}^N\|_{2,2}^{2\beta}\Big)^{1/\delta} \\ &\underset{(3.55)}{\overset{(3.10)}{\leq}} C\,. \end{aligned}$$

The last inequality holds provided that

$$1 = \frac{1}{\delta} + \frac{1}{\delta'} = \frac{\sigma r}{2\beta s} + \frac{r}{p}\Big(1 - \frac{\sigma}{s}\Big) = \frac{r}{p} + \sigma\frac{r}{s}\Big(\frac{p-2\beta}{2\beta p}\Big)\,,$$

which implies that

$$\sigma = \frac{2\beta s}{r}\,\frac{p-r}{p-2\beta}\,.$$

Thus, (3.58) is proved and the subcase (i) is finished. For $d \geq 4$, the proof of Theorem 3.4 is complete.

Ad (ii), $p < 2$ (which in fact reduces to the case $p \in (\frac{3d}{d+2}, 2)$ for $d = 2, 3$). As a simple consequence of (3.54) and (3.25) we obtain

$$\int_0^T \frac{\|D^2\mathbf{u}^N\|_p^2}{\big(1+\|\nabla\mathbf{u}^N\|_2^2\big)^\lambda \big(1+\|\nabla\mathbf{u}^N\|_p\big)^{2-p}}\, dt \leq C\,. \tag{3.59}$$

Let us first show that (3.59) implies

$$\begin{gathered}\int_0^T \|D^2\mathbf{u}^N\|_p^{2\beta}\, dt \leq C \\ \text{with} \quad \beta = \frac{p\big((d+2)p-3d\big)}{2\big(-p^2+(d+5)p-3d\big)}\,.\end{gathered} \tag{3.60}$$

Indeed, denoting the integrand in (3.59) by $\mathcal{K}(\mathbf{u}^N)$, we have

$$\begin{aligned}
&\int_0^T \|D^2\mathbf{u}^N\|_p^{2\beta}\, dt \\
&= \int_0^T \big(\mathcal{K}(\mathbf{u}^N)\big)^\beta \big(1+\|\nabla\mathbf{u}^N\|_2^2\big)^{\lambda\beta}\big(1+\|\nabla\mathbf{u}^N\|_p\big)^{(2-p)\beta}\, dt \\
&\overset{\text{Hölder}}{\underset{\text{ineq.}}{\leq}} \Big(\int_0^T \mathcal{K}(\mathbf{u}^N)\Big)^\beta \Big(\int_0^T \big(1+\|\nabla\mathbf{u}^N\|_2^2\big)^{\frac{\lambda\beta}{1-\beta}} \\
&\qquad\qquad \times\ \big(1+\|\nabla\mathbf{u}^N\|_p\big)^{(2-p)\frac{\beta}{1-\beta}}\, dt\Big)^{1-\beta} \\
&\overset{(3.59)}{\leq} C\Big(\int_0^T \big(1+\|\nabla\mathbf{u}^N\|_2^2\big)^{\frac{\lambda\beta}{1-\beta}}\big(1+\|\nabla\mathbf{u}^N\|_p\big)^{(2-p)\frac{\beta}{1-\beta}}\, dt\Big)^{1-\beta} \\
&\leq C\Big(\int_0^T \big(1+\|\nabla\mathbf{u}^N\|_p\big)^{(2-p)\frac{\beta}{1-\beta}}\, dt\Big)^{1-\beta} \\
&\quad + C\Big(\int_0^T \big(\|\nabla\mathbf{u}^N\|_2^2\big)^{\frac{\lambda\beta}{1-\beta}}\big(1+\|\nabla\mathbf{u}^N\|_p\big)^{(2-p)\frac{\beta}{1-\beta}}\, dt\Big)^{1-\beta} \\
&\equiv I_1^{1-\beta} + I_2^{1-\beta}\,.
\end{aligned} \tag{3.61}$$

Using the interpolation inequality

$$\|\mathbf{v}\|_2 \leq \|\mathbf{v}\|_p^{\frac{(d+2)p-2d}{2p}}\,\|\mathbf{v}\|_{\frac{dp}{d-p}}^{\frac{d(2-p)}{2p}}$$

and the continuous imbedding $W^{2,p}(\Omega)^d$ into $W^{1,\frac{dp}{d-p}}(\Omega)^d$, we find that I_2 in (3.61) is less than or equal to

$$C\int_0^T \big(1+\|\nabla \mathbf{u}^N\|_p\big)^{\left[(2-p)+\frac{(d+2)p-2d}{p}\lambda\right]\frac{\beta}{1-\beta}}\|D^2\mathbf{u}^N\|_p^{\frac{d(2-p)}{p}\frac{\lambda\beta}{1-\beta}}\,dt$$
$$\underset{\text{ineq.}}{\overset{\text{Hölder}}{\leq}} C\Big(\int_0^T \big(1+\|\nabla \mathbf{u}^N\|_p\big)^p\,dt\Big)^{\frac{1}{\delta}}\Big(\int_0^T \|D^2\mathbf{u}^N\|_p^{2\beta}\,dt\Big)^{\frac{1}{\delta'}}\,, \tag{3.62}$$

as far as

$$\begin{aligned}1=\frac{1}{\delta}+\frac{1}{\delta'}=\Big(\frac{2-p}{p}+\frac{(d+2)p-2d}{p^2}\lambda\Big)\frac{\beta}{1-\beta}\\ +\frac{\lambda}{1-\beta}\frac{d(2-p)}{2p}\,,\end{aligned} \tag{3.63}$$

where λ is defined in (3.47). It follows from (3.63) that

$$\beta=\frac{[(d+2)p-3d]\,p}{2(-p^2+(d+5)p-3d)}=\begin{cases}\frac{(5p-9)p}{2(-p^2+8p-9)} & \text{if } d=3\,,\\ \frac{(2p-3)p}{(p-1)(6-p)} & \text{if } d=2\,.\end{cases} \tag{3.64}$$

Since β must be positive, the lower bound for p, namely

$$p>\frac{3d}{d+2}$$

is obtained. Notice that

$$\begin{aligned}&\beta\in\big(0,\tfrac{1}{2}\big) &&\text{for } p\in\big(\tfrac{3}{2},2\big)\,, && d=2\,,\\ &\beta\in\big(0,\tfrac{1}{3}\big) &&\text{for } p\in\big(\tfrac{9}{5},2\big)\,, && d=3\,.\end{aligned}$$

Integral I_1 is finite, because of $(2-p)\frac{\beta}{1-\beta}\leq p$, and the first *a priori* estimates (3.10).

Thus we can finally conclude using also (3.61) and (3.62),

$$\int_0^T \|D^2\mathbf{u}^N\|_p^{2\beta}\,dt\leq C+\widetilde{C}\Big(\int_0^T \|D^2\mathbf{u}^N\|_p^{2\beta}\,dt\Big)^{\frac{1-\beta}{\delta'}}\,.$$

Applying Young's inequality, we obtain (3.60).

Proceeding similarly to the subcase (i), it is possible to choose $r\in(1,p)$ and to find $\sigma>0$ such that (3.60) implies the required estimate

$$\int_0^T \|\mathbf{u}^N\|_{1+\sigma,p}^r\,dt\leq C \qquad \text{with} \quad \sigma=\frac{2\beta(p-r)}{r(p-2\beta)}\,. \tag{3.65}$$

In order to prove (3.65), we start from the interpolation inequality

$$\|\mathbf{v}\|_{1+\sigma,p}\leq c\,\|\mathbf{v}\|_{1,p}^{1-\sigma}\|\mathbf{v}\|_{2,p}^{\sigma}\,.$$

Using (3.60) we obtain

$$\int_0^T \|\mathbf{u}^N\|_{1+\sigma,p}^r \, dt \le \int_0^T \|\mathbf{u}^N\|_{1,p}^{(1-\sigma)r} \|\mathbf{u}^N\|_{2,p}^{\sigma r} \, dt$$
$$\le \Big(\int_0^T \|\mathbf{u}^N\|_{1,p}^p \, dt \Big)^{1/\delta} \Big(\int_0^T \|\mathbf{u}^N\|_{2,p}^{2\beta} \, dt \Big)^{1/\delta'} \underset{(3.10)}{\overset{(3.60)}{\le}} C$$

provided that $1 = \frac{1}{\delta} + \frac{1}{\delta'} = \frac{(1-\sigma)r}{p} + \frac{\sigma r}{2\beta}$. This easily gives the formula for σ. The subcase (b) is finished.

The proof of Theorem 3.4 is now complete. ■

The following remark contains some other weak formulations equivalent to (3.7) which are satisfied by the weak solution $\mathbf{u}$ of the problem $(\mathrm{NS})_\mathrm{p}$. We will also show that whenever $p \ge \frac{3d}{d+2}$, we have $\mathbf{u} \in C_\omega\big(I; H\big)$ and $\frac{\partial \mathbf{u}}{\partial t} \in L^\gamma\big(I; Y^*\big)$ and whenever $p \ge \max\big(\frac{d+\sqrt{3d^2+4d}}{d+2}, \frac{3d}{d+2}\big)$, we have $\frac{\partial \mathbf{u}}{\partial t} \in L^1\big(I; V_p^*\big)$.

Let us recall that $Y \equiv \big\{\mathbf{u} \in W^{2,2}_{\mathrm{per}}(\Omega)^d; \operatorname{div} \mathbf{u} = 0, \int_\Omega \mathbf{u} \, dx = \mathbf{0}\big\}$. By $C_\omega\big(I; H\big)$ we denote the space of weakly continuous functions with values in H, i.e.,

$$C_\omega\big(I; H\big) \equiv \big\{\mathbf{u} \in L^\infty(I; H); \lim_{t \to t_0} \big(\mathbf{u}(t), \mathbf{h}\big) = \big(\mathbf{u}(t_0), \mathbf{h}\big)\,,$$
$$\forall\, \mathbf{h} \in H, \text{ for a.a. } t_0 \in I\big\}\,.$$

Remark 3.66 In the proof of Theorem 3.4, we have found the following *a priori* estimates:

$$\begin{aligned} \|\mathbf{u}^N\|_{L^p(I;V_p)} &\le C\,, \\ \|\mathbf{u}^N\|_{L^\infty(I;H)} &\le C\,, \\ \Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_{L^\gamma(I;Y^*)} &\le C\,, \qquad \gamma > 1\,, \\ \|\mathbf{u}^N\|_{L^r(I;W^{1+\sigma,r}_{\mathrm{per}}(\Omega)^d)} &\le C\,, \end{aligned} \tag{3.67}$$

which allow us to find a subsequence still labelled $\mathbf{u}^N$ such that

$$\begin{aligned} \mathbf{u}^N &\rightharpoonup \mathbf{u} && \text{weakly in } L^p(I; V_p)\,, \\ \mathbf{u}^N &\overset{*}{\rightharpoonup} \mathbf{u} && \text{weakly-}^* \text{ in } L^\infty(I; H)\,, \\ \mathbf{u}^N &\to \mathbf{u} && \text{strongly in } L^q(I; H)\,, \quad q \ge 1 \text{ arbitrary}\,, \\ \nabla \mathbf{u}^N &\to \nabla \mathbf{u} && \text{strongly in } L^r(I; L^r_{\mathrm{per}}(\Omega)^{d^2})\,, \end{aligned}$$

$r \in (1, p)$ arbitrary, and

$$\frac{\partial \mathbf{u}^N}{\partial t} \rightharpoonup \frac{\partial \mathbf{u}}{\partial t} \quad \text{weakly in } L^\gamma(I; Y^*)\,.$$

Then an equivalent form of (3.7) is also

$$\int_0^T \Big\langle \frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi} \Big\rangle_Y + \int_0^T \int_\Omega \Big[-u_j u_i \frac{\partial \varphi_i}{\partial x_j} + \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big) e_{ij}(\boldsymbol{\varphi}) \Big]\, dx\, dt = \int_0^T (\mathbf{f}, \boldsymbol{\varphi})\, dt \tag{3.68}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(0, T; \mathcal{V})$. Since $\mathcal{V}$ is dense in Y, and the integrals

$$\int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} \varphi_i\, dx\, dt \quad \text{and} \quad \int_0^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big) e_{ij}(\boldsymbol{\varphi})\, dx\, dt$$

are finite[§§] for $\mathbf{u} \in L^\infty\big(I; H\big) \cap L^p\big(I; V_p\big)$ and $\boldsymbol{\varphi} \in \mathcal{D}(0, T; Y)$ if $p \in \big(\frac{3d}{d+2}, \frac{2d}{d-2}\big)$, we obtain another equivalent formulation of (3.7), namely

$$\int_0^T \Big\langle \frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi} \Big\rangle_Y + \int_0^T \int_\Omega \Big[u_j \frac{\partial u_i}{\partial x_j} \varphi_i + \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big) e_{ij}(\boldsymbol{\varphi}) \Big]\, dx\, dt = \int_0^T (\mathbf{f}, \boldsymbol{\varphi})\, dt \tag{3.69}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(0, T; Y)$. Taking in (3.69) $\boldsymbol{\varphi} \in Y$ independent of t, we get

$$\begin{aligned} \int_0^T \left\langle \frac{\partial \mathbf{u}(t)}{\partial t}, \boldsymbol{\varphi} \right\rangle_Y dt + \int_0^T \int_\Omega u_j(t) \frac{\partial u_i(t)}{\partial x_j} \varphi_i\, dx\, dt \\ + \int_0^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}(t))\big) e_{ij}(\boldsymbol{\varphi})\, dx\, dt = \int_0^T \ (\mathbf{f}(t), \boldsymbol{\varphi})\ dt \end{aligned} \tag{3.70}$$

and due to integrability of all terms at least in $L^1(I)$, the weak solution $\mathbf{u} \in L^\infty(I; H) \cap L^p(I; V_p)$ with

$$\frac{\partial \mathbf{u}}{\partial t} \in L^\gamma(I; Y^*)$$

[§§] In order to verify the finiteness of $\int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} \varphi_i\, dx\, dt$, one uses the same approach as in the proof of the second assertion of Lemma 2.44.

satisfies for all $\boldsymbol{\varphi} \in Y$ the identity

$$\begin{aligned} \left\langle \frac{\partial \mathbf{u}(t)}{\partial t}, \boldsymbol{\varphi} \right\rangle_Y + \int_\Omega u_j(t) \frac{\partial u_i(t)}{\partial x_j} \varphi_i \, dx \\ + \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}(t))\big) e_{ij}(\boldsymbol{\varphi}) \, dx = (\mathbf{f}(t), \boldsymbol{\varphi}) \end{aligned} \tag{3.71}$$

for almost all $t \in I$.

Now, let us show that $\mathbf{u}$ satisfying (3.71) belongs to $C_\omega\big(I; H\big)$. Since $\mathbf{u} \in L^\infty(I; H)$, we know that

$$\|\mathbf{u}(t)\|_2 \le C \qquad \forall\, t \in I \setminus S\,, \;\text{ where } |S| = 0\,. \tag{3.72}$$

Integrating (3.71) between $t_0 \in I \setminus S$ and $t \in I \setminus S$, using (2.46) from Chapter 1, we obtain

$$\begin{aligned} (\mathbf{u}(t), \boldsymbol{\varphi}) - (\mathbf{u}(t_0), \boldsymbol{\varphi}) = \int_{t_0}^t (\mathbf{f}(s), \boldsymbol{\varphi}) \; ds \\ - \int_{t_0}^t \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}(s))\big) e_{ij}(\boldsymbol{\varphi}) \, dx \, ds \\ - \int_{t_0}^t \int_\Omega u_j(s) \frac{\partial u_i(s)}{\partial x_j} \varphi_i \, dx \, ds\,. \end{aligned} \tag{3.73}$$

However, the right-hand side of (3.72) vanishes as $t \to t_0$. Therefore,

$$(\mathbf{u}(t), \boldsymbol{\varphi}) \underset{t \to t_0}{\longrightarrow} (\mathbf{u}(t_0), \boldsymbol{\varphi}) \quad \forall\, t_0 \in I \setminus S \text{ and } \forall\, \boldsymbol{\varphi} \in Y\,. \tag{3.74}$$

But due to (3.72), we extend (3.74) also for all $\boldsymbol{\varphi} \in H$. Thus,

$$\mathbf{u} \in C_\omega\big(I; H\big)\,. \tag{3.75}$$

Finally, we want to show the validity of the weak formulation (3.7) for test functions $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; V_p)$. According to the second part of Lemma 2.44, the integral

$$\int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} \varphi_i \, dx \, dt < +\infty \tag{3.76}$$

for $\mathbf{u} \in L^\infty(I; H) \cap L^p(I; V_p)$ and $\boldsymbol{\varphi} \in L^\infty(I; V_p)$ whenever $p \ge \max\big(\frac{d+\sqrt{3d^2+4d}}{d+2}, \frac{3d}{d+2}\big)$. Therefore, we have on the one hand the weak formulation

$$\int_0^T \int_\Omega \Big[- u_i \frac{\partial \boldsymbol{\varphi}}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \varphi_i + \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big) e_{ij}(\boldsymbol{\varphi}) \Big] \, dx \, dt$$
$$= \int_0^T (\mathbf{f}, \boldsymbol{\varphi}) \, dt + (\mathbf{u}_0, \boldsymbol{\varphi}(0))$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; V_p)$. On the other hand, using (3.76), we immediately find that

$$\frac{\partial \mathbf{u}}{\partial t} \in L^1(I; V_p^*) \,. \tag{3.77}$$

The same argument as in the first part of this remark implies that $\mathbf{u} \in C_\omega\big(I; H\big)$. Since $\mathbf{u} \in L^p(I; V_p)$ and $\boldsymbol{\tau}$ satisfies (1.21), the operator $\mathbf{B}$ defined by

$$\langle \mathbf{B}(\mathbf{u}(t)), \boldsymbol{\varphi} \rangle_{V_p} \equiv \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}(t))\big) e_{ij}(\boldsymbol{\varphi}) \, dx \,, \qquad \boldsymbol{\varphi} \in V_p \,,$$

maps V_p onto V_p^*. We conclude: if $p \geq \max\big(\frac{d+\sqrt{3d^2+4d}}{d+2}, \frac{3d}{d+2}\big)$, then weak solutions of the problem $(\mathrm{NS})_\mathrm{p}$ satisfy

$$\mathbf{u} \in C_\omega\big(I; H\big) \cap L^p(I; V_p) \,,$$
$$\frac{\partial \mathbf{u}}{\partial t} \in L^1(I; V_p^*) \,,$$

and for almost all $t \in I$ and all $\boldsymbol{\varphi} \in V_p$ the identity

$$\Big\langle \frac{\partial \mathbf{u}(t)}{\partial t} + u_j(t) \frac{\partial \mathbf{u}(t)}{\partial x_j} + \mathbf{B}(\mathbf{u}(t)), \boldsymbol{\varphi} \Big\rangle_{V_p} = \langle \mathbf{f}(t), \boldsymbol{\varphi} \rangle_{V_p} \tag{3.78}$$

is fulfilled.

5.3.3 Extensions

In this section we will extend the validity of the assertion of Theorem 3.4 to a larger class of initial values and a broader classes of extra stresses. More precisely, we consider two kinds of extensions for the extra stress $\boldsymbol{\tau}^E$. Firstly, we will admit singular nonlinear tensor functions, as, e.g., $\boldsymbol{\tau}(\mathbf{e}(\mathbf{u})) = |\mathbf{e}(\mathbf{u})|^{p-2} \mathbf{e}(\mathbf{u})$, $p \in (1,2)$, using the assumption $(1.8)_1$ instead of $(1.8)_2$ considered in the problem $(\mathrm{NS})_\mathrm{p}$. Secondly, we will also admit perturbations of the constitutive law, cf. (1.58) in Chapter 1. We will assume that the extra stress $\boldsymbol{\tau}^E$ is given by the relation

$$\boldsymbol{\tau}^E(t,x) = \boldsymbol{\tau}\Big(\mathbf{e}\big(\mathbf{u}(t,x)\big)\Big) + \boldsymbol{\sigma}\Big(\mathbf{e}\big(\mathbf{u}(t,x)\big)\Big) \,,$$

where $\boldsymbol{\tau}$ satisfies (1.6), (1.7), $(1.8)_2$ and (1.9) and $\boldsymbol{\sigma}$ is, roughly speaking, an arbitrary tensor function controlled by $\boldsymbol{\tau}$. See (3.88)–(3.90) for precise assumptions on $\boldsymbol{\sigma}$. For $p \in (1,2)$, the assumption $(1.8)_1$ is in this case also possible.

The following theorem shows that the same result as in Theorem 3.4 holds under weaker assumptions on $\mathbf{u}_0$, namely $\mathbf{u}_0 \in H$.

Theorem 3.79 *Let $\mathbf{u}_0 \in H$ and $\mathbf{f}$ satisfy (3.3). Further, let p satisfy (3.5). Then there exists a weak solution $\mathbf{u} : Q_T \longrightarrow \mathbb{R}^d$ to the problem $(NS)_p$ such that*

$$\mathbf{u} \in L^\infty(I;H) \cap L^p(I;V_p)$$

and (3.7) holds for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty,T;\mathcal{V})$.

Proof : The proof coincides basically with the proof of Theorem 3.4 until the inequality (3.48) is derived. This inequality reads

$$\begin{aligned}\frac{1}{1-\lambda}\frac{d}{dt}\big(1+\|\nabla\mathbf{u}^N\|_2^2\big)^{1-\lambda} + C_1\mathcal{I}_p(\mathbf{u}^N)\big(1+\|\nabla\mathbf{u}^N\|_2^2\big)^{-\lambda}& \\ \le C\left(1+\|\nabla\mathbf{u}^N\|_p\right)^p .&\end{aligned} \tag{3.80}$$

Since $\mathbf{u}_0 \notin V_2$ we cannot immediately integrate (3.48) between $(0,t)$, $t \le T$. Let us first define for $\delta > 0$ a cut-off function $\xi \in C^1(I)$ such that $\xi(t) \in [0,1]$ on I and

$$\xi(t) = \begin{cases} 0 & \text{if } t \in [0,\frac{\delta}{2}]\,, \\ 1 & \text{if } t \in [\delta,T]\,, \end{cases}$$

and multiply (3.80) by ξ. We obtain

$$\begin{aligned}\frac{1}{1-\lambda}\frac{d}{dt}\big(\xi\big(1+\|\nabla\mathbf{u}^N\|_2^2\big)^{1-\lambda}\big) + C_1\xi\,\mathcal{I}_p(\mathbf{u}^N)\big(1+\|\nabla\mathbf{u}^N\|_2^2\big)^{-\lambda}& \\ \le C\left(1+\|\nabla\mathbf{u}^N\|_p\right)^p + \frac{1}{1-\lambda}\frac{d\xi}{dt}\big(1+\|\nabla\mathbf{u}^N\|_2^2\big)^{1-\lambda} .&\end{aligned} \tag{3.81}$$

Since $\lambda \le 1$ for $p \ge 1+\frac{2d}{d+2} \ge 2$, the last term in (3.81) is estimated as follows:

$$\frac{1}{1-\lambda}\frac{d\xi}{dt}\big(1+\|\nabla\mathbf{u}^N\|_2^2\big)^{(1-\lambda)} \le \begin{cases} \frac{1}{1-\lambda}\frac{c}{\delta}\big(1+\|\nabla\mathbf{u}^N\|_p^p\big)\,, & \text{if } \lambda < 1\,, \\ \frac{1}{\lambda-1}\frac{c}{\delta}\,, & \text{if } \lambda > 1\,. \end{cases}$$

Therefore, integrating (3.81) between 0 and t, $t \in [\delta,T]$, we obtain

$$\frac{1}{1-\lambda}\big(1+\|\nabla\mathbf{u}^N(t)\|_2^2\big)^{1-\lambda} + \int_\delta^t \mathcal{I}_p(\mathbf{u}^N)\big(1+\|\mathbf{u}^N\|_2^2\big)^{-\lambda}\,ds \le C\,.$$

Since the last inequality coincides with the inequality (3.50), we can proceed in the same way as in the proof of Theorem 3.4 in order to derive (for chosen $\tilde{p} \in (1,p)$ and appropriate $\sigma > 0$) the estimate

$$\int_\delta^T \|\mathbf{u}^N\|_{1+\sigma,\tilde{p}}^{\tilde{p}} \le C\,.$$

As we already know, the last estimate is crucial for proving

$$\int_\delta^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big) e_{ij}(\boldsymbol{\varphi})\,dx\,dt \underset{N\to\infty}{\longrightarrow} \int_\delta^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big) e_{ij}(\boldsymbol{\varphi})\,dx\,dt \tag{3.82}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$. However, $\nabla\mathbf{u}$ and $\nabla\mathbf{u}^N$ are uniformly bounded in $L^p((0,T)\times\Omega)$. Therefore, for all $\varepsilon > 0$ there is a $\delta > 0$ small enough such that for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$, $\|\nabla\boldsymbol{\varphi}\|_{\infty,Q_T} \le 1$,

$$\begin{aligned} &\Big|\int_0^\delta \int_\Omega \big(\tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big) - \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big)\big) e_{ij}(\boldsymbol{\varphi})\,dx\,dt\Big| \\ &\le 2\,C_4 \|\nabla\boldsymbol{\varphi}\|_{\infty,Q_T} \Big(\int_0^\delta \big(1 + \|\nabla\mathbf{u}^N\|_p\big)^p\,dt\Big)^{\frac{p-1}{p}} \big(|\Omega|\delta\big)^{1/p} \\ &\le \varepsilon. \end{aligned} \tag{3.83}$$

Finally, (3.82) and (3.83) imply that the limiting process

$$\int_0^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big) e_{ij}(\boldsymbol{\varphi})\,dx\,dt \longrightarrow \int_0^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big) e_{ij}(\boldsymbol{\varphi})\,dx\,dt$$

can be justified. The proof of Theorem 3.79 is finished. ■

As emphasized in Section 1.1.4, the extra stress $\boldsymbol{\tau}^E$ of the following form:

$$\boldsymbol{\tau}^E = \boldsymbol{\tau}(\mathbf{e}(\mathbf{u})) = |\mathbf{e}(\mathbf{u})|^{p-2}\mathbf{e}(\mathbf{u}) \tag{3.84}$$

with $p \in (1,2)$, plays an important role in modelling flows of non-Newtonian fluids, while the model (3.84) with $p > 2$ is of no interest from the point of view of mechanics of non-Newtonian fluids. However, the assumption $(1.8)_2$ does not allow us to take (3.84) into consideration. Let us therefore replace $(1.8)_2$ by $(1.8)_1$.

Then the following theorem holds.

Theorem 3.85 *Let $\mathbf{u}_0 \in H$ and $\mathbf{f}$ satisfy (3.3). Let us assume that the tensor function $\boldsymbol{\tau}$ satisfies (1.6), (1.7), $(1.8)_1$ and (1.9). Let $p \in \big(\frac{3d}{d+2}, 2\big)$. Then there exists a weak solution $\mathbf{u} : Q_T \longrightarrow \mathbb{R}^d$*

to the problem $(NS)_p$ *such that*

$$\mathbf{u} \in L^\infty(I;H) \cap L^p(I;V_p)\,,$$

and (3.7) holds for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$.

PROOF : If we change the definition of $\mathcal{I}_p(\mathbf{u})$ such that (instead of (3.22)) $\mathcal{I}_p(\mathbf{u})$ is given by

$$\mathcal{I}_p(\mathbf{u}) \equiv \int_\Omega \big|\mathbf{e}(\mathbf{u})\big|^{p-2} e_{ij}\Big(\frac{\partial \mathbf{u}}{\partial x_k}\Big) e_{ij}\Big(\frac{\partial \mathbf{u}}{\partial x_k}\Big)\, dx\,, \tag{3.86}$$

then the proof is identical with the proof of Theorem 3.4. This comes from the fact that the assertions (3.25) and (3.27) of Lemma 3.24 remain valid also for $\mathcal{I}_p(\mathbf{u})$ defined in (3.86). ■

In the remainder of this section we will consider incompressible fluids whose extra stress $\boldsymbol{\tau}^E$ is given by

$$\tau_{ij}^E(t,x) = \tau_{ij}\big(\mathbf{e}(\mathbf{u}(t,x))\big) + \sigma_{ij}\big(\mathbf{e}(\mathbf{u}(t,x))\big) \tag{3.87}$$

for all $t \geq 0$, $x \in \Omega$. Here, we assume again that $\boldsymbol{\tau}$ has a potential, i.e., that the assumptions (1.6)–(1.9) are fulfilled (with the convention that $(1.8)_1$ is considered only for $p \in (1,2)$), and $\boldsymbol{\sigma} : \mathbb{R}^{d^2}_{\text{sym}} \longrightarrow \mathbb{R}^{d^2}_{\text{sym}}$ fulfills

$$\sigma_{ij}(\mathbf{e})\, e_{ij} \geq 0 \qquad \forall\, \mathbf{e} \in \mathbb{R}^{d^2}_{\text{sym}}\,. \tag{3.88}$$

We shall also require that for $i,\, j = 1, \ldots, d$,

$$|\sigma_{ij}(\mathbf{e})| \leq C_{11}\big(1 + |\mathbf{e}|\big)^{p-1}\,. \tag{3.89}$$

Finally, we shall assume that there exists a constant $C_{12} > 0$ such that for $i,\, j,\, k,\, \ell = 1, \ldots, d$,

$$\Big|\frac{\partial \sigma_{k\ell}(\mathbf{e})}{\partial e_{ij}}\Big| \leq C_{12}\big(1 + |\mathbf{e}|\big)^{p-2}\,. \tag{3.90}$$

Now, we will define the corresponding extended problem and give a definition of weak solutions to it. Let $\mathbf{f} : Q_T \longrightarrow \mathbb{R}^d$ and $\mathbf{u}_0 : \Omega \longrightarrow \mathbb{R}^d$ be given and satisfy for some $p \in (1,\infty)$ the assumptions (1.6), (1.7), (1.9), (3.88)–(3.90) and $(1.8)_2$. (If $p \in (1,2)$, $(1.8)_2$ can be replaced by $(1.8)_1$.)

We look for $\mathbf{u} : Q_T \longrightarrow \mathbb{R}^d$ and $\pi : Q_T \longrightarrow \mathbb{R}$ solving in Q_T

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} + u_k \frac{\partial \mathbf{u}}{\partial x_k} &= -\nabla \pi + \operatorname{div} \boldsymbol{\tau}(\mathbf{e}(\mathbf{u})) + \operatorname{div} \boldsymbol{\sigma}(\mathbf{e}(\mathbf{u})) + \mathbf{f}\,, \\ \operatorname{div} \mathbf{u} &= 0\,,\end{aligned} \tag{3.91}$$

and satisfying the initial condition

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0 \qquad \text{in } \Omega\,, \tag{3.92}$$

and the space-periodicity requirements

$$\begin{gathered} \mathbf{u}|_{\Gamma_j} = \mathbf{u}|_{\Gamma_{j+d}}, \qquad \nabla\mathbf{u}|_{\Gamma_j} = \nabla\mathbf{u}|_{\Gamma_{j+d}}\,, \\ \pi|_{\Gamma_j} = \pi|_{\Gamma_{j+d}}\;. \end{gathered} \tag{3.93}$$

$j = 1,\ldots,d$. We will refer to the problem (3.91)–(3.93) as the **problem (NSext)$_\mathrm{p}$**.

Let us suppose that

$$\mathbf{u}_0 \in H \quad \text{ and } \quad \mathbf{f} \in \begin{cases} L^2(Q_T)^d & \text{if } p \geq 2\,, \\ L^{p'}(Q_T)^d & \text{if } p < 2\,. \end{cases} \tag{3.94}$$

Definition 3.95 *Let $\mathbf{u}_0$, $\mathbf{f}$ satisfy (3.94). A function*

$$\mathbf{u} \in C_\omega\big(I;H\big) \cap L^p(I;V_p)$$

with

$$\frac{\partial \mathbf{u}}{\partial t} \in L^1(I;Y^*)$$

is called a **weak solution** *to the problem (NSext)$_p$ if the identity*

$$\begin{aligned} \Big\langle \frac{\partial \mathbf{u}(t)}{\partial t}, \boldsymbol{\varphi}\Big\rangle_Y + \int_\Omega u_k(t)\frac{\partial u_i(t)}{\partial x_k}\varphi_i + \tau_{ij}\big(\mathbf{e}(\mathbf{u}(t))\big)e_{ij}(\boldsymbol{\varphi})\,dx \\ + \int_\Omega \sigma_{ij}\big(\mathbf{e}(\mathbf{u}(t))\big)e_{ij}(\boldsymbol{\varphi})\,dx = \int_\Omega f_i(t)\varphi_i\,dx \end{aligned} \tag{3.96}$$

is fulfilled almost everywhere in I and for all $\boldsymbol{\varphi} \in Y$. As before, $Y \equiv \big\{\mathbf{u} \in W^{2,2}_{\mathrm{per}}(\Omega)^d; \operatorname{div}\mathbf{u} = 0, \int_\Omega \mathbf{u}\,dx = \mathbf{0}\big\}$.

A general result is formulated in the following theorem.

Theorem 3.97 *Let $\mathbf{u}_0$, $\mathbf{f}$ satisfy (3.94). Let p satisfy (3.5). Let also*

$$C_{12}d^2 < C_1\,. \tag{3.98}$$

Then there exists a weak solution to the problem (NSext)$_p$.

PROOF : The proof follows the same lines as the proof of Theorem 3.4 except for small modifications. First let us note that we can assume $\mathbf{u}_0 \in V_2$ because of Theorem 3.79 and its proof. Even if the Galerkin system now contains a new term, namely

$\int_\Omega \sigma_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big)e_{ij}(\boldsymbol{\omega}^r)\,dx$, we notice that the first *a priori* inequality coincides with (2.27) due to the assumption (3.88). Also, the second energy inequality has the same form as (3.19), except for the coefficient in front of $\mathcal{I}_p(\mathbf{u}^N)$, where instead of C_1 we have $C_1 - C_{12}d^2$. This coefficient is positive due to (3.98).

The proof of Theorem 3.97 is finished. ■

5.4 Incompressible non-Newtonian fluids and strong solutions

We continue in the analysis of the properties of weak solutions $\mathbf{u}$ to the problem $(\mathrm{NS})_\mathrm{p}$; the existence of these has been proved for $p > \frac{3d}{d+2}$ in the previous section, if $d \le 4$. The aim of this section is to present three types of results:

1. To prove the global existence of strong solution (for $p \ge \frac{11}{5}$ if $d = 3$ and for $p > 1$ if $d = 2$) and its uniqueness for $p \ge 1 + \frac{2d}{d+2}$. A function $\mathbf{u}$ is said to be a **strong solution** if $\mathbf{u}$ is at least a weak solution to the problem $(\mathrm{NS})_\mathrm{p}$ and if $\mathbf{u}$ satisfies the following semiregularity properties:

$$\mathbf{u} \in C(I;H) \cap L^\infty(I;V_2) \cap L^2(I;Y)\,, \tag{4.1}$$

$$\mathbf{u} \in L^\infty(I;V_p)\,, \tag{4.2}$$

$$\frac{\partial \mathbf{u}}{\partial t} \in L^2(I;H)\,. \tag{4.3}$$

   Recall that $Y \equiv \{\mathbf{u} \in W^{2,2}_{\mathrm{per}}(\Omega)^d; \operatorname{div}\mathbf{u} = 0, \int_\Omega \mathbf{u}\,dx = \mathbf{0}\}$. If $p \in (1,2)$ then (4.1) is replaced by

$$\mathbf{u} \in C(I;H) \cap L^\infty(I;V_2) \cap L^2(I;W^{2,p}_{\mathrm{per}}(\Omega)^d \cap V_p)\,. \tag{4.4}$$

   In order to prove (4.1) and (4.4), the assumption $\mathbf{u}_0 \in V_2$ is needed, while for proving (4.2) and (4.3) we will have to assume $\mathbf{u}_0 \in V_p$.
2. To find conditions on the size of data (resp. on the length of time interval) under which the global in time existence of strong solutions for small data (resp. local in time existence of strong solutions for arbitrary data) can be proved. This problem will be studied for $p < \frac{11}{5}$ and $d = 3$ only, since otherwise strong solutions exist without any restriction on data. We will prove both kinds of conditional existence results for $p > \frac{5}{3}$.

3. To find fractional estimates for the L^2-norm of the time derivative of $\mathbf{u}^N$ as well as for the L^p-norm of the second derivatives of $\mathbf{u}^N$ for $p \in (\frac{3d}{d+2}, 1 + \frac{2d}{d+2})$.

All assertions and proofs will basically deal with two- or three-dimensional domains.

5.4.1 Global existence of strong solutions and uniqueness

This section consists of four theorems and their proofs. The first one, Theorem 4.5, deals with strong solutions to the problem $(\mathrm{NS})_\mathrm{p}$ if $p \geq 1 + \frac{2d}{d+2}$. Under the assumption $\mathbf{u}_0 \in V_p$ we will prove (4.1)–(4.3) without any restrictions on the size of data.

The next theorem, Theorem 4.21, strengthens the result in two dimensions. It will be shown that a strong solution exists for all $p > 1$. The space-periodic setting is essential for the proof.

The remaining two theorems give an answer to the following question: *When is a weak solution unique?* If $\mathbf{u}_0 \in H$, then Theorem 4.29 gives uniqueness for $p \geq \frac{d+2}{2}$. If $\mathbf{u}_0 \in V_2$, then the weak solution is unique according to Theorem 4.37 for $p \geq 1 + \frac{2d}{d+2}$. Notice that the lower bounds on p are the same in two space dimensions.

Theorem 4.5 *Let $\mathbf{u}_0 \in V_p$, $\mathbf{f} \in L^2(Q_T)$ and $p \geq 1 + \frac{2d}{d+2}$. Then the weak solution $\mathbf{u}$ of the problem $(NS)_p$ has the semiregularity properties (4.1)–(4.3), i.e. $\mathbf{u}$ is a strong solution.*

Remark 4.6 It is worth focusing upon changes of the assertion of Theorem 4.5 if the assumption on $\mathbf{u}_0$ is weakened. Assuming $\mathbf{u}_0 \in H$, we obtain

$$\frac{\partial \mathbf{u}}{\partial t} \in L^{p'}(I; V_p^*) \tag{4.7}$$

and by Lemma 2.45 from Chapter 1 and (3.6),

$$\mathbf{u} \in C(I; H)\,. \tag{4.8}$$

The other regularity properties in (4.1)–(4.3) are still valid provided that the interval I is replaced by $I_\delta \equiv [\delta, T)$, $\delta > 0$ arbitrary.

Finally, as far as $\mathbf{u}_0 \in V_2$ and $\mathbf{u}_0 \notin V_p$, (4.1) holds without any change, while (4.2) and (4.3) requires I to be replaced by I_δ. These modifications of Theorem 4.5 can be proved by means of the cut-off function technique explained in the proof of Theorem 3.79.

PROOF (of Theorem 4.5): Using the Galerkin approximations $\{\mathbf{u}^N\}$ defined in (2.21), we derived the first and second *a priori* estimates, see (3.9), (3.10), (3.51) and (3.53),

$$\|\mathbf{u}^N\|_{L^\infty(I;H)} + \|\mathbf{u}^N\|_{L^p(I;V_p)} \le C\,, \tag{4.9}$$

$$\|\mathbf{u}^N\|_{L^2(I;Y)} + \|\mathbf{u}^N\|_{L^\infty(I;V_2)} \le C\,, \tag{4.10}$$

which have been essential for proving the existence of weak solution $\mathbf{u}$ to the problem $(\mathrm{NS})_\mathrm{p}$. Since $\mathbf{u}$ was found as a weak limit of $\mathbf{u}^N$ and since balls in Banach spaces are weakly closed, we see that the *a priori* estimates (4.9) and (4.10) are valid also for $\mathbf{u}$. Thus,

$$\mathbf{u} \in L^\infty(I;H) \cap L^p(I;V_p) \cap L^\infty(I;V_2) \cap L^2(I;Y)\,. \tag{4.11}$$

According to the third part of Lemma 2.44, $u_j \frac{\partial \mathbf{u}}{\partial x_j} \in L^{p'}(I;V_p^*)$. Consequently, $\frac{\partial \mathbf{u}}{\partial t} \in L^{p'}(I;V_p^*)$ and by Lemma 2.45 in Chapter 1,

$$\Big\{\mathbf{u} \in L^p(I;V_p);\, \frac{\partial \mathbf{u}}{\partial t} \in L^{p'}(I;V_p^*)\Big\} \hookrightarrow C(I;H)\,. \tag{4.12}$$

Thus, (4.1) is proved.

In order to demonstrate (4.2) and (4.3), we will derive the third kind of *a priori* estimate for $\{\mathbf{u}^N\}$. For that purpose, multiply (2.21) by $\frac{d}{dt}c_r^N(t)$ and sum up over $r = 1,\dots,N$. As a result we obtain

$$\begin{aligned} \Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_2^2 &+ \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big)\frac{\partial}{\partial t} e_{ij}(\mathbf{u}^N)\,dx \\ &= -\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}\frac{\partial u_i^N}{\partial t}\,dx + \Big(\mathbf{f}, \frac{\partial \mathbf{u}^N}{\partial t}\Big)\,. \end{aligned} \tag{4.13}$$

Applying the Schwarz inequality to both terms at the right-hand side of (4.13) and using (1.6), we get

$$\begin{aligned} \frac{1}{2}\Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_2^2 &+ \frac{d}{dt}\int_\Omega U\big(\mathbf{e}(\mathbf{u}^N)\big)\,dx \\ &\le c\,\|\mathbf{f}\|_2^2 + c\int_\Omega |\mathbf{u}^N|^2 |\nabla \mathbf{u}^N|^2\,dx\,. \end{aligned} \tag{4.14}$$

Let us denote by I the integral $\int_\Omega |\mathbf{u}^N|^2|\nabla \mathbf{u}^N|^2\,dx$.

If $d = p = 2$, then using $\|\mathbf{v}\|_\infty \le c\|\mathbf{v}\|_2^{1/2}\|D^{(2)}\mathbf{v}\|_2^{1/2}$, we obtain

$$I \le \|\mathbf{u}^N\|_\infty^2 \|\nabla \mathbf{u}^N\|_2^2 \le c\|\mathbf{u}^N\|_2 \|D^{(2)}\mathbf{u}^N\|_2 \|\nabla \mathbf{u}^N\|_2^2\,.$$

Thus, due to (4.9) and (4.10), $\int_0^T I\,d\tau \le C$.

For $p \geq d$ ($p > 2$ if $d = 2$) we can estimate I as follows:

$$\begin{aligned}
\int_0^T I\,dt \overset{\text{Hölder}}{\underset{\text{ineq.}}{\leq}} & \int_0^T \|\mathbf{u}^N\|^2_{\frac{2p}{p-2}} \|\nabla \mathbf{u}^N\|_p^2\,dt \\
& \leq c \int_0^T \|\nabla \mathbf{u}^N\|_2^2 \|\nabla \mathbf{u}^N\|_p^2\,dt \qquad (4.15)\\
& \overset{(4.10)}{\leq} c \int_0^T \|\nabla \mathbf{u}^N\|_p^2\,dt \overset{(4.9)}{\leq} C\,,
\end{aligned}$$

where the second inequality expresses the imbedding $W^{1,2}(\Omega)^d \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)^d$ valid for $\frac{2p}{p-2} \leq \frac{2d}{d-2}$, which is $p \geq d$.

It remains to estimate I for $d = 3$ and $p \in [\frac{11}{5}, 3)$. We know (see (3.52)) that

$$\int_0^T \mathcal{I}_p(\mathbf{u}^N)\,dt \leq C\,. \qquad (4.16)$$

On the other hand, taking particularly $q = 2$ in (3.27), we have

$$\|\nabla \mathbf{u}^N\|_{3p}^p \leq \mathcal{I}_p(\mathbf{u}^N)\,. \qquad (4.17)$$

Hence

$$\begin{aligned}
\int_0^T I\,dt \overset{\text{Hölder}}{\underset{\text{ineq.}}{\leq}} & \int_0^T \|\nabla \mathbf{u}^N\|_{3p}^2 \|\mathbf{u}^N\|^2_{\frac{6p}{3p-2}}\,dt \\
& \leq c \int_0^T \|\nabla \mathbf{u}^N\|_2^2 \|\nabla \mathbf{u}^N\|_{3p}^2\,dt \qquad (4.18)\\
& \overset{(4.10)}{\leq} c\|\mathbf{u}^N\|_{L^\infty(I;V_2)} \int_0^T \|\nabla \mathbf{u}^N\|_{3p}^2\,dt \overset{(4.16)}{\underset{(4.17)}{\leq}} C\,.
\end{aligned}$$

Considering (4.14), (4.15) and (4.18), we obtain for all $t \leq T$

$$\int_0^t \Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_2^2\,dt + \int_\Omega U\big(\mathbf{e}(\mathbf{u}^N(t))\big)\,dx \leq C\,. \qquad (4.19)$$

Due to (1.37) and Korn's inequality (1.11),

$$\|\nabla \mathbf{u}^N(t)\|_p \leq C \qquad \forall\, t \leq T\,.$$

Therefore,

$$\begin{aligned}
\Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_{L^2(I;H)} &\leq C\,, \\
\|\mathbf{u}^N\|_{L^\infty(I;V_p)} &\leq C\,,
\end{aligned} \qquad (4.20)$$

and the same estimate is valid also for $\mathbf{u}$. Thus (4.2)–(4.3) hold. ■

Theorem 4.21 *Let $\mathbf{u}_0 \in V_p \cap V_2$, $\mathbf{f} \in L^2(Q_T)$ for $p \geq 2$ and $\mathbf{f} \in L^{p'}(Q_T)$ if $p < 2$. Provided that Ω is a two-dimensional cube, a strong solution to the problem $(NS)_p$ exists for all $p > 1$.*

PROOF : If $p \geq 2$ then the result follows from Theorem 4.5. Let $p \in (1, 2)$ in the sequel. According to (3.20)–(3.21),

$$\int_\Omega \frac{\partial u_j^N}{\partial x_k} \frac{\partial u_i^N}{\partial x_j} \frac{\partial u_i^N}{\partial x_k} \, dx = 0 \, .$$

Then (3.19) turns into

$$\frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}^N\|_2^2 + C_1 \mathcal{I}_p(\mathbf{u}^N) \leq |(\mathbf{f}, \Delta \mathbf{u}^N)| \, . \tag{4.22}$$

Integrating (4.22) between 0 and t and using Remark 3.29, we obtain for almost all $t \in I$,

$$\|\nabla \mathbf{u}^N(t)\|_2^2 + C \int_0^t \mathcal{I}_p(\mathbf{u}^N) \, d\tau \leq \|\nabla \mathbf{u}_0\|_2^2 + C \|\mathbf{f}\|_{L^{p'}(Q_T)}^2 \, . \tag{4.23}$$

This implies

$$\|\mathbf{u}^N\|_{L^\infty(I;V_2)} \leq C \tag{4.24}$$

and

$$\int_0^T \mathcal{I}_p(\mathbf{u}^N) \, dt \leq C \, . \tag{4.25}$$

By (3.25), (4.24) and (4.25),

$$\int_0^T \|D^{(2)} \mathbf{u}^N\|_p^2 \, dt \leq C \, . \tag{4.26}$$

Next,

$$\begin{aligned} \int_0^T \int_\Omega |\mathbf{u}^N|^2 |\nabla \mathbf{u}^N|^2 \, dx \, dt &\leq \int_0^T \|\nabla \mathbf{u}^N\|_{\frac{2p}{2-p}}^2 \|\mathbf{u}^N\|_{\frac{p}{p-1}}^2 \, dt \\ &\leq \int_0^T \|D^{(2)} \mathbf{u}^N\|_p^2 \|\nabla \mathbf{u}^N\|_2^2 \, dt \underset{(4.26)}{\overset{(4.24)}{\leq}} C \, . \end{aligned}$$

Then it follows from (4.14) that

$$\left\| \frac{\partial \mathbf{u}^N}{\partial t} \right\|_{L^2(I;L^2(\Omega)^d)} \leq C \, . \tag{4.27}$$

From Aubin-Lions Lemma 2.48 in Chapter 1, using (4.26), (4.27), we obtain the strong convergence

$$\nabla \mathbf{u}^N \to \nabla \mathbf{u} \quad \text{ in } L^p(Q_T)\,.$$

Thus $\mathbf{u}$ is a weak solution. Clearly, by Lemma 2.45 in Chapter 1, we have $\mathbf{u} \in C(I;H)$. Moreover, $\mathbf{u}$ satisfies all estimates (4.24), (4.26), (4.27). The proof of Theorem 4.21 is complete. ■

Now, we shall prove two uniqueness theorems under different assumptions on the regularity of the initial value $\mathbf{u}_0$. If $p \geq 1+\frac{2d}{d+2}$, then a weak solution $\mathbf{u}$ to the problem $(\mathrm{NS})_\mathrm{p}$ satisfies the weak formulation in the form

$$\begin{aligned} \Big\langle \frac{\partial \mathbf{u}(t)}{\partial t} + u_j(t)\frac{\partial \mathbf{u}(t)}{\partial x_j}, \boldsymbol{\varphi}\Big\rangle_{V_p} &+ \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}(t))\big)e_{ij}(\boldsymbol{\varphi})\,dx \\ &= (\mathbf{f}(t), \boldsymbol{\varphi}) \end{aligned} \tag{4.28}$$

for almost all $t \in I$ and for all $\boldsymbol{\varphi} \in V_p$, which follows from (3.78) and the semiregularity properties (4.1)–(4.3).

Theorem 4.29 *Let $\mathbf{u}_0 \in H$, $\mathbf{f} \in L^{p'}(Q_T)$. If*

$$p \geq \frac{d+2}{2}\,, \tag{4.30}$$

then there exists a unique weak solution to the problem $(NS)_p$.

Proof : Let us suppose that there exist two weak solutions $\mathbf{u}$, $\mathbf{v}$ to the problem $(\mathrm{NS})_\mathrm{p}$ with the same initial value $\mathbf{u}_0$. Put $\mathbf{w} \equiv \mathbf{u}-\mathbf{v}$. Subtracting the weak formulation for $\mathbf{v}$ from the one for $\mathbf{u}$ and taking $\mathbf{w}(t)$ as the test function in the resulting equation, we obtain¶¶

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_2^2 + \int_\Omega \Big(\tau_{ij}\big(\mathbf{e}(\mathbf{u})\big) - \tau_{ij}\big(\mathbf{e}(\mathbf{v})\big)\Big)e_{ij}(\mathbf{w})\,dx \\ = -\int_\Omega \Big(u_j\frac{\partial u_i}{\partial x_j} - v_j\frac{\partial v_i}{\partial x_j}\Big)w_i\,dx \\ = -\int_\Omega w_j\frac{\partial u_i}{\partial x_j}w_i\,dx - \int_\Omega v_j\frac{\partial w_i}{\partial x_j}w_i\,dx \\ = -\int_\Omega w_j\frac{\partial u_i}{\partial x_j}w_i\,dx\,, \end{aligned} \tag{4.31}$$

¶¶ We will not write the explicit dependence of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ on t.

where we have used the fact (see (2.12)) that

$$\int_\Omega v_j \frac{\partial w_i}{\partial x_j} w_i \, dx = 0.$$

By Hölder's inequality,

$$\Big| \int_\Omega w_j \frac{\partial u_i}{\partial x_j} w_i \, dx \Big| \le \int_\Omega |\mathbf{w}|^2 |\nabla \mathbf{u}| \, dx \le \|\nabla \mathbf{u}\|_p \|\mathbf{w}\|^2_{\frac{2p}{p-1}} . \tag{4.32}$$

Combining (4.31), (4.32), (1.25) and (1.11), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_2^2 + K_2^2 C_6 \|\nabla \mathbf{w}\|_2^2 \le \|\nabla \mathbf{u}\|_p \|\mathbf{w}\|^2_{\frac{2p}{p-1}} .$$

Because of the interpolation inequality (4.36) proved in Lemma 4.35 below, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_2^2 + K_2^2 C_6 \|\nabla \mathbf{w}\|_2^2 &\le \|\nabla \mathbf{u}\|_p \|\mathbf{w}\|_2^{\frac{2p-d}{p}} \|\nabla \mathbf{w}\|_2^{\frac{d}{p}} \\ &\overset{\text{Young}}{\underset{\text{ineq.}}{\le}} K_2^2 C_6 \|\nabla \mathbf{w}\|_2^2 \\ &\qquad + C \|\nabla \mathbf{u}\|_p^{\frac{2p}{2p-d}} \|\mathbf{w}\|_2^2 . \end{aligned} \tag{4.33}$$

Finally,

$$\frac{d}{dt}\|\mathbf{w}\|_2^2 \le C \|\nabla \mathbf{u}\|_p^{\frac{2p}{2p-d}} \|\mathbf{w}\|_2^2 . \tag{4.34}$$

Since $\mathbf{w}(0) = \mathbf{0}$ and $\mathbf{u} \in L^p(I; V_p)$, we can use Gronwall's lemma 3.5 from the Appendix if $\frac{2p}{2p-d} \le p$, which is equivalent to (4.30). As a conclusion of Gronwall's lemma we obtain

$$\|\mathbf{w}(t)\|_2 = 0 \qquad \forall\, t \le T .$$

Thus, $\mathbf{u}(t) = \mathbf{v}(t)$ almost everywhere in I.

Theorem 4.31 is proved. ■

Lemma 4.35 *Let $\mathbf{v} \in W^{1,2}_{\mathrm{per}}(\Omega)$ and $q \in [2, \frac{2d}{d-2}]$ for $d \ge 3$ and $q \in (2, +\infty)$ if $d = 2$. Then there exists $c > 0$ such that*

$$\|\mathbf{v}\|_q \le c \|\mathbf{v}\|_2^\alpha \|\nabla \mathbf{v}\|_2^{1-\alpha} , \tag{4.36}$$

with $\alpha = \frac{2d - q(d-2)}{2q}$.

PROOF : First let $d \geq 3$. Then $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2d}{d-2}}(\Omega)$ and therefore there exists $c > 0$ such that $\|\mathbf{v}\|_{\frac{2d}{d-2}} \leq c\|\nabla\mathbf{v}\|_{1,2}$. Combining this with the interpolation inequality (see Corollary 2.10 in Chapter 1)

$$\|\mathbf{v}\|_q \leq c\|\mathbf{v}\|_2^{\alpha}\|\mathbf{v}\|_{\frac{2d}{d-2}}^{1-\alpha}, \quad \alpha = \frac{2d-q(d-2)}{2q},$$

we obtain the assertion (4.36).

If $d = 2$, then $W^{1,2}(\Omega) \not\hookrightarrow L^\infty(\Omega)$. Nevertheless, for $s = \frac{q-2}{q}$, $W^{s,2}(\Omega) \hookrightarrow L^q(\Omega)$ and $\|\mathbf{v}\|_q \leq c\|\mathbf{v}\|_{s,2}$. Due to Lemma 2.18 in Chapter 1, we have

$$\|\mathbf{v}\|_{s,2} \leq c\|\mathbf{v}\|_2^{\alpha}\|\mathbf{v}\|_{1,2}^{1-\alpha}, \quad \alpha = \frac{2}{q}.$$

Putting both inequalities together gives (4.36). ■

Theorem 4.37 *Let $\mathbf{u}_0 \in V_2$, $\mathbf{f} \in L^{p'}(Q_T)$. If*

$$p \geq 1 + \frac{2d}{d+2}, \tag{4.38}$$

then there exists a unique weak solution to the problem $(NS)_p$.

Remark 4.39 In two dimensions, the lower bound $1 + \frac{2d}{d+2}$ coincides with $\frac{d+2}{2}$ and the uniqueness of a weak solution follows from Theorem 4.29. Thus, the assertion of Theorem 4.37 is new only for $d \geq 3$.

PROOF (of Theorem 4.37): Let us consider three-dimensional domains Ω only. Since $\mathbf{u}_0 \in V_2$, we know (see (4.1)) that

$$\nabla\mathbf{u} \in L^\infty(I; V_2). \tag{4.40}$$

Following the proof of Theorem 4.29, we can start with the inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_2^2 + K_2^2 C_6\|\nabla\mathbf{w}\|_2^2 &\overset{(4.36)}{\leq} \|\nabla\mathbf{u}\|_2\|\mathbf{w}\|_2^{\frac{1}{2}}\|\nabla\mathbf{w}\|_2^{\frac{3}{2}} \\ &\underset{\text{ineq.}}{\overset{\text{Young}}{\leq}} K_2^2 C_6\|\nabla\mathbf{w}\|_2^2 + C\|\nabla\mathbf{u}\|_2^4\|\mathbf{w}\|_2^2 .\end{aligned}$$

Hence,

$$\frac{d}{dt}\|\mathbf{w}\|_2^2 \leq \|\nabla\mathbf{u}\|_2^4\|\mathbf{w}\|_2^2$$

together with (4.40) and Gronwall's lemma finishes the proof. ■

5.4.2 Existence of a strong solution under some restriction on data

This section deals with the conditional existence of strong solutions in three dimensions. We will first show the existence of such solutions for $p \in (\frac{5}{3}, \frac{11}{5})$ provided that some norms of $\mathbf{f}$ and $\mathbf{u}_0$ are small enough. Later, we will formulate an assertion saying that it is possible to find a (short) time interval on which strong solutions exist for arbitrarily large $\mathbf{f}$ and $\mathbf{u}_0$. Throughout this section, we will assume that

$$\mathbf{u}_0 \in V_p \cap V_2 \quad \text{and} \quad \mathbf{f} \in \begin{cases} L^2(Q_T) & \text{if } p \geq 2\,, \\ L^{p'}(Q_T) & \text{if } p < 2\,. \end{cases} \tag{4.41}$$

The method of the proof is again based on the second *a priori* estimate which finally reads (compare with (3.48))

$$\begin{aligned} &\frac{1}{1-\lambda}(1 + \|\nabla \mathbf{u}^N(t)\|_2^2)^{1-\lambda} \\ &\qquad + \int_0^t \mathcal{I}_p(\mathbf{u}^N)(1 + \|\mathbf{u}^N\|_2^2)^{-\lambda}\, d\tau \\ &\qquad \leq C \int_0^t \|\nabla \mathbf{u}^N\|_p^p\, d\tau + c_{\mathbf{f}} + \frac{1}{1-\lambda}(1 + \|\nabla \mathbf{u}_0\|_2^2)^{1-\lambda}\,. \end{aligned} \tag{4.42}$$

By (3.47), we know that $\lambda = \frac{2(3-p)}{3p-5} > 1$ for $p < \frac{11}{5}$. For 'global' estimates the negative term

$$\frac{1}{1-\lambda}(1 + \|\nabla \mathbf{u}^N(t)\|_2^2)^{1-\lambda} \tag{4.43}$$

was moved to the left-hand side and estimated by $\frac{1}{\lambda-1}$ there. In order to obtain the results mentioned above, the term (4.43) will enter essentially into the proof.

More precisely, we rewrite (4.42) as follows:

$$\begin{aligned} \frac{1}{\lambda-1}\frac{1}{(1 + \|\nabla \mathbf{u}_0\|_2^2)^{\lambda-1}} \leq & c_p \int_0^t \|\nabla \mathbf{u}^N\|_p^p\, d\tau + c_{\mathbf{f}} \\ & + \frac{1}{\lambda-1}\frac{1}{(1 + \|\nabla \mathbf{u}^N(t)\|_2^2)^{\lambda-1}}\,. \end{aligned} \tag{4.44}$$

The first *a priori* estimates, see (2.28), give the existence of a constant c_a depending on $\|\mathbf{u}_0\|_2$, $\|\mathbf{f}\|_{L^{p'}(Q_T)}$, T and $|\Omega|$ such that

$$\int_0^T \|\nabla \mathbf{u}^N\|_p^p\, dt \leq c_a\,. \tag{4.45}$$

Then it follows from (4.44) that

$$\|\nabla \mathbf{u}^N(t)\|_2^2 \leq \frac{1+\|\nabla \mathbf{u}_0\|_2^2}{\left[1-(c_p c_a + c_{\mathbf{f}})(\lambda-1)(1+\|\nabla \mathbf{u}_0\|_2^2)^{\lambda-1}\right]^{1/(\lambda-1)}}\,. \tag{4.46}$$

Now, whenever

$$c_p c_a + c_{\mathbf{f}} < \frac{1}{(\lambda-1)(1+\|\nabla \mathbf{u}_0\|_2^2)^{\lambda-1}} \tag{4.47}$$

then we will get an uniform bound of $\{\mathbf{u}^N\}$ in $L^\infty(I;V_2)$, which is a springboard to a proof of the existence of a strong solution. Since this last statement has been clarified for example in the proof of Theorem 4.21, the remaining task is to show that c_a can be made as small as needed and depending only on norms of $\mathbf{u}_0$ and $\mathbf{f}$ (and not on $|\Omega|$!).

The reason why Ω appears at the right-hand side of (2.28) comes from the coercivity condition (1.20). If one uses the condition (1.23) for $p \in (1,2)$ or (1.24) for $p \geq 2$ instead of $(1.20)_2$, the dependence of the right-hand side of (2.28) on $|\Omega|$ can be removed. Let us illustrate it only for the more complicated case, $p \in (1,2)$. Then we have from (2.26) and (1.23) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^N\|_2^2 + c_3\int_\Omega \min\left(|\mathbf{e}(\mathbf{u}^N)|^2, |\mathbf{e}(\mathbf{u}^N)|^p\right)dx&\\ \leq \int_\Omega |\mathbf{f}\cdot\mathbf{u}^N|\,dx\,.&\end{aligned} \tag{4.48}$$

Denoting $B^N \equiv \{x \in \Omega, |\mathbf{e}(\mathbf{u}^N)| \leq 1\}$, $C^N \equiv \{x \in \Omega, |\mathbf{e}(\mathbf{u}^N)| > 1\}$ we obtain from (4.48) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^N\|_2^2 + c_3\int_{B^N}|\mathbf{e}(\mathbf{u}^N)|^2\,dx + c_3\int_{C^N}|\mathbf{e}(\mathbf{u}^N)|^p\,dx\\ &\leq c\|\mathbf{f}\|_{p'}\left(\|\mathbf{e}(\mathbf{u}^N)\|_{p,B^N} + \|\mathbf{e}(\mathbf{u}^N)\|_{p,C^N}\right)\\ &\leq c\|\mathbf{f}\|_{p'}^{p'} + \frac{c_3}{2}\|\mathbf{e}(\mathbf{u}^N)\|_{p,C^N}^p + c\|\mathbf{f}\|_{p'}^2 + \frac{c_3}{2}\|\mathbf{e}(\mathbf{u}^N)\|_{2,B^N}^2\end{aligned}$$

and thus

$$\begin{aligned}\|\mathbf{u}^N(t)\|_2^2 + c_3\int_0^t\left(\|\mathbf{e}(\mathbf{u}^N)\|_{p,C^N}^p + \|\mathbf{e}(\mathbf{u}^N)\|_{2,B^N}^2\right)ds&\\ \leq \|\mathbf{u}_0\|_2^2 + c\int_0^T \|\mathbf{f}\|_{p'}^{p'} + \|\mathbf{f}\|_{p'}^2\,dt \equiv c(\mathbf{u}_0,\mathbf{f})\,.&\end{aligned} \tag{4.49}$$

Since $p < 2$, we have

$$\int_0^T \|\nabla \mathbf{u}^N\|_p^p \, dt \le \int_0^T \|\nabla \mathbf{u}^N\|_{p,C^N}^p \, dt + c \left(\int_0^T \|\nabla \mathbf{u}^N\|_{2,B^N}^2 \, dt \right)^{p/2}$$
$$\overset{(4.49)}{\le} \frac{1}{c_3} \Big(c(\mathbf{u}_0, \mathbf{f}) + c\big(c(\mathbf{u}_0, \mathbf{f})\big)^{p/2} \Big) \equiv c_a \, . \tag{4.50}$$

As a conclusion of the above considerations we obtain the following theorem.

Theorem 4.51 *Let $\mathbf{u}_0$ and $\mathbf{f}$ satisfying (4.41) be such that (4.47) holds. Then there exists a strong solution to the problem $(NS)_p$ for $p > \frac{5}{3}$ $(d = 3)$.*

The local existence of a strong solution is treated in the following theorem.

Theorem 4.52 *Let $\mathbf{u}_0$, $\mathbf{f}$ satisfy (4.41). If $p > \frac{5}{3}$ $(d = 3)$ then there exists a $t^* > 0$ such that a strong solution to the problem $(NS)_{\mathrm{p}}$ exists on $(0, t^*)$. This solution is unique for $p \ge 2$.*

Proof : We will proceed similarly to before. For simplicity, however, we assume that $\mathbf{f}$ is independent of t and we denote a given $p > \frac{5}{3}$ by p_0. Let us assume that instead of (4.44) we have

$$\begin{aligned} \frac{1}{\lambda_\varepsilon - 1} \frac{1}{(1 + \|\nabla \mathbf{u}_0\|_2^2)^{\lambda_\varepsilon - 1}} &\le c_p \int_0^t \|\nabla \mathbf{u}^N\|_p^{\frac{p}{1+\varepsilon}} \, d\tau + t\, c_{\mathbf{f}} \\ &+ \frac{1}{\lambda_\varepsilon - 1} \frac{1}{(1 + \|\nabla \mathbf{u}^N(t)\|_2^2)^{\lambda_\varepsilon - 1}} \, . \end{aligned} \tag{4.53}$$

where $\lambda_\varepsilon \equiv \lambda_\varepsilon(p) \to \lambda$ as $\varepsilon \to 0$ and $\lambda_\varepsilon(p_0) > 0$. By Hölder's inequality,

$$\int_0^t \|\nabla \mathbf{u}^N\|_p^{\frac{p}{1+\varepsilon}} \, d\tau \le \left(\int_0^t \|\nabla \mathbf{u}^N\|_p^p \, d\tau \right)^{\frac{1}{1+\varepsilon}} t^{\frac{\varepsilon}{1+\varepsilon}} \le c_a t^{\frac{\varepsilon}{1+\varepsilon}} \, .$$

As in (4.46), we have

$$\|\nabla \mathbf{u}^N\|_2^2 \le C \tag{4.54}$$

whenever

$$c_p c_a t^{\frac{\varepsilon}{1+\varepsilon}} + t c_{\mathbf{f}} \le \frac{1}{(\lambda_\varepsilon - 1)(1 + \|\nabla \mathbf{u}_0\|_2^2)^{\lambda_\varepsilon - 1}} \, . \tag{4.55}$$

Thus, for all t^* satisfying (4.55), the existence of a strong solution on $(0, t^*)$ follows.

In order to derive (4.53) we have to return to Section 5.3, and to modify the calculations between (3.42) and (3.48). Instead of (3.44), we require the condition

$$\Big(Q_2 + \frac{2-q}{2} Q_3\Big)\,\delta' = \frac{p}{1+\varepsilon}$$

to hold. It is clear that for ε sufficiently small, $\lambda_\varepsilon(p_0) > 0$.

The uniqueness proof coincides with the proof of Theorem 4.37. The restriction $p \geq 2$ is due to (1.25).

The details are left to the reader. ■

5.4.3 Fractional derivative estimates

In this section we will find exponents β, γ such that

$$\begin{aligned} \int_0^T \|D^{(2)}\mathbf{u}^N\|_r^{2\beta} < +\infty\,, \\ \int_0^T \|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^{2\gamma} < +\infty\,. \end{aligned} \tag{4.56}$$

Here, $\mathbf{u}^N$ denotes Galerkin approximations to the problem $(\mathrm{NS})_{\mathrm{p}}$.

We have already proved several estimates of the type (4.56) in Section 5.3. Let us recall them:

- If $p \geq 1 + \frac{2d}{d+2}$ then (4.56) holds with $r = 2$ and $\beta = \gamma = 1$.
- If moreover $d = 2$ and $p \in (1,2)$, then (4.56) holds with $r = p$ and $\beta = \gamma = 1$.
- If $d = 3$, then $(4.56)_1$ holds with

$$\begin{aligned} &r = 2 \quad \text{and} \quad \beta = \frac{3p-5}{p+1} && \text{if} \quad p \in \big[2, \tfrac{11}{5}\big)\,, \\ &r = p \quad \text{and} \quad \beta = \frac{p}{2}\,\frac{5p-9}{(-p^2+8p-9)} && \text{if} \quad p \in \big(\tfrac{9}{5}, 2\big)\,. \end{aligned}$$

Compare these results with (3.60) for $p < 2$ and (3.55) for $p \in [2, \frac{11}{5})$.

In this section we will show the estimates of the type $(4.56)_2$.

Lemma 4.57 *Let $d = 3$ and $p \in [2, \frac{11}{5})$. Then the Galerkin approximations $\mathbf{u}^N$ to the problem $(NS)_p$ satisfy $(4.56)_2$ with*

$$\gamma = \frac{(3p-5)\,p}{5p^2 - 15p + 16}\,. \tag{4.58}$$

Remark 4.59 Taking $p = 2$ in (3.55) and (4.58), we obtain

$$\int_0^T \|D^{(2)}\mathbf{u}^N\|_2^{2/3} < +\infty\,, \tag{4.60}$$

$$\int_0^T \Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_2^{2/3} < +\infty\,. \tag{4.61}$$

As emphasized in Example 1.62 in Chapter 1, the classical Navier-Stokes system is a particular subcase of the model $(\mathrm{NS})_\mathrm{p}$, when $p = 2$. For a weak solution of that system, the estimates (4.60)–(4.61) were derived in FOIAS, GUILLOPÉ AND TEMAM [1981] (for the space-periodic problem) and in DUFF [1990b] (for the Dirichlet problem).

PROOF (of Lemma 4.57): In order to prove $(4.56)_2$ with γ defined in (4.58), we multiply the rth member of the Galerkin system by $\frac{d}{dt}c_r^N(t)$ and sum up. Following the same lines as in (4.13)–(4.14), we obtain, applying Hölder's inequality,

$$\begin{aligned}\frac{1}{2}\Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_2^2 &+ \frac{d}{dt}\|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1 \\ &\le c\|\mathbf{f}\|_2^2 + \|\mathbf{u}^N\|^2_{\frac{3p}{3-p}}\|\nabla\mathbf{u}^N\|^2_{\frac{6p}{5p-6}}\,.\end{aligned} \tag{4.62}$$

By the interpolation lemma (see Corollary 2.10 in Chapter 1), we have

$$\|\nabla\mathbf{u}^N\|_{\frac{6p}{5p-6}} \le \|\nabla\mathbf{u}^N\|_p^{\frac{5p-8}{4}}\|\nabla\mathbf{u}^N\|_{3p}^{\frac{12-5p}{4}}\,. \tag{4.63}$$

Taking $q = 2$ in (3.27), we obtain

$$\|\nabla\mathbf{u}^N\|_{3p}^p \le \mathcal{I}_p(\mathbf{u}^N)\,. \tag{4.64}$$

Then (4.62)–(4.64) imply

$$\begin{aligned}\frac{1}{2}\Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_2^2 &+ \frac{d}{dt}\|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1 \\ &\le c\|\mathbf{f}\|_2^2 + \|\nabla\mathbf{u}^N\|_p^{\frac{5p-4}{2}}\Big\{\mathcal{I}_p(\mathbf{u}^N)\Big\}^{\frac{12-5p}{2p}}\,.\end{aligned} \tag{4.65}$$

By (3.54) we know that

$$\int_0^T \mathcal{I}_p(\mathbf{u}^N)(\tau)\big(1+\|\nabla\mathbf{u}^N(\tau)\|_2^2\big)^{-\lambda}\,d\tau \le C\,, \tag{4.66}$$

where $\lambda = 2\,\frac{3-p}{3p-5}$. According to (1.36) and Korn's inequality (1.11) and after some algebraic manipulation it is possible to find a constant $c > 0$ such that

$$\|\nabla \mathbf{v}\|_p^p \leq c\big(1 + \|U\big(\mathbf{e}(\mathbf{v})\big)\|_1\big)\,. \tag{4.67}$$

Using $\big(1 + \|\nabla \mathbf{u}^N\|_2^2\big) \leq c\big(1 + \|\nabla \mathbf{u}^N\|_p^p\big)$, (4.66) and (4.67), the inequality (4.65) turns into

$$\begin{aligned} &\frac{1}{2}\Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_2^2 + \frac{d}{dt}\big(1 + \|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1\big) \\ &\qquad \leq c_{\mathbf{f}} + c\left\{\frac{\mathcal{I}_p(\mathbf{u}^N)}{\big(1 + \|\nabla \mathbf{u}^N\|_2^2\big)^{\lambda}}\right\}^{\frac{12-5p}{2p}} \\ &\qquad\qquad \times \big(1 + \|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1\big)^{\lambda\frac{12-5p}{2p} + \frac{5p-4}{2p}}\,. \end{aligned} \tag{4.68}$$

Let $\mu > 0$. Multiplying (4.68) by $\big(1 + \|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1\big)^{-\mu}$, integrating between 0 and t, $t \leq T$, we get

$$\begin{aligned} &\int_0^t \frac{\|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^2}{\big(1 + \|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1\big)^{\mu}}\,d\tau + \frac{1}{1-\mu}\big(1 + \|U\big(\mathbf{e}(\mathbf{u}^N(t))\big)\|_1\big)^{1-\mu} \\ &\qquad \leq c_{\mathbf{f},\mathbf{u}_0} + c\int_0^t \left\{\frac{\mathcal{I}_p(\mathbf{u}^N)}{\big(1 + \|\nabla \mathbf{u}^N\|_2^2\big)^{\lambda}}\right\}^{\frac{12-5p}{2p}} \\ &\qquad\qquad \times \big(1 + \|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1\big)^{-\mu+\lambda\frac{12-5p}{2p} + \frac{5p-4}{2p}}\,d\tau\,. \end{aligned} \tag{4.69}$$

Because of the uniform estimates (4.66), we apply Hölder's inequality on the last term in (4.69) together with the requirements

$$\begin{aligned} \frac{1}{\delta} + \frac{1}{\delta'} &= 1\,, \\ \Big(-\mu + \lambda\frac{12-5p}{2p} + \frac{5p-4}{2p}\Big)\,\delta &= 1\,, \\ \frac{12-5p}{2p}\,\delta' &= 1\,. \end{aligned} \tag{4.70}$$

Solving (4.70) we get

$$\mu = \frac{2(p^2 - 5p + 8)}{p(3p-5)}\,. \tag{4.71}$$

Since

$$\begin{aligned}\int_0^t \big(1+\|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1\big)\,d\tau &\overset{(1.36)}{\leq} c\int_0^t \big(1+\|\nabla\mathbf{u}^N\|_p^p\big)\,d\tau \\ &\overset{(3.10)}{\leq} C\,,\end{aligned} \tag{4.72}$$

it follows from (4.69)–(4.72) that

$$\int_0^T \frac{\|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^2}{\big(1+\|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1\big)^{\mu}}\,d\tau \leq C\,. \tag{4.73}$$

Next,

$$\begin{aligned}\int_0^T \|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^{2\gamma}\,dt \leq \int_0^T &\Big(\|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^2\big(1+\|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1\big)^{-\mu}\Big)^{\gamma} \\ &\times \big(1+\|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1\big)^{\mu\gamma}\,dt\,.\end{aligned}$$

By Hölder's inequality, using (4.73) and the same argument as in (4.72), we obtain

$$\int_0^T \|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^{2\gamma} \leq C \qquad \text{with} \qquad \gamma = \frac{p(3p-5)}{5p^2-15p+16}\,. \tag{4.74}$$

Thus, (4.58) is found and Lemma 4.57 is proved. ∎

5.5 Compressible non-Newtonian gases and measure-valued solutions

The objective of this final section is to prove the existence of measure-valued solutions to an initial-boundary value problem for a flow of compressible gases undergoing isothermal processes in bounded domains.

The result that we will present here has been proved by Matušů and Novotný [1994], using considerations from Nečas, Novotný and Šilhavý [1989, 1990] and Novotný [1992]. A different kind of measure-valued solution has been studied in Neustupa [1993] for barotropic flows and by Kröner and Zajaczkowski [1996] for the Euler equations of compressible fluids.

Let us emphasize that the global existence of weak solutions for the problem studied in this section is not known. Very inspiring in this direction is the paper of Padula (see Padula [1986]); unfortunately some conclusions are not correct as pointed out in Padula [1988]. For perfect isentropic gases, where the pressure is given by $p=\kappa\rho^{\gamma}, \gamma>1$, the global existence of weak solutions has been proved very recently by Lions [1993a, 1993b] for $\gamma \geq \frac{3d}{d+2}$. A promising method in the direction of the investigation of the properties of weak solutions seems to be the method of decomposition proposed in Novotný and Padula [1994]. This method has already been successfully applied for studying steady flows of the isentropic perfect gases, see Novotný [1995, 1996].

Let us now formulate the problem. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\partial\Omega \in C^{0,1}$, $d = 2$ or 3. For $T \in (0, \infty)$, we use the notation $I \equiv (0,T)$ and $Q_T \equiv I \times \Omega$. We also set $\Omega_t \equiv \{(t,x) \in Q_T; x \in \Omega\}$.

Let $\mathbf{f} : Q_T \to \mathbb{R}^d$, $\rho_0 : \Omega \to \mathbb{R}$, $\mathbf{u}_0 : \Omega \to \mathbb{R}^d$ and $\boldsymbol{\tau} : \mathbb{R}^{d^2}_{\text{sym}} \to \mathbb{R}^{d^2}_{\text{sym}}$ be given. Assume that there exists a $p \in (1, \infty)$ and $c_1, c_2 > 0$ such that for all $\boldsymbol{\eta} \in \mathbb{R}^{d^2}_{\text{sym}}$,

$$\boldsymbol{\tau}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq c_1 |\boldsymbol{\eta}|^p \,, \tag{5.1}$$

$$|\boldsymbol{\tau}(\boldsymbol{\eta})| \leq c_2 \big(1 + |\boldsymbol{\eta}|\big)^{p-1} \,. \tag{5.2}$$

By the **problem** $\mathbf{(CF)}_{\mathrm{p}}$ we denote the initial-boundary value problem to find $\mathbf{u} = (u_1, \ldots, u_d) : Q_T \to \mathbb{R}^d$ and $\rho : Q_T \to \mathbb{R}$ solving the system of $(d+1)$-equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0 \,, \tag{5.3}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_j u_i) + \frac{\partial \rho}{\partial x_i} - \frac{\partial}{\partial x_j} \tau_{ij}(\mathbf{e}(\mathbf{u})) = \rho f_i \,, \tag{5.4}$$

$i = 1, \ldots, d$, and satisfying the initial and boundary conditions

$$\mathbf{u}(0,x) = \mathbf{u}_0(x) \,, \qquad \rho(0,x) = \rho_0(x) \,, \qquad \forall\, x \in \Omega \,, \tag{5.5}$$

$$\mathbf{u}(t,x) = \mathbf{0} \qquad \forall\, (t,x) \in I \times \partial\Omega \,. \tag{5.6}$$

Recall that $2\,\mathbf{e} = \nabla \mathbf{u} + (\nabla \mathbf{u})^T$. We refer to Section 1.1.4 for more comments on the system of equations.

Let

$$\rho_0 \in C^1(\overline{\Omega}),\ \rho_0 > 0 \text{ in } \overline{\Omega} \text{ and } \mathbf{u}_0 \in W^{3,2}_0(\Omega)^d \,, \tag{5.7}$$

$$\mathbf{f} \in C(I; L^\infty(\Omega)^d) \,. \tag{5.8}$$

It is possible to weaken the assumptions on ρ_0, $\mathbf{u}_0$ and $\mathbf{f}$, if we regularize the data in appropriate parts of proofs. Nevertheless, we prefer to put stronger assumptions on ρ_0, $\mathbf{u}_0$ and $\mathbf{f}$ in order to retain the main ideas.

Besides Sobolev, Bochner, Lebesgue spaces, and spaces of Radon measures, we will use also Orlicz spaces here. See Section 1.2.5 for definition and an exposition of basic properties of these spaces.

Let $\Psi(t) \equiv (1+t)\ln(1+t) - t$ be a Young function.

Definition 5.9 *Let $\rho_0, \mathbf{u}_0$ and $\mathbf{f}$ satisfy the assumptions (5.7), (5.8). The triple $(\rho, \mathbf{u}, \nu)$ is called a* **measure-valued solution** *to the problem $(CF)_{\mathrm{p}}$ if*

$$\rho \in L^\infty(I; L_\Psi(\Omega)) \,, \quad \rho \geq 0 \,, \tag{5.10}$$

$$\mathbf{u} \in L^p(I; W_0^{1,p}(\Omega)^d) \,, \tag{5.11}$$

$$\nu \in L^\infty_w(Q_T; \mathrm{Prob}(\mathbb{R}^{d^2})) \,, \tag{5.12}$$

satisfy

$$\int_{\mathbb{R}^{d^2}} \boldsymbol{\lambda} \, d\nu_{t,x}(\boldsymbol{\lambda}) = \nabla \mathbf{u}(t,x) \quad \text{for a.a. } (t,x) \in Q_T, \tag{5.13}$$

$$\begin{gathered}\int_0^T \int_\Omega \rho \frac{\partial \psi}{\partial t} \, dx \, dt + \int_\Omega \rho u_j \frac{\partial \psi}{\partial x_j} \, dx \, dt + \int_\Omega \rho_0 \psi(0) \, dx = 0 \,, \\ \forall \, \psi \in \mathcal{D}\big(-\infty, T; \mathcal{D}(\Omega)\big) \,,\end{gathered} \tag{5.14}$$

and

$$\begin{aligned}&- \int_0^T \left(\int_\Omega \rho u_i \frac{\partial \varphi_i}{\partial t} \, dx \right) dt - \int_0^T \left(\int_\Omega \rho u_j u_i \frac{\partial \varphi_i}{\partial x_j} \, dx \right) dt \\ &+ \int_0^T \int_\Omega e_{ij}(\boldsymbol{\varphi}) \int_{\mathbb{R}^{d^2}} \tau_{ij}(\mathbf{e}(\boldsymbol{\lambda})) \, d\nu_{t,x}(\boldsymbol{\lambda}) \, dx \, dt - \int_0^T \int_\Omega \rho \frac{\partial \varphi_i}{\partial x_i} \, dx \, dt \\ &= \int_0^T \int_\Omega \rho f_i \varphi_i \, dx \, dt + \int_\Omega \rho_0 u_{0i} \varphi_i(0) \, dx \,,\end{aligned} \tag{5.15}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}\big(-\infty, T, \mathcal{D}(\Omega)^d\big)$.

Theorem 5.16 *Let $\rho_0, \mathbf{u}_0, \mathbf{f}$ satisfy (5.7), (5.8). If $p > d$ then there exists a measure-valued solution in the sense of Definition 5.9.*

The proof of Theorem 5.16 will be presented in the rest of this section. For $\mu > 0$ we will first define a singular perturbation to the problem $(\mathrm{CF})_{\mathrm{p}}$, denoted by $(\mathrm{CFpert})^\mu_{\mathrm{p}}$. Then we will construct an approximation of the problem $(\mathrm{CFpert})^\mu_{\mathrm{p}}$ and we will show the existence of a solution by a fixed point argument combined with the method of characteristics and the Galerkin method. By passing to the limit, we will prove the existence of a weak solution (global in time) to the problem $(\mathrm{CFpert})^{\mu}_{\mathrm{p}}$. Finally, letting $\mu \to 0+$, we will demonstrate the existence of a measure-valued solution to the problem $(\mathrm{CF})_{\mathrm{p}}$ and the proof will be finished.

We will split the exposition of the proof into the following four steps:

- Definition of the problem $(\mathrm{CFpert})^{\mu}_{\mathrm{p}}$ and its approximations.
- The solvability of the approximate problems.
- The existence of a weak solution to the weak problem $(\mathrm{CFpert})^{\mu}_{\mathrm{p}}$.
- The existence of a measure-valued solution.

• *Definition of the problem* $(\mathit{CFpert})^{\mu}_{\mathrm{p}}$ *and its approximations*

Let $\mu > 0$ and let $(\!(\cdot,\cdot)\!)_3$ denote a scalar product in $W^{3,2}_0(\Omega)^d$. Then $\|\cdot\|_{3,2} = (\!(\cdot,\cdot)\!)_3^{\frac{1}{2}}$. We say that a couple $(\rho^{\mu}, \mathbf{u}^{\mu})$ is a **weak solution to the problem $(\mathbf{CFpert})^{\mu}_{\mathrm{p}}$**, if

$$\mathbf{u}^{\mu} \in L^{\infty}(I; W^{3,2}_0(\Omega)^d) \cap L^p(I; W^{1,p}_0(\Omega)^d)\,, \tag{5.17}$$

$$\frac{\partial \mathbf{u}^{\mu}}{\partial t} \in L^{p'}(I, L^{p'}(\Omega)^d)\,, \tag{5.18}$$

$$\rho^{\mu} \in L^{\infty}(Q_T) \cap L^2(I; W^{1,2}(\Omega))\,, \tag{5.19}$$

$$\frac{\partial \rho^{\mu}}{\partial t} \in L^2(Q_T)\,, \tag{5.20}$$

and if the following identities are satisfied:

$$\int_{Q_T} \rho^{\mu} \frac{\partial \psi}{\partial t}\, dx\, dt + \int_{Q_T} \rho^{\mu} u^{\mu}_j \frac{\partial \psi}{\partial x_j}\, dx\, dt + \int_{\Omega} \rho^{\mu}_0 \psi(0,x)\, dx = 0\,, \tag{5.21}$$

for all $\psi \in \mathcal{D}\big(-\infty, T, \mathcal{D}(\Omega)\big)$, and

$$\begin{aligned} &- \int_0^T \left(\int_{\Omega} \rho^{\mu} u^{\mu}_i \frac{\partial \varphi_i}{\partial t}\, dx \right) dt - \int_0^T \left(\int_{\Omega} \rho^{\mu} u^{\mu}_j u^{\mu}_i \frac{\partial \varphi_i}{\partial x_j}\, dx \right) dt \\ &\qquad\qquad + \mu (\!(\mathbf{u}^{\mu}, \boldsymbol{\varphi})\!)_3 \\ &\qquad + \int_0^T \int_{\Omega} \tau_{ij}(\mathbf{e}(\mathbf{u}^{\mu})) e_{ij}(\boldsymbol{\varphi}) dx\, dt - \int_0^T \int_{\Omega} \rho^{\mu} \frac{\partial \varphi_i}{\partial x_i}\, dx\, dt \\ &= \int_0^T \int_{\Omega} \rho^{\mu} f_i \varphi_i\, dx\, dt + \int_{\Omega} \rho^{\mu}_0 u^{\mu}_{0i} \varphi_i(0)\, dx\,, \end{aligned} \tag{5.22}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}\big(-\infty, T, \mathcal{D}(\Omega)^d\big)$.

Remark 5.23 There are many possibilities for defining the perturbation $(\mathrm{CFpert})^{\mu}_{\mathrm{p}}$. Basically, we add to the term $\frac{\partial}{\partial x_j}\tau_{ij}(\mathbf{e}(\mathbf{u}))$ another linear operator of sixth order and of course we have to increase the number of boundary conditions. For our presentation the precise form of the boundary conditions (which disappear when $\mu \to 0+$) and the form of the elliptic operator are not important. We refer to NEČAS AND ŠILHAVÝ [1991] for introducing **multipolar fluids** that could be convenient for that purpose. See also NEČAS, NOVOTNÝ AND ŠILHAVÝ [1989, 1990] or NOVOTNÝ [1992].

Let us fix $\mu > 0$ and for simplicity, let us denote $(\rho^{\mu}, \mathbf{u}^{\mu})$ by $(\rho, \mathbf{u})$. Then $(\rho^N, \mathbf{u}^N)$ will denote the solution of the approximation to the problem $(\mathrm{CFpert})^{\mu}_{\mathrm{p}}$. As presented in Theorem 4.11 and Remark 4.14 of the Appendix, we can construct a basis $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty} \subset W_0^{3,2}(\Omega)^d$ consisting of solutions to the eigenvalue problem

$$((\boldsymbol{\omega}^r, \boldsymbol{\xi}))_3 = \lambda_r(\boldsymbol{\omega}^r, \boldsymbol{\xi}), \quad \forall\, \boldsymbol{\xi} \in W_0^{3,2}(\Omega)^d .$$

Then $\boldsymbol{\omega}^r$ are orthonormal in $L^2(\Omega)^d$, and due to the regularity of elliptic systems we can assume that $\boldsymbol{\omega}^r \in C^{\infty}(\Omega)^d$, if $\partial\Omega$ is smooth enough. Defining $P^N\mathbf{u} \equiv \sum_{k=1}^{N}(\boldsymbol{\omega}^r, \mathbf{u})\boldsymbol{\omega}^r$, we know, from (6.4.12)–(6.4.13), that

$$\begin{aligned} \|P^N\mathbf{u}\|_{3,2} &\le \|\mathbf{u}\|_{3,2}\,, \\ \|P^N\mathbf{u}\|_2 &\le \|\mathbf{u}\|_2\,. \end{aligned} \tag{5.24}$$

Let us put $\mathbf{u}^N(t,x) \equiv \sum_{k=1}^{N} c_k^N(t)\boldsymbol{\omega}^k(x)$. Then functions $(\rho^N, \mathbf{u}^N)$ are called **approximations of the problem** $(\mathbf{CFpert})^{\mu}_{\mathrm{p}}$ if they satisfy

$$\frac{\partial\rho^N}{\partial t} + \frac{\partial}{\partial x_i}(\rho^N u_i^N) = 0 \quad \text{in } \Omega\,, \tag{5.25}$$

$$\begin{aligned} &\int_{\Omega}\Big(\rho^N\frac{\partial u_i^N}{\partial t} + \rho^N u_j^N\frac{\partial u_i^N}{\partial x_j} + \frac{\partial\rho^N}{\partial x_i} - \rho^N f_i\Big)w_i^r\,dx \\ &+ \int_{\Omega}\tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(\boldsymbol{\omega}^r)\,dx + \mu((\mathbf{u}^N, \boldsymbol{\omega}^r))_3 = 0\,, \end{aligned} \tag{5.26}$$

$r = 1, \dots, N$, with initial conditions

$$\rho^N(0) = \rho_0\,, \tag{5.27}$$

and

$$\mathbf{u}^N(0) = P^N \mathbf{u}_0 \,, \tag{5.28}$$

respectively.

• *The solvability of the approximate problems*

Lemma 5.29 *Let $(\rho_0, \mathbf{u}_0, \mathbf{f})$ satisfy (5.7), (5.8). Then, for fixed N, the system (5.25)–(5.28) has a solution $(\rho^N, \mathbf{u}^N)$, where*

$$\begin{aligned} \rho^N &\in C^1(\overline{Q}_T) \,, \\ \mathbf{u}^N &\in C^1(I; W_0^{3,2}(\Omega)^d) \,. \end{aligned} \tag{5.30}$$

In order to prove the existence of $(\rho^N, \mathbf{u}^N)$ solving (5.25)–(5.28) we will proceed as follows: For fixed $N \in \mathbb{N}$ and $L > 0$, we define

$$B_L(0) \equiv \{\mathbf{c} \in C(\overline{I})^N ; \|\mathbf{c}\|_{C(\overline{I})^N} \le L\} \,. \tag{5.31}$$

Let $\overline{\mathbf{c}} \in B_L(0)$ be such that

$$\overline{c}_k(0) \equiv (\mathbf{u}_0, \boldsymbol{\omega}^k) \ , \quad k = 1, 2, \ldots, N \,. \tag{5.32}$$

Setting

$$\overline{\mathbf{u}}^N(t,x) \equiv \sum_{i=1}^N \overline{c}_k(t) \boldsymbol{\omega}^k(x) \,, \tag{5.33}$$

we will seek ρ^N, by the method of characteristics, as a solution of

$$\frac{\partial \rho^N}{\partial t} + \frac{\partial}{\partial x_i}(\rho^N \overline{u}_i^N) = 0 \ , \quad \rho^N(0) = \rho_0 \ . \tag{5.34}$$

Having ρ^N, we will finally look for $\mathbf{u}^N \in C^1(I; C^\infty(\overline{\Omega})^d)$ in the form

$$\mathbf{u}^N(t,x) = \sum_{k=1}^N c_k(t) \boldsymbol{\omega}^k(x) \,, \tag{5.35}$$

solving the Galerkin system

$$\begin{aligned} &\int_\Omega \Big(\rho^N \frac{\partial u_i^N}{\partial t} + \rho^N \overline{u}_j^N \frac{\partial u_i^N}{\partial x_j} + \frac{\partial \rho^N}{\partial x_i} - \rho^N f_i \Big) w_i^r \\ &+ \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N)) e_{ij}(\boldsymbol{\omega}^r) \, dx + \mu (\!(\mathbf{u}^N, \boldsymbol{\omega}^r)\!)_3 = 0 \,, \qquad r = 1, \ldots, N \,. \end{aligned} \tag{5.36}$$

Getting $\mathbf{u}^N$ as a solution of (5.36), we define

$$S : \overline{\mathbf{c}}^N \in B_L(0) \mapsto \mathbf{c}^N .$$

If we verify that S satisfies the assumptions of the Schauder Fixed Point Theorem, we will finally find a solution of (5.25)–(5.28).

Let us denote

$$k(N) \equiv \sup_{k=1,\dots,N} \max_{x\in\overline{\Omega}} |\nabla \boldsymbol{\omega}^k(x)| . \tag{5.37}$$

Further, let $\overline{\mathbf{u}}^N \in C^1(I; C^\infty(\overline{\Omega})^d)$ be given by (5.33), $\overline{\mathbf{c}} \in B_L(0)$. Then the characteristics $\mathbf{z}^N$ to (5.34) are determined as a solution of

$$\begin{aligned} \frac{d\mathbf{z}^N(\tau;t,x)}{d\tau} &= \overline{\mathbf{u}}^N(\tau, \mathbf{z}^N(\tau;t,x)) , \\ \mathbf{z}^N(0;t,x) &= x , \end{aligned} \tag{5.38}$$

or in the equivalent integral form

$$\mathbf{z}^N(\tau;t,x) = x + \int_\tau^t \overline{\mathbf{u}}^N(s, \mathbf{z}^N(s;t,x))\, ds . \tag{5.39}$$

Since $\|\overline{\mathbf{c}}\|_{C(I)} \leq L$, the method of successive approximations will provide the existence of a solution for all $t \in I$; see any book on ordinary differential equations for details. Taking (5.38) into consideration, equation (5.34) becomes

$$\frac{d}{d\tau}\big(\ln \rho^N(\tau, \mathbf{z}^N(\tau;t,x))\big) = -\operatorname{div} \overline{\mathbf{u}}^N(\tau, \mathbf{z}^N(\tau;,t,x)) . \tag{5.40}$$

Hence, for $t \leq \tau$,

$$\begin{aligned} \rho^N(t,x) =& \rho_0(\mathbf{z}^N(0;t,x)) \\ & \times \exp\left(-\int_0^t \operatorname{div} \overline{\mathbf{u}}^N(\tau, \mathbf{z}^N(\tau;t,x))\, d\tau \right) . \end{aligned} \tag{5.41}$$

Thus $\rho^N \in C^1(Q_T)$ is the uniquely defined solution of (5.34). Moreover,

$$0 < \rho_1 \leq \rho^N(t,x) \leq \rho_2 , \tag{5.42}$$

where

$$\begin{aligned} \rho_1 &\equiv \min_{y\in\overline{\Omega}} \rho_0(y) \exp(-L\, k(N) T) , \\ \rho_2 &\equiv \max_{y\in\overline{\Omega}} \rho_0(y) \exp(L\, k(N) T) . \end{aligned}$$

Taking $(\rho^N, \overline{\mathbf{u}}^N)$, we will find the coefficients $(c_k^N)_{k=1}^N$, such that (5.36) holds. The system (5.36) is a system of ordinary differential equations, that can be rewritten as

$$\mathbf{A}\frac{d\mathbf{c}^N}{dt} = \mathbf{F}(\mathbf{c}^N)\,, \qquad \mathbf{c}^N(0) = \mathbf{c}_0\,, \tag{5.43}$$

where $\mathbf{A} = (a_{ij})_{i,j=1}^N$ has components

$$a_{ij} = \int_\Omega \rho^N(t,x)\boldsymbol{\omega}^i(x)\boldsymbol{\omega}^j(x)\,dx$$

and the vectors $\mathbf{F}(\mathbf{c}^N) = (F_r(\mathbf{c}^N))_{r=1}^N$ and $\mathbf{c}_0 = (c_{0r})_{r=1}^N$ are determined by

$$\begin{aligned} F_r(\mathbf{c}^N) =& - c_k^N(t)\overline{c}_s^N(t)\int_\Omega \rho^N \frac{\partial w_i^k}{\partial x_j} w_j^s w_i^r\,dx - \int_\Omega \frac{\partial \rho^N}{\partial x_i} w_i^r\,dx \\ &+ \int_\Omega \rho^N f_i w_i^r\,dx - \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(\boldsymbol{\omega}^r)\,dx \\ &- \mu c_r^N(\!(\boldsymbol{\omega}^r, \boldsymbol{\omega}^r)\!)_3\,, \\ c_{0r} =& (\mathbf{u}_0, \boldsymbol{\omega}^r)\,. \end{aligned}$$

Since $\mathbf{A}$ is invertible, we get, from (5.43)

$$\frac{d\mathbf{c}^N}{dt} = \mathbf{A}^{-1}\mathbf{F}(\mathbf{c}^N)\,, \qquad \mathbf{c}(0) = \mathbf{c}_0\,. \tag{5.44}$$

The local solvability follows from Theorem 3.4 in the Appendix, the global solvability is obtained exactly in the same way as in Section 5.2, where we proved the solvability of the Galerkin system (2.21). This means that global *a priori* estimates for $\mathbf{c}^N$ have to be derived. We will find them first for the equations (5.25)–(5.26) and then for the equations (5.36). Let us recall that $\Psi(\xi) \equiv (1+\xi)\ln(1+\xi) - \xi$.

Lemma 5.45 *Let $(\rho^N, \mathbf{u}^N)$ be a solution of (5.25)–(5.28). Then*

- *there exists a constant $C > 0$ independent of μ and N such that*

$$\int_{\Omega_t} \rho^N\,dx = \int_{\Omega_0} \rho_0\,dx, \tag{5.46}$$

$$\begin{aligned} \int_{\Omega_t} \rho^N\,|\mathbf{u}^N|^2\,dx + \int_{\Omega_t} \Psi(\rho^N)\,dx \le C\,, \\ \mu\int_0^t \|\mathbf{u}^N\|_{3,2}^2\,d\tau + K_p^p\int_0^t \|\nabla\mathbf{u}^N\|_p^p\,d\tau \le C\,; \end{aligned} \tag{5.47}$$

• *there exists a constant* $C(\frac{1}{\mu}) > 0$ *and* $\rho^* = \rho^*(\frac{1}{\mu}) > 0$ *such that for* $t \leq T$

$$\|\rho^N\|_{\infty,Q_t} + \int_0^t \|\frac{\partial \rho^N}{\partial t}\|_2^2 + \int_0^t \|\nabla \rho^N\|_2^2 \leq C(\tfrac{1}{\mu}), \quad (5.48)$$

$$\rho^N(t,x) \geq \rho^*, \qquad \forall\,(t,x) \in Q_T, \quad (5.49)$$

$$\int_0^T \|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^2 + \mu \|\mathbf{u}^N\|_{L^\infty(I;W_0^{3,2}(\Omega)^d)} \leq C(\tfrac{1}{\mu}). \quad (5.50)$$

Proof : Let g be any function (scalar, vector or tensor) defined on Q_T. Then, $\frac{dg}{dt} = \frac{\partial g}{\partial t} + u_k \frac{\partial g}{\partial x_k}$ and by the Transport Theorem (see Chorin and Marsden [1992] or Feistauer [1993]),

$$\frac{d}{dt}\int_{\Omega_t} g\,dx = \int_{\Omega_t} \left(\frac{dg}{dt} + g \operatorname{div} \mathbf{u}\right) dx. \quad (5.51)$$

Thus, from (5.25),

$$\frac{d}{dt}\int_{\Omega_t} \rho^N\,dx = 0, \quad (5.52)$$

and (5.46) follows.
Also,

$$\frac{d}{dt}\int_{\Omega_t} (1+\rho^N)\ln(1+\rho^N)\,dx = -\int_{\Omega_0} \rho^N \frac{\partial u_i^N}{\partial x_i}\,dx + \int_{\Omega_t} \ln(1+\rho^N)\frac{\partial u_j^N}{\partial x_j}\,dx. \quad (5.53)$$

Now, if we multiply (5.26) by $c_r^N(t)$, sum up over $r = 1,\ldots,N$, use (5.1) and the generalized Korn inequality (1.11), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \rho^N |\mathbf{u}^N|^2\,dx + K_p^p \int_{\Omega_t} |\nabla \mathbf{u}^N|^p\,dx + \mu(\!(\mathbf{u}^N,\mathbf{u}^N)\!)_3 \leq \int_{\Omega_t} \rho^N \frac{\partial u_i^N}{\partial x_i}\,dx + \int_{\Omega_t} \rho^N f_i u_i^N\,dx. \quad (5.54)$$

Let us denote

$$E^N(t) \equiv \frac{1}{2}\int_{\Omega_t} \left(\rho^N|\mathbf{u}^N|^2 + \Psi(\rho^N)\right) dx\ ,$$

$$D^N(t) \equiv \frac{K_p^p}{2}\|\nabla \mathbf{u}^N\|_p^p + \mu\|\mathbf{u}^N\|_{3,2}^2.$$

If we add (5.53) and (5.54) and subtract (5.52), we obtain

$$\begin{aligned}\frac{d}{dt}E^N(t) + D^N(t) \le \int_{\Omega_t} \rho^N f_i u_i^N \, dx \\ + \int_{\Omega_t} \ln(1+\rho^N) \frac{\partial u_i^N}{\partial x_i} \, dx \ .\end{aligned} \tag{5.55}$$

Since

$$\begin{aligned}\Big| \int_{\Omega_t} \rho^N f_i u_i^N \, dx \Big| &\le \|\mathbf{f}\|_\infty \Big(\int_{\Omega_t} \rho^N |\mathbf{u}^N|^2 \, dx \Big)^{\frac{1}{2}} \Big(\int_{\Omega_t} \rho^N \, dx \Big)^{\frac{1}{2}} \\ &\overset{(5.46)}{\le} \frac{1}{2}\|\mathbf{f}\|_\infty^2 \int_{\Omega_t} \rho_0 \, dx + \frac{1}{2} \int_{\Omega_t} \rho^N |\mathbf{u}^N|^2 \, dx\end{aligned}$$

and

$$\begin{aligned}&\Big| \int_{\Omega_t} \ln(1+\rho^N) \frac{\partial u_i^N}{\partial x_i} \, dx \, \Big| \\ &\qquad = \Big| \int_{\Omega_t} \big(\ln(1+\rho^N)\big)^{\frac{1}{2}} \big(\ln(1+\rho^N)\big)^{\frac{1}{2}} \frac{\partial u_i^N}{\partial x_i} \, dx \Big| \\ &\qquad \le \frac{1}{e} \int_{\Omega_t} \big[(1+\rho^N)\ln(1+\rho^N)\big]^{\frac{1}{2}} |\nabla \mathbf{u}^N| \, dx \\ &\qquad \le \int_{\Omega_t} \Psi(\rho^N) \, dx + \int_{\Omega} \rho_0 \, dx + \frac{K_p^p}{2} \|\nabla \mathbf{u}^N\|_p^p + C \, .\end{aligned}$$

We finally obtain, for all $t \le T$,

$$E^N(t) + \int_0^t D^N(\tau) \, d\tau \le c_{\mathbf{f},\rho_0} + \int_0^t E^N(\tau) \, d\tau + E^N(0) \, .$$

The Gronwall lemma 3.5 in the Appendix finishes the proof of (5.47).

In order to prove (5.48)–(5.50), let us first observe, from (5.47), that

$$\int_0^t \|\mathbf{u}^N\|_{3,2}^2 \, d\tau \le \frac{C}{\mu} \, , \qquad t \le T \, . \tag{5.56}$$

By (5.41),

$$\begin{aligned}&\|\rho_0\|_\infty \exp\left(- \int_0^T \|\operatorname{div} \mathbf{u}^N\|_\infty \, dt \right) \\ &\qquad \le \|\rho^N\|_\infty \le \|\rho_0\|_\infty \exp\left(\int_0^T \|\operatorname{div} \mathbf{u}^N\|_\infty \, dt \right) \, .\end{aligned}$$

Since for $t \le T$,

$$\int_0^T \|\operatorname{div} \mathbf{u}^N\|_\infty \, dt \le \int_0^T \|\mathbf{u}^N\|_{3,2} \, dt \le \Big(\frac{C}{\mu}\Big)^{1/2} \sqrt{T} \,,$$

we obtain

$$\begin{aligned} \rho^* &\equiv \|\rho_0\|_\infty \exp\left(-\Big(\frac{C}{\mu}\Big)^{1/2} \sqrt{T} \right) \\ &\le \|\rho^N\|_\infty \le \|\rho_0\|_\infty \exp\left(\Big(\frac{C}{\mu}\Big)^{1/2} \sqrt{T} \right) \,, \end{aligned} \tag{5.57}$$

which is (5.49) and a part of (5.48). The further estimates in (5.48) follow from the formulae (5.41) differentiated with respect to t and x_i, (5.25), and the estimates (5.56) and (5.57). We will skip the details.

Note that, due to (5.49),

$$\rho^* \Big\| \frac{\partial \mathbf{u}^N}{\partial t} \Big\|_2^2 \le \int_{\Omega_t} \rho^N \Big| \frac{\partial \mathbf{u}^N}{\partial t} \Big|^2 \, dx \,. \tag{5.58}$$

Let us finally multiply the rth equation in (5.26) by $\frac{dc_r^N}{dt}$ and sum up over $r = 1, \ldots, N$. One of the terms we obtain reads as

$$\int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big) e_{ij}\Big(\frac{\partial \mathbf{u}^N}{\partial t}\Big) \, dx \,.$$

In order to avoid estimating this integral, we will strengthen the assumptions on $\boldsymbol{\tau}$ slightly. Namely, we will assume that there exists a non-negative potential $U : \mathbb{R}^{d^2}_{\mathrm{sym}} \longrightarrow \mathbb{R}$ for $\boldsymbol{\tau}$, i.e., that $\frac{\partial U}{\partial e_{ij}} = \tau_{ij}$ holds. See Section 1.1.5 for some examples of such $\boldsymbol{\tau}$. Then,

$$\begin{aligned} \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big) e_{ij}\Big(\frac{\partial \mathbf{u}^N}{\partial t}\Big) \, dx &= \frac{d}{dt} \int_\Omega U\big(\mathbf{e}(\mathbf{u}^N)\big) \, dx \\ &= \frac{d}{dt} \|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1 \,. \end{aligned}$$

Therefore, the inequality which is derived from (5.26) has the form:

$$\begin{aligned} \frac{\rho^*}{2} \Big\| \frac{\partial \mathbf{u}^N}{\partial t} \Big\|_2^2 &+ \frac{\mu}{2} \frac{d}{dt} \|\mathbf{u}^N\|_{3,2}^2 + \frac{d}{dt} \|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1 \\ &\le C \Big(\|\nabla \rho^N\|_2^2 + \int_{\Omega_t} |\rho^N|^2 |\mathbf{f}|^2 \, dx \\ &\qquad + \int_{\Omega_t} |\rho^N|^2 |\mathbf{u}^N|^2 |\nabla \mathbf{u}^N|^2 \, dx \Big) \,, \end{aligned} \tag{5.59}$$

where we have used (5.58) and Young's inequality many times. Since all terms on the right-hand side of (5.59) can be estimated by means of (5.56) and (5.48), the proof of Lemma 5.45 is complete. ■

Lemma 5.60 *Let N be fixed. Let $\overline{\mathbf{u}}^N$ be in the form (5.33) with $\overline{\mathbf{c}}^N \in B_L(0)$. Then for L big enough there exists a t^* such that the coefficients $\mathbf{c}^N$ satisfy*

$$\|\mathbf{c}^N\|_{C([0,t^*])} \leq L\,, \tag{5.61}$$

$$\|\frac{d\mathbf{c}^N}{dt}\|_{C([0,t^*])} \leq C\,. \tag{5.62}$$

PROOF : Due to the orthonormality of $\boldsymbol{\omega}^r$ in $L^2(\Omega)^d$,

$$|\mathbf{c}^N|^2 = (\mathbf{u}^N, \mathbf{u}^N) \quad \text{and} \quad \Big|\frac{d\mathbf{c}^N}{dt}\Big|^2 = \Big(\frac{\partial \mathbf{u}^N}{\partial t}, \frac{\partial \mathbf{u}^N}{\partial t}\Big)\,.$$

From (5.49) we have

$$|\mathbf{c}^N|^2 \leq \frac{1}{\rho^*}\int_{\Omega_t} \rho^N |\mathbf{u}^N|^2\,dx$$

and

$$\Big|\frac{d\mathbf{c}^N}{dt}\Big|^2 \leq \frac{1}{\rho^*}\int_{\Omega_t} \rho^N \Big|\frac{\partial \mathbf{u}^N}{\partial t}\Big|^2\,dx\,.$$

Since the proof of (5.61)–(5.62) is based on multiplying (5.36) by $c_r^N(t)$ and $\frac{dc_r^N(t)}{dt}$, respectively, we proceed in the same way as in the proof of Lemma 5.45. Moreover, we can use the fact that N is fixed, i.e., $k(N)$ defined in (5.37) is finite, and that $\|\overline{\mathbf{c}}^N\|_{C([0,T])} \leq L$. Essentially, it is possible to obtain an estimate

$$|\mathbf{c}^N(t)|^2 \leq c(\mathbf{u}_0, \rho^*) + c(T, \mathbf{f})\,,$$

where $c(T,\mathbf{f}) \to 0$ if $T \to 0$. Thus, taking $L > c(\mathbf{u}_0, \rho^*)$, we see that it is possible to find a t^* such that (5.61) holds. We leave the details to the reader. ■

It follows from (5.61)–(5.62) that

$$S : \overline{\mathbf{c}}^N \mapsto \mathbf{c}^N$$

satisfies all assumptions of the Schauder Fixed Point Theorem. Thus (5.25)–(5.28) has a solution at least on $(0, t^*)$. But the *a priori* estimates (5.46)–(5.50) hold on the whole interval $(0, T)$.

Let us therefore assume that the maximal solution to (5.25)–(5.28) exists on $[0, t_0)$. We want to show that t_0 must be T. Let us assume that $t_0 < T$. Then there exist a solution $\mathbf{c}^N : C([0, t_0)) \to C([0, t_0))$ and some $L > 0$ such that $|\mathbf{c}^N(t)| \le L$ for all $t < t_0$. But then also $\mathbf{c}^N(t_0) \equiv \lim_{t\to t_0} \mathbf{c}^N(t)$ satisfies

$$|\mathbf{c}^N(t_0)| \le L\,,$$

and we extend the above procedure to some interval $(t_0, t_0 + t^*)$, which gives the contradiction.

• *The existence a of weak solution to the problem (CFpert)*$^\mu_{\mathrm{p}}$

Now, $\mu > 0$ is fixed and we will first look for $(\rho^\mu, \mathbf{u}^\mu)$ solving (5.21)–(5.22). Then we will derive other estimates on $(\rho^\mu, \mathbf{u}^\mu)$ independent of μ.

For simplicity we will drop the index μ.

Lemma 5.63 *Let $p > 1$ and $\mu > 0$. Let $(\rho_0, \mathbf{u}_0)$ satisfy (5.7), (5.8). Then there exists a weak solution to the problem (CFpert)$^\mu_{\mathrm{p}}$.*

Proof : Since approximations $(\rho^N, \mathbf{u}^N)$ satisfy the estimates (5.48)–(5.50), we can extract a subsequence, labelled again by $(\rho^N, \mathbf{u}^N)$, such that

$$\rho^N \rightharpoonup \rho \qquad \text{weakly}^* \text{ in } L^\infty(Q_T)\,, \tag{5.64}$$

$$\frac{\partial \rho^N}{\partial t} \rightharpoonup \frac{\partial \rho}{\partial t} \qquad \text{weakly in } L^2(Q_T)\,, \tag{5.65}$$

$$\nabla\rho^N \rightharpoonup \nabla\rho \qquad \text{weakly in } L^2(I; L^2(\Omega)^d)\,, \tag{5.66}$$

$$\mathbf{u}^N \rightharpoonup \mathbf{u} \qquad \text{weakly in } L^2(I; W^{3,2}_0(\Omega)^d)\,, \tag{5.67}$$

$$\frac{\partial \mathbf{u}^N}{\partial t} \rightharpoonup \frac{\partial \mathbf{u}}{\partial t} \qquad \text{weakly in } L^2(I; L^2(\Omega)^d)\,, \tag{5.68}$$

and by Aubin-Lions Lemma 2.48 in Chapter 1,

$$\rho^N \to \rho \qquad \text{strongly in } L^2(Q_T)\,, \tag{5.69}$$

$$\nabla\mathbf{u}^N \to \nabla\mathbf{u} \qquad \text{strongly in } L^p(Q_T)\,. \tag{5.70}$$

Let $\varphi \in \mathcal{D}(-\infty, T)$. By Vitali's theorem 2.11 in Chapter 1, see also (3.11)–(3.14), we obtain

$$\begin{aligned}\int_0^T \int_{\Omega_t} \tau_{ij}(\mathbf{e}(\mathbf{u}^N)) e_{ij}(\boldsymbol{\omega}^r)\, dx\, \varphi\, d\tau \\ \to \int_0^T \int_{\Omega_t} \tau_{ij}(\mathbf{e}(\mathbf{u})) e_{ij}(\boldsymbol{\omega}^r)\, dx\, \varphi\, d\tau.\end{aligned} \tag{5.71}$$

Also,

$$\mu \int_0^T ((\mathbf{u}^N, \boldsymbol{\omega}^r))_3 \varphi \, dt \overset{(5.67)}{\to} \mu \int_0^T ((\mathbf{u}, \boldsymbol{\omega}^r))_3 \varphi \, dt \,. \tag{5.72}$$

The other terms are treated by combining strong and weak convergences. Since these limiting processes will be also studied later (even under worse circumstances), when we will prove the existence of measure-valued solution to the problem $(\mathrm{CF})_{\mathrm{p}}$, we will skip these limiting processes now. ∎

Let us set $\Phi(\xi) \equiv e^{\xi} - \xi - 1$, $\Phi_2(\xi) \equiv e^{\xi^2} - 1$, $\Psi(\xi) \equiv (1+\xi)\ln(1+\xi) - \xi$, and let $\Psi_{1/2}$ denote the complementary Young function to Φ_2. Since $p > d$, there exist $\gamma < 1$ and $\varepsilon > 0$ such that $\gamma(p-\varepsilon) > d$. Then by Remark 2.34 and Example 2.39 from Chapter 1,

$$\begin{aligned} W_0^{3,2}(\Omega) \hookrightarrow\hookrightarrow W_0^{1,p}(\Omega) &\hookrightarrow\hookrightarrow W_0^{\gamma, p-\varepsilon}(\Omega) \\ &\hookrightarrow L_{\Phi_2}(\Omega) \hookrightarrow L_{\Phi}(\Omega) \,, \end{aligned} \tag{5.73}$$

and

$$\begin{aligned} L_{\Psi}(\Omega) \hookrightarrow L_{\Psi_{1/2}}(\Omega) &\hookrightarrow \left[W^{\gamma, p-\varepsilon}(\Omega)\right]^* \\ &\hookrightarrow\hookrightarrow W^{-1,p}(\Omega) \hookrightarrow\hookrightarrow W^{-3,2}(\Omega) \,. \end{aligned} \tag{5.74}$$

Lemma 5.75 *Let $(\rho^\mu, \mathbf{u}^\mu)$ be weak solutions to the problem $(CFpert)_{\mathrm{p}}^{\mu}$. Then there exists a constant $C > 0$ independent of μ such that*

$$\begin{gathered} \int_{\Omega_t} \rho^\mu \, |\mathbf{u}^\mu|^2 \, dx + \|\rho^\mu\|_{L^\infty(I; L_\Psi(\Omega))} \le C \,, \\ \mu \int_0^t \|\mathbf{u}^\mu\|_{3,2}^2 \, d\tau + K_p^p \int_0^t \|\nabla \mathbf{u}^\mu\|_p^p \, d\tau \le C \,, \end{gathered} \tag{5.76}$$

$$\|\rho^\mu \mathbf{u}^\mu\|_{L^\infty(I; L_{\Psi_{1/2}}(\Omega)^d)} \le C \,, \tag{5.77}$$

$$\|\rho^\mu u_i^\mu u_j^\mu\|_{L^2(I; L_{\Psi_{1/2}}(\Omega))} \le C \,, \qquad i,j = 1, \dots, d \,, \tag{5.78}$$

$$\left\| \frac{\partial \rho^\mu}{\partial t} \right\|_{L^2(I; W^{-3,2}(\Omega))} \le C \,, \tag{5.79}$$

$$\left\| \frac{\partial (\rho^\mu \mathbf{u}^\mu)}{\partial t} \right\|_{L^2(I; W^{-3,2}(\Omega)^d)} \le C \,. \tag{5.80}$$

Moreover,

$$\rho^\mu \ge 0 \quad \text{a.e. in } Q_T \,. \tag{5.81}$$

PROOF : Because $(\rho^\mu, \mathbf{u}^\mu)$ are obtained as limits of approximations $(\rho^N, \mathbf{u}^N)$, (more precisely $(\rho^{N\mu}, \mathbf{u}^{N\mu})$), the estimate (5.76) follows immediately from (5.47), and (5.81) follows from (5.49). Further, let us take $\varphi \in L_{\Phi_2}(\Omega)$, $d(\Phi_2, \varphi) \le 1$. Then from the definition of Orlicz spaces it follows that $\varphi^2 \in L_\Phi(\Omega)$, $d(\Phi, \varphi^2) \le 1$. Therefore, for $i = 1, \dots, d$,

$$\begin{aligned}
\int_\Omega |\rho^\mu u_i^\mu \varphi|\, dx &\le \left(\int_\Omega \rho^\mu |u_i^\mu|^2\, dx\right)^{1/2} \left(\int_\Omega \rho^\mu |\varphi|^2\, dx\right)^{1/2} \\
&\overset{(5.76)}{\le} C \left(\int_\Omega \rho^\mu |\varphi|^2\, dx\right)^{1/2} \\
&\overset{(2.27),\ \mathrm{Ch.1}}{\le} C\big(d(\Psi;\rho^\mu) + d(\Phi;\varphi^2)\big) \overset{(5.76)}{\le} C\,.
\end{aligned}$$

Thus,

$$\|\rho^\mu u_i^\mu\|_{L^\infty(I;L_{\Psi_{1/2}}(\Omega))} = \operatorname*{ess\,sup}_{t\in I} \sup_{\substack{\varphi\in L_{\Phi_2}(\Omega)\\ d(\Phi_2;\varphi)\le 1}} \int_\Omega |\rho^\mu u_i^\mu \varphi|\, dx \le C\,,$$

and (5.77) is proved. Next,

$$\begin{aligned}
&\sup_{\substack{v\in L^2(I;L_{\Phi_2}(\Omega))\\ d(\Phi_2,v(t))\le 1}} \int_0^T \int_\Omega |\rho^\mu u_i^\mu u_j^\mu v|\, dx\, dt \\
&\qquad \overset{(2.27),\ \mathrm{Ch.1}}{\le} \int_0^T \|\mathbf{u}^\mu\|_\infty \Big(\|\rho^\mu \mathbf{u}^\mu\|_{L_{\Psi_{1/2}}(\Omega)} + 1\Big) \\
&\qquad \overset{(5.77)}{\le} C\ \big(\|\mathbf{u}^\mu\|_{L^2(I;W^{1,p}(\Omega)^d)} + 1\big) \overset{(5.76)}{\le} C\,,
\end{aligned}$$

where we have used $W^{1,p}(\Omega)^d \hookrightarrow L^\infty(\Omega)^d$, $p > d$. So (5.78) holds. In order to prove (5.79) and (5.80), let us take $\psi \in L^2(I; W_0^{3,2}(\Omega))$, $\|\psi\|_{L^2(I;W_0^{3,2}(\Omega))} \le 1$. Then it follows from equation (5.21) that

$$\begin{aligned}
\Big|\int_0^T \int_\Omega \frac{\partial \rho^\mu}{\partial t}\psi\, dx\, dt\Big| &\le c(\rho_0) + \int_0^T \int_\Omega \rho^\mu |\mathbf{u}^\mu|\, |\nabla\psi|\, dx \\
&\le c(\rho_0) + \int_0^T \|\rho^\mu \mathbf{u}^\mu\|_{L_{\Psi_{1/2}}} \|\nabla\psi\|_{L_{\Phi_2}} \\
&\overset{(5.73)}{\le} c(\rho_0) + \|\rho^\mu \mathbf{u}^\mu\|_{L^2(I;L_{\Psi_{1/2}}(\Omega)^d)} \|\psi\|_{L^2(I;W_0^{3,2}(\Omega))} \\
&\overset{(5.77)}{\le} C\,,
\end{aligned}$$

which implies (5.79).

Let us consider $\boldsymbol{\varphi} \in L^2(I; W_0^{3,2}(\Omega)^d)$, $\|\boldsymbol{\varphi}\|_{L^2(I;W_0^{3,2}(\Omega)^d)} \leq 1$. Taking (5.22) into consideration, we have

$$\begin{aligned}
&\Big|\int_0^T \int_\Omega \frac{\partial \rho^\mu u_i^\mu}{\partial t} \varphi_i \,dx\,dt\Big| = \Big|\int_0^T \int_\Omega \rho^\mu u_i^\mu \frac{\partial \varphi_i}{\partial t}\,dx\,dt\Big| \\
&\overset{(5.22)}{\leq} \int_0^T \int_\Omega |\rho^\mu u_j^\mu u_i^\mu \frac{\partial \varphi_i}{\partial x_j}| + |\rho^\mu \frac{\partial \varphi_i}{\partial x_i}| + |\rho^\mu f_i \varphi_i|\,dx\,dt \\
&\quad + \int_0^T \int_\Omega |\tau_{ij}(\mathbf{e}(\mathbf{u}^\mu)) e_{ij}(\boldsymbol{\varphi})|\,dx\,dt + \mu \int_0^T (\!(\mathbf{u}^\mu, \boldsymbol{\varphi})\!)_3 dt + c(\rho_0, \mathbf{u}_0) \\
&\leq C \int_0^T \left(\|\rho^\mu |\mathbf{u}^\mu|^2\|_1 + \|\rho^\mu\|_1 + (1 + \|\nabla \mathbf{u}^\mu\|_p)^{p-1}\right) \|\nabla \boldsymbol{\varphi}\|_\infty \,dt \\
&\quad + \int_0^T \|\mathbf{f}\|_\infty \|\boldsymbol{\varphi}\|_\infty \|\rho^\mu\|_1 \,dt + \mu \int_0^T \|\mathbf{u}^\mu\|_{3,2} \|\boldsymbol{\varphi}\|_{3,2}\,dt + c(\rho_0, \mathbf{u}_0) \\
&\overset{(5.76)}{\leq} C \int_0^T \|\boldsymbol{\varphi}\|_{3,2}^2 \,dt \leq C.
\end{aligned}$$

Thus, (5.80) is valid and the proof of Lemma 5.75 is complete. ■

• *The existence of a measure-valued solution*

Let $p > d$. Letting $\mu \to 0+$, and using the estimates (5.76)–(5.80), we will finish the proof of Theorem 5.16. We will first find a subsequence of $(\rho^\mu, \mathbf{u}^\mu)$, denoted again by $(\rho^\mu, \mathbf{u}^\mu)$ such that

$$\rho^\mu \rightharpoonup \rho \qquad \text{weakly-}^* \text{ in } L^\infty(I; L_\Psi(\Omega)), \tag{5.82}$$
$$\mathbf{u}^\mu \rightharpoonup \mathbf{u} \qquad \text{weakly in } L^p(I; W_0^{1,p}(\Omega)^d), \tag{5.83}$$
$$\rho^\mu \to \rho \qquad \text{strongly in } L^2(I; W^{-1,p}(\Omega)), \tag{5.84}$$
$$\rho^\mu \mathbf{u}^\mu \to \rho \mathbf{u} \qquad \text{strongly in } L^2(I; W^{-1,p}(\Omega)^d), \tag{5.85}$$
$$\rho^\mu u_i^\mu u_j^\mu \rightharpoonup \rho u_i u_j \qquad \text{weakly-}^* \text{ in } L^2(I; L_{\Psi_{1/2}}(\Omega)), \tag{5.86}$$

as $\mu \to 0+$.

Let us prove (5.82)–(5.86). The first two assertions follow immediately from (5.76) and the fact that $(L_\Phi(\Omega))^* = L_\Psi(\Omega)$. By (5.74),

$$L_\Psi(\Omega) \hookrightarrow\hookrightarrow W^{-1,p}(\Omega) \hookrightarrow W^{-3,2}(\Omega)\,.$$

Then the Aubin-Lions lemma 2.48 in Chapter 1, (5.76) and (5.79) imply (5.84). Similarly, because of (5.77), (5.80) and

$$L_{\Psi_{1/2}}(\Omega)^d \hookrightarrow\hookrightarrow W^{-1,p}(\Omega)^d \hookrightarrow W^{-3,2}(\Omega)^d\,,$$

we obtain

$$\rho^\mu \mathbf{u}^\mu \to \rho \mathbf{u} \quad \text{strongly in } L^2(I; W^{-1,p}(\Omega)^d)\,.$$

In general we obtain only the existence of $\mathbf{z} \in L^2(I; W^{-1,p}(\Omega)^d)$ such that $\rho^\mu \mathbf{u}^\mu \to \mathbf{z}$. However, for all $\varphi \in \mathcal{D}(I; W_0^{1,p}(\Omega)^d)$,

$$\begin{aligned}\int_0^T \int_\Omega \big(\rho^\mu u_i^\mu - \rho u_i\big)\varphi_i \, dx &= \int_0^T \int_\Omega \big(\rho^\mu - \rho\big) u_i^\mu \varphi_i \, dx \\ &\quad + \int_0^T \int_\Omega \rho\big(u_i^\mu - u_i^\mu\big)\varphi_i \, dx = I_1 + I_2\,.\end{aligned} \tag{5.87}$$

Letting $\mu \to 0+$, $|I_1| \to 0$ due to (5.84) and (5.76), while $I_2 \to 0$ owing to (5.83). Thus, (5.85) holds.

Finally, $\big(L_{\Psi_{1/2}}(\Omega)\big)^* = L_{\Phi_{1/2}}(\Omega)$ and (5.78) gives for $i, j = 1, \dots, d$,

$$\rho^\mu u_i^\mu u_j^\mu \rightharpoonup c_{ij} \text{ weakly-}^* \text{ in } L^2(I; L_{\Psi_{1/2}}(\Omega))\,.$$

Let us show that $c_{ij} = \rho u_i u_j$. Taking $\varphi \in C^\infty(I; \mathcal{D}(\Omega))$, we have

$$\begin{aligned}\int_0^T \int_\Omega &(\rho^\mu u_i^\mu u_j^\mu - \rho u_i u_j)\varphi \, dx \\ &= \int_0^T \int_\Omega (\rho^\mu u_i^\mu - \rho u_i) u_j^\mu \varphi \, dx \\ &\quad + \int_0^T \int_\Omega \rho u_i (u_j^\mu - u_j)\varphi \, dx \\ &\equiv I_1 + I_2\,.\end{aligned} \tag{5.88}$$

Again, $|I_1| \to 0$ due to (5.85) and $\|\mathbf{u}^\mu\|_{L^p(I; W_0^{1,p}(\Omega)^d)} \le c$, and $I_2 \to 0$ due to (5.83) and $\rho u_i \varphi \in W^{-1,p}(\Omega)$. Thus (5.86) holds.

Take $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{D}(\Omega)^d)$, and consider the system (5.21)–(5.22) under the limit as $\mu \to 0+$. Then

$$\begin{aligned}\left|\mu \int_0^T ((\mathbf{u}^\mu, \boldsymbol{\varphi}))_3 \, dt\right| &\le \sqrt{\mu}\left(\mu \int_0^T \|\mathbf{u}^\mu\|_{3,2}^2 \, dt\right)^{1/2} \|\boldsymbol{\varphi}\|_{L^2(I; W_0^{3,2}(\Omega)^d)} \\ &\overset{(5.76)}{\le} c\sqrt{\mu}\,.\end{aligned}$$

Thus,

$$\left|\mu \int_0^T ((\mathbf{u}^\mu, \boldsymbol{\varphi}))_3 \, dt\right| \to 0 \qquad \text{as } \mu \to 0+\,.$$

In order to characterize the limit of the nonlinear term given by $\boldsymbol{\tau}$, we will apply Corollary 2.10 from Chapter 4. Because $\{\nabla \mathbf{u}^{\mu}\}$ is bounded uniformly in $L^p(Q_T)^{d^2}$ and the components of $\boldsymbol{\tau}$ satisfy (5.2), we put in Corollary 2.10 from Chapter 4 $z^j = \nabla \mathbf{u}^{\mu}$, $\boldsymbol{\tau} = \tau_{ij}$, $q = p - 1$, $s = d^2$ and $Q = Q_T$, and we obtain the existence of

$$\nu \in L^{\infty}(Q_T; M(\mathbb{R}^{d^2}))$$

such that

$$\begin{aligned}\lim_{N\to\infty} \int_0^T \int_{\Omega} \tau_{ij}(\mathbf{e}(\mathbf{u}^{\mu})) e_{ij}(\boldsymbol{\varphi})\, dx\, d\tau \\ = \int_0^T \int_{\Omega} e_{ij}(\boldsymbol{\varphi}) \int_{\mathbb{R}^{n^2}} \tau_{ij}(\mathbf{e}(\boldsymbol{\lambda}))\, d\nu_{t,x}(\boldsymbol{\lambda})\, dx\, d\tau\end{aligned}$$

and

$$\nabla \mathbf{u}(t,x) \overset{\text{a.e.}}{=} \int_{\mathbb{R}^{d^2}} \boldsymbol{\lambda}\, d\nu_{t,x}(\boldsymbol{\lambda})\,.$$

The passage to the limit in the other terms appearing in (5.21)–(5.22) follow from (5.82)–(5.86). Justifications of these limiting processes in fact use only modifications of (5.87)–(5.88) and we will leave this to the reader. The existence of a measure-valued solution to the problem $(\mathrm{CF})_{\mathrm{p}}$ is proved.

The proof of Theorem (5.16) is complete.