CHAPTER 3

Young measures and scalar conservation laws

3.1 Introduction

In the previous chapter we have seen how the general strategy of vanishing viscosity method (introduced in the 1950s by LAX [1954] and HOPF [1950]) was successfully applied to obtain a unique weak entropy solution to the Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x_j} f_j(u) = 0\,, &\qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^d\,, \\ u(0,\cdot) = u_0\,, &\qquad \text{in } \mathbb{R}^d\,, \end{aligned} \tag{1.1}$$

for $u_0 \in L^\infty(\mathbb{R}^d)$, $f_j \in C^1(\mathbb{R})$, $j = 1, \ldots, d$, (see Theorem 5.1 in Chapter 2). In this chapter we want to have a closer look at the Young measure technique in the vanishing viscosity method.

Recall that assuming $u_0 \in L^\infty(\mathbb{R}^d)$, $f_j \in C^1(\mathbb{R})$, $j = 1, \ldots, d$, one knows that there are $u^\varepsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ solutions to the parabolic perturbation of (1.1),

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + \frac{\partial}{\partial x_j} f_j(u^\varepsilon) = \varepsilon \Delta u^\varepsilon, &\qquad \varepsilon > 0\,, \\ u^\varepsilon(0,\cdot) = u_0\,, & \end{aligned} \tag{1.2}$$

satisfying

$$\|u^\varepsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^d)} \le c \tag{1.3}$$

(see Theorem 4.48 in Chapter 2).

This uniform estimate implies that for a subsequence (here we use $u^k \equiv u^{\varepsilon_k}$, $\varepsilon_k \to 0+$ as $k \to \infty$),

$$u^k \overset{*}{\rightharpoonup} u \qquad \text{in } L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)\,. \tag{1.4}$$

It is not difficult to show that a weak convergence of the type (1.4) is enough to ensure that

$$\begin{aligned} \frac{\partial u^k}{\partial t} &\to \frac{\partial u}{\partial t} \\ \varepsilon_k \Delta u^k &\to 0 \end{aligned} \qquad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)\,.$$

Thus, there is the second (nonlinear) term in (1.2) which remains to be studied as $\varepsilon_k \to 0+$.

Typically, there are no *a priori* estimates independent of ε on derivatives of u^ε. Hence, the classical compactness argument cannot be applied and one must deal with the weak convergence (1.4). In our situation, we have $f_j \in C^1(\mathbb{R})$, therefore (1.3) implies

$$\|f_j(u^\varepsilon)\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)} \le c\,,$$

and consequently, for a subsequence,

$$f_j(u^k) \overset{*}{\rightharpoonup} \overline{f_j} \qquad \text{in } L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)\,.$$

Unfortunately, for a nonlinear (smooth) function f_j (1.4) is in general not enough to ensure the desired equality

$$\overline{f_j} = f_j(u)\,. \tag{1.5}$$

To see this, let us consider the following example.

Example 1.6 Let $u^n(x) = \sin nx$, $x \in [0, 2\pi]$, $f(y) = y^2$. In this situation we use the following lemma.

Lemma 1.7 *Let $v \in L^2(0, 2\pi)$ be 2π-periodic. Let $v^n(x) \equiv v(nx)$. Then $v^n \rightharpoonup \dfrac{a_0}{2}$ in $L^2(0, 2\pi)$, where $a_0 = \frac{1}{\pi}\int_0^{2\pi} v(x)\,dx$.*

Proof : We use the Fourier series expansion for v to express v^n as

$$v^n(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(knx) + b_k \sin(knx)\right)\,.$$

Then for $0 \le a < b \le 2\pi$,

$$\begin{aligned} \Big| \int_a^b \sum_{k=1}^{\infty} a_k \cos(knx)\,dx \Big| &= \Big| \sum_{k=1}^{\infty} a_k \int_a^b \cos(knx)\,dx \Big| \\ &\le \sum_{k=1}^{\infty} |a_k| \frac{c}{kn} \le \frac{c}{n} \left(\sum_{k=1}^{\infty} |a_k|^2\right)^{\frac12} \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{\frac12} \to 0 \end{aligned}$$

as $n \to \infty$ and the same holds when replacing cos by sin. Therefore

$$\int_0^{2\pi} v^n(x)\chi_{[a,b]}(x)\,dx \longrightarrow \int_0^{2\pi} \frac{a_0}{2}\chi_{[a,b]}(x)\,dx \tag{1.8}$$

as $n \to \infty$. The proof now follows by approximating $\varphi \in L^2(0, 2\pi)$ by step-functions and using (1.8). ∎

Applying this lemma to both u^n and $f \circ u^n$, we get:

$$u^n \rightharpoonup \frac{1}{2\pi}\int_0^{2\pi} \sin x\,dx = 0 \qquad \text{in } L^2(0, 2\pi)\,,$$

$$(u^n)^2 = f(u^n) \rightharpoonup \frac{1}{2\pi}\int_0^{2\pi} \sin^2 x\,dx = \frac{1}{2} \qquad \text{in } L^2(0, 2\pi).$$

Note that also in $L^\infty(0, 2\pi)$ we have $u^n \overset{*}{\rightharpoonup} 0$, $(u^n)^2 \overset{*}{\rightharpoonup} 1/2$ (at least for some subsequence), since bounded functions on bounded domains are taken into account. Thus we see that

$$f(\text{weak-}^*\lim_{n\to\infty} u^n) < \text{weak-}^*\lim_{n\to\infty} f(u^n)\,. \tag{1.9}$$

This example shows that the lack of strong convergence (here especially the oscillation of the bounded sequence $\{\sin nx\}_{n=1}^\infty$) can cause undesired results even in the case of smooth nonlinear f.

In what follows we will see that the composite limits $\overline{f_j}$ can be described by a family of compactly supported probability measures $\{\nu_{t,x}\}_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d}$, called Young measures. Namely, we will see that

$$\overline{f_j}(t, x) = \langle \nu_{t,x}, f_j\rangle \qquad \text{for a.e. } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between continuous functions and measures (to be discussed later).

In this way, the desired equality (1.5) can be obtained by showing that the support of Young measures reduces to a point:

$$\nu_{t,x} = \delta_{u(t,x)} \qquad \text{for a.e. } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d\,. \tag{1.10}$$

In this chapter we will follow the described strategy to prove the existence of a weak entropy solution to scalar conservation laws in one space dimension.

Even if the existence result proved in this chapter (see Theorem 4.15) is weaker than that obtained in the previous chapter (see Theorem 5.1 in Chapter 2), we present it here, since it nicely describes the Young measure approach. Also, in the following chapters the concept of Young measures will be used in a slightly more complicated situation.

3.2 Young measures

Before we formulate the main theorem on the existence of Young measures, we recall some basic notions (see Chapter 1 for more details).

By $C_C(\mathbb{R}^s)$ we denote the space of continuous functions with compact support in $\mathbb{R}^s$ and put $C_0(\mathbb{R}^s) \equiv \overline{C_C(\mathbb{R}^s)}^{\|\cdot\|_\infty}$. Further, $M(\mathbb{R}^s)$ denotes the space of bounded Radon measures on $\mathbb{R}^s$ and

$$\langle \mu, f \rangle \equiv \mu(f) = \int_{\mathbb{R}^s} f\, d\mu\,, \quad \mu \in M(\mathbb{R}^s),\, f \in C_0(\mathbb{R}^s)\,. \tag{2.1}$$

The space of probability measures is then defined as follows:

$$\mathrm{Prob}(\mathbb{R}^s) \equiv \{\mu \in M(\mathbb{R}^s),\, \mu \text{ non-negative}, \mu(\mathbb{R}^s) = 1\}\,. \tag{2.2}$$

The following theorem introduces the concept of Young measures which turns out to be an appropriate tool for describing composite limits of smooth nonlinearities with weakly convergent sequences.

Note that Theorem 2.3 will be proved in a more general setting than will in fact be needed in this chapter. Thus, this theorem indicates that the concept of Young measures as a technical tool for describing the above-mentioned weak composite limits would also make sense in the system case ($s > 1$), if of course L^∞ *a priori* estimates of the type (1.3) were available for solutions of the perturbed parabolic problem. Unfortunately, as far as the authors know, in the general case of systems it is still an open problem to establish uniform control on the amplitude of viscous approximations $\mathbf{u}^\varepsilon$ (see for example DiPerna [1985, p.247]).

Theorem 2.3 (Existence of the Young measures) *Let $\mathbf{u}^n : \mathbb{R}^m \to \mathbb{R}^s$ be an arbitrary sequence of measurable functions for which*

$$\|\mathbf{u}^n\|_{L^\infty(\mathbb{R}^m)^s} \leq c\,. \tag{2.4}$$

Then there exists a (weakly-* convergent) subsequence $\mathbf{u}^{n_k}$ of $\mathbf{u}^n$ and a family of probability measures $\{\nu_y\}_{y\in\mathbb{R}^m}$, called **Young measures***, supported uniformly in a compact set $K \subset \mathbb{R}^s$:*

$$\{\nu_y\}_{y\in\mathbb{R}^m} \subseteq \mathrm{Prob}(\mathbb{R}^s)\,, \quad \mathrm{supp}(\nu_y) \subseteq K \qquad \textit{for a.e. } y \in \mathbb{R}^m\,,$$

which represents the subsequence $\mathbf{u}^{n_k}$ in the following sense:
For any $\mathbf{g} \in C(\mathbb{R}^s)^p$ we have

$$\mathbf{g} \circ \mathbf{u}^{n_k} \overset{*}{\rightharpoonup} \overline{\mathbf{g}} \quad \textit{in } L^\infty(\mathbb{R}^m)^p\,, \tag{2.5}$$

and

$$\overline{\mathbf{g}}(y) = \int_{\mathbb{R}^s} \mathbf{g}(\lambda)\, d\nu_y(\lambda) = \langle \nu_y, \mathbf{g} \rangle \qquad \text{for a.e. } y \in \mathbb{R}^m \,. \tag{2.6}$$

The proof we give is based on the duality Theorem 2.11. First, we have to introduce some useful notions.

Definition 2.7 *Let $Q \subseteq \mathbb{R}^m$ be an open set. The mapping $\nu : Q \to M(\mathbb{R}^s)$ is said to be weak-* measurable, if for all $F \in L^1(Q; C_0(\mathbb{R}^s))$ the function*[†]

$$x \mapsto \langle \nu_x, F(x,\cdot) \rangle = \int_{\mathbb{R}^s} F(x,\lambda)\, d\nu_x(\lambda) \tag{2.8}$$

is measurable. Further, let us define[‡]

$$\|\nu\|_{L^\infty_\omega(Q;M(\mathbb{R}^s))} = \operatorname*{ess\,sup}_{x\in Q} \|\nu_x\|_{M(\mathbb{R}^s)} \,. \tag{2.9}$$

Finally, let

$$\begin{aligned} L^\infty_\omega(Q; M(\mathbb{R}^s)) = \big\{ \nu : Q \to M(\mathbb{R}^s);\, \nu \text{ weak-}^* \text{ measurable}\,, \\ \|\nu\|_{L^\infty_\omega(Q;M(\mathbb{R}^s))} < \infty \big\} \,. \end{aligned} \tag{2.10}$$

Then the following theorem holds.

Theorem 2.11 *Let $Q \subseteq \mathbb{R}^m$ be open. Let $\Phi \in \big(L^1(Q; C_0(\mathbb{R}^s))\big)^*$ be a linear bounded functional. Then there exists a unique $\nu \in L^\infty_\omega(Q; M(\mathbb{R}^s))$ such that*

$$\Phi(F) = \int_Q \langle \nu_x, F(x) \rangle\, dx \qquad \forall\, F \in L^1(Q; C_0(\mathbb{R}^s)) \,,$$

$$\|\Phi\|_{(L^1(Q;C_0(\mathbb{R}^s)))^*} = \|\nu\|_{L^\infty_\omega(Q;M(\mathbb{R}^s))} \,.$$

Proof : Throughout this proof we will use the notation

$$X = L^1(Q; C_0(\mathbb{R}^s)) \,,$$
$$Y = L^\infty_\omega(Q; M(\mathbb{R}^s)) \,.$$

Uniqueness: Let there exist ν^1, $\nu^2 \in Y$, both representing the functional $\Phi \in X^*$. Define $\widetilde{\nu} \equiv \nu^1 - \nu^2$. Let $R > 0$ and $g \in C_0(\mathbb{R}^s)$.

[†] Here and in the sequel we use the standard notation $\nu_x \equiv \nu(x)$, as if measures ν_x were 'parametrized' by x.

[‡] Note that since $\|\nu_x\|_{M(\mathbb{R}^s)} = \sup_{\|f\|_\infty \le 1} |\langle \nu_x, f\rangle|$, the function $x \mapsto \|\nu_x\|_{M(\mathbb{R}^s)}$ is measurable.

Let $\varphi \in \mathcal{D}(\mathbb{R}^m)$ be such that $\varphi = 1$ on $B_R(0)$, $\varphi = 0$ outside $B_{R+1}(0)$, $\varphi \le 1$ on $\mathbb{R}^m$. Then, from the inequality

$$\int_Q |\langle \widetilde{\nu}_x, g\rangle \varphi(x)| \, dx \le \operatorname{meas}(\operatorname{supp}(\varphi)) \, \|\widetilde{\nu}\|_Y \, \|g\|_{C_0(\mathbb{R}^s)} < \infty \,,$$

it follows that the function

$$x \mapsto \langle \widetilde{\nu}_x, g\rangle \, \varphi(x)$$

is an element of the space $L^1(Q)$. Hence, according to the theorem on Lebesgue points, we conclude that for almost all $x_0 \in B_R(0) \cap Q$ we have

$$\lim_{r\to 0} \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \langle \widetilde{\nu}_x, g\rangle \; \varphi(x) \, dx = \langle \widetilde{\nu}_{x_0}, g\rangle \;. \tag{2.12}$$

Note that the null set of x_0s for which (2.12) does not hold depends on g, and that for r small enough $B_r(x_0) \subset B_R(0)$ and hence $\varphi = 1$ on $B_r(x_0)$. Now, for such x_0 and r we define $F_r(x) \equiv g$ whenever $x \in B_r(x_0)$ and $F_r(x) \equiv 0$ otherwise. We have

$$\begin{aligned} \int_{B_r(x_0)} \langle \widetilde{\nu}_x, g\rangle \, \varphi(x) \, dx &= \int_Q \left\langle \nu_x^1 - \nu_x^2, F_r(x) \right\rangle \, dx \\ &= \Phi(F_r) - \Phi(F_r) = 0. \end{aligned}$$

This together with (2.12) gives us for $g \in C_0(\mathbb{R}^s)$ fixed, that

$$\langle \widetilde{\nu}_{x_0}, g\rangle = 0 \tag{2.13}$$

for almost all $x_0 \in B_R(0) \cap Q$. At this point, we use the separability of the function space $C_0(\mathbb{R}^s)$. Namely, we choose a countable dense set $S \subset C_0(\mathbb{R}^s)$. Using the same argument as before we obtain (2.13) for every $g \in S$ and for almost all $x_0 \in B_R(0) \cap Q$. Now, since $\widetilde{\nu}_{x_0} \in M(\mathbb{R}^s)$, as an element of dual space to $C_0(\mathbb{R}^s)$ is uniquely determined by the values on the dense set $S \subset C_0(\mathbb{R}^s)$, we have $\widetilde{\nu}_{x_0} = 0 \in M(\mathbb{R}^s)$ for almost all $x_0 \in B_R(0) \cap Q$. Now take $R \to \infty$ to get $\tilde{\nu}_{x_0} = 0 \in M(\mathbb{R}^s)$ for almost all $x_0 \in Q$ and so $\widetilde{\nu} = 0$ as an element of the space Y, which proves uniqueness.

Existence: As before, we will work with a countable dense set $S \subset C_0(\mathbb{R}^s)$. Let $\Phi \in X^*$. Choose $g \in S$, $h \in L^1(Q)$ and define

$$\begin{aligned} &F_{hg} : Q \to C_0(\mathbb{R}^s)\,, \\ &F_{hg} : x \mapsto h(x) g \quad \text{ for a.e. } x \in Q\,. \end{aligned} \tag{2.14}$$

We show that $F_{hg} \in X$. Indeed, for all $\mu \in M(\mathbb{R}^s)$ the function

$$x \mapsto \langle \mu, F_{hg}(x)\rangle = h(x) \, \langle \mu, g\rangle \tag{2.15}$$

is measurable, which means F_{hg} is weakly measurable and hence, since $C_0(\mathbb{R}^s)$ is separable, we conclude, according to the Pettis theorem on weak and strong measurability (see YOSIDA [1965, p. 130]) that F_{hg} is measurable. Moreover,

$$\|F_{hg}\|_X = \int_Q \|h(x)\, g\|_{C_0(\mathbb{R}^s)}\, dx = \|g\|_{C_0(\mathbb{R}^s)} \int_Q |h(x)|\, dx < \infty \,.$$

Consequently, $F_{hg} \in X$. Now,

$$\begin{aligned} |\Phi(F_{hg})| &\le \|\Phi\|_{X^*} \|F_{hg}\|_X \\ &\le \|\Phi\|_{X^*} \|g\|_{C_0(\mathbb{R}^s)} \|h\|_{L^1(Q)} \\ &\le c(g)\, \|h\|_{L^1(Q)} \,. \end{aligned} \tag{2.16}$$

Hence, defining the mapping

$$\psi_g : h \mapsto \Phi(F_{hg}) \,,$$

we have from (2.16) that $\psi_g \in \big(L^1(Q)\big)^*$. So, there exists a $u_g \in L^\infty(Q)$ such that

$$\Phi(F_{hg}) = \psi_g(h) = \int_Q u_g(x)\, h(x)\, dx \qquad \forall\, h \in L^1(Q) \,. \tag{2.17}$$

Since $u_g \in L^\infty(Q)$ and S is a countable set, there is a null set N such that for all $x \in Q \setminus N$ and all $g \in S$ it makes sense to consider the value $u_g(x)$ at a point x. For any $x \in Q \setminus N$, the mapping

$$g \mapsto u_g(x)$$

is linear and continuous on $C_0(\mathbb{R}^s)$: indeed, using (2.16) and (2.17) it follows that

$$\begin{aligned} |u_g(x)| \le \|u_g\|_{L^\infty(Q)} &= \sup_{\|h\|_{L^1(Q)} \le 1} \left| \int_Q u_g(x)\, h(x)\, dx \right| \\ &\le \sup_{\|h\|_{L^1(Q)} \le 1} |\Phi(F_{hg})| \le \|\Phi\|_{X^*}\, \|g\|_{C_0(\mathbb{R}^s)} \,. \end{aligned}$$

Hence, for any $x \in Q \setminus N$ there exists a $\nu_x \in M(\mathbb{R}^s) = (C_0(\mathbb{R}^s))^*$ such that

$$\langle \nu_x, g \rangle = u_g(x) \qquad \forall\, g \in S \,.$$

This gives us the family of measures $\{\nu_x\}_{x \in Q}$. Now, employing the countability of the set S we conclude that

$$\langle \nu_x, g \rangle = u_g(x) \qquad \forall\, g \in C_0(\mathbb{R}^s) \,, \quad \text{for a.a. } x \in Q \,. \tag{2.18}$$

It follows that $\Phi(F_{hg})$ equals

$$\int_Q u_g(x)\, h(x)\, dx = \int_Q \langle \nu_x, g\rangle\; h(x)\, dx = \int_Q \langle \nu_x, F_{hg}(x)\rangle\; dx\,, \tag{2.19}$$

which is the desired representation formula for all functions of the type $F_{hg} = h\, g \in X$, $h \in L^1(Q)$, $g \in C_0(\mathbb{R}^s)$. But, finite sums of the type

$$\sum_{i=1}^{m} h_i\, g_i\,, \qquad h_i \in L^1(Q)\,,\; g_i \in C_0(\mathbb{R}^s)$$

are dense in X and consequently the representation formula (2.19) holds true for all $F \in X$.

It remains to verify that the mapping

$$\nu : x \mapsto \nu_x$$

is an element of the space Y. For $F_{hg} \in X$ of the type (2.14) the function

$$x \mapsto \langle \nu_x, F_{hg}(x)\rangle = h(x)\; \langle \nu_x, g\rangle = h(x)\, u_g(x)$$

is measurable and the same holds if we use any $F \in X$, because of the density mentioned above. Consequently, ν is weak-* measurable. Moreover,

$$\|\nu\|_Y = \operatorname*{ess\,sup}_{x\in Q} \|\nu_x\|_{M(\mathbb{R}^s)} = \operatorname*{ess\,sup}_{x\in Q} \sup_{\|g\|_{C_0(\mathbb{R}^s)}\le 1} |\,\langle \nu_x, g\rangle\,|$$

and

$$|\,\langle \nu_x, g\rangle\,| = |u_g(x)| \le \|\Phi\|_{X^*}\, \|g\|_{C_0(\mathbb{R}^s)}\,.$$

This implies

$$\|\nu\|_Y \le \|\Phi\|_{X^*}\,. \tag{2.20}$$

Therefore, $\nu \in Y$. Finally,

$$\begin{aligned} \|\Phi\|_{X^*} &= \sup_{\|F\|_X\le 1} \int_Q |\,\langle \nu_x, F(x)\rangle\,|\, dx \\ &\le \sup_{\|F\|_X\le 1} \int_Q \|\nu_x\|_{M(\mathbb{R}^s)} \|F(x)\|_{C_0(\mathbb{R}^s)}\, dx \le \|\nu\|_Y\,, \end{aligned}$$

which together with (2.20) finishes the proof. ■

Now we are ready to prove the existence of the Young measure.

PROOF (of Theorem 2.3):

First note that since $\mathbf{g}$ is continuous and the sequence $\{\mathbf{u}^n\}$ is uniformly bounded in $L^\infty(\mathbb{R}^m)^s$, a uniform bound in $L^\infty(\mathbb{R}^m)^p$ holds for the sequence $\{\mathbf{g}\circ\mathbf{u}^n\}$. Therefore, for any $\mathbf{g}$ we always have a subsequence $\mathbf{u}^{n_k}$ of $\mathbf{u}^n$ such that (2.5) holds. In fact, we show that there is a subsequence $\mathbf{u}^{n_k}$ of $\mathbf{u}^n$ such that (2.5) holds for all $\mathbf{g}\in C(\mathbb{R}^s)^p$.

Note also that (2.6) is equivalent to

$$\overline{g_j}(y) = \int_{\mathbb{R}^s} g_j(\lambda)\,d\nu_y(\lambda) \equiv \langle \nu_y, g_j\rangle \qquad \text{for a.e. } y\in\mathbb{R}^m\,, \tag{2.21}$$

$j = 1,\ldots,p$. Therefore we can prove (2.6) componentwise, i.e., consider $p = 1$ and $g\in C(\mathbb{R}^s)$ without loss of generality.

Note finally that since $\|\mathbf{u}^n\|_{L^\infty}\le c$ uniformly in n, we can consider $g\in C_0(\mathbb{R}^s)$ (instead of $g\in C(\mathbb{R}^s)$) without loss of generality.

Now, define the sequence of probability measures

$$\nu_y^n \equiv \delta_{\mathbf{u}^n(y)} \qquad \forall\, y\in\mathbb{R}^m\,,$$

where δ_a stands for the Dirac measure at the point $a\in\mathbb{R}^s$. Then, for any $F\in L^1(\mathbb{R}^m; C_0(\mathbb{R}^s))$, we find that

$$\left\langle \nu_y^n, F(y)\right\rangle = \int_{\mathbb{R}^s} F(y,\lambda)\,d\nu_y^n(\lambda) = F(y,\mathbf{u}^n(y))$$

is a measurable function of y. Moreover, if we define

$$\nu^n : y\mapsto \nu_y^n\,,$$

we obtain

$$\|\nu^n\|_{L^\infty_\omega(\mathbb{R}^m;M(\mathbb{R}^s))} = \operatorname*{ess\,sup}_{y\in\mathbb{R}^m}\|\delta_{\mathbf{u}^n(y)}\|_{M(\mathbb{R}^s)} = 1\,.$$

Therefore, $\{\nu^n\}$ is uniformly bounded in $L^\infty_\omega(\mathbb{R}^m;M(\mathbb{R}^s))$. Hence, there exists a subsequence still labelled ν^n, such that

$$\nu^n \overset{*}{\rightharpoonup} \nu \quad \text{in } L^\infty_\omega(\mathbb{R}^m;M(\mathbb{R}^s))\,.$$

This means that for all $F\in L^1(\mathbb{R}^m;C_0(\mathbb{R}^s))$

$$\int_{\mathbb{R}^m}\left\langle \nu_y^n, F(y)\right\rangle dy \to \int_{\mathbb{R}^m}\langle \nu_y, F(y)\rangle\, dy, \qquad \text{as } n\to\infty\,. \tag{2.22}$$

Let us now for $\phi\in L^1(\mathbb{R}^m)$ and $g\in C_0(\mathbb{R}^s)$ define the function F

by $F(y) \equiv \phi(y)g$. Then $F \in L^1(\mathbb{R}^m; C_0(\mathbb{R}^s))$. Using this particular F we obtain from (2.22)

$$\int_{\mathbb{R}^m} \phi(y)\, g(\mathbf{u}^n(y))\, dy \to \int_{\mathbb{R}^m} \phi(y)\ \langle \nu_y, g\rangle\ dy, \qquad \text{as } n \to \infty\,. \tag{2.23}$$

But since the function '$y \mapsto \langle \nu_y, g\rangle$' is an element of the space $L^\infty(\mathbb{R}^m)$ for any $g \in C_0(\mathbb{R}^s)$, (2.23) means nothing other than

$$g \circ \mathbf{u}^n \overset{*}{\rightharpoonup} \langle \nu_{(\cdot)}, g\rangle \equiv \overline{g} \qquad \text{in } L^\infty(\mathbb{R}^m)\,.$$

Furthermore, for $g \in C_0(\mathbb{R}^s)$, $g \geq 0$, $\phi \in L^1(\mathbb{R}^m)$, $\phi \geq 0$ we have, according to (2.23),

$$0 \leq \int_{\mathbb{R}^m} \phi(y)\, g(\mathbf{u}^n(y))\, dy \to \int_{\mathbb{R}^m} \phi(y)\ \langle \nu_y, g\rangle\ dy, \qquad \text{as } n \to \infty \tag{2.24}$$

and consequently $\nu_y \geq 0$ for almost all $y \in \mathbb{R}^m$.

Moreover, thanks to the estimate $\|\mathbf{u}^n\|_{L^\infty} \leq c$, we have $\langle \nu_y, g\rangle = 0$ for all g such that $\operatorname{supp} g \cap \overline{B_{c+1}(0)} = \emptyset$. Hence, $\operatorname{supp} \nu_y \subseteq \overline{B_{c+1}(0)} \equiv K$ (uniformly) for almost all $y \in \mathbb{R}^m$.

Finally, choosing $g_0 \equiv 1$ on $\overline{B_{c+2}(0)}$, g_0 continuous with compact support, $|g_0| \leq 1$, we obtain from (2.6)

$$\overline{g_0}(y) = 1 = \langle \nu_y, g_0\rangle \qquad \text{for a.e. } y \in \mathbb{R}^m\,.$$

Since $\|\nu_y\|_{M(\mathbb{R}^s)}$ is a supremum of expressions of this kind, we see that $\|\nu_y\|_{M(\mathbb{R}^s)} \geq 1$. On the other hand, the weak-* lower semicontinuity of the norm implies that

$$\|\nu_y\|_{M(\mathbb{R}^s)} \leq \liminf \|\nu_y^n\|_{M(\mathbb{R}^s)} = 1\,.$$

In such a way, ν_y is a probability measure on $\mathbb{R}^s$, for almost all $y \in \mathbb{R}^m$, and thus the proof concludes. ∎

Remark 2.25 The Young measure ν_y can intuitively be thought of as giving the limit probability distribution of the values $\mathbf{u}^n$ in the neighbourhood of y as $n \to \infty$. More precisely, if $B_\delta(y)$ is a ball centred at y with radius $\delta > 0$, we can define $\nu_y^{n,\delta}$ as

$$\nu_y^{n,\delta}(A) \equiv \operatorname{meas}\big(B_\delta(y)\big)^{-1}\ \operatorname{meas}\{x \in B_\delta(y)\,,\ \mathbf{u}^n(x) \in A\}\,,$$

i.e., '$\nu_y^{n,\delta}$ is the probability distribution of values $\mathbf{u}^n(x)$ for $x \in B_\delta(y)$'. Then it can be shown (see BALL [1989]) that

$$\nu_y = \lim_{\delta \to 0} \lim_{n \to \infty} \nu_{y,\delta}^n\,,$$

where the convergence is weak-* in the sense of measures. We emphasize the word 'neighbourhood' above: the Young measure is not obtained when limiting $\delta \to 0$ first and then (not taking 'neighbourhood' into account) $n \to \infty$.

Example 2.26 In the case of the sequence $\{\sin nx\}$ (see Example 1.6) it can be shown that the corresponding Young measure can be expressed explicitly as (see DiPerna [1985] for details):

$$d\nu_y(\lambda) = \frac{1}{\pi}\frac{d\lambda}{\sqrt{1-\lambda^2}}\chi_{(-1,1)}(\lambda)\,.$$

Then for all $g \in C(\mathbb{R})$, (2.6) gives

$$g(\sin nx) \overset{*}{\rightharpoonup} \overline{g} \quad \text{in } L^\infty(\mathbb{R}), \qquad \text{where}$$
$$\overline{g} = \text{const.} = \frac{1}{\pi}\int_{-1}^{1}\frac{g(\lambda)}{\sqrt{1-\lambda^2}}\,d\lambda\,.$$

In fact, a simple calculation shows that the integral on the right-hand side is equal to

$$\frac{1}{2\pi}\int_0^{2\pi} g(\sin x)\,dx\,,$$

which corresponds to the result of Lemma 1.7.

Further, we have seen in Example 1.6 that the lack of strong convergence of the sequence $\{\sin nx\}$ implies the inequality (1.9) to occur for $f(u) = u^2$ instead of the desired equality. The following lemma shows that this result holds for any strictly convex function g. Moreover, Lemma 2.27 shows that the inequality of the type (1.9) is typical when strictly convex nonlinearities are superposed to a weak-* converging sequence $\mathbf{u}^n \in L^\infty(\mathbb{R}^m)^s$ which does not converge strongly. More precisely, it will be proved that the equality in (1.9) occurs if and only if the Young measure representing the sequence $\{\mathbf{u}^n\}$ is the family of Dirac ones. As we will see in Theorem 2.31, the latter statement holds if and only if $\mathbf{u}^n$ converges strongly in $L^2_{\text{loc}}(\mathbb{R}^m)^s$.

Lemma 2.27 (generalized Jensen's inequality) *Let g be a strictly convex function, $g : \mathbb{R}^s \to \mathbb{R}$. Let μ be a probability measure on $\mathbb{R}^s$ with compact support. Then*

$$\langle \mu, g\rangle \geq g(\langle \mu, \text{Id}\rangle)\,, \tag{2.28}$$

with equality occurring if and only if μ is a Dirac measure.

PROOF : Note that in our notation $\langle \mu, g \rangle$ is a scalar

$$\int_{\mathbb{R}^s} g(\lambda)\, d\mu(\lambda)\,,$$

while $\langle \mu, \mathrm{Id} \rangle$ denotes a vector

$$\left(\int_{\mathbb{R}^s} \lambda_1\, d\mu(\lambda), \int_{\mathbb{R}^s} \lambda_2\, d\mu(\lambda), \ldots, \int_{\mathbb{R}^s} \lambda_s\, d\mu(\lambda) \right) \in \mathbb{R}^s\,.$$

We start the proof by recalling that for any strictly convex $g : \mathbb{R}^s \to \mathbb{R}$ there exists a constant $\beta \in \mathbb{R}^s$ such that

$$g(\lambda) > g(y) + \beta_i(\lambda - y)_i \quad \forall\, \lambda \neq y\,. \tag{2.29}$$

Putting

$$y \equiv \langle \mu, \mathrm{Id} \rangle = \int_{\mathbb{R}^s} \lambda\, d\mu \ \in \mathbb{R}^s$$

we get (since μ is a probability measure):

$$\begin{aligned} \int_{\mathbb{R}^s} \big[g(y) + \beta_i(\lambda_i - y_i) \big]\, d\mu(\lambda) \\ &= g(y) + \beta_i \int_{\mathbb{R}^s} (\lambda_i - y_i)\, d\mu(\lambda) \\ &= g(y) = g(\langle \mu, \mathrm{Id} \rangle)\,. \end{aligned} \tag{2.30}$$

Hence, if $\operatorname{supp} \mu \neq \{y\}$, (2.29) and (2.30) give

$$\langle \mu, g \rangle > g(\langle \mu, \mathrm{Id} \rangle)\,.$$

Otherwise, if $\mu = \delta_y$ then

$$\langle \mu, g \rangle = \int_{\mathbb{R}^s} g(\lambda)\, d\mu(\lambda) = g(y) = g(\langle \mu, \mathrm{Id} \rangle)$$

and equality in (2.28) occurs. ■

We will see in the following theorem that Young measures turn out to be a proper tool in distinguishing between strong and weak convergences of the sequences they represent.

Before the theorem is formulated, let us recall the following fact: if $u^n \overset{*}{\rightharpoonup} u$ weak-* in $L^\infty(\Omega)$, where $\Omega \subseteq \mathbb{R}^m$ is a bounded domain, then clearly $u^n \rightharpoonup u$ weakly in $L^p(\Omega)$, $p \in [1, \infty)$. Moreover, if there exists $r \in [1, \infty)$ such that $u^n \to v$ in $L^r(\Omega)$, then necessarily $u = v$ and $u^n \to u$ in all $L^p(\Omega)$, $p \in [1, \infty)$, cf. EVANS [1990]. We conclude that for uniformly bounded sequences in $L^\infty(\Omega)$ there are just two possibilities: either $u^n \rightharpoonup u$ weakly in $L^p(\Omega)$ for all $p \in [1,\infty)$, or $u^n \to u$ strongly in these spaces. Similarly, if $\Omega \subseteq \mathbb{R}^m$ is an unbounded domain and u^n is a sequence bounded uniformly in $L^\infty(\Omega)$ then a strong convergence of u^n in $L^q_{\rm loc}(\Omega)$ for some $q \in [1,\infty)$ is equivalent to a strong convergence of u^n in $L^p_{\rm loc}(\Omega)$ for all $p \in [1,\infty)$. See for example DiPerna [1985, p. 233] or Evans [1990].

The two cases just mentioned are fully characterized by the properties of the corresponding family of Young measures:

Theorem 2.31 *Let $\|\mathbf{u}^n\|_{L^\infty(\mathbb{R}^m)^s} \le c$, suppose (without loss of generality) that $\mathbf{u}^n \overset{*}{\rightharpoonup} \mathbf{u}$ weak-* in $L^\infty(\mathbb{R}^m)^s$. Let a family $\{\nu_y\}_{y\in\mathbb{R}^m} \subset \mathrm{Prob}(\mathbb{R}^s)$ correspond to this weak-* convergent sequence. Then $\mathbf{u}^n$ converges strongly in $L^2_{\rm loc}(\mathbb{R}^m)^s$ if and only if ν_y reduces to a Dirac measure concentrated at $\mathbf{u}(y)$, i.e.,*

$$\nu_y = \delta_{\mathbf{u}(y)} \qquad \text{for a.e. } y \in \mathbb{R}^m\,. \tag{2.32}$$

Proof : Assume that (2.32) holds. Then (2.5) and (2.6) give

$$\overline{g} \equiv \text{weak-}^*\lim_{n\to\infty} g(\mathbf{u}^n) = g(\mathbf{u})\,, \tag{2.33}$$

for any $g \in C(\mathbb{R}^s)$. Choosing auxiliary functions $g_i(x) = x_i^2$, $i = 1,\dots,s$, one can write for any compact subset $K \subset \mathbb{R}^m$:

$$\begin{aligned}\int_K |\mathbf{u}^n - \mathbf{u}|^2\,dy &= \sum_{i=1}^s \int_K (u_i^n - u_i)^2\,dy \\ &= \sum_{i=1}^s \left[\int_K (u_i^n)^2\,dy - 2\int_K u_i\,(u_i^n - u_i)\,dy - \int_K (u_i)^2\,dy\right]\,.\end{aligned} \tag{2.34}$$

Now, the first integral in the sum converges to $\int_K (u_i)^2\,dy$, since (2.33) gives $(u_i^n)^2 \overset{*}{\rightharpoonup} (u_i)^2$ in $L^\infty(K)$. The second integral converges to zero, since $u_i \in L^\infty(K) \hookrightarrow L^1(K)$ can be viewed as a test function for the sequence $(u_i^n - u_i)$. Hence $\mathbf{u}^n \to \mathbf{u}$ in $L^2_{\rm loc}(\mathbb{R}^m)^s$.

Conversely, if $\{\mathbf{u}^n\}$ converges strongly in $L^2_{\rm loc}(\mathbb{R}^m)^s$ and at the same time is bounded in $L^\infty(\mathbb{R}^m)^s$, one can pass to the limit in the superposition $g \circ \mathbf{u}^n$ for arbitrary $g \in C(\mathbb{R}^s)$. Hence, we obtain from (2.5) and (2.6) that for almost all $y \in \mathbb{R}^m$ it holds

$$\langle \delta_{\mathbf{u}(y)}, g\rangle = g(\mathbf{u}(y)) = \overline{g}(y) = \langle \nu_y, g\rangle\,. \tag{2.35}$$

Repeating this process for all $g \in S$, where S is a countable dense set in $C(\mathbb{R}^s)$, we find that for almost all $y \in \mathbb{R}^m$ measures $\delta_{\mathbf{u}(y)}$ and ν_y coincide on S and (2.32) follows. ∎

Hence, as we have claimed (see (1.10) and earlier claims), the existence proof for hyperbolic equations via the vanishing viscosity method and the Young measure technique consists in showing that the support of the Young measure, representing a weak-* convergent sequence in L^∞, is a point.

At the end of this chapter we show how to prove such a statement using the so-called Murat-Tartar relation, to which we devote the following section.

3.3 The Murat-Tartar relation for non-convex entropies

Up to this point, the systems in more than one space dimension were also permitted. From now on, we restrict ourselves to the case of a scalar equation in one space dimension ($s = d = 1$). We start by proving the following theorem.

Theorem 3.1 (Murat-Tartar relation) *Let $u_0 \in L^\infty(\mathbb{R})$, $f \in C^2(\mathbb{R})$. Let $u^\varepsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ be solutions to (1.2), satisfying a uniform estimate (1.3). Further, let $\{\nu_{t,x}\}$ be the family of Young measures, corresponding to some weak-* converging subsequence u^k of u^ε, $u^k \equiv u^{\varepsilon_k}$, $\varepsilon_k \to 0+$ as $k \to \infty$. Finally, let η, $q \in C^2(\mathbb{R})$ be any (not necessarily convex) entropy-entropy flux pair, i.e. $q' = f'\eta'$ pointwise. Then the Murat-Tartar relation:*

$$\langle \nu_{t,x}, \mathrm{Id}\; q - f\eta \rangle = \langle \nu_{t,x}, \mathrm{Id} \rangle \, \langle \nu_{t,x}, q \rangle - \langle \nu_{t,x}, f \rangle \, \langle \nu_{t,x}, \eta \rangle \,, \tag{3.2}$$

holds for almost all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$.

To prove this theorem, we will need two auxiliary technical lemmas.

Lemma 3.3 (div-curl) *Let $Q \subset \mathbb{R}^2$ be a bounded domain, $Q \in C^{1,1}$. Let*

$$\begin{aligned} w_1^k &\rightharpoonup \overline{w}_1 \,, \qquad & w_2^k &\rightharpoonup \overline{w}_2 \,, \\ v_1^k &\rightharpoonup \overline{v}_1 \,, \qquad & v_2^k &\rightharpoonup \overline{v}_2 \,, \end{aligned} \tag{3.4}$$

weakly in $L^2(Q)$, as $k \to \infty$. Suppose that[§]

$$\{\mathrm{div}(v_1^k, v_2^k), \mathrm{curl}(w_1^k, w_2^k)\} \subset E \,,$$

where E is a compact set in $W^{-1,2}(Q)$. Then, for a subsequence,

$$v_1^k\, w_1^k + v_2^k\, w_2^k \to \overline{v}_1\, \overline{w}_1 + \overline{v}_2\, \overline{w}_2 \qquad \text{in } \mathcal{D}'(Q) \text{ as } k \to \infty \,. \tag{3.5}$$

[§] Recall that $\mathrm{curl}(w_1^k, w_2^k) \equiv \frac{\partial w_2^k}{\partial x_1} - \frac{\partial w_1^k}{\partial x_2}$.

PROOF : Since we have $w_i^k \in L^2(Q)$, it follows from the theory of linear elliptic equations that there are uniquely determined solutions $u_i^k \in W^{2,2}(Q) \cap W_0^{1,2}(Q)$ of Dirichlet problems,

$$\begin{aligned} -\Delta u_i^k &= w_i^k && \text{in } Q\,, \\ u_i^k &= 0 && \text{on } \partial Q\,, \quad i = 1, 2\,. \end{aligned} \tag{3.6}$$

Moreover, u_i^k are bounded in $W^{2,2}(Q)$, since weakly convergent sequences w_i^k are bounded in $L^2(Q)$. Consequently,

$$f^k \equiv \frac{\partial u_1^k}{\partial x_1} + \frac{\partial u_2^k}{\partial x_2}$$

is bounded in $W^{1,2}(Q)$ and therefore

$$f^k \rightharpoonup f \qquad \text{in } W^{1,2}(Q) \tag{3.7}$$

for a subsequence still denoted f^k.

Further we set

$$g_i^k \equiv w_i^k + \frac{\partial f^k}{\partial x_i} = -\Delta u_i^k + \frac{\partial}{\partial x_i} \operatorname{div}(u_1^k, u_2^k) \tag{3.8}$$

to obtain, after straightforward calculation,

$$g_1^k = \frac{\partial r^k}{\partial x_2}, \qquad g_2^k = -\frac{\partial r^k}{\partial x_1}$$

for

$$r^k \equiv \frac{\partial u_2^k}{\partial x_1} - \frac{\partial u_1^k}{\partial x_2} = \operatorname{curl}(u_1^k, u_2^k)\,.$$

Now, from (3.6) one can see that $r^k \in W^{1,2}(Q)$ solve

$$\begin{aligned} -\Delta r^k &= \operatorname{curl}(w_1^k, w_2^k) && \text{in } Q\,, \\ r^k &= s^k && \text{on } \partial Q\,, \end{aligned} \tag{3.9}$$

where s^k can be expressed in terms of traces of partial derivatives of u_i^k. Consequently, due to $W^{1,2}(Q) \hookrightarrow\hookrightarrow L^2(\partial Q)$, s^k lie in a compact set of $L^2(\partial Q)$. But, according to our assumptions, $\operatorname{curl}(w_1^k, w_2^k)$ lie in a compact set of $W^{-1,2}(Q)$. Hence, the linearity of the problem (3.9) together with the continuous dependence on the data

$$\|r^k\|_{1,2} \le c\ \left(\|\operatorname{curl}(w_1^k, w_2^k)\|_{-1,2} + \|s^k\|_{L^2(\partial Q)}\right)$$

gives us that r^k lie in a compact set of $W^{1,2}(Q)$. Finally, it follows that g_i^k lie in a compact set in $L^2(Q)$ and consequently

$$g_i^k \to g_i \qquad \text{in } L^2(Q) \tag{3.10}$$

for a subsequence. It is worth noting at this point that uniqueness of the weak limit in $L^2(Q)$ gives us, together with (3.4), (3.7), (3.8), (3.10), that

$$\overline{w}_i = g_i - \frac{\partial f}{\partial x_i}, \qquad i = 1, 2. \tag{3.11}$$

Now we are ready to prove the convergence (3.5). For $\varphi \in \mathcal{D}(Q)$ we have

$$\int_Q v_i^k w_i^k \varphi \, dx = \int_Q v_i^k (g_i^k - \frac{\partial f^k}{\partial x_i}) \varphi \, dx \equiv I_1 - I_2,$$

where due to (3.10), (3.4),

$$I_1 = \int_Q v_i^k g_i^k \varphi \, dx \longrightarrow \int_Q \overline{v}_i g_i \varphi \, dx.$$

Further, because of (3.4), (3.7) together with Rellich's compactness theorem, and the fact that $\operatorname{div}(v_1^k, v_2^k)$ converges strongly in $W^{-1,2}(Q)$ necessarily to $\operatorname{div}(\overline{v}_1, \overline{v}_2)$ (which follows from the assumptions of the theorem), we have

$$\begin{aligned} I_2 &= \int_Q v_i^k \frac{\partial f^k}{\partial x_i} \varphi \, dx \\ &= -\left\langle \operatorname{div}(v_1^k, v_2^k), f^k \varphi \right\rangle_{W_0^{1,2}(Q)} - \int_Q v_i^k f^k \frac{\partial \varphi}{\partial x_i} \, dx \\ &\longrightarrow \; -\langle \operatorname{div}(\overline{v}_1, \overline{v}_2), f\varphi \rangle_{W_0^{1,2}(Q)} - \int_Q \overline{v}_i f \frac{\partial \varphi}{\partial x_i} \, dx \\ &= \int_Q \overline{v}_i \frac{\partial f}{\partial x_i} \varphi \, dx. \end{aligned}$$

Thus, due to (3.11) we have

$$\int_Q v_i^k w_i^k \varphi \, dx \longrightarrow \int_Q \overline{v}_i (g_i - \frac{\partial f}{\partial x_i}) \varphi \, dx = \int_Q \overline{v}_i \overline{w}_i \varphi \, dx,$$

which proves the assertion. ∎

Lemma 3.12 (Murat) *Let $Q \subset \mathbb{R}^2$ be a bounded domain, $Q \in C^{1,1}$. Let A be a compact set in $W^{-1,2}(Q)$, B be a bounded set in*

$M(Q)$ and C be a bounded set in $W^{-1,p}(Q)$ for some $2 < p \leq \infty$. Further, let $D \subset \mathcal{D}'(Q)$ be such that

$$D \subset (A+B) \cap C\,.$$

Then there exists E, a compact set in $W^{-1,2}(Q)$ such that

$$D \subset E\,.$$

Here $M(Q)$ denotes the space of bounded Radon measures on Q.

PROOF : Let $f^k \in D$, then $f^k = g^k + h^k$, $g^k \in A$, $h^k \in B$, $f^k \in C$. Firstly, solving the equations

$$\begin{aligned} -\Delta v^k &= g^k \qquad \text{in } Q\,,\\ v^k &= 0 \qquad \text{on } \partial Q\,, \end{aligned} \tag{3.13}$$

we get from $g^k \in A$ that v^k lie in a compact set of $W_0^{1,2}(Q)$. Then, using the compactness theorem for measures (cf. Lemma 2.55 in Chapter 1), we deduce from $h^k \in B$ that h^k lie in a compact set of $W^{-1,q}(Q)$ for $1 \leq q < 2$. This, and the L^p-theory of elliptic equations (SIMADER [1972, Section 7]), implies that solutions w^k of the equations

$$\begin{aligned} -\Delta w^k &= h^k \qquad \text{in } Q\,,\\ w^k &= 0 \qquad \text{on } \partial Q \end{aligned} \tag{3.14}$$

lie in a compact set of $W_0^{1,q}(Q)$ for $1 \leq q < 2$. In such a way,

$$u^k \equiv v^k + w^k$$

lie in a compact set of $W_0^{1,q}(Q)$ for $1 \leq q < 2$. Now, since $-\Delta u^k = f^k$, we have

$$\|f^k\|_{-1,q} = \sup\left\{ |\left\langle f^k, \varphi \right\rangle|;\ \varphi \in W_0^{1,q'}(Q), \|\varphi\|_{1,q'} \leq 1 \right\} \leq \|u^k\|_{1,q}$$

and consequently f^k lie in a compact set of $W^{-1,q}(Q)$ for $1 \leq q < 2$. But general interpolation theory provides us with an inequality (TRIEBEL [1978, Section 1.11])

$$\|f^k\|_{-1,2} \leq c\,\|f^k\|_{-1,p}^{\theta}\,\|f^k\|_{-1,q}^{1-\theta} \qquad \text{for} \qquad \frac{1}{2} = \frac{1-\theta}{p'} + \frac{\theta}{q'}\,,$$

which together with $f^k \in C$ (a bounded set in $W^{-1,p}(Q)$) proves the lemma. ■

Now we are ready to prove the Murat-Tartar relation (3.2).

Proof (of Theorem 3.1): Recall that by $u^k \equiv u^{\varepsilon_k}$, $\varepsilon_k \to 0+$ as $k \to \infty$, we denoted some weak-* converging subsequence of u^ε. Multiplying (1.2) by $\eta'(u^k)$, we obtain

$$\frac{\partial \eta(u^k)}{\partial t} + \frac{\partial q(u^k)}{\partial x} = \varepsilon_k \frac{\partial^2 \eta(u^k)}{\partial x^2} - \eta''(u^k)\left(\sqrt{\varepsilon_k}\,\frac{\partial u^k}{\partial x}\right)^2 . \quad (3.15)$$

At this point we use the results on parabolic equations of type (1.2) (cf. Ladyzhenskaya, Solonnikov and Uraltzeva [1968], see Theorem 2.9 in Chapter 2). Namely, we know that for any open bounded set $Q \subset \mathbb{R}^+ \times \mathbb{R}$ there exists constant c, independent of k such that

$$\|\sqrt{\varepsilon_k}\,\frac{\partial u^k}{\partial x}\|_{L^2(Q)} \leq c(Q)\,. \quad (3.16)$$

Now, since $\eta \in C^2$, (1.3) implies $\|\eta''(u^k)\|_{L^\infty(Q)} \leq c$, which together with (3.16) yields

$$\|\eta''(u^k)\left(\sqrt{\varepsilon_k}\,\frac{\partial u^k}{\partial x}\right)^2\|_{L^1(Q)} \leq c.$$

But the same estimate also holds for the $M(Q)$-norm, since $L^1(Q)$ is imbedded into $M(Q)$ isometrically. Hence,

$$\left\{\eta''(u^k)\left(\sqrt{\varepsilon_k}\,\frac{\partial u^k}{\partial x}\right)^2\right\}_{k\in\mathbb{N}} \subset B\,, \quad (3.17)$$

where B is a bounded set in $M(Q)$.

Further, (3.16) gives

$$\begin{aligned}\|\sqrt{\varepsilon_k}\,\frac{\partial \eta(u^k)}{\partial x}\|_{L^2(Q)} &= \|\sqrt{\varepsilon_k}\,\eta'(u^k)\,\frac{\partial u^k}{\partial x}\|_{L^2(Q)} \\ &\leq c\,\|\sqrt{\varepsilon_k}\,\frac{\partial u^k}{\partial x}\|_{L^2(Q)} \leq c.\end{aligned} \quad (3.18)$$

Consequently, $\varepsilon_k \dfrac{\partial \eta(u^k)}{\partial x} \to 0$ in $L^2(Q)$ and therefore we have

$$\varepsilon_k \frac{\partial^2 \eta(u^k)}{\partial x^2} \to 0 \text{ in } W^{-1,2}(Q)\,.$$

In particular,

$$\left\{\varepsilon_k \frac{\partial^2 \eta(u^k)}{\partial x^2}\right\}_{k\in\mathbb{N}} \subset A\,, \quad (3.19)$$

where A is a compact set in $W^{-1,2}(Q)$. Note that for $\eta \equiv \mathrm{Id}$ we immediately obtain from (3.19) that

$$\left\{ \varepsilon_k \frac{\partial^2 u^k}{\partial x^2} \right\}_{k\in\mathbb{N}} \subset E\,, \tag{3.20}$$

where E is a compact set in $W^{-1,2}(Q)$.

Finally, since (1.3) implies $\|\eta(u^k)\|_\infty \leq c$, $\|q(u^k)\|_\infty \leq c$, the left-hand side of (3.15) lies in a bounded set of $W^{-1,\infty}(Q)$. But this is nothing other than

$$\left\{ \varepsilon_k \eta'(u^k) \frac{\partial^2 u^k}{\partial x^2} \right\}_{k\in\mathbb{N}} \subset C\,, \tag{3.21}$$

where C is a bounded set in $W^{-1,\infty}(Q)$.

So, from (3.17), (3.19), (3.21) it follows that we can apply Lemma 3.12 to (3.15). This gives us

$$\begin{aligned} \left\{ \varepsilon_k \eta'(u^k) \frac{\partial^2 u^k}{\partial x^2} \right\}_{k\in\mathbb{N}} &= \left\{ \frac{\partial \eta(u^k)}{\partial t} + \frac{\partial q(u^k)}{\partial x} \right\}_{k\in\mathbb{N}} \\ &= \left\{ \mathrm{curl}_{(t,x)} \left(-q(u^k), \eta(u^k) \right) \right\}_{k\in\mathbb{N}} \subset E\,, \end{aligned} \tag{3.22}$$

where E is some compact set in $W^{-1,2}(Q)$. Eventually we obtain, directly from (3.20), without using Lemma 3.12, that

$$\begin{aligned} \left\{ \varepsilon_k \frac{\partial^2 u^k}{\partial x^2} \right\}_{k\in\mathbb{N}} &= \left\{ \frac{\partial u^k}{\partial t} + \frac{\partial f(u^k)}{\partial x} \right\}_{k\in\mathbb{N}} \\ &= \left\{ \mathrm{div}_{(t,x)} \left(u^k, f(u^k) \right) \right\}_{k\in\mathbb{N}} \subset E\,, \end{aligned} \tag{3.23}$$

where E can be considered to be the same as above. Applying Lemma 3.3 to $w^k = \left(u^k, f(u^k)\right)$, and $v^k = \left(-q(u^k), \eta(u^k)\right)$ we obtain:

$$u^k q(u^k) - f(u^k)\eta(u^k) \to \overline{u}\;\overline{q} - \overline{f}\;\overline{\eta}\,, \tag{3.24}$$

as $k \to \infty$, in the sense of distributions. (Overlines denote the weak limits in $L^2(Q)$.) Since we work with bounded sequences on bounded sets, the left-hand side of (3.24) (at least a subsequence) also converges weak-* in $L^\infty(Q)$ and then necessarily to the right-hand side of (3.24). Using Young measures, this can be expressed as

$$\langle \nu_{t,x}, \mathrm{Id}\; q - f\eta \rangle = \langle \nu_{t,x}, \mathrm{Id} \rangle\; \langle \nu_{t,x}, q \rangle - \langle \nu_{t,x}, f \rangle\; \langle \nu_{t,x}, \eta \rangle\;, \tag{3.25}$$

which is exactly the Murat-Tartar identity (3.2). ∎

In the next section we will see how (3.25) can help us to reduce the support of $\nu_{t,x}$ to a point in the case of scalar equations in one space dimension.

3.4 Scalar hyperbolic equations in one space dimension

In this particular case ($s = d = 1$) we have for $f \in C^1(\mathbb{R})$, $u_0 \in L^\infty(\mathbb{R})$ and $\varepsilon > 0$ a viscous perturbation

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + \frac{\partial f(u^\varepsilon)}{\partial x} &= \varepsilon \frac{\partial^2 u^\varepsilon}{\partial x^2} \qquad && \text{in } \mathbb{R}^+ \times \mathbb{R}\,, \\ u^\varepsilon(0,\cdot) &= u_0 && \text{in } \mathbb{R}, \end{aligned} \tag{4.1}$$

of the scalar hyperbolic equation

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} &= 0 \qquad && \text{in } \mathbb{R}^+ \times \mathbb{R}\,, \\ u(0,\cdot) &= u_0 && \text{in } \mathbb{R}\,. \end{aligned} \tag{4.2}$$

Recall that if we prove the Young measure $\nu_{t,x}$ corresponding to the sequence $\{u^k\} \subset \{u^\varepsilon\}$, $u^k \overset{*}{\rightharpoonup} u$ in $L^\infty(\mathbb{R}^+ \times \mathbb{R})$ to be a Dirac one supported at $u(t,x)$ for almost all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, we obtain two pieces of additional information: firstly, according to Theorem 2.31, u^k converges strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$ and secondly, due to (2.5) and (2.6), $f(u^k) \overset{*}{\rightharpoonup} f(u)$ for any $f \in C^1(\mathbb{R})$. This in fact implies the existence result for the scalar equation in one space dimension, which is formulated and proved at the end of this section.

Note that, since subsequences of u^ε were taken into account, we do not obtain the uniqueness result via the Young measure technique.

Now we show how to use the Murat-Tartar identity (3.2) to obtain the desired result

$$\nu_{t,x} = \delta_{u(t,x)} \qquad \text{for a.e. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}\,. \tag{4.3}$$

There are various approaches how to prove (4.3) using (3.2). In Tartar [1983] and Vecchi [1989] it is shown that if there is no interval $[a,b]$, $a < b$, on which f' is constant (we say that f is **genuinely nonlinear**), (4.3) holds. This is done by applying (3.2) to an infinite family of smooth, strictly convex entropy-entropy flux pairs, which in the scalar 1D case can be expressed explicitly.

Moreover, in Tartar [1983] it is shown that if the scalar flux is a strictly convex function, then it is sufficient to apply (3.2) to a single smooth strictly convex entropy to obtain (4.3).

Another proof is given by Vecchi—see VECCHI [1989, 1990]. He was able to show that even if a genuinely nonlinear function f is not convex, (4.3) holds. This was done by using just one smooth entropy-entropy flux pair. We will follow this approach to obtain the following theorem.

Theorem 4.4 *Let $f \in C^2(\mathbb{R})$ be a genuinely nonlinear function. Let $\eta, q \in C^2(\mathbb{R})$ satisfy $q' = \eta' f'$ pointwise. Let $\nu \in \mathrm{Prob}(\mathbb{R})$ be a probability measure with compact support $K \subset \mathbb{R}$, such that the Murat-Tartar relation*

$$\langle \nu, \lambda q(\lambda) - f(\lambda)\eta(\lambda)\rangle = \langle \nu, \lambda\rangle \, \langle \nu, q(\lambda)\rangle - \langle \nu, f(\lambda)\rangle \, \langle \nu, \eta(\lambda)\rangle \tag{4.5}$$

holds. Then the support of ν is a point.

PROOF (Vecchi): The basic idea of the proof is as follows: Suppose we have a compactly supported probability measure $\nu \in \mathrm{Prob}(\mathbb{R})$ such that $\operatorname{supp} \nu = K$. Then we define an auxiliary product measure $\nu \otimes \nu$ on $\mathbb{R} \times \mathbb{R}$ (points in $\mathbb{R} \times \mathbb{R}$ being denoted by (λ, τ)), $\operatorname{supp}(\nu \otimes \nu) = K \times K$, a rectangle. Then, if we find an integrable function $F(\lambda, \tau)$ such that

$$\begin{gathered} F(\lambda,\tau) > 0 \quad (\lambda \neq \tau)\,, \qquad F(\lambda,\lambda) = 0\,, \\ \int_{K\times K} F(\lambda,\tau)\, d(\nu(\lambda) \otimes \nu(\tau)) = 0\,, \end{gathered} \tag{4.6}$$

clearly the support of $\nu \otimes \nu$ must lie on the line $\{\lambda = \tau\}$. Now, if $\operatorname{supp} \nu$ consists of more than one point, then in the set $\operatorname{supp}(\nu \otimes \nu) = \operatorname{supp} \nu \times \operatorname{supp} \nu$ there are points outside the line $\{\lambda = \tau\}$, which is a contradiction. Hence, $\operatorname{supp} \nu$ is a point. The rigorous proof of this idea (slightly generalized) will be given in Lemma 4.13. Now we prove the theorem by constructing the function F, satisfying (4.6).

The right-hand side of (4.5) can be written (using Fubini's theorem—note that $\nu \otimes \nu$ is a product measure) as

$$\begin{aligned} &\frac{1}{2}\Big[\int_K \tau\, d\nu(\tau) \int_K q(\lambda)\, d\nu(\lambda) + \int_K \lambda\, d\nu(\lambda) \int_K q(\tau)\, d\nu(\tau) \\ &- \int_K \eta(\tau)\, d\nu(\tau) \int_K f(\lambda)\, d\nu(\lambda) - \int_K \eta(\lambda)\, d\nu(\lambda) \int_K f(\tau)\, d\nu(\tau)\Big] \\ &= \frac{1}{2} \int_{K\times K} \big(\tau q(\lambda) + \lambda q(\tau) - \eta(\tau) f(\lambda) - \eta(\lambda) f(\tau)\big)\; d(\nu(\lambda) \otimes \nu(\tau)). \end{aligned} \tag{4.7}$$

On the other hand, since $\int_K d\nu(\lambda) = 1$, the left-hand side is equal to

$$\frac{1}{2}\int_{K\times K}\Big(\tau q(\tau) + \lambda q(\lambda) - \eta(\tau) f(\tau) - \eta(\lambda) f(\lambda)\Big)\; d(\nu(\lambda)\otimes\nu(\tau))\,. \tag{4.8}$$

Denote $[\,a\,]_\lambda^\tau \equiv a(\tau) - a(\lambda)$. Subtracting (4.7) from (4.8) we can write (4.5) as

$$\int_{K\times K}\left((\tau-\lambda)\,[\,q\,]_\lambda^\tau - [\,\eta\,]_\lambda^\tau\,[\,f\,]_\lambda^\tau\right)\, d(\nu(\lambda)\otimes\nu(\tau)) = 0\,. \tag{4.9}$$

Now, we can choose $\eta \equiv f$ and $q(u) \equiv \int_{u_0}^u [f'(s)]^2\, ds$. Denoting the integrand in (4.9) by $F(\lambda,\tau)$, we obtain:

$$F(\lambda,\tau) = (\tau-\lambda)\left(\int_\lambda^\tau \left[f'(s)\right]^2 ds\right) - \big(f(\tau) - f(\lambda)\big)^2\,. \tag{4.10}$$

Clearly, $F(\lambda,\lambda) = 0$, while for $\lambda \neq \tau$

$$F(\lambda,\tau) = (\tau-\lambda)\left(\int_\lambda^\tau \left[f'(s)\right]^2 ds\right) - \left(\int_\lambda^\tau f'(s)\, ds\right)^2\,. \tag{4.11}$$

We recall the classical Jensen's inequality (see e.g. RUDIN [1974]): for any strictly convex function g and $h \in L^1((\lambda,\tau))$ there is:

$$\frac{1}{\tau-\lambda}\int_\lambda^\tau g(h(s))\, ds \geq g\left(\frac{1}{\tau-\lambda}\int_\lambda^\tau h(s)\, ds\right)\,,$$

with equality occurring if and only if $h(s) = \text{const}$ on (λ,τ). Therefore, for $g(s) = s^2$, $h(s) = f'(s) \neq \text{const}$ on (λ,τ), one gets $F(\lambda,\tau) > 0 \quad (\lambda \neq \tau)$. In such a way, the function F satisfies the properties (4.6) and the proof will be finished by proving Lemma 4.13. ■

Remark 4.12 If $f'(s) = \text{const}$ on $[a,b]$, it follows from (4.11) that $F(\lambda,\tau) = 0$ on $[a,b]\times[a,b]$. Consequently, ν can be concentrated either at a point or on the interval, where $f'(s) = 0$, which is in correspondence with the result of TARTAR [1983].

Now we generalize the basic idea used in the proof of Theorem 4.4 for the case of more than one dimension. Note that the expression 'more than one dimension' in this context refers to the dimension s of the space on which the measure μ is defined. Hence, Lemma 4.13 would help to reduce the support of the Young measure to a point in the case of system of s hyperbolic equations, if, of course, there was a function F, satisfying (4.14). However, the problem of finding such a function in general seems to be as difficult as solving the corresponding system of equations. In view of this, the following lemma, even if it holds generally, is of some use only for $s = 1$.

Lemma 4.13 *Let $\mu \in M(\mathbb{R}^s)$ be a non-negative bounded Radon measure,* $\operatorname{supp}\mu = K$, *a compact set. Let $F \in L^1(\mu \otimes \mu)$ satisfy:*

$$\begin{aligned} &F(x,x) = 0 \quad \forall\, x \in \mathbb{R}^s\,, \\ &F(x,y) > 0 \quad \forall\, x,y \in \mathbb{R}^s\,, \quad x \neq y\,, \\ &\int_{\mathbb{R}^{2s}} F(\lambda,\tau)\, d(\mu(\lambda)\otimes\mu(\tau)) = 0\,. \end{aligned} \tag{4.14}$$

Then $\operatorname{supp}\mu$ *is at most a point.*

PROOF : Let us assume that there exist $x \neq y$, $x \in K$, $y \in K$ such that $\{x,y\} \subset \operatorname{supp}\mu$. Then there exist compact disjoint neighbourhoods $U(x)$ and $V(y)$ of x and y, respectively. It follows that $F(\lambda,\tau) > 0$ on $U(x)\times V(y)$. But then we get

$$\begin{aligned} 0 &< \int_{U(x)\times V(y)} F(\lambda,\tau)\, d(\mu(\lambda)\otimes\mu(\tau)) \\ &\leq \int_{\mathbb{R}^{2s}} F(\lambda,\tau)\, d(\mu(\lambda)\otimes\mu(\tau)) = 0\,, \end{aligned}$$

which is a contradiction. ∎

Finally, we formulate the main theorem of this section.

Theorem 4.15 *Let $f \in C^2(\mathbb{R})$ be genuinely nonlinear, $u_0 \in L^\infty(\mathbb{R})$. Then there exists a weak entropy solution $u \in L^\infty(\mathbb{R}^+\times\mathbb{R})$ to the scalar conservation law (4.2).*

PROOF : Let u^ε be solutions to the parabolic perturbations (4.1) of (4.2), $u^k \overset{*}{\rightharpoonup} u$ in $L^\infty(\mathbb{R}^+\times\mathbb{R})$, $u^k = u^{\varepsilon_k}$, $\varepsilon_k \to 0+$ as $k \to +\infty$. Denote by $\{\nu_{t,x}\}$ the family of Young measures corresponding to u^k, which exists according to Theorem 2.3. Then we have for all $g \in C(\mathbb{R})$:

$$g \circ u^k \overset{*}{\rightharpoonup} \overline{g}\,, \quad \overline{g}(t,x) = \langle \nu_{t,x}, g\rangle \qquad \text{for a.e. } (t,x) \in \mathbb{R}^+\times\mathbb{R}\,. \tag{4.16}$$

According to Theorem 3.1, the measures $\nu_{t,x}$ satisfy for almost all $(t,x) \in \mathbb{R}^+\times\mathbb{R}$ the Murat-Tartar relation (3.2) for some entropy-entropy flux pair $\eta, q \in C^2(\mathbb{R})$ and therefore, according to Theorem

4.4 there exists $a = a(t,x)$ such that

$$\nu_{t,x} = \delta_a \qquad \text{for a.e. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}\,. \tag{4.17}$$

But (2.5) for $g = \mathrm{Id}$ implies

$$u(t,x) = \langle \delta_a, \mathrm{Id} \rangle \qquad \text{for a.e. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}\,,$$

and therefore

$$\nu_{t,x} = \delta_{u(t,x)} \qquad \text{for a.e. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}\,. \tag{4.18}$$

Thus, using (2.5) again, we have

$$f(u^k) \overset{*}{\rightharpoonup} f(u)\,, \tag{4.19}$$

which gives that u is a weak solution to (4.2).

Finally, for a convex entropy-entropy flux pair η, $q \in C^2(\mathbb{R})$ we deduce from (3.15) that

$$\frac{\partial \eta(u^k)}{\partial t} + \frac{\partial q(u^k)}{\partial x} \le \varepsilon_k \, \frac{\partial^2 \eta(u^k)}{\partial x^2}\,. \tag{4.20}$$

Similarly to (4.19), (4.18) now implies

$$\eta(u^k) \overset{*}{\rightharpoonup} \eta(u)\,,$$
$$q(u^k) \overset{*}{\rightharpoonup} q(u)\,.$$

Therefore, passing to the limit in (4.20) one finds that u is an entropy solution. Moreover, due to (4.18) and Theorem 2.31, $u^k \to u$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R})$ for all $p \in [1, \infty)$. ∎

Remark 4.21 Young measures were given their name after L.C. Young, who in his pioneering works (YOUNG [1937, 1938, 1942]) studied nonlinear problems developing oscillations of solutions. Since the fundamental papers of TARTAR [1979, 1983], see also TARTAR [1990], the theory of measure-valued solutions became important in the context of partial differential equations. For an extensive study of the Young measure technique we also refer to the monograph by ROUBÍČEK [1996].

The Young measure approach can also be successfully used in numerical analysis, namely to prove the convergence of so-called finite volume methods for scalar conservation laws in d space dimensions—for details see, e.g., COCKBURN, COQUEL, LEFLOCH AND SHU [1991] and KRÖNER AND ROKYTA [1994].