2.6 Conservation laws in bounded domains

In the following sections we would like to show similar results concerning the existence and uniqueness of solutions to the hyperbolic conservation law as in the sections before, but now for bounded smooth domains. Namely, we show that even when boundary and initial data u^D and u^0, respectively, are only in the space L^∞, it is possible to define a well-posed initial-boundary value problem for the scalar hyperbolic conservation law which admits a unique (entropy) solution $u \in L^\infty$. More precisely, uniqueness is established for $\mathbf{f} \in C^1$ (see Theorem 7.28), whereas existence is obtained for $\mathbf{f} \in C^2$ (see Theorem 8.20).

Moreover, here we will also use the **vanishing viscosity** method to obtain the desired result. Our aim is to search for a solution $u \in L^\infty$ even in the case of bounded domains, i.e. in the same class as for the Cauchy problem in the whole space. The main problem in this approach is that in general an L^∞-function does not admit a trace at the boundary. This difficulty was overcome in the elegant paper by OTTO [1992], who introduced the so-called **boundary entropy-entropy flux pairs**, which turned out to be a proper tool to establish the well-posedness of the problem. In a sense, the remainder of this chapter is nothing other than a treatment of a special case from the unpublished PhD thesis of OTTO [1993]. In particular, a part of the exposition and all proofs we give here follow the lines of Otto's thesis.

Let us note that all previous results concerning bounded domains need, to the knowledge of the authors, the solution u of the problem to admit a trace in some sense. For more details we refer to LE ROUX [1977] (he obtained a unique solution $u \in BV \cap L^\infty$, under the assumptions $d = 1$, $u^0, u^D \in BV$, $\mathbf{f} \in C^1$), BARDOS, LE ROUX AND NEDELEC [1979] (they obtained a unique solution $u \in BV$, under the assumptions $u^0, u^D, \mathbf{f} \in C^2$ for $d \geq 1$), LE FLOCH [1988] (he obtained a solution u which is piecewise continuous, under the assumptions $u^0, u^D \in L^\infty$, $\mathbf{f} \in C^1$, $\mathbf{f}$ strictly convex, for u^D depending on $\mathbf{f}$ and $d = 1$. This solution was proved to be unique in the class of piecewise C^1 functions).

In the rest of this chapter, $\Omega \subseteq \mathbb{R}^d$ will always be a bounded smooth domain. Let us denote by $\partial\Omega$ the boundary of Ω and by $\boldsymbol{\nu}(x)$ the outer normal vector to $\partial\Omega$ at a point $x \in \partial\Omega$. As before, we denote $Q_T \equiv (0,T) \times \Omega$ and $\Gamma \equiv (0,T) \times \partial\Omega$.

To denote the closed interval between points a and b, the following notation will be used throughout the rest of this chapter:

$$\mathcal{I}[a,b] \equiv [\min(a,b), \max(a,b)]\,.$$

In what follows we will look for solutions $u \in L^\infty(Q_T)$ of the initial-boundary value problem for the scalar conservation law

$$\begin{aligned} \frac{\partial u}{\partial t} + \operatorname{div}\mathbf{f}(u) = 0 \qquad & \text{in } Q_T, \\ u(0,\cdot) = u^0 \qquad & \text{in } \Omega\,, \\ u = u^D \qquad & \text{on } \Gamma\,. \end{aligned} \tag{6.1}$$

Let $\mathbf{f} \in C^1(\mathbb{R})^d$. We say that $(\eta, \mathbf{q})$, $\mathbf{q} = (q_1, \dots, q_d)$, η, $q_j \in C^2(\mathbb{R})$, is a convex **entropy-entropy flux pair**, if

$$\eta''(z) \geq 0 \quad \text{and} \quad q_j'(z) = \eta'(z)\, f_j'(z)$$

for all $z \in \mathbb{R}$, $j = 1, \dots, d$. By an entropy-entropy flux pair we will always mean the convex one, at least in the rest of this chapter.

We will consider weak entropy solutions $u \in L^\infty(Q_T)$ of the scalar conservation law, i.e., bounded measurable functions u satisfying both the integral identity

$$\int_0^T \int_\Omega u \frac{\partial \psi}{\partial t} + f_j(u) \frac{\partial \psi}{\partial x_j}\, dx\, dt = 0 \tag{6.2}$$

for all $\psi \in \mathcal{D}(Q_T)$, and the entropy inequality

$$\int_0^T \int_\Omega \eta(u) \frac{\partial \varphi}{\partial t} + q_j(u) \frac{\partial \varphi}{\partial x_j}\, dx\, dt \geq 0 \tag{6.3}$$

for all $\varphi \in \mathcal{D}(Q_T)$, $\varphi \geq 0$, and all entropy-entropy flux pairs $(\eta, \mathbf{q})$.

Before we formulate the sense in which boundary data on Γ for solutions of (6.2) and (6.3) will be assumed, let us recall some specific difficulties occurring in the boundary-value problem for hyperbolic conservation laws.

Consider for a moment the parabolic perturbation of (6.1) given by

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + \operatorname{div}\mathbf{f}(u^\varepsilon) - \varepsilon \Delta u^\varepsilon = 0 \qquad & \text{in } Q_T, \\ u^\varepsilon(0,\cdot) = u^0 \qquad & \text{in } \Omega\,, \\ u^\varepsilon = u^D \qquad & \text{on } \Gamma\,, \end{aligned} \tag{6.4}$$

$\varepsilon > 0$. It is well known that for sufficiently smooth $\mathbf{f}$ there exists a unique smooth solution u^ε to given sufficiently smooth initial and boundary value data u^0 and u^D, respectively, satisfying suitable compatibility conditions at $\partial\Omega$—see Section 2.8 or the Appendix for more details.

The situation drastically changes for $\varepsilon = 0$. Let us consider for a moment that there exists some smooth solution u of an equation

$$\frac{\partial u}{\partial t} + \operatorname{div} \mathbf{f}(u) = 0 \qquad \text{in } Q_T\,. \tag{6.5}$$

This solution automatically satisfies both (6.2) and (6.3). Let u have the value $u(p)$ at the point $p = (t, x) \in Q_T$. It is well known (from the method of characteristics) that u is constant along the maximal segment of the characteristic line in Q_T containing point p, i.e., along the segment

$$\sigma \mapsto (t + \sigma, x + \sigma\, \mathbf{f}'(u(p)))\ . \tag{6.6}$$

Now, if this segment intersects both $\{0\} \times \Omega$ and Γ, the solution u would in general be overdetermined if both (independently prescribed) initial and boundary conditions were expected to be assumed in a classical sense. In connection with this observation, we note that since the slope of the segment (6.6) depends on $\mathbf{f}$, it would be appropriate to require a formulation of boundary conditions involving in some sense a function $\mathbf{f}$.

Let us therefore modify the notion of a boundary value problem for (6.1) starting again with smooth solutions to (6.5). Let $(t_0, x_0) \in \Gamma$ be a boundary point for which all the characteristics

$$\sigma \mapsto (t_0 + \sigma, x_0 + \sigma\, \mathbf{f}'(k))$$

emanating from (t_0, x_0) enter into Q_T, as σ increases, independently of the value k they transport, i.e., where

$$\mathbf{f}'(k) \cdot \boldsymbol{\nu}(x_0) < 0 \qquad \forall\, k \in \mathbb{R}\,. \tag{6.7}$$

In such a point the prescribed boundary data should be assumed, i.e.,

$$u(t_0, x_0) = u^D(t_0, x_0)\ .$$

If, on the other hand, the opposite takes place, i.e.,

$$\mathbf{f}'(k) \cdot \boldsymbol{\nu}(x_0) \geq 0 \qquad \forall\, k \in \mathbb{R}\,, \tag{6.8}$$

the prescribed boundary data should be ignored. Of course, for nonlinear $\mathbf{f}$ there can be boundary points $(t_0, x_0) \in \Gamma$ for which neither (6.7) nor (6.8) happens. At these points we would like the following condition to hold: if the prescribed value $u^D(t_0, x_0)$ is not assumed, then the characteristic line, carrying the actual boundary value $u(t_0, x_0)$, should emerge from the interior of Q_T, i.e.,

$$u(t_0, x_0) \neq u^D(t_0, x_0) \quad \text{should imply} \quad \mathbf{f}'(u(t_0, x_0)) \cdot \boldsymbol{\nu}(x_0) \geq 0\,.$$

Unfortunately, the above listed set of 'boundary conditions', derived from the behaviour of a smooth solution to (6.5), is not suitable for the generally non-smooth solutions. (In fact, with these conditions, the uniqueness of, in general, non-smooth solutions, would not be guaranteed—cf. OTTO [1993].)

Let us therefore consider the one-dimensional **Riemann problem** as a model problem for non-smooth solutions. That is, let $u \in L^\infty((0,\infty) \times \mathbb{R})$ satisfy both

$$\int_0^\infty \int_{\mathbb{R}} u \frac{\partial \varphi}{\partial t} + f(u) \frac{\partial \varphi}{\partial x}\, dx\, dt + \int_{\mathbb{R}} u_0 \varphi(0) = 0 \tag{6.9}$$

for all $\varphi \in \mathcal{D}\big((-\infty,\infty) \times \mathbb{R}\big)$, and

$$\int_0^\infty \int_{\mathbb{R}} \eta(u) \frac{\partial \varphi}{\partial t} + q(u) \frac{\partial \varphi}{\partial x}\, dx\, dt \geq 0 \tag{6.10}$$

for all $\varphi \in \mathcal{D}((0,\infty) \times \mathbb{R})$, $\varphi \geq 0$, and all entropy-entropy flux pairs η, $q \in C^2(\mathbb{R})$. Here, the **Riemann initial data** u_0 are defined as

$$u_0(x) = \begin{cases} u_-^0 & \text{for } x < 0\,, \\ u_+^0 & \text{for } x > 0\,, \end{cases} \tag{6.11}$$

u_+^0, u_-^0 being given constants. Therefore, $u_0 \in L^\infty(\mathbb{R})$, and there is a unique $u \in L^\infty\big((0,+\infty) \times \mathbb{R}\big)$ satisfying (6.9)–(6.11), cf. Theorem 5.1.

Now, consider a function $\psi(t,x) \equiv \varphi(at, ax)$ for fixed $a > 0$. We see that ψ is a test function for (6.9) (or (6.10)) if and only if φ is a test function, too. Thanks to this and the particular structure of initial data (6.11) it follows that the weak formulation (6.9)–(6.10) is invariant under transformations of the form

$$(t,x) \mapsto (at, ax)\,, \qquad a > 0\,.$$

Therefore, if u is a weak entropy solution to (6.9)–(6.11), so is $v(t,x) \equiv u(at, ax)$, $a > 0$. Then, the uniqueness result gives us for $a > 0$,

$$u(at, ax) = u(t,x) \quad \text{for a.e. } (t,x) \in (0,\infty) \times \mathbb{R}\,.$$

In other words, u depends on the ratio x/t only. Thus, there exists a $\mu \in L^\infty(\mathbb{R})$ such that

$$u(t,x) = \mu\left(\frac{x}{t}\right). \tag{6.12}$$

Relation (6.12) allows us to construct explicitly a uniquely determined weak entropy solution to the Riemann problem (6.9)–(6.11). We will outline this only for given $u_-^0 < u_+^0$ and $f \in C^2(\mathbb{R})$ with finitely many inflection points. For more general discussion we refer to Godlewski and Raviart [1991].

Let us consider $\widetilde{f} : [u_-^0, u_+^0] \to \mathbb{R}$, the lower convex envelope of f on $[u_-^0, u_+^0]$. Then the interval $[u_-^0, u_+^0]$ splits into finitely many subintervals on which either

1. $\widetilde{f} = f$ and f is convex; or
2. $\widetilde{f}$ is affine and $\widetilde{f} < f$ in the interior of such an interval.

We can number the subintervals in such a way that $[u_{2i}, u_{2i+1}]$ always corresponds to an interval of the first type and $[u_{2i+1}, u_{2i+2}]$ always corresponds to an interval of the second type.

Consider the interval $[u_{2i}, u_{2i+1}]$. In the interior of this interval we have $f'' > 0$. Hence, $(f')^{-1}$ is well-defined and we can put

$$\mu(\alpha) \equiv (f')^{-1}(\alpha) \qquad \text{for } \alpha \in \big(f'(u_{2i}), f'(u_{2i+1})\big). \tag{6.13}$$

In such a way, μ is defined on the whole interval $\big(f'(u_-^0), f'(u_+^0)\big)$ except for the points $\alpha_i = f'(u_{2i+1}) = f'(u_{2i+2})$ (to which the remaining intervals $[f'(u_{2i+1}), f'(u_{2i+2})]$ reduce). At those points the function μ has a jump discontinuity such that

$$\begin{aligned} \mu(\alpha_i-) &= u_{2i+1}, \\ \mu(\alpha_i+) &= u_{2i+2}. \end{aligned} \tag{6.14}$$

Since $\widetilde{f}$ is affine on the intervals $[u_{2i+1}, u_{2i+2}]$ of the second type,

$$\alpha_i = f'(u_{2i+1}) = \frac{f(u_{2i+2}) - f(u_{2i+1})}{u_{2i+2} - u_{2i+1}},$$

which means that for the jump (6.14) the Rankine-Hugoniot condition is fulfilled as the solution $u(t,x)$ jumps across $\{x/t = \alpha_i\}$.

Finally, we define on the remainder of the real axis

$$\begin{aligned} \mu(\alpha) &= u_-^0 \qquad \forall\, \alpha \in (-\infty, f'(u_-^0)], \\ \mu(\alpha) &= u_+^0 \qquad \forall\, \alpha \in [f'(u_+^0), \infty). \end{aligned} \tag{6.15}$$

These are the only intervals on which μ is constant.

Now, (6.13) and (6.15) imply that on $\big(f'(u_{2i}), f'(u_{2i+1})\big)$ we have the equality $f'(\mu(\alpha)) - \alpha = 0$, while on $(-\infty, f'(u_-^0))$ and $(f'(u_+^0), \infty)$ we have $\mu'(\alpha) = 0$. Combining both, we obtain that on $\mathbb{R} \setminus \{\alpha_i\}$ it holds

$$f'\big(\mu(\alpha)\big)\mu'(\alpha) - \alpha\mu'(\alpha) = 0\,. \tag{6.16}$$

From (6.12) we deduce that (taking into account $\alpha = x/t$)

$$\begin{aligned} -\alpha\mu'(\alpha) &= t\frac{\partial u}{\partial t}(t,x)\,, \\ \mu'(\alpha) &= t\frac{\partial u}{\partial x}(t,x)\,, \end{aligned}$$

which converts (6.16) into

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} = 0\,.$$

This means that u, defined by (6.12), (6.13), (6.15), is a classical solution on $(\mathbb{R}^+ \times \mathbb{R}) \setminus \{t = \alpha_i x\}$. Since the Rankine-Hugoniot condition is satisfied on all jumps, we get that u is a weak solution to our Riemann problem.

We show that u is also an entropy solution. Since u is a piecewise C^1 function, the entropy inequality is trivially satisfied in domains where $u \in C^1$. If u jumps from the value u_L to the value u_R, $u_L < u_R$, as x increases, the Rankine-Hugoniot condition

$$s(u_R - u_L) = f(u_R) - f(u_L) \tag{6.17}$$

is fulfilled. Moreover, our construction ensures that the graph of f lies above the chord connecting the values $f(u_L)$ and $f(u_R)$, i.e.,

$$f\big(\lambda u_L + (1-\lambda)u_R\big) \geq \lambda f(u_L) + (1-\lambda)f(u_R)\,, \tag{6.18}$$

for $\lambda \in [0,1]$. Setting $k \equiv \lambda u_L + (1-\lambda)u_R$ we obtain

$$\begin{aligned} f(u_R) + f(u_L) - 2f(k) &\overset{(6.18)}{\leq} (2\lambda - 1)\big(f(u_R) - f(u_L)\big) \\ &\overset{(6.17)}{=} s\,(2\lambda-1)(u_R - u_L) \\ &= s(u_L + u_R - 2k) \\ &= s\left(|u_R - k| - |u_L - k|\right). \end{aligned}$$

Since $u_L \leq k \leq u_R$, the left-hand side can be written as

$$\operatorname{sgn}(u_R - k)\big(f(u_R) - f(k)\big) - \operatorname{sgn}(u_L - k)\big(f(u_L) - f(k)\big)\,.$$

Altogether we have shown that

$$s\big(\widetilde{\eta}(u_R) - \widetilde{\eta}(u_L)\big) \geq \widetilde{q}(u_R) - \widetilde{q}(u_L) \tag{6.19}$$

for a non-smooth entropy-entropy flux pair (cf. (5.2))

$$\begin{aligned} \widetilde{\eta}(u) &= |u-k|\,, \\ \widetilde{q}(u) &= \operatorname{sgn}(u-k)\big(f(u)-f(k)\big)\,, \end{aligned} \tag{6.20}$$

with $u_L \leq k \leq u_R$. Further, for $k \leq u_L$ or $u_R \leq k$, the equality in (6.19) follows trivially from (6.17). Therefore, the entropy inequality is satisfied for all $\widetilde{\eta}(u) = |u-k|$, $k \in \mathbb{R}$, and corresponding $\tilde{q}$s. As we have already observed (see e.g. Remark 5.7), this implies that u is an entropy solution. We conclude that we have constructed the only weak entropy solution of the problem (6.9)–(6.11).

From the construction one easily deduces that μ is a monotone function, so that $\lim_{\alpha\to 0-}\mu(\alpha)$ exists. Moreover, this limit is equal to the lowest point at which a global minimum of f on the interval $[u_-^0, u_+^0]$ is achieved. In other words, in the case of $u_-^0 \leq u_+^0$

$$\begin{aligned} \mu(0-) &\equiv \lim_{\alpha\to 0-}\mu(\alpha) \\ &= \min\{z \in \mathcal{I}\big[u_-^0, u_+^0\big] : f(z) \leq f(k)\ \forall\, k \in \mathcal{I}\big[u_-^0, u_+^0\big]\}\,. \end{aligned}$$

Similarly, in the case of $u_-^0 \geq u_+^0$ it can be shown that

$$\mu(0-) = \max\{z \in \mathcal{I}\big[u_+^0, u_-^0\big] : f(k) \leq f(z)\ \forall\, k \in \mathcal{I}\big[u_+^0, u_-^0\big]\}\,.$$

In both cases we particularly have that for all $k \in \mathcal{I}\big[\mu(0-), u_+^0\big]$,

$$\operatorname{sgn}(\mu(0-) - u_+^0)\big(f(\mu(0-)) - f(k)\big) \geq 0\,. \tag{6.21}$$

Let $\widetilde{u}$ be the solution of the Riemann problem above, but now with initial data given by $u_-^0, \widetilde{u}_+^0$. Let us assume that also in this situation a condition analogous to (6.21) is fulfilled, namely that

$$\operatorname{sgn}(\widetilde{\mu}(0-) - u_+^0)\big(f(\widetilde{\mu}(0-)) - f(k)\big) \geq 0 \tag{6.22}$$

for all $k \in \mathcal{I}\big[\widetilde{\mu}(0-), u_+^0\big]$, where

$$\widetilde{\mu}\Big(\frac{x}{t}\Big) = \widetilde{u}(t,x)\,. \tag{6.23}$$

From the definition of $\mu(0-)$ and $\widetilde{\mu}(0-)$ and assumption (6.22) it can be seen that

$$\widetilde{\mu}(0-) = \mu(0-)\,. \tag{6.24}$$

Let us show this equality in the case of $u_-^0 < \widetilde{u}_+^0 < u_+^0$.

Suppose that (6.24) does not hold. Since $\widetilde{\mu}(0-) \in [u_-^0, \widetilde{u}_+^0]$ and $\mu(0-) \in [u_-^0, u_+^0]$ are the lowest points (of the respective intervals) at which a global minimum of f is achieved, the inequality in (6.24) would imply that $\mu(0-) \notin [u_-^0, \widetilde{u}_+^0]$. Consequently, $u_-^0 \le \widetilde{\mu}(0-) \le \widetilde{u}_+^0 \le \mu(0-) \le u_+^0$ and $f(\mu(0-)) < f(\widetilde{\mu}(0-))$. On the other hand, (6.22) implies $f(\widetilde{\mu}(0-)) \le f(k)$ for all $k \in [\widetilde{\mu}(0-), u_+^0]$, which is a contradiction for $k = \mu(0-)$. All remaining cases can be discussed similarly.

Now, since μ and $\widetilde{\mu}$ are constructed using the same nonlinearity f (cf. (6.13)–(6.15)), and achieving the same value both in $-\infty$ and $0-$, we conclude that

$$\widetilde{\mu}(\alpha) = \mu(\alpha) \qquad \forall\, \alpha \in (-\infty, 0)$$

independently on the difference between u_+^0 and $\widetilde{u}_+^0$.

It follows that the solution u of the Riemann problem corresponding to the initial data u_-^0, u_+^0 coincides in $(0,\infty) \times (-\infty, 0)$ with the solution $\widetilde{u}$ of the Riemann problem with initial data $u_-^0, \widetilde{u}_+^0$ if and only if

$$\operatorname{sgn}\big(\widetilde{u}(t,0-) - u_+^0\big)\big(f(\widetilde{u}(t,0-)) - f(k)\big) \ge 0 \tag{6.25}$$

holds for all $k \in \mathcal{I}\big[\widetilde{u}(t,0-), u_+^0\big]$ and $t \in (0,\infty)$ (cf. (6.23)).

Thus we see that for the 'Riemann' initial-boundary problem

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} &= 0 \qquad \text{in } (0,\infty) \times (-\infty, 0)\,, \\ u(0,x) &= u^0 \qquad \text{on } (-\infty, 0)\,, \\ u(t,0) &= u^D \qquad \text{on } (0,\infty)\,, \end{aligned} \tag{6.26}$$

$u^0 = \text{const.}$, $u^D = \text{const.}$, the boundary condition $u(t,0) = u^D$ considered in the sense of (6.25) ensures the uniqueness of the entropy solution. In this special case there are only two possibilities: if $u(t,0) = u^D$ for all $t \in (0,\infty)$, (6.25) is satisfied trivially. On the other hand, if $u(t,0) \ne u^D$, our construction ensures that $u(t,0)$ is determined by (6.25) uniquely, independently of the prescribed value u^D (cf. Lemma 7.24).

The aim of this section was to convince the reader that the heuristically developed condition (6.25) plays a crucial role in the problem of uniqueness. We will see that in the general situation the formulation of a well-posed boundary value problem will be an infinitesimal copy of the one above.

As it turns out, the problem of uniqueness is more important than the problem of existence in the context of bounded domains. Therefore, we will study the problem of uniqueness in the following section. After that, an existence result will be presented.

2.7 Uniqueness in bounded domains

First of all we need a new notion, namely that of a so-called boundary entropy-entropy flux pair.

Definition 7.1 *The pair $(H,\mathbf{Q})$, $\mathbf{Q} = (Q_1, \dots, Q_d)$, $H, Q_j \in C^2(\mathbb{R}^2)$ is called a* **boundary entropy-entropy flux pair** *if and only if for all $w \in \mathbb{R}$, $(H(\cdot,w), \mathbf{Q}(\cdot,w))$ is an entropy-entropy flux pair in the sense of Definition 3.22 and $H, \mathbf{Q}$ satisfy*

$$H(w,w) = 0\,, \quad \mathbf{Q}(w,w) = 0\,, \quad \partial_1 H(w,w) = 0\,.$$

Here, $\partial_1 H$ denotes the partial derivative with respect to the first variable.

Next we will give a definition of a weak solution to the initial-boundary value problem (6.1). This definition as well as the interpretation of boundary entropy-entropy flux pairs will be discussed later. In particular, Lemma 7.24 will show that Definition 7.2 is really motivated by the considerations from the previous section.

Definition 7.2 *Let $u^0 \in L^\infty(\Omega)$, $u^D \in L^\infty(\Gamma)$, $\mathbf{f} \in C^1(\mathbb{R})^d$. We say that u is a weak solution of the initial-boundary value problem (6.1) if and only if*

$$u \in L^\infty(Q_T)\,, \tag{7.3}$$

and satisfies:

- *the conservation law and the entropy condition in the sense*

$$\int_{Q_T} \eta(u)\frac{\partial \varphi}{\partial t} + q_i(u)\frac{\partial \varphi}{\partial x_i}\,dx\,dt \geq 0 \tag{7.4}$$

  for all $\varphi \in \mathcal{D}_0^\infty(Q_T)$, $\varphi \geq 0$ and all entropy-flux pairs $(\eta, \mathbf{q})$;
- *the boundary condition $u^D \in L^\infty(\Gamma)$ in the sense*

$$\operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathbf{Q}(u(r + s\,\boldsymbol{\nu}(r)), u^D(r)) \cdot \boldsymbol{\nu}(r)\beta(r)\,dr \geq 0 \tag{7.5}$$

  for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere, and all boundary entropy-entropy flux pairs $(H, \mathbf{Q})$;

- *the initial condition* $u^0 \in L^\infty(\Omega)$ *in the sense*

$$\operatorname*{ess\,lim}_{t\to 0+} \int_\Omega |u(t,x) - u^0(x)|\,dx = 0 \ . \tag{7.6}$$

Remark 7.7 Note that the weak formulation (6.2) of the conservation law $(6.1)_1$ is included in (7.4). Indeed, for a special choice of $\eta = \pm\,\mathrm{Id}$ one obtains (6.2) for all non-negative test functions. Equation (6.2) then follows by the usual method of splitting any test function into its positive and negative parts which are mollified appropriately. For further details see also Remark 7.22. The condition (7.6) is quite usual (see for example KRUŽKOV [1970, Definition 1]). On the other hand, the boundary condition (7.5) was introduced by OTTO [1992]. The formulation of (7.5) is motivated heuristically (see previous section). However, we will see that it has a precise mathematical interpretation, which corresponds to our understanding of boundary conditions (see Lemma 7.24, Lemma 7.26).

Remark 7.8

- Let $k \in \mathbb{R}$ be arbitrary but fixed. Define, for $\ell \in \mathbb{N}$, the entropy-entropy flux pair $(\eta_\ell, \mathbf{q}_\ell)$ by

$$\begin{aligned} \eta_\ell(z) &\equiv \left((z-k)^2 + \left(\frac{1}{\ell}\right)^2 \right)^{1/2} - \frac{1}{\ell}\,, \\ \mathbf{q}_\ell(z) &\equiv \int_k^z \eta_\ell'(r)\mathbf{f}'(r)\,dr \ . \end{aligned}$$

  Obviously, $(\eta_\ell, \mathbf{q}_\ell)$ converges uniformly, as $\ell \to \infty$, to a non-smooth entropy-entropy flux pair

$$(|z-k|, \mathbf{F}(z,k))\,,$$

  where

$$\mathbf{F}(z,k) \equiv \begin{cases} \mathbf{f}(k) - \mathbf{f}(z) & \text{for } z \le k\,, \\ \mathbf{f}(z) - \mathbf{f}(k) & \text{for } z \ge k\,. \end{cases} \tag{7.9}$$

  Define a boundary entropy-entropy flux pair $(H_\ell, \mathbf{Q}_\ell)$ by

$$\begin{aligned} H_\ell(z,w) &\equiv \left(\big(\operatorname{dist}(z, \mathcal{I}[w,k])\big)^2 + \left(\frac{1}{\ell}\right)^2 \right)^{1/2} - \frac{1}{\ell}\,, \\ \mathbf{Q}_\ell(z,w) &\equiv \int_w^z \partial_1 H_\ell(\lambda, w)\mathbf{f}'(\lambda)\,d\lambda \ , \end{aligned} \tag{7.10}$$

where $\mathcal{I}[w,k]$ is the closed interval with end points w, k. Obviously, this sequence converges uniformly, as $\ell \to \infty$, to

$$\big(\operatorname{dist}\big(z, \mathcal{I}[w,k]\big), \boldsymbol{\mathcal{F}}(z,w,k)\big)$$

where $\boldsymbol{\mathcal{F}} \in C(\mathbb{R}^3)^d$ is given by

$$\boldsymbol{\mathcal{F}}(z,w,k) \equiv \begin{cases} \left.\begin{array}{ll} \mathbf{f}(w) - \mathbf{f}(z) & \text{for } z \le w \\ \mathbf{0} & \text{for } w \le z \le k \\ \mathbf{f}(z) - \mathbf{f}(k) & \text{for } k \le z \end{array}\right\} & \text{and } w \le k\,, \\ \left.\begin{array}{ll} \mathbf{f}(k) - \mathbf{f}(z) & \text{for } z \le k \\ \mathbf{0} & \text{for } k \le z \le w \\ \mathbf{f}(z) - \mathbf{f}(w) & \text{for } w \le z \end{array}\right\} & \text{and } w \ge k\,. \end{cases} \tag{7.11}$$

- Assume $g \in L^\infty(0,\delta)$. Then we have

$$\liminf_{n\to\infty}\ n\int_0^{1/n} g(t)\,dt \ge \operatorname{ess}\liminf_{t\to 0+}\ g(t)\,,$$

  where $\operatorname{ess}\liminf_{t\to 0+} g(t) \equiv \lim_{n\to\infty} \operatorname{ess}\inf_{(0,1/n)} g(t)$. Indeed, we have for almost every $t \in (0, 1/n)$

$$g(t) \ge \operatorname*{ess\,inf}_{(0,1/n)}\ g(t)$$

  and thus

$$n\int_0^{1/n} g(t)\,dt \ge \operatorname*{ess\,inf}_{(0,1/n)}\ g(t)$$

  and the assertion follows.
- Similarly we get

$$\limsup_{n\to\infty}\ n\int_0^{1/n} g(t)\,dt \le \operatorname{ess}\limsup_{t\to 0+}\ g(t).$$

  Therefore, combining the above results, we obtain the following assertion: if for $g \in L^\infty(0,\delta)$ there exists $\operatorname{ess}\lim_{t\to 0+} g(t)$, we have

$$\lim_{n\to\infty} n\int_0^{1/n} g(t)\,dt = \operatorname{ess}\lim_{t\to 0+}\ g(t).$$

Some interesting consequences of (7.4)–(7.6) are collected in the following lemma.

Lemma 7.12 *Let* $\mathbf{F}$ *and* $\mathcal{F}$ *be defined by (7.9) and (7.11), respectively.*

- *Let* $u \in L^\infty(Q_T)$ *satisfy (7.4); then*

$$\int_0^T \int_\Omega |u-k|\frac{\partial \varphi}{\partial t} + F_i(u,k)\frac{\partial \varphi}{\partial x_i}\,dx\,dt \geq 0 \tag{7.13}$$

  for all $\varphi \in \mathcal{D}(Q_T)$, $\varphi \geq 0$, *and all* $k \in \mathbb{R}$.

- *Let* $u \in L^\infty(Q_T)$ *satisfy (7.4), (7.6); then*

$$\begin{aligned} -\int_0^T \int_\Omega |u-k|\frac{\partial \varphi}{\partial t} + F_i(u,k)\frac{\partial \varphi}{\partial x_i}\,dx\,dt &\leq \int_\Omega |u^0-k|\varphi(0)\,dx \\ &- \operatorname*{ess\,lim\,inf}_{s\to 0-} \int_\Gamma \mathbf{F}(u(r+s\boldsymbol{\nu}),k)\cdot \boldsymbol{\nu}(r)\varphi(r)\,dr \end{aligned} \tag{7.14}$$

  for all $\varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^d)$, $\varphi \geq 0$, *and all* $k \in \mathbb{R}$.

- *Let* $u \in L^\infty(Q_T)$ *satisfy (7.5); then*

$$\operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathcal{F}\big(u(r+s\,\boldsymbol{\nu}(r)), u^D(r), k\big)\cdot \boldsymbol{\nu}(r)\,\beta(r)\,dr \geq 0 \tag{7.15}$$

  for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ *almost everywhere, and all* $k \in \mathbb{R}$.

Proof : The proofs of the first and the third assertion follow from the first part of Remark 7.8.

It remains to show the second statement. In order to make the argument simpler, we restrict ourselves to the case of a half-space, i.e.,

$$\begin{aligned} \Omega &= \{x=(x',x_d)\in \mathbb{R}^{d-1}\times\mathbb{R}; x_d<0\}\,, \\ \boldsymbol{\nu} &= (0,\dots,0,1)\,, \\ \Gamma &= (0,T)\times\mathbb{R}^{d-1}\,, \qquad r=(t,x')\in\Gamma\,, \\ Q_T &= \{p=(r,x_d); r\in\Gamma, x_d<0\}. \end{aligned}$$

The general case can be obtained by the usual covering argument, i.e., by considering that $\partial\Omega$ can be locally replaced by the border of a half-space. Notice that in this case we get an additional dependence of the nonlinearity on the space variable x, but this implies no additional substantial difficulties and the argument follows the same lines.

Let $(\eta, \mathbf{q})$ be an entropy-entropy flux pair and let us define $\beta_n, n \in \mathbb{N}$, by

$$\beta_n(t,x) \equiv \begin{cases} -n x_d & \text{for } t \geq -x_d, \quad x_d \in (-1/n, 0)\,, \\ nt & \text{for } x_d \leq -t, \quad t \in (0, 1/n)\,, \\ 0 & \text{for } x_d = 0, \quad t \geq 0 \text{ or } t = 0, \quad x_d \leq 0\,, \\ 1 & \text{for } x_d \leq -1/n, \quad t \geq 1/n\,. \end{cases}$$

Let $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$, $\varphi \geq 0$ and use $\varphi_n \equiv \varphi \beta_n$ as a test function in (7.4). We obtain:

$$\begin{aligned} 0 &\leq \int_0^T \int_\Omega \eta(u) \Big[\frac{\partial \varphi}{\partial t} \beta_n + \varphi \frac{\partial \beta_n}{\partial t} \Big] + q_i(u) \Big[\frac{\partial \varphi}{\partial x_i} \beta_n + \varphi \frac{\partial \beta_n}{\partial x_i} \Big]\, dx\, dt \\ &\equiv I_1 + \cdots + I_4 . \end{aligned} \tag{7.16}$$

Note that β_n is not smooth enough to be a test function in a classical sense. Nevertheless, (7.16) is meaningful, as the derivatives of β_n are defined almost everywhere on Q_T.

Since $\beta_n \overset{\text{loc}}{\rightrightarrows} 1$ (uniformly on compact sets, as $n \to \infty$) and φ has compact support, we have, for the first and third integral in (7.16),

$$\begin{aligned} \lim_{n \to \infty} \int_0^T \int_\Omega \eta(u) \frac{\partial \varphi}{\partial t} \beta_n \, dx\, dt &= \int_0^T \int_\Omega \eta(u) \frac{\partial \varphi}{\partial t}\, dx\, dt\,, \\ \lim_{n \to \infty} \int_0^T \int_\Omega q_i(u) \frac{\partial \varphi}{\partial x_i} \beta_n \, dx\, dt &= \int_0^T \int_\Omega q_i(u) \frac{\partial \varphi}{\partial x_i}\, dx\, dt. \end{aligned} \tag{7.17}$$

From the definition of β_n it is clear that we can consider

$$\frac{\partial \beta_n}{\partial t} = \begin{cases} n & \text{for } x_d \leq -t, t \in (0, 1/n)\,, \\ 0 & \text{elsewhere,} \end{cases}$$

and therefore

$$\begin{aligned} I_2 &= n \int_0^{1/n} \int_{-\infty}^{-t} \int_{\mathbb{R}^{d-1}} \eta(u(t,x))\, \varphi(t,x)\, dx'\, dx_d\, dt \\ &= n \int_0^{1/n} \int_{-\infty}^{0} \int_{\mathbb{R}^{d-1}} \eta(u(t,x))\, \varphi(t,x)\, dx'\, dx_d\, dt \\ &\qquad - n \int_0^{1/n} \int_{-t}^{0} \int_{\mathbb{R}^{d-1}} \eta(u(t,x))\, \varphi(t,x)\, dx'\, dx_d\, dt \\ &\equiv I_5 + I_6 . \end{aligned}$$

We show that the integral I_6 tends to zero for $n \to \infty$. Indeed,

$$
\begin{aligned}
|I_6| &\le n \int_0^{1/n} \int_{-1/n}^0 \int_{\mathbb{R}^{d-1}} |\eta(u(t,x))\, \varphi(t,x)|\, dx'\, dx_d\, dt \\
&\le n\, \frac{1}{n^2} c_0 \;\to 0 \qquad \text{as } n \to \infty,
\end{aligned}
\tag{7.18}
$$

where

$$
c_0 \equiv c_0(\eta, \varphi, \|u\|_\infty) < \infty\,.
$$

Further we have

$$
\begin{aligned}
I_5 &= n \int_0^{1/n} \int_\Omega \eta(u(t,x))\, \varphi(t,x)\, dx\, dt \\
&\equiv n \int_0^{1/n} g(t)\, dt\,.
\end{aligned}
$$

Note that $\operatorname{ess\,lim}_{t\to 0+} g(t) = \int_\Omega \eta(u^0)\, \varphi(0)\, dx$ exists thanks to (7.6) and therefore due to the third part of Remark 7.8 we have

$$
\lim_{n\to\infty} n \int_0^{1/n} \int_\Omega \eta(u(t,x))\, \varphi(t,x)\, dx\, dt = \int_\Omega \eta(u^0) \varphi(0)\, dx. \tag{7.19}
$$

The integral I_4 can be treated in a similar way. Using

$$
\frac{\partial \beta_n}{\partial x_i} = \begin{cases} -n\nu_i & \text{for } t \ge -x_d, \quad x_d \in (-1/n, 0)\,, \\ 0 & \text{elsewhere,} \end{cases}
$$

we find

$$
\begin{aligned}
I_4 &= -n \int_{-1/n}^0 \int_{-x_d}^T \int_{\mathbb{R}^{d-1}} \nu_i q_i(u(t,x))\, \varphi(t,x)\, dx'\, dt\, dx_d \\
&= -n \int_{-1/n}^0 \int_0^T \int_{\mathbb{R}^{d-1}} \nu_i q_i(u(t,x))\, \varphi(t,x)\, dx'\, dt\, dx_d \\
&\qquad - n \int_{-1/n}^0 \int_0^{-x_d} \int_{\mathbb{R}^{d-1}} \nu_i q_i(u(t,x))\, \varphi(t,x)\, dx'\, dt\, dx_d \\
&\equiv I_7 + I_8\,.
\end{aligned}
$$

Again we can show that (cf. (7.18))

$$
\lim_{n\to\infty} I_8 = 0 \tag{7.20}
$$

and rewrite I_7 as

$$I_7 = -n \int_{-1/n}^{0} h(x_d)\, dx_d\, .$$

From the second part of Remark 7.8 and $\Gamma = (0,T) \times \mathbb{R}^{d-1}$ it follows that

$$\begin{aligned} \liminf_{n\to\infty} & \int_{-1/n}^{0} \int_0^T \int_{\mathbb{R}^{d-1}} n\, \nu_i q_i(u(t,x',x_d))\varphi(t,x',x_d)\, dx'\, dt\, dx_d \\ & \geq \operatorname{ess}\liminf_{s\to 0-} \int_\Gamma q_i(u(r+s\boldsymbol{\nu}))\nu_i \varphi(r)\, dr \qquad (7.21) \\ & + \operatorname{ess}\liminf_{s\to 0-} \int_\Gamma q_i(u(r+s\boldsymbol{\nu}))\nu_i \big(\varphi(r+s\boldsymbol{\nu}) - \varphi(r)\big)\, dr\, , \end{aligned}$$

where the last term is zero due to the smoothness of φ. Hence, from (7.16)–(7.21) we have

$$\begin{aligned} 0 \leq & \int_0^T \int_\Omega \eta(u)\frac{\partial\varphi}{\partial t} + q_i(u)\frac{\partial\varphi}{\partial x_i}\, dx\, dt \\ & + \int_\Omega \eta(u^0)\varphi(0)\, dx - \operatorname{ess}\liminf_{s\to 0-} \int_\Gamma q_i(u(r+s\boldsymbol{\nu}))\nu_i\varphi(r)\, dr\, , \end{aligned}$$

which is (7.14) for smooth entropy flux pairs. Using now the approximation of $(|z-k|, \mathbf{F}(z,k))$ defined in Remark 7.8, we easily obtain the assertion. ■

Remark 7.22 Note that also in the case of a bounded domain, (7.13) together with $u \in L^\infty(Q_T)$ imply the integral identity

$$\int_0^T \int_\Omega u\frac{\partial\psi}{\partial t} + f_i(u)\frac{\partial\psi}{\partial x_i}\, dx\, dt = 0 \qquad (7.23)$$

for all $\psi \in \mathcal{D}(Q_T)$. Indeed, since the smooth function $\varphi \geq 0$ in (7.13) can be chosen arbitrarily, it is obvious that (7.13) for $k = \pm\|u\|_\infty$ implies (7.23).

The following lemma connects our understanding of the boundary conditions in the sense of (7.5) with the condition (6.21), cf. assertion no. 5 of the following lemma.

Lemma 7.24 *Let $u^D \in L^\infty(\Gamma)$ and $u \in L^\infty(Q_T)$ be given. Further, let u assume some boundary value $u^\tau \in L^\infty(\Gamma)$ in the sense*

$$\operatorname{ess}\lim_{s\to 0-} \int_\Gamma |u(r+s\,\boldsymbol{\nu}(r)) - u^\tau(r)|\, dr = 0\, .$$

Then the following statements are equivalent:

1. (u, u^D) *satisfy (7.5);*
2. (u, u^D) *satisfy (7.15);*
3. *for* $\mathcal{H}^d$*-almost all*[‖] $r \in \Gamma$ *and all entropy-entropy flux pairs* $(\eta, \mathbf{q})$ *we have*
$$\mathbf{q}(u^\tau(r)) \cdot \boldsymbol{\nu}(r) \geq 0$$
provided that $\eta'(u^D(r)) = 0$ *and* $\mathbf{q}(u^D(r)) = 0$*;*
4. *for* $\mathcal{H}^d$*-almost all* $r \in \Gamma$ *we have*
$$\boldsymbol{\mathcal{F}}(u^\tau(r), u^D(r), k) \cdot \boldsymbol{\nu}(r) \geq 0$$
for all $k \in \mathbb{R}$*;*
5. *for* $\mathcal{H}^d$*-almost all* $r \in \Gamma$ *we have*
$$\operatorname{sgn}(u^\tau(r) - u^D(r))(\mathbf{f}(u^\tau(r)) \cdot \boldsymbol{\nu}(r) - \mathbf{f}(k) \cdot \boldsymbol{\nu}(r)) \geq 0$$
for all $k \in \mathcal{I}\big[u^\tau(r), u^D(r)\big]$.

PROOF :

The implication '1 ⇒ 2' was proved in Lemma 7.12.

'2 ⇒ 4': From the assumptions and (7.15) it follows for fixed $k \in \mathbb{R}$ that
$$\begin{aligned}&\int_\Gamma \boldsymbol{\mathcal{F}}(u^\tau, u^D(r), k) \cdot \boldsymbol{\nu}(r)\, \beta(r)\, dr \\ &= \operatorname*{ess\,lim}_{s \to 0-} \int_\Gamma \boldsymbol{\mathcal{F}}(u(r + s\,\boldsymbol{\nu}(r)), u^D(r), k) \cdot \boldsymbol{\nu}(r)\, \beta(r)\, dr \geq 0\end{aligned}$$
for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere. Therefore, there is a set E, $\mathcal{H}^d(E) = 0$, such that
$$\boldsymbol{\mathcal{F}}(u^\tau(r), u^D(r), k) \cdot \boldsymbol{\nu}(r) \geq 0 \quad \forall\, k \in \mathbb{Q}, \quad \forall\, r \in \Gamma \setminus E\ .$$
This and properties of $\boldsymbol{\mathcal{F}}$ imply 4.

The implication '4 ⇒ 5' follows immediately from the definition of $\boldsymbol{\mathcal{F}}$.

'5 ⇒ 3': For an entropy-entropy flux pair $(\eta, \mathbf{q})$ with $\eta'(w) = 0$, $\mathbf{q}(w) = 0$ we get
$$\mathbf{q}(z) = \int_w^z \eta''(\lambda)(\mathbf{f}(z) - \mathbf{f}(\lambda))\, d\lambda\,.$$
This formula shows that 5 implies 3.

[‖] By $\mathcal{H}^d$ we denote the d-dimensional Hausdorff measure.

'3 ⇒ 1': For all boundary entropy fluxes $\mathbf{Q}$ and $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere, we obviously get

$$\begin{aligned} 0 &\leq \int_\Gamma \mathbf{Q}(u^\tau(r), u^D(r)) \cdot \boldsymbol{\nu}(r)\, \beta(r)\, dr \\ &= \operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathbf{Q}(u(r + s\,\boldsymbol{\nu}(r)), u^D(r)) \cdot \boldsymbol{\nu}(r)\, \beta(r)\, dr\,, \end{aligned}$$

which is (7.5). ∎

Remark 7.25 The formulation of statement no. 5 of the previous lemma corresponds to the formulation of the boundary value problem in Bardos, Le Roux and Nedelec [1979]; compare it also to (6.21). Statement no. 3 is a special case of the 'boundary entropy condition' due to Dubois and Le Floch [1988, Theorem 1.1]. Note that both 3 and 5 need the existence of a trace u^τ.

All statements of the previous lemma are equivalent only in the case that the existence of a trace u^τ is assumed, despite the fact that some of the statements do not involve u^τ. In the following two lemmas we will see that there is a subset $\Gamma_D \subset \Gamma$, depending on $\mathbf{f}$, on which the existence of $u^\tau = u^D$ can be deduced from (7.5). Moreover, there is a subset $\Gamma_N \subset \Gamma$, depending on $\mathbf{f}$, on which the boundary condition (7.5) is trivially satisfied (see Lemma 7.27). Compare also the definition of Γ_D and Γ_N to (6.7), (6.8), respectively.

Let $\mathbf{f} \in C^1(\mathbb{R})^d$. Define the Borel sets $\Gamma_D, \Gamma_N \subset \Gamma$ by

$$\Gamma_N = \{r \in \Gamma, \text{the function '}x \mapsto \mathbf{f}(x)\cdot\boldsymbol{\nu}(r)\text{' is non-decreasing}\}\,,$$
$$\Gamma_D = \{r \in \Gamma, \text{the function '}x \mapsto \mathbf{f}(x)\cdot\boldsymbol{\nu}(r)\text{' is decreasing}\}\,.$$

Lemma 7.26 *For every $u \in L^\infty(Q_T)$, $u^D \in L^\infty(\Gamma)$ satisfying (7.5), we have*

$$\operatorname*{ess\,lim}_{s\to 0-} \int_{\Gamma_D} |u(r + s\,\boldsymbol{\nu}(r)) - u^D(r)|\, dr = 0\,,$$

with Γ_D defined above.

Proof : Let us use (7.5) with the boundary entropy-entropy flux pair $(H_\ell, \mathbf{Q}_\ell)$ defined by

$$\begin{aligned} H_\ell(z,w) &\equiv \left((z-w)^2 + \left(\frac{1}{\ell}\right)^2\right)^{1/2} - \frac{1}{\ell}\,, \\ \mathbf{Q}_\ell(z,w) &\equiv \int_w^z \partial_1 H_\ell(\lambda, w)\, \mathbf{f}'(\lambda)\, d\lambda. \end{aligned}$$

Letting $\ell \to \infty$ we obtain

$$\operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathbf{F}(u(r+s\,\boldsymbol{\nu}(r)), u^D(r)) \cdot \boldsymbol{\nu}(r)\,\beta(r)\,dr \geq 0$$

for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere. Using the characteristic function of the set Γ_D as a test function β we obtain

$$\mathbf{F}(z,w)\cdot\boldsymbol{\nu}(r) = -|\mathbf{f}(z)\cdot\boldsymbol{\nu}(r) - \mathbf{f}(w)\cdot\boldsymbol{\nu}(r)|$$

for all $(z,w) \in \mathbb{R}^2$ and all $r \in \Gamma_D$. Therefore, there is an $\mathcal{H}^1$-null set E such that

$$\lim_{\substack{s\to 0-\\ s\notin E}} \int_{\Gamma_D} |\mathbf{f}(u(r+s\,\boldsymbol{\nu}(r)))\cdot\boldsymbol{\nu}(r) - \mathbf{f}(u^D(r))\cdot\boldsymbol{\nu}(r)|\,dr = 0$$

and for a subsequence of $\{s_\ell\}_\ell \subset (-\infty, 0)\setminus E$, still denoted s_ℓ,

$$\lim_{\ell\to\infty} \mathbf{f}(u(r+s_\ell\,\boldsymbol{\nu}(r)))\cdot\boldsymbol{\nu}(r) = \mathbf{f}(u^D(r))\cdot\boldsymbol{\nu}(r) \quad \text{for a.a. } r \in \Gamma_D\ .$$

Due to the monotonicity of $\mathbf{f}\cdot\boldsymbol{\nu}(r)$ on Γ_D we get

$$\lim_{\ell\to\infty} u(r+s_\ell\,\boldsymbol{\nu}(r)) = u^D(r) \quad \text{for a.a. } r\in\Gamma_D\,.$$

Using the Lebesgue dominated convergence theorem we conclude that

$$\lim_{\ell\to\infty}\int_{\Gamma_D} |u(r+s_\ell\,\boldsymbol{\nu}(r)) - u^D(r)|\,dr = 0.$$

■

Lemma 7.27 *For every $u \in L^\infty(Q_T)$ and $u^D \in L^\infty(\Gamma)$ we have*

$$\operatorname*{ess\,lim}_{s\to 0-}\int_{\Gamma_N} \mathbf{Q}(u(r+s\,\boldsymbol{\nu}(r)), u^D(r))\cdot\boldsymbol{\nu}(r)\,\beta(r)\,dr \geq 0$$

for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere, and all boundary entropy-entropy flux pairs $(H,\mathbf{Q})$. Here, Γ_N is defined before Lemma 7.26.

Proof : Due to the definition of a boundary entropy-entropy flux pair we have

$$\mathbf{Q}(w,w) = 0\ ,\quad \partial_1 H(w,w) = 0\ ,\quad \partial^2_{11}H(\lambda,w) \geq 0$$

and therefore

$$\mathbf{Q}(z,w) = \int_w^z \partial_1 H(\lambda,w)\,\mathbf{f}'(\lambda)\,d\lambda\,,$$
$$\operatorname{sgn}(\lambda - w)\,\partial_1 H(\lambda,w) \geq 0\,.$$

Finally, if $r \in \Gamma_N$, we have $\mathbf{f}'(\lambda) \cdot \boldsymbol{\nu}(r)$ for all $\lambda \in \mathbb{R}$. Hence,

$$\mathbf{Q}(z,w) \cdot \boldsymbol{\nu}(r) = \int_w^z \partial_1 H(\lambda, w)\, \mathbf{f}'(\lambda) \cdot \boldsymbol{\nu}(r)\, d\lambda \geq 0 \ ,$$

which implies the assertion. ∎

Now we will formulate two theorems which we will prove in the remainder of this section. The L^1-contraction property, proved in the first theorem, implies uniqueness of weak solutions and the second theorem gives an equivalent formulation of (7.4)–(7.6).

Theorem 7.28 (Uniqueness) *Let $\mathbf{f} \in C^1(\mathbb{R})^d$ and let $\mathbf{F}$ be defined by (7.9). Let $(u_i, u_i^D, u_i^0) \in \big(L^\infty(Q_T) \times L^\infty(\Gamma) \times L^\infty(\Omega)\big)$, $i = 1, 2$, satisfy (7.3)–(7.6). Then*

$$\begin{aligned} -\int_0^T \int_\Omega |u_1 - u_2| \frac{\partial \beta}{\partial t} + F_i(u_1, u_2) \frac{\partial \beta}{\partial x_i}\, dx\, dt \\ \leq \int_\Omega |u_1^0 - u_2^0| \beta(0)\, dx + \int_\Gamma \operatorname{diam}\big(\mathbf{f} \cdot \boldsymbol{\nu}, \mathcal{I}\big[u_1^D, u_2^D\big]\big) \beta\, dr \end{aligned} \tag{7.29}$$

holds for all $\beta \in \mathcal{D}((-\infty, T) \times \mathbb{R}^d)$, where

$$\begin{aligned} &\operatorname{diam}\big(\mathbf{f} \cdot \boldsymbol{\nu}(r), \mathcal{I}\big[u_1^D(r), u_2^D(r)\big]\big) \\ &\quad = \sup\big\{|\mathbf{f}(z_1) \cdot \boldsymbol{\nu}(r) - \mathbf{f}(z_2) \cdot \boldsymbol{\nu}(r)|; z_1, z_2 \in \mathcal{I}\big[u_1^D(r), u_2^D(r)\big]\big\} \,. \end{aligned}$$

Moreover, for almost all $t \in (0,T)$ we have

$$\begin{aligned} \int_\Omega |u_1(t) - u_2(t)|\, dx \leq \int_\Omega |u_1^0 - u_2^0|\, dx \\ + M \int_0^t \int_{\partial\Omega} |u_1^D - u_2^D|\, dr\, ds \,, \end{aligned} \tag{7.30}$$

where M is the constant of Lipschitz continuity of $\mathbf{f}$, restricted on the ball with radius $\max\{\|u_i\|_\infty, \|u_i^D\|_\infty, \|u_i^0\|_\infty\}$.

Theorem 7.31 *Provided that $(u, u^D, u^0) \in \big(L^\infty(Q_T) \times L^\infty(\Gamma) \times L^\infty(\Omega)\big)$, the formulation (7.4)–(7.6) is equivalent to the statement that*

$$\begin{aligned} -\int_0^T \int_\Omega H(u,k) \frac{\partial \beta}{\partial t} + Q_i(u,k) \frac{\partial \beta}{\partial x_i}\, dx\, dt \\ \leq \int_\Omega H(u^0, k)\, \beta(0)\, dx + M \int_\Gamma H(u^D, k)\, \beta\, dr \end{aligned} \tag{7.32}$$

holds for all $\beta \in \mathcal{D}((-\infty, T) \times \mathbb{R}^d)$, $\beta \geq 0$, $k \in \mathbb{R}$, *and all boundary entropy-entropy flux pairs* $(H, \mathbf{Q})$.

Again, M *is the constant of Lipschitz continuity of* $\mathbf{f}$, *restricted on the ball with radius* $\max\{\|u\|_\infty, \|u^D\|_\infty, \|u^0\|_\infty\}$.

Remark 7.33

- Let us emphasize that due to Theorem 7.31, the statements (7.4)–(7.6) are formulated weakly enough to be stable under L^1-convergence. That is, if $(u_\ell, u_\ell^D, u_\ell^0)$ is a sequence of solutions which converges in L^1 as $\ell \to \infty$, then the limit (u, u^D, u^0) also satisfies (7.4)–(7.6). On the other hand, the formulation (7.4)–(7.6) is strong enough to ensure uniqueness and even more, continuous dependence on the data (see Theorem 7.28).
- Theorem 7.31 implies the following maximum principle:

$$\sup_{Q_T} u \leq \max(\sup_{\Gamma} u^D, \sup_{\Omega} u^0)\,,$$
$$\inf_{Q_T} u \geq \min(\inf_{\Gamma} u^D, \inf_{\Omega} u^0)\,.$$

Indeed, using (7.32) with the boundary entropy-entropy flux pair

$$H_\ell(z,k) \equiv \left(\left(\max(z-k,0)\right)^2 + \left(\frac{1}{\ell}\right)^2\right)^{1/2} - \frac{1}{\ell}\,,$$
$$\mathbf{Q}_\ell(z,k) \equiv \int_k^z \partial_1 H_\ell(\lambda,k)\,\mathbf{f}'(\lambda)\,d\lambda$$

where

$$k = \max\left(\sup_{\Gamma} u^D, \sup_{\Omega} u^0\right)$$

and letting $\ell \to \infty$, we obtain

$$\int_0^T \int_\Omega \max(u-k,0)\frac{\partial\beta}{\partial t} + F_i^+(u,k)\frac{\partial\beta}{\partial x_i}\,dx\,dt \geq 0,$$

where

$$\mathbf{F}^+(z,k) \equiv \begin{cases} \mathbf{f}(z) - \mathbf{f}(k) & \text{for } z \geq k\,, \\ \mathbf{0} & \text{for } z \leq k\,. \end{cases}$$

In particular, for $\beta(t,x) = \alpha(t)\gamma(x), \gamma = 1$ on Ω, we have

$$\int_0^T \int_\Omega \max(u(t,x)-k,0)\,dx\,\alpha'(t)\,dt \geq 0$$

for all $\alpha \in \mathcal{D}((-\infty, T))$, $\alpha \geq 0$, and thus

$$\int_\Omega \max(u(t,x) - k, 0)\, dx = 0 \quad \text{for a.a. } t \in (0,T).$$

- If the existence of a weak solution to (7.4)–(7.6) for smooth boundary and initial data has already been proved, we will be able to show that the problem is solvable under only the assumption that u^D, u^0 are L^∞-functions. Let us sketch the proof of this statement: for given $u^D \in L^\infty(\Gamma)$, $u^0 \in L^\infty(\Omega)$, let $u_\ell^D \in C_0^\infty(\Gamma)$, $u_\ell^0 \in C_0^\infty(\Omega)$ be such that

$$\lim_{\ell\to\infty} u_\ell^D = u^D \qquad \text{in } L^1(\Gamma)\,,$$
$$\lim_{\ell\to\infty} u_\ell^0 = u^0 \qquad \text{in } L^1(\Omega)\,.$$

Knowing that our problem is solvable for smooth data, we also know that due to the second part of Theorem 7.28, the solutions $\{u_\ell\}$ form a Cauchy sequence in $L^1(Q_T)$. Moreover, due to the maximum principle, they are bounded in $L^\infty(Q_T)$. Therefore, there is a $u \in L^\infty(Q_T)$ such that

$$\lim_{\ell\to\infty} u_\ell = u \quad \text{in } L^1(Q_T)\,.$$

Due to the first part of Remark 7.33, this u solves (7.4)–(7.6).

In order to prove Theorems 7.28 and 7.31, we need some lemmas.

Lemma 7.34 *Let $(H, \mathbf{Q})$ be a boundary entropy-entropy flux pair and let $u \in L^\infty(Q_T)$ satisfy*

$$\int_0^T \int_\Omega H(u,k)\frac{\partial\gamma}{\partial t} + Q_i(u,k)\frac{\partial\gamma}{\partial x_i}\, dx\, dt \geq 0 \tag{7.35}$$

for all $\gamma \in \mathcal{D}(Q_T)$, $\gamma \geq 0$ and all $k \in \mathbb{R}$. Then there is a set E of Lebesgue measure zero, such that

$$\lim_{\substack{s\to 0-\\ s\notin E}} \int_\Gamma \mathbf{Q}(u(r + s\,\boldsymbol{\nu}(r)), v^D(r)) \cdot \boldsymbol{\nu}(r)\, \beta(r)\, dr$$

exists for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere, and all $v^D \in L^\infty(\Gamma)$.

Moreover, if u additionally satisfies for some $u^D \in L^\infty(\Gamma)$

$$-\int_0^T \int_\Omega H(u,k)\frac{\partial\gamma}{\partial t} + Q_i(u,k)\frac{\partial\gamma}{\partial x_i}\, dx\, dt \leq M \int_\Gamma H(u^D,k)\,\gamma\, dr \tag{7.36}$$

for all $\gamma \in \mathcal{D}((0,T)\times\mathbb{R}^d)$, $\gamma \geq 0$ and all $k \in \mathbb{R}$, where M is constant of Lipschitz continuity of $\mathbf{f}$, then

$$\operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathbf{Q}(u(r+s\,\boldsymbol{\nu}(r)), u^D(r)) \cdot \boldsymbol{\nu}(r)\,\beta(r)\,dr \geq 0$$

for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere.

Proof : In order to simplify the argument, we again restrict ourselves to the case of half-space, i.e.,

$$\Omega = \{x = (x', s) \in \mathbb{R}^{d-1} \times \mathbb{R}; s \equiv x_d < 0\} \ .$$

Then we have (cf. the proof of Lemma 7.12):

$$\begin{aligned} &\boldsymbol{\nu} = (0, \ldots, 0, 1) \in \mathbb{R}^d \,, \\ &\partial\Omega = \mathbb{R}^{d-1} \times \{0\}, \quad \Gamma = (0,T) \times \mathbb{R}^{d-1}, \quad r = (t, x') \in \Gamma \,, \\ &Q_T = \{p = (r,s); r \in \Gamma \ , \ s < 0\} \,. \end{aligned}$$

Let $w \in \mathbb{Q}$ and $\varphi \in \mathcal{D}(\Gamma)$, $\varphi \geq 0$, be arbitrary but fixed. We denote

$$\eta(z) \equiv H(z,w) \ , \quad \mathbf{q}(z) \equiv \mathbf{Q}(z,w) \ .$$

Then for $\gamma(r,s) = \beta(r)\,\alpha(s)$ we obtain from (7.35):

$$-\int_{-\infty}^0 \int_\Gamma \mathbf{q}(u(r,s)) \cdot \boldsymbol{\nu}\,\beta(r)\,dr\,\alpha'(s)\,ds \leq C \int_{-\infty}^0 \alpha(s)\,ds \tag{7.37}$$

for all $\alpha \in \mathcal{D}((-\infty,0))$, $\alpha \geq 0$, where

$$C = \operatorname*{ess\,sup}_{Q_T} |\eta(u)| \int_\Gamma \left|\frac{\partial\beta}{\partial t}\right| dr + \operatorname*{ess\,sup}_{Q_T} |\mathbf{q}(u)| \int_\Gamma |\nabla_{x'}\beta|\,dr \,.$$

Using partial integration of the left-hand side of (7.37) and $\alpha \geq 0$ we see that the function

$$s \mapsto \int_\Gamma \mathbf{q}(u(r,s)) \cdot \boldsymbol{\nu}\,\beta(r)\,dr - Cs \,, \tag{7.38}$$

after a possible modification on a set of zero measure, is non-increasing on $(-\infty, 0)$. On the other hand we have

$$\operatorname*{ess\,lim\,inf}_{s\to 0-} \int_\Gamma \mathbf{q}(u(r,s)) \cdot \boldsymbol{\nu}\,\beta(r)\,dr \geq -\operatorname*{ess\,sup}_{Q_T} |\mathbf{q}(u)| \int_\Gamma \beta(r)\,dr \,.$$

From the monotonicity and the boundedness from below we infer that

$$\operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathbf{q}(u(r,s)) \cdot \boldsymbol{\nu}\, \beta(r)\, dr \tag{7.39}$$

exists. Provided that (7.36) is fulfilled, we get

$$\begin{aligned} -\int_{-\infty}^{0} \int_\Gamma \mathbf{q}(u(r,s)) \cdot \boldsymbol{\nu}\, \beta(r)\, dr\, \alpha'(s)\, ds \\ \le C \int_{-\infty}^{0} \alpha(s)\, ds + M \int_\Gamma \eta(u^D(r))\, \beta(r)\, dr\, \alpha(0) \end{aligned}$$

for all $\alpha \in \mathcal{D}(\mathbb{R})$, $\alpha \geq 0$. Now the special choice of $\alpha_n(s) = (ns + 1)\chi_{(1/n,0)}$ (mollified properly) and (7.39) imply for $n \to \infty$ that

$$\operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathbf{q}(u(r,s)) \cdot \boldsymbol{\nu}\, \beta(r)\, dr \geq -M \int_\Gamma \eta(u^D(r))\, \beta(r)\, dr\ .$$

In what follows the consequences of this inequality will appear in parentheses.

Let $J \subset \mathcal{D}(\Gamma)$ be a countable set of non-negative functions such that for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere, there is a sequence $\{\beta_\ell\}_\ell \subset J$ such that

$$\lim_{\ell\to\infty} \beta_\ell = \beta \quad \text{in } L^1(\Gamma).$$

Thus

$$\lim_{\ell\to\infty} \int_\Gamma \mathbf{q}(u(r,s)) \cdot \boldsymbol{\nu}\, \beta_\ell(r)\, dr = \int_\Gamma \mathbf{q}(u(r,s)) \cdot \boldsymbol{\nu}\, \beta(r)\, dr$$

uniformly in $s \in (-\infty, 0)$ and

$$\lim_{\ell\to\infty} \int_\Gamma \eta(u^D(r))\, \beta_\ell(r)\, dr = \int_\Gamma \eta(u^D(r))\, \beta(r)\, dr\ .$$

Due to (7.39) there is a set E_w of measure zero (recall that η and $\mathbf{q}$ depend on w), such that for all $\beta \in J$

$$\lim_{\substack{s\to 0- \\ s\notin E_w}} \int_\Gamma \mathbf{q}(u(r,s)) \cdot \boldsymbol{\nu}\, \beta(r)\, dr \left(\geq -M \int_\Gamma \eta(u^D(r))\, \beta(r)\, dr \right)$$

exists. Of course, this can be extended to functions $\beta \in L^1(\Gamma)$,

$\beta \geq 0$ almost everywhere, and therefore

$$\lim_{\substack{s\to 0- \\ s\notin E}} \int_\Gamma \mathbf{Q}(u(r,s),w)\cdot\boldsymbol{\nu}\,\beta(r)\,dr \qquad \left(\geq -M\int_\Gamma H(u^D(r),w)\,\beta(r)\,dr\right) \tag{7.40}$$

exists for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere, and all $w \in \mathbb{Q}$.

Now let $v^D \in L^\infty(\Gamma)$ and $\beta \in L^1(\Gamma)$, $\beta \geq 0$ be given. Further, let $\{v_\ell^D\}$ be a sequence of simple functions with values in $\mathbb{Q}$ such that

$$\lim_{\ell\to\infty} v_\ell^D = v^D \quad \text{a.e. in } \Gamma.$$

Thus (7.40) holds for all $w = v_\ell^D$. On the other hand we have

$$\lim_{\ell\to\infty}\int_\Gamma \mathbf{Q}(u(r,s),v_\ell^D(r))\cdot\boldsymbol{\nu}\,\beta(r)\,dr = \int_\Gamma \mathbf{Q}(u(r,s),v^D(r))\cdot\boldsymbol{\nu}\,\beta(r)\,dr$$

uniformly in $s \in (-\infty,0)$ and

$$\lim_{\ell\to\infty}\int_\Gamma H(u^D(r),v_\ell^D(r))\,\beta(r)\,dr = \int_\Gamma H(u^D(r),v^D(r))\,\beta(r)\,dr\,.$$

Altogether we get that

$$\lim_{\substack{s\to 0- \\ s\notin E}} \int_\Gamma \mathbf{Q}(u(r,s),v^D(r))\cdot\boldsymbol{\nu}\,\beta(r)\,dr \qquad \left(\geq -M\int_\Gamma H(u^D(r),v^D(r))\beta(r)\,dr\right)$$

exists, which implies the assertions of Lemma 7.34. ■

Lemma 7.41 *Let $(u,u^0) \in L^\infty(Q_T)\times L^\infty(\Omega)$ satisfy*

$$-\int_0^T\int_\Omega |u-k|\frac{\partial\beta}{\partial t} + F_i(u,k)\frac{\partial\beta}{\partial x_i}\,dx\,dt \leq \int_\Omega |u^0-k|\,\beta(0)\,dx$$

for all $\beta \in \mathcal{D}((-\infty,T)\times\Omega)$, $\beta \geq 0$, and all $k \in \mathbb{R}$. Then we have

$$\operatorname*{ess\,lim}_{t\to 0+}\int_\Omega |u(t,x)-u^0(x)|\,dx = 0\,.$$

PROOF : With the same methods as in the previous lemma we show

$$\operatorname*{ess\,lim\,sup}_{t\to 0+} \int_\Omega |u(t,x)-k|\beta(x)\,dx \le \int_\Omega |u^0(x)-k|\beta(x)\,dx$$

for all $\beta \in L^1(\Omega)$, $\beta \ge 0$ almost everywhere, and all $k \in \mathbb{Q}$. Thus, we conclude that

$$\operatorname*{ess\,lim\,sup}_{t\to 0+} \int_\Omega |u(t,x)-v^0(x)|\,dx \le \int_\Omega |u^0(x)-v^0(x)|\,dx$$

for all $v^0 \in L^\infty(\Omega)$ and the assertion follows. ∎

Now we are ready to prove both main theorems of this section.

PROOF (of Theorem 7.28): Again we will show the theorem only in the case of a half-space, where $\boldsymbol{\nu} \equiv \boldsymbol{\nu}(r)$ for all $r \in \Gamma$. We use the same notation as in the proof of Lemma 7.34.

Let u be a solution of (7.4) and let

$$H_\ell(z,w) = \left((z-w)^2 + \frac{1}{\ell^2}\right)^{1/2} - \frac{1}{\ell}\,,$$
$$\mathbf{Q}_\ell(z,w) = \int_w^z \partial_1 H_\ell(\lambda,w)\,\mathbf{f}'(\lambda)\,d\lambda$$

be a boundary entropy-entropy flux pair. Lemma 7.34 now implies that there is a set E of measure zero such that the limit

$$\lim_{\substack{s\to 0-\\ s\notin E}} \int_\Gamma \mathbf{F}(u(r,s),v^D(r))\cdot\boldsymbol{\nu}\,\beta(r)\,dr \tag{7.42}$$

exists for all $\beta \in L^1(\Gamma)$, $\beta \ge 0$ almost everywhere, and all $v^D \in L^\infty(\Gamma)$. Now let (u,u^D) satisfy (7.4), (7.5) and therefore, due to Lemma 7.12, also (7.13), (7.15). From the obvious relation

$$2\boldsymbol{\mathcal{F}}(z,w,k) = \mathbf{F}(z,w) - \mathbf{F}(k,w) + \mathbf{F}(z,k)\,, \tag{7.43}$$

and (7.42), (7.15) we get

$$\begin{aligned}
-\operatorname*{ess\,lim}_{s\to 0-} &\int_\Gamma \mathbf{F}(u(r,s),k)\cdot\boldsymbol{\nu}\beta(r)\,dr \\
&\le \operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathbf{F}(u(r,s),u^D(r))\cdot\boldsymbol{\nu}\,\beta(r)\,dr \\
&\quad - \int_\Gamma \mathbf{F}(k,u^D(r))\cdot\boldsymbol{\nu}\,\beta(r)\,dr
\end{aligned}$$

for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere, and all $k \in \mathbb{R}$. This and (7.13) give

$$
\begin{aligned}
-\int_0^T \int_\Omega |u-k| \frac{\partial \beta}{\partial t} + F_i(u,k) \frac{\partial \beta}{\partial x_i} \, dx \, dt \\
\leq \operatorname*{ess\,lim}_{s \to 0-} \int_\Gamma \mathbf{F}(u(r,s), u^D(r)) \cdot \boldsymbol{\nu} \, \beta(r) \, dr \qquad (7.44) \\
- \int_\Gamma \mathbf{F}(k, u^D(r)) \cdot \boldsymbol{\nu} \beta(r) \, dr
\end{aligned}
$$

for all $\beta \in \mathcal{D}((0,T) \times \mathbb{R}^d)$, $\beta \geq 0$ and all $k \in \mathbb{R}$.

Now let $(u_i, u_i^D, u_i^0)_{i=1,2}$ be two solutions of (7.3)–(7.6). From (7.42) we infer that there exist $\theta_{i,j} \in L^\infty(\Gamma)$, such that for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere,

$$
\operatorname*{ess\,lim}_{s \to 0-} \int_\Gamma \mathbf{F}(u_i(r,s), u_j^D(r)) \cdot \boldsymbol{\nu} \, \beta(r) \, dr = \int_\Gamma \theta_{i,j}(r) \, \beta(r) \, dr \ .
$$

Note that we are using the convention $\beta(r) = \beta(r,0)$ on Γ (see also (7.44)). Let $\rho \in \mathcal{D}(\mathbb{R}^{d+1})$ be a symmetric mollifier and set

$$
\beta_\varepsilon(p, \tilde{p}) = \beta\Big(\frac{p+\tilde{p}}{2}\Big) \rho_\varepsilon(p - \tilde{p}) \quad \forall \, p, \tilde{p} \in Q_T
$$

for given $\beta \in \mathcal{D}((0,T) \times \mathbb{R}^d)$, $\beta \geq 0$. Hold $\tilde{p} \in Q_T$ fixed and replace in (7.44) (u, u^D) by (u_1, u_1^D), k by $u_2(\tilde{p})$ and β by $\beta_\varepsilon(\cdot, \tilde{p})$. After integration over $\tilde{p} \in Q_T$ we get

$$
\begin{aligned}
-\frac{1}{2} \int_{Q_T} \int_{Q_T} \Big\{ |u_1(p) - u_2(\tilde{p})| \frac{\partial \beta}{\partial t} \Big(\frac{p+\tilde{p}}{2}\Big) \\
+ F_i(u_1(p), u_2(\tilde{p})) \frac{\partial \beta}{\partial x_i} \Big(\frac{p+\tilde{p}}{2}\Big) \Big\} \rho_\varepsilon(p-\tilde{p}) \, dp \, d\tilde{p} \\
- \int_{Q_T} \int_{Q_T} \Big\{ |u_1(p) - u_2(\tilde{p})| \frac{\partial \rho_\varepsilon}{\partial t}(p - \tilde{p}) \\
+ F_i(u_1(p), u_2(\tilde{p})) \frac{\partial \rho_\varepsilon}{\partial x_i}(p-\tilde{p}) \Big\} \beta\Big(\frac{p+\tilde{p}}{2}\Big) \, dp \, d\tilde{p} \\
\leq \int_{Q_T} \int_\Gamma \theta_{1,1}(r) \, \beta\Big(\frac{r+\tilde{p}}{2}\Big) \rho_\varepsilon(r - \tilde{p}) \, dr \, d\tilde{p} \\
- \int_{Q_T} \int_\Gamma \mathbf{F}(u_2(\tilde{p}), u_1^D(r)) \cdot \boldsymbol{\nu} \, \beta\Big(\frac{r+\tilde{p}}{2}\Big) \rho_\varepsilon(r-\tilde{p}) \, dr \, d\tilde{p} \ .
\end{aligned}
$$

Changing the role of u_1 and u_2, and of p and $\tilde{p}$, we get

$$\begin{aligned}
-\frac{1}{2}\int_{Q_T}\int_{Q_T}\Big\{&|u_2(\tilde{p})-u_1(p)|\frac{\partial\beta}{\partial t}\Big(\frac{p+\tilde{p}}{2}\Big)\\
&+F_i(u_2(\tilde{p}),u_1(p))\frac{\partial\beta}{\partial x_i}\Big(\frac{p+\tilde{p}}{2}\Big)\Big\}\rho_\varepsilon(p-\tilde{p})\,d\tilde{p}\,dp\\
&+\int_{Q_T}\int_{Q_T}\Big\{|u_2(\tilde{p})-u_1(p)|\frac{\partial\rho_\varepsilon}{\partial t}(p-\tilde{p})\\
&+F_i(u_2(\tilde{p}),u_1(p))\frac{\partial\rho_\varepsilon}{\partial x_i}(p-\tilde{p})\Big\}\beta\Big(\frac{p+\tilde{p}}{2}\Big)\,d\tilde{p}\,dp\\
\le\int_{Q_T}\int_\Gamma&\theta_{2,2}(\tilde{r})\,\beta\Big(\frac{p+\tilde{r}}{2}\Big)\rho_\varepsilon(p-\tilde{r})\,d\tilde{r}\,dp\\
&-\int_{Q_T}\int_\Gamma\mathbf{F}(u_1(p),u_2^D(\tilde{r}))\cdot\boldsymbol{\nu}\,\beta\Big(\frac{p+\tilde{r}}{2}\Big)\rho_\varepsilon(p-\tilde{r})\,d\tilde{r}\,dp\ .
\end{aligned}$$

Adding these two inequalities we note that the integrals obtaining $\frac{\partial\rho_\varepsilon}{\partial t}$ and $\frac{\partial\rho_\varepsilon}{\partial x_i}$ vanish. Now, we are about to show that

$$\begin{aligned}
\lim_{\varepsilon\to 0+}\int_{Q_T}\int_{Q_T}\Big\{&|u_1(p)-u_2(\tilde{p})|\frac{\partial\beta}{\partial t}\Big(\frac{p+\tilde{p}}{2}\Big)\\
&+F_i(u_1(p),u_2(\tilde{p}))\frac{\partial\beta}{\partial x_i}\Big(\frac{p+\tilde{p}}{2}\Big)\Big\}\rho_\varepsilon(p-\tilde{p})\,dp\,d\tilde{p}\\
&=\int_{Q_T}|u_1(p)-u_2(p)|\frac{\partial\beta}{\partial t}(p)+F_i(u_1(p),u_2(p))\frac{\partial\beta}{\partial x_i}(p)\,dp\ .
\end{aligned}\tag{7.45}$$

Denote

$$\begin{aligned}
\mathcal{T}_1\equiv\int_{Q_T}\int_{Q_T}\Big\{&|u_1(p)-u_2(\tilde{p})|\frac{\partial\beta}{\partial t}\Big(\frac{p+\tilde{p}}{2}\Big)\\
&+F_i(u_1(p),u_2(\tilde{p}))\frac{\partial\beta}{\partial x_i}\Big(\frac{p+\tilde{p}}{2}\Big)\Big\}\rho_\varepsilon(p-\tilde{p})\,dp\,d\tilde{p}\\
&-\int_{Q_T}|u_1(p)-u_2(p)|\frac{\partial\beta}{\partial t}(p)+F_i(u_1(p),u_2(p))\frac{\partial\beta}{\partial x_i}(p)\,dp\,.
\end{aligned}$$

Further, denote

$$\Delta p\equiv p-\tilde{p}$$

and

$$Q^{\Delta p}\equiv\{p\in Q_T : p-\Delta p\in Q_T\}\,.$$

Then

$$\begin{aligned}
\mathcal{T}_1 = &\int_{\mathbb{R}^{d+1}} \rho_\varepsilon(\Delta p) \int_{Q^{\Delta p}} |u_1(p) - u_2(p-\Delta p)| \\
&\qquad\qquad \times \left\{ \frac{\partial\beta}{\partial t}\Big(p - \frac{1}{2}\Delta p\Big) - \frac{\partial\beta}{\partial t}(p) \right\} dp\, d\Delta p \\
&+ \int_{\mathbb{R}^{d+1}} \rho_\varepsilon(\Delta p) \int_{Q^{\Delta p}} F_i(u_1(p), u_2(p-\Delta p)) \\
&\qquad\qquad \times \left\{ \frac{\partial\beta}{\partial x_i}\Big(p - \frac{1}{2}\Delta p\Big) - \frac{\partial\beta}{\partial x_i}(p) \right\} dp\, d\Delta p \\
&+ \int_{\mathbb{R}^{d+1}} \rho_\varepsilon(\Delta p) \int_{Q^{\Delta p}} \Big\{ |u_1(p) - u_2(p-\Delta p)| \\
&\qquad\qquad - |u_1(p) - u_2(p)| \Big\} \frac{\partial\beta}{\partial t}(p)\, dp\, d\Delta p \\
&+ \int_{\mathbb{R}^{d+1}} \rho_\varepsilon(\Delta p) \int_{Q^{\Delta p}} \Big\{ F_i(u_1(p), u_2(p-\Delta p)) \\
&\qquad\qquad - F_i(u_1(p), u_2(p)) \Big\} \frac{\partial\beta}{\partial x_i}(p)\, dp\, d\Delta p \\
&+ \int_{Q_T} \left\{ |u_1(p) - u_2(p)| \frac{\partial\beta}{\partial t}(p) + F_i(u_1(p), u_2(p)) \frac{\partial\beta}{\partial x_i}(p) \right\} \\
&\qquad\qquad\qquad \times \left\{ \int_{Q_T} \rho_\varepsilon(p - \tilde{p})\, d\tilde{p} - 1 \right\} dp\ .
\end{aligned}$$

Therefore we get the estimate

$$\begin{aligned}
|\mathcal{T}_1| \le &\operatorname*{ess\,sup}_{(p,\tilde{p})\in Q_T^2} |u_1(p) - u_2(\tilde{p})| \sup_{|\Delta p|<\varepsilon} \int_Q \Big| \frac{\partial\beta}{\partial t}(p - \frac{1}{2}\Delta p) - \frac{\partial\beta}{\partial t}(p) \Big| \\
&+ M \Big| \nabla\beta(p - \frac{1}{2}\Delta p) - \nabla\beta(p) \Big|\, dp \\
&+ \sup_{|\Delta p|<\varepsilon} \int_{Q^{\Delta p}} |u_2(p-\Delta p) - u_2(p)| \left\{ \Big| \frac{\partial\beta}{\partial t}(p) \Big| + M |\nabla\beta(p)| \right\} dp \\
&+ \operatorname*{ess\,sup}_{Q_T} |u_1 - u_2| \int_{Q_\varepsilon} \Big| \frac{\partial\beta}{\partial t}(p) \Big| + M |\nabla\beta|\, dx\, dt\ ,
\end{aligned}$$

where

$$Q_\varepsilon = \{(r,s) \in Q_T : -\varepsilon < s < 0\}\,.$$

This gives (7.45).

The same argument implies that

$$\begin{aligned}
\lim_{\varepsilon\to 0+} \int_{Q_T}\int_{Q_T} \Big\{ & |u_2(\tilde p) - u_1(p)| \frac{\partial\beta}{\partial t}\Big(\frac{p+\tilde p}{2}\Big) \\
& + F_i(u_2(\tilde p), u_1(p)) \frac{\partial\beta}{\partial x_i}\Big(\frac{p+\tilde p}{2}\Big)\Big\} \rho_\varepsilon(p-\tilde p)\, d\tilde p\, dp \\
& = \int_{Q_T} |u_2(p) - u_1(p)| \frac{\partial\beta}{\partial t}(p) \\
& + F_i(u_2(p), u_1(p)) \frac{\partial\beta}{\partial x_i}(p)\, dp\ .
\end{aligned} \tag{7.46}$$

Next, we have that

$$\lim_{\varepsilon\to 0+} \int_{Q_T} \beta\Big(\frac{r+\tilde p}{2}\Big)\, \rho_\varepsilon(r-\tilde p)\, d\tilde p = \frac{1}{2}\, \beta(r)\,,$$

uniformly in $r \in \Gamma$. Note that the factor $1/2$ on the right-hand side occurs since $r \in \Gamma$ are boundary points of a half-space.

Using this result, we can calculate the following two limits:

$$\begin{aligned}
\lim_{\varepsilon\to 0+} \int_{Q_T}\int_{\Gamma} \theta_{1,1}(r)\, \beta\Big(\frac{r+\tilde p}{2}\Big)\, \rho_\varepsilon(r-\tilde p)\, dr\, d\tilde p \\
= \frac{1}{2}\int_\Gamma \theta_{1,1}(r)\, \beta(r)\, dr
\end{aligned} \tag{7.47}$$

and

$$\begin{aligned}
\lim_{\varepsilon\to 0+} \int_{Q_T}\int_{\Gamma} \theta_{2,2}(\tilde r)\, \beta\Big(\frac{p+\tilde r}{2}\Big)\, \rho_\varepsilon(p-\tilde r)\, d\tilde r\, dp \\
= \frac{1}{2}\int_\Gamma \theta_{2,2}(r)\, \beta(r)\, dr\ .
\end{aligned} \tag{7.48}$$

Further, we show that

$$\begin{aligned}
\lim_{\varepsilon\to 0+} \int_{Q_T}\int_{\Gamma} \mathbf{F}(u_1(p), u_2^D(\tilde r))\cdot\boldsymbol{\nu}\, \beta\Big(\frac{p+\tilde r}{2}\Big)\, \rho_\varepsilon(p-\tilde r)\, d\tilde r\, dp \\
= \frac{1}{2}\int_\Gamma \theta_{1,2}(r)\, \beta(r)\, dr\ .
\end{aligned} \tag{7.49}$$

Indeed, we can write

$$
\begin{aligned}
\mathcal{T}_2 \equiv \int_{Q_T}\int_\Gamma & \mathbf{F}(u_1(p), u_2^D(\tilde r))\cdot\boldsymbol{\nu}\,\beta\Big(\frac{p+\tilde r}{2}\Big)\,\rho_\varepsilon(p-\tilde r)\,d\tilde r\,dp \\
&\qquad - \frac{1}{2}\int_\Gamma \theta_{1,2}(r)\,\beta(r)\,dr \\
= \int_{-\infty}^0 \int_{\mathbb{R}^d} & \rho_\varepsilon(s,\Delta r)\int_\Gamma \mathbf{F}(u_1(r,s), u_2^D(r-\Delta r))\cdot\boldsymbol{\nu} \\
&\qquad \times\Big\{\beta\Big(r-\frac{1}{2}\Delta r, \frac{1}{2}s\Big) - \beta(r,0)\Big\}\,dr\,d\Delta r\,ds \\
+ \int_{-\infty}^0 \int_{\mathbb{R}^d} & \rho_\varepsilon(s,\Delta r)\int_\Gamma \big\{\mathbf{F}(u_1(r,s), u_2^D(r-\Delta r)) \\
&\qquad - \mathbf{F}(u_1(r,s), u_2^D(r))\big\}\cdot\boldsymbol{\nu}\,\beta(r,0)\,dr\,d\Delta r\,ds \\
+ \int_{-\infty}^0 \int_{\mathbb{R}^d} & \rho_\varepsilon(s,\Delta r)\,d\Delta r\Big\{\int_\Gamma \mathbf{F}(u_1(r,s), u_2^D(r))\cdot\boldsymbol{\nu} \\
&\qquad \times\beta(r,0)\,dr - \int_\Gamma \theta_{1,2}(r)\beta(r)\,dr\Big\}\,ds
\end{aligned}
$$

and thus we get that $|\mathcal{T}_2|$ is less than or equal to

$$
\begin{aligned}
&\frac{M}{2}\operatorname*{ess\,sup}_{p\in Q_T, r\in\Gamma}|u_1(p)-u_2^D(r)|\sup_{|\Delta r|<\varepsilon}\int_\Gamma \Big|\beta(r-\frac{1}{2}\Delta r)-\beta(r)\Big|\,dr \\
&+\frac{M}{2}\sup_{|\Delta r|<\varepsilon}\int_\Gamma |u_2^D(r-\Delta r)-u_2^D(r)|\,\beta(r)\,dr \\
&+\sup_{s<0}\int_{\mathbb{R}^d}\rho(s,\Delta r)\,d\Delta r\frac{1}{\varepsilon}\int_{-\varepsilon}^0\Big|\int_\Gamma \mathbf{F}(u_1(r,s), u_2^D(r))\cdot\boldsymbol{\nu} \\
&\qquad\qquad \times\beta(r,0)\,dr - \int_\Gamma \theta_{1,2}(r)\,\beta(r)\,dr\Big|\,ds
\end{aligned}
$$

which proves (7.49).

Similarly, we get

$$
\begin{aligned}
\lim_{\varepsilon\to 0+}\int_{Q_T}\int_\Gamma \mathbf{F}(u_2(\tilde p), u_1^D(r))\cdot\boldsymbol{\nu}\,\beta\Big(\frac{r+\tilde p}{2}\Big)\rho(r-\tilde p)\,dr\,d\tilde p& \\
= \frac{1}{2}\int_\Gamma \theta_{2,1}(r)\,\beta(r)\,dr\ .&
\end{aligned}
\tag{7.50}
$$

Altogether from (7.45)–(7.50) we obtain

$$-\int_0^T \int_\Omega |u_1-u_2|\frac{\partial\beta}{\partial t}+F_i(u_1,u_2)\frac{\partial\beta}{\partial x_i}\,dx\,dt \\ \leq \frac{1}{2}\int_\Gamma \big(\theta_{1,1}(r)-\theta_{2,1}(r)+\theta_{2,2}(r)-\theta_{1,2}(r)\big)\beta(r)\,dr\ .$$

If one discusses the cases, one easily sees that for all $z_1,z_2,w_1,w_2\in\mathbb{R}$ the inequality

$$\Big|\sum_{i,j=1}^{2}(-1)^{i+j}\mathbf{F}(z_i,w_j)\cdot\boldsymbol{\nu}\Big| \leq 2\,\mathrm{diam}\,\big(\mathbf{f}\cdot\boldsymbol{\nu},\mathcal{I}[w_1,w_2]\big)$$

holds and thus we have for all $s\in(-\infty,0)$

$$\Big|\sum_{i,j=1}^{2}(-1)^{i+j}\int_\Gamma \mathbf{F}(u_i(r,s),u_j^D(r))\,\boldsymbol{\nu}\,\beta(r)\,dr\Big| \\ \leq 2\int_\Gamma \mathrm{diam}\,\big(\mathbf{f}\cdot\boldsymbol{\nu},\mathcal{I}[u_1^D,u_2^D]\big)\,\beta\,dr\,.$$

Letting $s\to 0-$ we get

$$\Big|\sum_{i,j=1}^{2}(-1)^{i+j}\int_\Gamma \theta_{i,j}(r)\,\beta(r)\,dr\Big| \leq 2\int_\Gamma \mathrm{diam}\,\big(\mathbf{f}\cdot\boldsymbol{\nu},\mathcal{I}[u_1^D,u_2^D]\big)\,\beta\,dr\,,$$

and therefore we have

$$-\int_0^T \int_\Omega |u_1-u_2|\frac{\partial\beta}{\partial t}+F_i(u_1,u_2)\frac{\partial\beta}{\partial x_i}\,dx\,dt \\ \leq \int_\Gamma \mathrm{diam}\,\big(\mathbf{f}\cdot\boldsymbol{\nu},\mathcal{I}[u_1^D,u_2^D]\big)\,\beta(r)\,dr$$

for all $\beta\in\mathcal{D}((0,T)\times\mathbb{R}^d)$, $\beta\geq 0$. Assertion (7.29) is obtained by this inequality and (7.6). Note that the integral term with $\beta(0)$ is obtained by the same procedure as in Lemma 7.12.

If we now restrict ourselves in (7.29) to test functions α which are functions of t only and if we use

$$\mathrm{diam}\,\big(\mathbf{f}(r)\cdot\boldsymbol{\nu},\mathcal{I}[u_1^D(r),u_2^D(r)]\big)\leq M|u_1^D(r)-u_2^D(r)|\quad\forall\, r\in\Gamma\,,$$

we obtain

$$-\int_0^T \int_\Omega |u_1(t,x) - u_2(t,x)|\, dx\, \alpha'(t)\, dt$$
$$\leq \int_\Omega |u_1^0(x) - u_2^0(x)|\, dx\, \alpha(0)$$
$$+ M \int_0^T \int_{\partial\Omega} |u_1^D(t,y) - u_2^D(t,y)|\, dy\, \alpha(t)\, dt$$

for all $\alpha \in \mathcal{D}((-\infty,T))$, $\alpha \geq 0$. This obviously gives (7.30), if one uses the sequence of test functions $\alpha_n(0) = \alpha_n(t+1/n) = 0, \alpha_n(s) \equiv 1$ on $[1/n, t]$, α_n piecewise linear, appropriately mollified. ∎

Proof (of Theorem 7.31):

The implication '(7.32) $\Rightarrow$ (7.4)–(7.6)': Let $(\eta, \mathbf{q})$ be an entropy-entropy flux pair. The function u is bounded and therefore we can assume that η attains a minimum at some $k_0 \in \mathbb{R}$. Then of course

$$H(z,k) = \eta(z - k + k_0) - \eta(k_0)\,,$$
$$\mathbf{Q}(z,k) = \mathbf{q}(z - k + k_0) - \mathbf{q}(k_0)\,,$$

is a boundary entropy-entropy flux pair and thus we get

$$-\int_0^T \int_\Omega \eta(u)\frac{\partial\gamma}{\partial t} + q_i(u)\frac{\partial\gamma}{\partial x_i}\, dx\, dt$$
$$= -\int_0^T \int_\Omega H(u,k)\frac{\partial\gamma}{\partial t} + Q_i(u,k)\frac{\partial\gamma}{\partial x_i}\, dx\, dt \leq 0$$

for all $\gamma \in \mathcal{D}(Q_T)$, $\gamma \geq 0$. Therefore u satisfies (7.4). Due to Lemma 7.34, (u, u^D) satisfy (7.5). In order to show (7.6) we use

$$H_\ell(z,k) \equiv \Big((z-k)^2 + \frac{1}{\ell^2}\Big)^{1/2} - \frac{1}{\ell}\,,$$
$$\mathbf{Q}_\ell(z,k) = \int_k^z \partial_1 H_\ell(\lambda,k)\, \mathbf{f}'(\lambda)\, d\lambda$$

in (7.32) and obtain

$$-\int_0^T \int_\Omega H_\ell(u,k)\frac{\partial\beta}{\partial t} + Q_{\ell i}(u,k)\frac{\partial\beta}{\partial x_i}\, dx\, dt \leq \int_\Omega H_\ell(u^0,k)\, \beta(0)\, dx$$

for all $\beta \in \mathcal{D}((-\infty,T) \times \Omega)$, $\beta \geq 0$ and all $k \in \mathbb{R}$. Letting $\ell \to \infty$

we obtain

$$-\int_0^T \int_\Omega |u-k| \frac{\partial \beta}{\partial t} + F_i(u,k) \frac{\partial \beta}{\partial x_i} \, dx \, dt \le \int_\Omega |u^0 - k| \, \beta(0) \, dx \, .$$

This enables us to use Lemma 7.41, which gives that (u, u^0) satisfy (7.6).

The implication '(7.4)–(7.6) $\Rightarrow$ (7.32)': Let (u, u^D, u^0) satisfy (7.4)–(7.6). Let $(H, \mathbf{Q})$ be a boundary entropy-entropy flux pair and $k \in \mathbb{R}$. We put

$$\eta(z) \equiv H(z,k) \ , \quad \mathbf{q}(z) \equiv \mathbf{Q}(z,k) \ .$$

Let us define (notice that $\eta(k) = 0$)

$$\widetilde{H}(z,w) \equiv \begin{cases} \left.\begin{array}{ll} \eta(z) - \eta(w) & \text{if } z \le w \\ 0 & \text{if } w \le z \le k \\ \eta(z) & \text{if } k \le z \end{array}\right\} & \text{and } w \le k \, , \\ \left.\begin{array}{ll} \eta(z) & \text{if } z \le k \\ 0 & \text{if } k \le z \le w \\ \eta(z) - \eta(w) & \text{if } w \le z \end{array}\right\} & \text{and } w \ge k \end{cases}$$

and

$$\widetilde{\mathbf{Q}}(z,w) \equiv \begin{cases} \left.\begin{array}{ll} \mathbf{q}(z) - \mathbf{q}(w) & \text{if } z \le w \\ \mathbf{0} & \text{if } w \le z \le k \\ \mathbf{q}(z) & \text{if } k \le z \end{array}\right\} & \text{and } w \le k \, , \\ \left.\begin{array}{ll} \mathbf{q}(z) & \text{if } z \le k \\ \mathbf{0} & \text{if } k \le z \le w \\ \mathbf{q}(z) - \mathbf{q}(w) & \text{if } w \le z \end{array}\right\} & \text{and } w \ge k \, . \end{cases}$$

Thus $(\widetilde{H}, \widetilde{\mathbf{Q}}) \in C(\mathbb{R}^2) \times C(\mathbb{R}^2)^d$ and $(\widetilde{H}, \widetilde{\mathbf{Q}})$ can be locally uniformly approximated by $(\widetilde{H}_\ell, \widetilde{\mathbf{Q}}_\ell)$ which are defined as

$$\begin{aligned} \widetilde{H}_\ell(z,w) &\equiv \int_{\mathbb{R}} H_\ell(\lambda, w) \, \rho_{1/\ell}(z-\lambda) \, d\lambda \, , \\ \widetilde{\mathbf{Q}}_\ell(z,w) &\equiv \int_w^z \partial_1 \, \widetilde{H}_\ell(z,w) \, \mathbf{f}'(\lambda) \, d\lambda \, , \end{aligned}$$

where $\rho_{1/\ell}$ is a usual mollifier, and $H_\ell(z,w)$ is defined as

$$\begin{cases} \left.\begin{array}{ll} \eta(z)-\eta(w-\frac{1}{\ell}) & \text{if } z\le w-\frac{1}{\ell} \\ 0 & \text{if } w-\frac{1}{\ell}\le z\le k+\frac{1}{\ell} \\ \eta(z)-\eta(k+\frac{1}{\ell}) & \text{if } k+\frac{1}{\ell}\le z \end{array}\right\} \text{ and } w\le k, \\ \left.\begin{array}{ll} \eta(z)-\eta(k-\frac{1}{\ell}) & \text{if } z\le k-\frac{1}{\ell} \\ 0 & \text{if } k-\frac{1}{\ell}\le z\le w+\frac{1}{\ell} \\ \eta(z)-\eta(w+\frac{1}{\ell}) & \text{if } w+\frac{1}{\ell}\le z \end{array}\right\} \text{ and } w\ge k. \end{cases}$$

It follows from (7.5) that

$$\operatorname*{ess\,lim}_{s\to 0-}\int_\Gamma \widetilde{\mathbf{Q}}_\ell(u(r+s\,\boldsymbol{\nu}(r)),u^D(r))\cdot\boldsymbol{\nu}(r)\,\beta(r)\,dr\ge 0$$

for all $\beta\in L^1(\Gamma)$, $\beta\ge 0$ almost everywhere. For $\ell\to\infty$ we have

$$\operatorname*{ess\,lim}_{s\to 0-}\int_\Gamma \widetilde{\mathbf{Q}}(u(r+s\,\boldsymbol{\nu}(r)),u^D(r))\cdot\boldsymbol{\nu}(r)\,\beta(r)\,dr\ge 0\ .$$

Discussing the cases in the definition of $\widetilde{\mathbf{Q}}$ and using the properties of η we obtain

$$|\widetilde{\mathbf{Q}}(z,w)-\mathbf{q}(z)|\le M\eta(w)$$

and therefore for all $\beta\in C^1(\Gamma)$, $\beta\ge 0$,

$$\operatorname*{ess\,lim\,inf}_{s\to 0-}\int_\Gamma \mathbf{q}(u(r+s\,\boldsymbol{\nu}(r)))\cdot\boldsymbol{\nu}(r)\,\beta(r)\,dr \ge -M\int_\Gamma \eta(u^D(r))\,\beta(r)\,dr\ .$$

Because of (7.6), we have for all $\beta\in L^1(\Omega)$, $\beta\ge 0$ almost everywhere,

$$\operatorname*{ess\,lim}_{t\to 0+}\int_\Omega \eta(u(t,x))\,\beta(x)\,dx=\int_\Omega \eta(u^0(x))\,\beta(x)\,dx\,.$$

It follows from (7.4) that

$$-\int_0^T\int_\Omega \eta(u)\frac{\partial\gamma}{\partial t}+q_i(u)\frac{\partial\gamma}{\partial x_i}\,dx\,dt\le 0$$

for all $\gamma\in\mathcal{D}(Q_T)$, $\gamma\ge 0$. Altogether we have (cf. again proof of

Lemma 7.12)

$$-\int_0^T \int_\Omega \eta(u)\frac{\partial \beta}{\partial t} + q_i(u)\frac{\partial \beta}{\partial x_i}\, dx\, dt \le \int_\Omega \eta(u^0)\,\beta(0)\, dx \\ + M\int_\Gamma \eta(u^D)\,\beta(r)\, dr$$

for all $\beta \in \mathcal{D}((-\infty, T) \times \mathbb{R}^d)$, $\beta \ge 0$, which is (7.32). ∎

2.8 Existence in bounded domains via parabolic approximation

In order to show the existence of the solution $u \in L^\infty(Q_T)$ satisfying (7.4)–(7.6), we again use the parabolic perturbation

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + \operatorname{div} \mathbf{f}(u^\varepsilon) - \varepsilon\, \Delta u^\varepsilon &= 0 \qquad \text{in } Q_T\,, \\ u^\varepsilon(0,\cdot) &= u^{\varepsilon 0} \quad \text{in } \Omega\,, \\ u^\varepsilon &= u^{\varepsilon D} \quad \text{on } \Gamma\,. \end{aligned} \tag{8.1}$$

For smooth $u^{\varepsilon 0}$, $u^{\varepsilon D}$, satisfying compatibility conditions on $\overline{\Gamma} \cap \overline{\Omega}$, and $\varepsilon > 0$ fixed, there exists a unique smooth solution u^ε of (8.1) (to prove this, we can use the same method as in the proof of Lemma 2.3, namely, apply Theorems 2.7 and 2.10 from the Appendix and then proceed to the nonlinear case using $v = ue^{-\lambda t}$). The main goal of this section is to show that if $u^{\varepsilon D}$, $u^{\varepsilon 0}$ approximate $u^D \in L^\infty(\Gamma)$, $u^0 \in L^\infty(\Omega)$, respectively, then the sequence of solutions u^ε of (8.1) approaches a function $u \in L^\infty(Q_T)$, satisfying (7.4)–(7.6).

Let us start with the following construction: for $\delta > 0$ sufficiently small we define

$$s(x) \equiv \begin{cases} \min(\operatorname{dist}(x, \partial\Omega), \delta) & \text{for } x \in \Omega\,, \\ -\min(\operatorname{dist}(x, \partial\Omega), \delta) & \text{for } x \in \mathbb{R}^d \setminus \Omega. \end{cases}$$

This function is Lipschitz continuous in $\mathbb{R}^d$ and smooth on the closure of $\{x \in \mathbb{R}^d : |s(x)| < \delta\}$. For $\varepsilon > 0$ we define ξ_ε by

$$\xi_\varepsilon(x) \equiv 1 - \exp\Big(-\frac{M + \varepsilon\, L}{\varepsilon} s(x)\Big)\,, \tag{8.2}$$

where $L \equiv \sup_{0<s(x)<\delta} |\Delta s(x)|$ and $M > 0$. This function satisfies the weak differential inequality

$$M\int_\Omega |\nabla \xi_\varepsilon|\,\beta \le \varepsilon \int_\Omega \nabla\xi_\varepsilon\,\nabla\beta + (M + L\,\varepsilon)\int_{\partial\Omega} \beta\, dr \tag{8.3}$$

for all $\beta \in \mathcal{D}(\mathbb{R}^d)$, $\beta \geq 0$. Indeed, define for $\kappa > 0$

$$w_\kappa(x) = 1 - \exp\Big(-\frac{1}{\kappa} s(x)\Big) \ .$$

Since for $x \in \overline{\Omega}$,

$$\nabla w_\kappa(x) = \frac{1}{\kappa} \exp\Big(-\frac{1}{\kappa} s(x)\Big) \nabla s(x) \,,$$
$$|\nabla s(x)| = \begin{cases} 1 & \text{for } 0 \leq s(x) < \delta \,, \\ 0 & \text{for } s(x) = \delta \end{cases}$$

and

$$\Delta w_\kappa = -\frac{1}{\kappa^2} \exp\Big(-\frac{1}{\kappa} s(x)\Big) + \frac{1}{\kappa} \exp\Big(-\frac{1}{\kappa} s(x)\Big) \Delta s(x) \,,$$

we obtain $w_\kappa \in W^{1,\infty}(\Omega) \cap C^2(\overline{\{0 < s < \delta\}})$. Moreover, for $\beta \in \mathcal{D}(\mathbb{R}^d)$, $\beta \geq 0$ we have

$$\begin{aligned}
\int_\Omega \nabla w_\kappa \, \nabla \beta \, dx = & \int_{\{0<s<\delta\}} \nabla w_\kappa \, \nabla \beta \, dx \\
= & - \int_{\{0<s<\delta\}} \Delta w_\kappa \, \beta \, dx - \int_{\{s=0+\}} \nabla w_\kappa \, \nabla s \, \beta \, dr \\
& \qquad + \int_{\{s=\delta-\}} \nabla w_\kappa \, \nabla s \, \beta \, dr \\
= & \frac{1}{\kappa^2} \int_{\{0<s<\delta\}} \exp\Big(-\frac{1}{\kappa} s(x)\Big) \beta \, dx \\
& \qquad - \frac{1}{\kappa} \int_{\{0<s<\delta\}} \exp\Big(-\frac{1}{\kappa} s(x)\Big) \, \Delta s \, \beta \, dx \\
& \qquad - \frac{1}{\kappa} \int_{\{s=0+\}} \beta \, dr + \frac{1}{\kappa} \exp\Big(-\frac{1}{\kappa} \delta\Big) \int_{\{s=\delta-\}} \beta \, dr \\
\geq & \frac{1}{\kappa^2} \int_{\{0<s<\delta\}} \exp\Big(-\frac{1}{\kappa} s(x)\Big) \, \beta(x) \, dx \\
& \qquad - \frac{L}{\kappa} \int_{\{0<s<\delta\}} \exp\Big(-\frac{1}{\kappa} s(x)\Big) \, \beta(x) \, dx \\
& \qquad - \frac{1}{\kappa} \int_{\{s=0+\}} \beta \, dr
\end{aligned}$$

and therefore we obtain

$$\kappa \int_\Omega \nabla w_\kappa \, \nabla \beta + \int_{\partial\Omega} \beta \, dr \geq (1 - \kappa \, L) \int_\Omega |\nabla w_\kappa| \, \beta \, dx$$

and if $1 - \kappa L > 0$,

$$M \int_\Omega |\nabla w_\kappa|\, \beta\, dx \le \frac{M\kappa}{(1-\kappa L)} \int_\Omega \nabla w_\kappa\, \nabla \beta + \frac{M}{1-\kappa\, L} \int_{\partial\Omega} \beta\, dr\ .$$

For $\kappa = \frac{\varepsilon}{M+\varepsilon\, L}$ we obtain (8.3).

The following theorem investigates the properties of the solution u^ε to the problem (8.1) for fixed $\varepsilon > 0$. For simplicity we drop the superscript ε in u^ε, $u^{\varepsilon 0}$, $u^{\varepsilon D}$.

Theorem 8.4 *Let u, u_1, u_2 be solutions of (8.1) corresponding to the smooth initial data u^0, u_1^0, u_2^0, and boundary data u^D, u_1^D, u_2^D, respectively. Let M be a constant of Lipschitz continuity of $\mathbf{f} \in C^1(\mathbb{R})^d$, restricted to the ball with radius defined as the maximum of $\{\|u^0\|_\infty, \|u_i^0\|_\infty, \|u^D\|_\infty, \|u_i^D\|_\infty\}$. Let ξ_ε, L be defined by (8.2). Then we have:*

- $$\begin{aligned} &- \int_0^T \int_\Omega \Big\{ H(u,k) \frac{\partial \beta}{\partial t} + Q_i(u,k) \frac{\partial \beta}{\partial x_i} + \varepsilon\, H(u,k)\, \Delta\beta \Big\} \xi_\varepsilon\, dx\, dt \\ &\le \int_\Omega H(u^0,k)\, \beta(0)\, \xi_\varepsilon\, dx + (M + L\,\varepsilon) \int_\Gamma H(u^D,k)\, \beta\, dr \\ &\quad + 2\varepsilon \int_0^T \int_\Omega H(u,k) \frac{\partial \beta}{\partial x_i} \frac{\partial \xi_\varepsilon}{\partial x_i}\, dx\, dt \end{aligned} \tag{8.5}$$

holds for all $\beta \in \mathcal{D}((-\infty, T) \times \mathbb{R}^d)$, $\beta \ge 0$, all $k \in \mathbb{R}$ and all boundary entropy-entropy flux pairs $(H, \mathbf{Q})$;

- $$\begin{aligned} \int_\Omega |u_1(t) - u_2(t)|\, \xi_\varepsilon\, dx \le &\int_\Omega |u_1^0 - u_2^0|\, \xi_\varepsilon\, dx \\ &+ (M + L\,\varepsilon) \int_\Gamma |u_1^D - u_2^D|\, dr \end{aligned} \tag{8.6}$$

holds for all $t \in (0,T)$;

- $$\begin{aligned} &\sup_{Q_T} u \le \max\,(\sup_\Omega u^0,\ \sup_\Gamma u^D)\ , \\ &\inf_{Q_T} u \ge \min\,(\inf_\Omega u^0,\ \inf_\Gamma u^D)\ . \end{aligned} \tag{8.7}$$

In particular,

$$\sup_{Q_T} |u| \le \max\,(\sup_\Omega |u^0|,\ \sup_\Gamma |u^D|)\ . \tag{8.8}$$

- *Moreover, if* $\mathbf{f} \in C^2(\mathbb{R})^d$, *we have*

$$\sup_{t\in(0,T)} \int_\Omega \Big|\frac{\partial u}{\partial t}(t)\Big| + |\nabla u(t)|\, dx \le \lambda\,, \tag{8.9}$$

where $\lambda = \lambda(\|u^0\|_\Omega, \|u^D\|_\Gamma, T, \Omega, \mathbf{f})$ *does not depend on* $\varepsilon \in (0,1)$. *Here, we use the notation*

$$\begin{aligned}
\|u^0\|_\Omega &\equiv \int_\Omega |\Delta u^0| + |\nabla u^0| + |u^0|\, dx\,,\\
\|u^D\|_\Gamma &= \sup_{Q_T}\{|\Delta u^D| + \Big|\frac{\partial u^D}{\partial t}\Big| + |\nabla u^D| + |u^D|\}\\
&\quad + \int_0^T \int_\Omega \Big|\nabla^2 \frac{\partial u^D}{\partial t}\Big| + |\nabla^3 u^D| + \Big|\frac{\partial^2 u^D}{\partial t^2}\Big|\\
&\qquad + \Big|\nabla \frac{\partial u^D}{\partial t}\Big| + |\nabla^2 u^D|\, dx\, dt\,,
\end{aligned}$$

where u^D *is identified with its smooth extension to* $\overline{Q_T}$.

PROOF : • Let $(H, \mathbf{Q})$ be a boundary entropy-entropy flux pair and let $k \in \mathbb{R}$ be fixed. We again use the abbreviation

$$\eta(z) \equiv H(z,k)\,, \qquad \mathbf{q}(z) \equiv \mathbf{Q}(z,k)\,.$$

Multiplying (8.1) by $\eta'(u)$ and using the same argument as in Section 2.3, we obtain

$$\frac{\partial \eta(u)}{\partial t} + \operatorname{div} \mathbf{q}(u) \le \varepsilon\, \Delta \eta(u)\,. \tag{8.10}$$

Now we multiply (8.10) by $\xi_\varepsilon\, \beta$, integrate over Q_T, and after partial integration we obtain

$$\begin{aligned}
&- \int_0^T \int_\Omega \Big\{\eta(u) \frac{\partial \beta}{\partial t} + q_i(u) \frac{\partial \beta}{\partial x_i} + \varepsilon\, \eta(u)\, \Delta\beta\Big\} \xi_\varepsilon\, dx\, dt\\
&\le \int_\Omega \eta(u^0)\, \beta(0)\, \xi_\varepsilon\, dx + \int_0^T \int_\Omega q_i(u)\, \beta \frac{\partial \xi_\varepsilon}{\partial x_i}\, dx\, dt\\
&\quad - \varepsilon \int_0^T \int_\Omega \frac{\partial\big(\eta(u)\beta\big)}{\partial x_i}\, \frac{\partial \xi_\varepsilon}{\partial x_i} - 2\eta(u) \frac{\partial \beta}{\partial x_i} \frac{\partial \xi_\varepsilon}{\partial x_i}\, dx\, dt\\
&\le \int_\Omega \eta(u^0)\, \beta(0)\, \xi_\varepsilon\, dx + M \int_0^T \int_\Omega \eta(u)\, \beta\, |\nabla \xi_\varepsilon|\, dx\, dt\\
&\quad - \varepsilon \int_0^T \int_\Omega \frac{\partial\big(\eta(u)\, \beta\big)}{\partial x_i} \frac{\partial \xi_\varepsilon}{\partial x_i}\, dx\, dt + 2\, \varepsilon \int_0^T \int_\Omega \eta(u) \frac{\partial \beta}{\partial x_i} \frac{\partial \xi_\varepsilon}{\partial x_i}\, dx\, dt\,,
\end{aligned}$$

where we used the fact that

$$|\mathbf{q}(z)| \le M\,\eta(z)\,.$$

Using (8.3) with $\eta(u)\beta$ instead of β in the last but one inequality, we obtain (8.5).

- Let us denote

$$w \equiv u_1 - u_2\,, \qquad w^D \equiv u_1^D - u_2^D\,, \qquad w^0 \equiv u_1^0 - u_2^0\,.$$

Multiplying the equation

$$\frac{\partial w}{\partial t} + \operatorname{div}(\mathbf{f}(u_1) - \mathbf{f}(u_2)) - \varepsilon\,\Delta w = 0$$

by $\varphi_\delta'(w)\xi_\varepsilon$, where $\varphi_\delta(z) \equiv (z^2+\delta^2)^{1/2}$, and integrating over $(0,t)\times\Omega$ we obtain

$$\begin{aligned}
\int_\Omega \varphi_\delta(w(t))\,\xi_\varepsilon\,dx - \int_\Omega \varphi_\delta(w^0)\xi_\varepsilon\,dx \\
&- \int_0^t \int_\Omega (f_i(u_1) - f_i(u_2))\,\varphi_\delta''(w)\frac{\partial w}{\partial x_i}\xi_\varepsilon\,dx\,d\tau \\
&- \int_0^t \int_\Omega (f_i(u_1) - f_i(u_2))\,\varphi_\delta'(w)\frac{\partial \xi_\varepsilon}{\partial x_i}\,dx\,d\tau \\
&+ \varepsilon \int_0^t \int_\Omega |\nabla w|^2\,\varphi_\delta''(w)\,\xi_\varepsilon\,dx\,d\tau \\
&+ \varepsilon \int_0^t \int_\Omega \frac{\partial \varphi_\delta(w)}{\partial x_i}\,\frac{\partial \xi_\varepsilon}{\partial x_i}\,dx = 0\,.
\end{aligned}$$

Using the Lipschitz continuity of $\mathbf{f}$, Young's inequality and the fact that $z^2\varphi_\delta''(z) = z^2\delta^2(z^2+\delta^2)^{-3/2} < \delta$, we get

$$\begin{aligned}
-\big(f_i(u_1) - f_i(u_2)\big)\frac{\partial w}{\partial x_i}\varphi_\delta''(w)\,\xi_\varepsilon + \varepsilon\,|\nabla w|^2\,\varphi_\delta''(w)\xi_\varepsilon \\
&\ge \big\{-M\,|w|\,|\nabla w| + \varepsilon\,|\nabla w|^2\big\}\varphi_\delta''(w)\,\xi_\varepsilon \\
&\ge -\,\frac{M^2}{4\,\varepsilon}w^2\,\varphi_\delta''(w)\,\xi_\varepsilon \\
&\ge -\,\frac{M^2\,\delta}{4\,\varepsilon}\xi_\varepsilon\,.
\end{aligned}$$

Moreover, observing that $|z|\,|\varphi_\delta'(z)| \le \varphi_\delta(z)$, we obtain

$$\begin{aligned}
-\big(f_i(u_1) - f_i(u_2)\big)\frac{\partial \xi_\varepsilon}{\partial x_i}\,\varphi_\delta'(w) &\ge -\,M\,|w|\,|\varphi_\delta'(w)|\,|\nabla \xi_\varepsilon| \\
&\ge -\,M\,\varphi_\delta(w)\,|\nabla \xi_\varepsilon|\,.
\end{aligned}$$

Finally,

$$\int_\Omega \varphi_\delta(w(t))\xi_\varepsilon\,dx - \int_\Omega \varphi_\delta(w^0)\xi_\varepsilon\,dx - \frac{M^2\,T\,\delta}{4\,\varepsilon}\int_\Omega \xi_\varepsilon\,dx$$
$$- M\int_0^t\int_\Omega \varphi_\delta(w)\,|\nabla\xi_\varepsilon|\,dx\,d\tau + \varepsilon\int_0^t\int_\Omega \frac{\partial\varphi_\delta(w)}{\partial x_i}\frac{\partial\xi_\varepsilon}{\partial x_i}\,dx\,d\tau \le 0$$

and therefore, taking $\beta = \varphi_\delta$ in (8.3),

$$\begin{aligned}\int_\Omega \varphi_\delta(w(t))\,\xi_\varepsilon\,dx \le &\int_\Omega \varphi_\delta(w^0)\,\xi_\varepsilon\,dx\\ &+ (M + L\,\varepsilon)\int_0^t\int_{\partial\Omega}\varphi_\delta(w^D)\,dr\,d\tau \qquad (8.11)\\ &+ \frac{M^2\,T}{4\,\varepsilon}\delta\int_\Omega \xi_\varepsilon\,dx\ .\end{aligned}$$

Letting $\delta \to 0+$ in (8.11) we obtain (8.6).

- Let us multiply (8.1) by $\varphi'_\delta(u)$, where

$$\varphi_\delta(z) = \begin{cases} ((z-m)^2 + \delta^2)^{1/2} - \delta & \text{for } z \ge m\,,\\ 0 & \text{for } z \le m,\end{cases}$$
$$m = \max\{\sup_\Gamma u^D, \sup_\Omega u^0\}\ .$$

Using the properties of φ_δ, we obtain

$$\int_\Omega \varphi_\delta(u(t))\,dx - \int_0^t\int_\Omega (f_i(u) - f_i(m))\frac{\partial u}{\partial x_i}\varphi''_\delta(u)\,dx\,d\tau$$
$$+ \varepsilon\int_0^t\int_\Omega |\nabla u|^2\varphi''_\delta(u)\,dx\,d\tau = 0\ .$$

Now, the estimate

$$\begin{aligned}-\big(f_i(u) - f_i(m)\big)\frac{\partial u}{\partial x_i}&\varphi''_\delta(u) + \varepsilon|\nabla u|^2\varphi''_\delta(u)\\ &\ge \big\{-M\,|u-m|\,|\nabla u| + \varepsilon\,|\nabla u|^2\big\}\varphi''_\delta(u)\\ &\ge -\frac{M^2}{4\,\varepsilon}(u-m)^2\varphi''_\delta(u)\\ &\ge -\frac{M^2\delta}{4\,\varepsilon}\end{aligned}$$

follows by an analogous argument as above, and the limit $\delta \to 0+$

gives

$$\int_\Omega \max\,(u(t)-m,0)\,dx \le 0$$

which is $(8.7)_1$. Similarly, we get $(8.7)_2$ and therefore we have (8.8).

• Let u^D be the smooth extension of u^D onto $\overline{Q_T}$. Denoting for $\mathbf{f} \in C^2(\mathbb{R})^d$,

$$\begin{aligned} v &\equiv \frac{\partial u}{\partial t} - \frac{\partial u^D}{\partial t}\,, \\ e &\equiv \frac{\partial^2 u^D}{\partial t^2} + \operatorname{div}\Big(\mathbf{f}'(u)\frac{\partial u^D}{\partial t}\Big) - \varepsilon\,\Delta \frac{\partial u^D}{\partial t}\,, \end{aligned}$$

we have

$$\frac{\partial v}{\partial t} + \operatorname{div}(\mathbf{f}'(u)v) - \varepsilon\,\Delta v = -e\,. \tag{8.12}$$

Multiplying (8.12) by $\varphi'_\delta(v)$, where $\varphi_\delta(z) \equiv (z^2+\delta^2)^{1/2}$, and integrating over $(0,t)\times\Omega$, we obtain

$$\begin{aligned} \int_\Omega \varphi_\delta(v(t))\,dx - \int_\Omega \varphi_\delta(v(0))\,dx - \int_0^t\int_\Omega \mathbf{f}'(u)\cdot\nabla v\, v\,\varphi''_\delta(v)\,dx\,d\tau \\ + \varepsilon\int_0^t\int_\Omega |\nabla v|^2\varphi''_\delta(v)\,dx\,d\tau = -\int_0^t\int_\Omega e\,\varphi'_\delta(v)\,dx\,d\tau\,, \end{aligned} \tag{8.13}$$

where we used $\varphi'_\delta(v) = 0$ on Γ. Further we have

$$\begin{aligned} -\mathbf{f}'(u)\cdot\nabla v\, v\,\varphi''_\delta(v) + \varepsilon\,|\nabla v|^2\varphi''_\delta(v) &\ge -\frac{1}{4\,\varepsilon}|\mathbf{f}'(u)|^2\, v^2\,\varphi''_\delta(v) \\ &\ge -\frac{M^2\,\delta}{4\,\varepsilon} \end{aligned}$$

and thus if we let $\delta \to 0+$ in (8.13), we obtain

$$\int_\Omega |v(t)|\,dx \le \int_\Omega |v(0)|\,dx + \int_0^t\int_\Omega |e|\,dx\,d\tau\ .$$

Using

$$v(0) = \frac{\partial u}{\partial t}(0) - \frac{\partial u^D}{\partial t}(0) = -\operatorname{div}\mathbf{f}(u^0) + \varepsilon\,\Delta u^0 - \frac{\partial u^D}{\partial t}(0)$$

we obtain

$$\int_\Omega \Big|\frac{\partial u}{\partial t}\Big|\,dx \le \lambda\Big(1 + \int_0^t\int_\Omega |\nabla u|\,dx\,d\tau\Big) \tag{8.14}$$

for all $t \in [0,T]$, with λ depending on T, Ω, $\mathbf{f}$, $\|u^0\|_\Omega$, $\|u^D\|_\Gamma$. Further we have

$$\frac{\partial w}{\partial t} + \operatorname{div}(\mathbf{f}(u) - \mathbf{f}(u^D)) - \varepsilon\,\Delta w = -g\,, \tag{8.15}$$

where

$$w \equiv u - u^D\,, \qquad g \equiv \frac{\partial u^D}{\partial t} + \operatorname{div}\mathbf{f}(u^D) - \varepsilon\,\Delta u^D\ .$$

Now we multiply (8.15) by $\varphi_\delta'(w)\,\beta$, where $\beta \in \mathcal{D}(\mathbb{R}^d)$, $\beta \geq 0$, depends only on the space variable and where

$$\varphi_\delta(z) \equiv (z^2 + \delta^2)^{1/2} - \delta\ .$$

After integration over $(0,t)\times\Omega$, partial integration and using the fact that on Γ

$$\varphi_\delta'(w) = 0\,, \qquad \varphi_\delta(w) = 0\,, \qquad \nabla\varphi_\delta(w)\cdot\boldsymbol{\nu} = 0\,,$$

we arrive at

$$\begin{aligned}
\int_\Omega \varphi_\delta(w(t))\,\beta\,dx - \int_\Omega \varphi_\delta(w(0))\,\beta\,dx - \varepsilon\int_0^t\int_\Omega \varphi_\delta(w)\,\Delta\beta\,dx\,d\tau \\
- \int_0^t\int_\Omega \varphi_\delta'(w)(f_i(u) - f_i(u^D))\frac{\partial\beta}{\partial x_i}\,dx\,d\tau \\
- \int_0^t\int_\Omega (f_i(u) - f_i(u^D))\frac{\partial w}{\partial x_i}\varphi_\delta''(w)\,\beta\,dx\,d\tau \\
+ \varepsilon\int_0^t\int_\Omega |\nabla w|^2\,\varphi_\delta''(w)\,\beta\,dx\,d\tau = -\int_0^t\int_\Omega \varphi_\delta'(w)\,g\,\beta\,dx\,d\tau\,.
\end{aligned}$$

Now we let $\delta \to 0+$ and obtain (F_i are as in Remark 7.8)

$$\begin{aligned}
\int_\Omega |w(t)|\,dx - \int_\Omega |w(0)|\,dx - \int_0^t\int_\Omega F_i(u,u^D)\frac{\partial\beta}{\partial x_i}\,dx\,d\tau \\
- \varepsilon\int_0^t\int_\Omega |w|\,\Delta\beta\,dx\,d\tau \leq -\int_0^t\int_\Omega \operatorname{sgn}(w)\,g\,\beta\,dx\,d\tau\ .
\end{aligned}$$

We put

$$\beta(x) = \beta_\rho(x) \equiv \gamma\Big(\frac{s(x)}{\rho}\Big)\,,$$

where $s(x)$ is as before and $\gamma \in C^\infty(\mathbb{R})$ is a fixed non-negative function such that

$$\gamma(0) = 0\,, \qquad \gamma(\sigma) = 1 \quad \text{for } \sigma \geq 1.$$

Due to $\beta_\rho = 0$ on Γ we have

$$\begin{aligned}
&\lim_{\rho\to 0+} \int_0^t \int_\Omega \mathbf{F}(u,u^D)\,\nabla\beta_\rho\,dx\,d\tau \\
&= \lim_{\rho\to 0+} \int_0^{\frac{\delta}{\rho}} \gamma'(\sigma) \int_0^t \int_{\partial\Omega} \mathbf{F}(u(\tau, r+\rho\sigma), u^D(\tau, r+\rho\sigma)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot\boldsymbol{\nu}(r)\,dr\,d\tau\,d\sigma \\
&= \int_0^\infty \gamma'(\sigma)\,d\sigma \int_0^t \int_{\partial\Omega} \mathbf{F}(u,u^D)\cdot\boldsymbol{\nu}\,dr\,d\tau \\
&= \int_0^t \int_{\partial\Omega} \mathbf{F}(u,u^D)\cdot\boldsymbol{\nu}\,dr\,d\tau = 0\,,
\end{aligned}$$

and if $r(x)$ is the nearest boundary point to x, then similarly

$$\begin{aligned}
\lim_{\rho\to 0+} &\int_0^t \int_\Omega |w|\,\Delta\beta_\rho\,dx\,d\tau \\
&= \lim_{\rho\to 0+} \int_0^t \int_\Omega |w(t,x) - w(t,r(x))|\,\Delta\beta_\rho(x)\,dx\,d\tau \\
&= \int_0^\infty \sigma\gamma''(\sigma)\,d\sigma \int_0^t \int_{\partial\Omega} |\nabla w\cdot\boldsymbol{\nu}|\,dr\,d\tau \\
&= -\int_0^t \int_{\partial\Omega} |\nabla w\cdot\boldsymbol{\nu}|\,dr\,d\tau\,.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\int_\Omega |w(t)|\,dx + \varepsilon &\int_0^t \int_{\partial\Omega} |\nabla w\cdot\boldsymbol{\nu}|\,dr\,d\tau \\
&\le \int_\Omega |w(0)|\,dx + \int_0^t \int_\Omega |g|\,dx\,d\tau\ .
\end{aligned} \tag{8.16}$$

Let us denote $w_i \equiv \frac{\partial w}{\partial x_i}$, $i = 1,\dots,d$, $\mathbf{w} \equiv (w_1,\dots,w_d)$. Then we have (again note that $\mathbf{f} \in C^2(\mathbb{R})^d$)

$$\frac{\partial w_i}{\partial t} + \operatorname{div}(\mathbf{f}'(u)w_i) - \varepsilon\,\Delta w_i = -h_i\,, \tag{8.17}$$

$i = 1,\dots,d$, where

$$h_i = \frac{\partial^2 u^D}{\partial x_i \partial t} + \operatorname{div}\Big(\mathbf{f}'(u)\frac{\partial u^D}{\partial x_i}\Big) - \varepsilon\,\Delta\frac{\partial u^D}{\partial x_i}\ .$$

Let us multiply the ith equation of (8.17) by $\frac{\partial}{\partial \xi_i}\phi_\delta(\mathbf{w})$, where

$\phi_\delta(\xi) = (|\xi|^2 + \delta^2)^{1/2}$, add it up and integrate over $(0,t) \times \Omega$. Let us investigate the resulting terms:

$$\begin{aligned}
&\int_0^t \int_\Omega \frac{\partial w_i}{\partial t} \frac{\partial \phi_\delta}{\partial \xi_i}(\mathbf{w})\, dx\, d\tau \\
&\qquad = \int_\Omega \phi_\delta(\mathbf{w}(t))\, dx - \int_\Omega \phi_\delta(\mathbf{w}(0))\, dx\,, \\
&-\varepsilon \int_0^t \int_\Omega \Delta w_i \frac{\partial \phi_\delta}{\partial \xi_i}(\mathbf{w})\, dx\, d\tau \\
&\qquad = \varepsilon \int_0^t \int_\Omega \frac{\partial w_i}{\partial x_j} \frac{\partial^2 \phi_\delta}{\partial \xi_i \partial \xi_k}(\mathbf{w}) \frac{\partial w_k}{\partial x_j}\, dx\, d\tau \\
&\qquad - \varepsilon \int_0^t \int_{\partial\Omega} \frac{\partial w_i}{\partial x_j} \nu_j \frac{\partial \phi_\delta}{\partial \xi_i}(\mathbf{w})\, dr\, d\tau\,, \\
&\int_0^t \int_\Omega \operatorname{div}(\mathbf{f}'(u) w_i) \frac{\partial \phi_\delta}{\partial \xi_i}(\mathbf{w})\, dx\, d\tau \\
&\qquad = -\int_0^t \int_\Omega f_j'(u)\, w_i \frac{\partial^2 \phi_\delta}{\partial \xi_i \partial \xi_k} \frac{\partial w_k}{\partial x_j}\, dx\, d\tau \\
&\qquad + \int_0^t \int_{\partial\Omega} f_j'(u)\, \nu_j\, w_i \frac{\partial \phi_\delta}{\partial \xi_i}(\mathbf{w})\, dr\, d\tau\ .
\end{aligned}$$

Due to the estimate

$$\begin{aligned}
&\varepsilon \frac{\partial w_i}{\partial x_j} \frac{\partial^2 \phi_\delta}{\partial \xi_i \partial \xi_k}(\mathbf{w}) \frac{\partial w_k}{\partial x_j} - f_j'(u)\, w_i \frac{\partial^2 \phi_\delta}{\partial \xi_i \partial \xi_k}(\mathbf{w}) \frac{\partial w_k}{\partial x_j} \\
&\qquad = \frac{\delta^2}{(|\mathbf{w}|^2 + \delta^2)^{3/2}} \left[\varepsilon |\nabla \mathbf{w}|^2 - f_j'(u)\, w_i \frac{\partial w_i}{\partial x_j} \right] \\
&\qquad \geq -\frac{1}{4\,\varepsilon} |\mathbf{f}'(u)|^2 \frac{\delta^2 |\mathbf{w}|^2}{(|\mathbf{w}|^2 + \delta^2)^{3/2}} \geq -\frac{M^2\, \delta}{4\, \varepsilon}
\end{aligned}$$

we obtain for $\delta \to 0+$

$$\begin{aligned}
\int_\Omega |\mathbf{w}(t)|\, dx &\leq \int_\Omega |\mathbf{w}(0)|\, dx + \int_0^t \int_\Omega |\mathbf{h}|\, dx\, d\tau \\
&+ \limsup_{\delta \to 0+} \int_0^t \int_{\partial\Omega} \Big| \mathbf{f}'(u) \cdot \boldsymbol{\nu}\, w_j \frac{\partial \phi_\delta}{\partial \xi_j}(\mathbf{w}) \qquad (8.18) \\
&- \varepsilon\, \nabla w_i \cdot \boldsymbol{\nu} \frac{\partial \phi_\delta}{\partial \xi_i}(\mathbf{w}) \Big|\, dr\, d\tau\ .
\end{aligned}$$

Due to $w = 0$ on Γ, we have on Γ

$$\mathbf{w} = \nabla w = (\nabla w \cdot \boldsymbol{\nu})\boldsymbol{\nu}\,, \qquad \Delta w = D^2 w(\boldsymbol{\nu},\boldsymbol{\nu}) + \Delta s\, \nabla w \cdot \boldsymbol{\nu}\ ,$$

where $D^2 w$ is the bilinear form of the second differential of w. Therefore, the integrand can be rewritten as

$$\begin{aligned}
&\mathbf{f}'(u)\cdot\boldsymbol{\nu}\, w_j \frac{\partial \phi_\delta}{\partial \xi_j}(\mathbf{w}) - \varepsilon\, \nabla w_i \cdot \boldsymbol{\nu} \frac{\partial \phi_\delta}{\partial \xi_i}(\mathbf{w}) \\
&= \mathbf{f}'(u)\cdot \boldsymbol{\nu} \frac{|\nabla w|^2}{(|\nabla w|^2+\delta^2)^{1/2}} - \varepsilon\, D^2 w\Big(\boldsymbol{\nu}, \frac{\nabla w}{(|\nabla w|^2+\delta^2)^{1/2}}\Big) \\
&= (\mathbf{f}'(u)\cdot\nabla w - \varepsilon\, D^2 w(\boldsymbol{\nu},\boldsymbol{\nu}))\frac{\nabla w\cdot\boldsymbol{\nu}}{(|\nabla w|^2+\delta^2)^{1/2}} \\
&= \Big(\frac{\partial w}{\partial t} + \operatorname{div}(\mathbf{f}(u) - \mathbf{f}(u^D)) - \varepsilon\,\Delta w \\
&\qquad\qquad + \varepsilon\,\Delta s\,\nabla w\cdot\boldsymbol{\nu}\Big)\frac{\nabla w\cdot\boldsymbol{\nu}}{(|\nabla w|^2+\delta^2)^{1/2}} \\
&= (-g + \varepsilon\,\Delta s\,\nabla w\cdot\boldsymbol{\nu})\frac{\nabla w\cdot\boldsymbol{\nu}}{(|\nabla w|^2+\delta^2)^{1/2}}\ ,
\end{aligned}$$

where g is defined after (8.15). From this and (8.18) we get

$$\begin{aligned}
\int_\Omega |\nabla w(t)|\,dx \le \int_\Omega |\nabla w(0)|\,dx + \int_0^t \int_\Omega |\mathbf{h}|\,dx\,d\tau \\
+ \int_0^t \int_{\partial\Omega} |g| + \varepsilon\, L\,|\nabla w\cdot\boldsymbol{\nu}|\,dr\,d\tau\ ,
\end{aligned}$$

which, together with (8.16), gives

$$\begin{aligned}
\int_\Omega |\nabla w(t)|\,dx + L\int_\Omega |w(t)|\,dx \le \int_\Omega |\nabla w(0)|\,dx + L\int_\Omega |w(0)|\,dx \\
+ \int_0^t \int_{\partial\Omega} |g|\,dr\,d\tau \\
+ \int_0^t \int_\Omega \big(|\mathbf{h}| + L|g|\big)\,dx\,d\tau\ .
\end{aligned}$$

This implies

$$\int_\Omega |\nabla u(t)|\,dx \le \lambda\Big(1 + \int_0^t \int_\Omega |\nabla u|\,dx\,d\tau\Big) \tag{8.19}$$

for all $t \in [0,T]$, where λ depends on T, Ω, $\mathbf{f}$, $\|u^0\|_\Omega$, $\|u^D\|_\Gamma$.

Assertion (8.9) now follows from Gronwall's lemma, (8.19) and (8.14). ∎

Now we are ready to prove the main theorem of this section. We note that since the existence result uses the last assertion of Theorem 8.4, we need the nonlinearity $\mathbf{f}$ to be in the class C^2, whereas the uniqueness can be established for $\mathbf{f} \in C^1$ (see Theorem 7.31).

Theorem 8.20 (Existence) *Let $\mathbf{f} \in C^2(\mathbb{R})^d$. Let u^ε, $\varepsilon > 0$, be the solutions of (8.1) corresponding to the smooth boundary and initial data $u^{\varepsilon D}$, $u^{\varepsilon 0}$, satisfying suitable compatibility conditions on $\partial\Omega$. Let $u^{\varepsilon D}$, $u^{\varepsilon 0}$ be uniformly bounded in the respective L^∞ norms and let*

$$\lim_{\varepsilon\to 0+} u^{\varepsilon D} = u^D \quad \text{in } L^1(\Gamma)\,,$$
$$\lim_{\varepsilon\to 0+} u^{\varepsilon 0} = u^0 \quad \text{in } L^1(\Omega)\,,$$

where $u^D \in L^\infty(\Gamma)$, $u^0 \in L^\infty(\Omega)$. Then the sequence $\{u^\varepsilon\}$ is also uniformly bounded and converges in $C^0([0,T], L^1(\Omega))$ to some function $u \in L^\infty(Q_T)$ which solves (7.4) and satisfies (7.5), (7.6).

Proof : Let $\varepsilon_\ell > 0$ be an arbitrary sequence which converges to zero as $\ell \to \infty$. Denote

$$(u_\ell, u_\ell^D, u_\ell^0) \equiv (u^{\varepsilon_\ell}, u^{\varepsilon_\ell D}, u^{\varepsilon_\ell 0})\,.$$

In order to use Theorem 8.4 we need a smooth extension of u_ℓ^D and u_ℓ^0 to $\overline{Q_T}$. We define $u_\ell^{D,0}$ by

$$\begin{aligned}
u_\ell^{D,0}(t, r + s\,\boldsymbol{\nu}(r)) &\equiv u_\ell^D(t,r) && t \in (0,T),\ r \in \partial\Omega,\\
& && |s| < \min(t,\delta)\,,\\
u_\ell^{D,0}(t,x) &\equiv u_\ell^0(x) && -\delta < t < \min(\operatorname{dist}(x,\partial\Omega),\delta)\,,\\
& && x \in \Omega,\\
u_\ell^{D,0}(t,x) &\equiv 0 && \text{elsewhere}
\end{aligned}$$

and then we mollify for $h > 0$ to get

$$u_{\ell,h}^{D,0}(p) \equiv \int_{\mathbb{R}^{d+1}} u_\ell^{D,0}(\widetilde{p})\,\phi_h(p-\widetilde{p})\,d\widetilde{p}\,,$$

where ϕ_h is the usual mollifier. Let us now denote by $u_{\ell,h}^D$ and $u_{\ell,h}^0$ the restriction of $u_{\ell,h}^{D,0}$ to Γ and $\{0\}\times\Omega$, respectively. Let $u_{\ell,h}$ be the solution of (8.1) with $\varepsilon = \varepsilon_\ell$ and initial and boundary data $u_{\ell,h}^0$ and $u_{\ell,h}^D$, respectively. The uniform boundedness of u_ℓ^0, u_ℓ^D implies the uniform boundedness of $u^0_{\ell,h}$, $u^D_{\ell,h}$, which (see (8.8)) gives the uniform boundedness of u_ℓ, $u_{\ell,h}$. Obviously, we also get

$$\lim_{h\to 0+} u^D_{\ell,h} = u^D_\ell \quad \text{in } L^1(\Gamma)\,,$$
$$\lim_{h\to 0+} u^0_{\ell,h} = u^0_\ell \quad \text{in } L^1(\Omega)\,,$$

uniformly with respect to ℓ. This, together with (8.6), implies

$$\lim_{h\to 0+} u_{\ell,h} = u_\ell \quad \text{in } C^0([0,T], L^1(\Omega))\,, \tag{8.21}$$

uniformly in $\ell \in \mathbb{N}$. On the other hand, it follows from the boundedness of $u^D_\ell \in L^1(\Gamma)$ and $u^0_\ell \in L^1(\Omega)$, that

$$\|u^D_{\ell,h}\|_\Gamma \leq \frac{c}{h^3}\,, \qquad \|u^0_{\ell,h}\|_\Omega \leq \frac{c}{h^2}\,,$$

respectively. For fixed $h > 0$ it follows from (8.9) that the sequences

$$\left\{\frac{\partial u_{\ell,h}}{\partial t}\right\}, \left\{\nabla u_{\ell,h}\right\} \text{ are bounded in } C^0([0,T], L^1(\Omega)). \tag{8.22}$$

Let $\alpha > 0$ be given. From (8.21) we get the existence of some $h > 0$ such that

$$2\int_\Omega |u_{\ell,h}(t) - u_\ell(t)|\,dx < \frac{\alpha}{2} \qquad \forall\, t \in [0,T]\,, \quad \ell \in \mathbb{N}$$

and due to (8.22) there is some $\delta > 0$ such that

$$\delta \int_\Omega \left|\frac{\partial u_{\ell,h}}{\partial t}(t)\right| dx < \frac{\alpha}{2} \qquad \forall\, t \in [0,T]\,, \quad \ell \in \mathbb{N}.$$

Thus for all $\ell \in \mathbb{N}$ and all $t_1, t_2 \in [0,T]$ such that $|t_1 - t_2| \leq \delta$, we have

$$\begin{aligned} \int_\Omega |u_\ell(t_1) - u_\ell(t_2)|\,dx &\leq \sum_{i=1}^2 \int_\Omega |u_{\ell,h}(t_i) - u_\ell(t_i)|\,dx \\ + \int_\Omega |u_{\ell,h}(t_1) - u_{\ell,h}(t_2)|\,dx &\leq \sum_{i=1}^2 \int_\Omega |u_{\ell,h}(t_i) - u_\ell(t_i)|\,dx \\ &+ |t_1 - t_2| \sup_{[t_1,t_2]} \int_\Omega \left|\frac{\partial u_{\ell,h}}{\partial t}(t)\right| dx \leq \alpha\,. \end{aligned} \tag{8.23}$$

This means that u_ℓ is uniformly continuous in $C^0([0,T]; L^1(\Omega))$.

On the other hand, it follows from (8.22) that there is a $\delta > 0$ such that

$$\delta \int_\Omega |\nabla u_{\ell,h}(t)|\,dx < \frac{\alpha}{2} \qquad \forall\, t \in [0,T]\,, \quad \ell \in \mathbb{N}\,.$$

Therefore, we get uniformly in $t \in [0,T]$ and $\ell \in \mathbb{N}$,

$$\begin{aligned}\int_{\Omega^{\Delta x}} |u_\ell(t,x+\Delta x) - u_\ell(t,x)|\,dx &\le 2\int_\Omega |u_{\ell,h}(t,x) - u_\ell(t,x)|\,dx \\ &\quad + |\Delta x| \int_\Omega |\nabla u_{\ell,h}(t,x)|\,dx < \alpha\,,\end{aligned}$$

for $|\Delta x| < \delta$ and

$$\Omega^{\Delta x} \equiv \{x \in \Omega; x + \Delta x \in \Omega\}\,.$$

This, together with the uniform boundedness of u_ℓ, gives that

$$\{u_\ell(t)\}_{t\in[0,T],\ell\in\mathbb{N}} \text{ is precompact in } L^1(\Omega).$$

This and (8.23) imply, according to the Arzelà-Ascoli Theorem, that

$$\{u_\ell\}_\ell \text{ is precompact in } C^0([0,T];L^1(\Omega))$$

and therefore (for a subsequence)

$$\lim_{\ell\to\infty} u_\ell = u \quad \text{in } C^0([0,T];L^1(\Omega))\,.$$

Finally, $u \in L^\infty(Q_T)$ (see (8.8)).

Let us now show that u satisfies (7.4)–(7.6). From (8.5) we have for all boundary entropy-entropy flux pairs $(H,\mathbf{Q})$, all $k \in \mathbb{R}$ and $\beta \in \mathcal{D}((-\infty,T)\times\mathbb{R}^d)$, $\beta \ge 0$,

$$\begin{aligned}&-\int_0^T\int_\Omega \Big\{H(u_\ell,k)\frac{\partial\beta}{\partial t} + Q_i(u_\ell,k)\frac{\partial\beta}{\partial x_i} + \varepsilon_\ell H(u_\ell,k)\,\Delta\beta\Big\}\xi_{\varepsilon_\ell}\,dx\,dt \\ &\quad\le \int_\Omega H(u_\ell^0,k)\,\beta(0)\,dx + (M + L\,\varepsilon_\ell)\int_\Gamma H(u_\ell^D,k)\,\beta\,dr \\ &\quad\quad + 2\,\varepsilon_\ell\int_0^T\int_\Omega H(u_\ell,k)\frac{\partial\beta}{\partial x_i}\frac{\partial\xi_{\varepsilon_\ell}}{\partial x_i}\,dx\,dt\end{aligned} \tag{8.24}$$

for all $\ell \in \mathbb{N}$. Due to

$$\lim_{\ell\to\infty}\int_\Omega |1 - \xi_{\varepsilon_\ell}|\,dx = 0\,,$$

$$\lim_{\ell\to\infty}\varepsilon_\ell\int_\Omega |\nabla\xi_{\varepsilon_\ell}|\,dx = 0\ ,$$

where in the last limit (8.3) is used, we can let $\ell \to \infty$ in (8.24) and obtain

$$-\int_0^T \int_\Omega H(u,k)\frac{\partial \beta}{\partial t} + Q_i(u,k)\frac{\partial \beta}{\partial x_i}\,dx\,dt \le \int_\Omega H(u^0,k)\,\beta(0)\,dx + M\int_\Gamma H(u^D,k)\,\beta\,dr\ .$$

Now from Theorem 7.31 it follows that (u, u^D, u^0) satisfy (7.4)–(7.6). From Theorem 7.28 it follows that there is only one such $u \in L^\infty(Q_T)$ and therefore

$$\lim_{\varepsilon \to 0+} u^\varepsilon = u \quad \text{in } C^0([0,T]; L^1(\Omega)).$$

■