CHAPTER 2

Scalar conservation laws

2.1 Introduction

If one looks at the basic equations of physics, one sees that a lot of them can be written in the form of conservation law, i.e.,

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x_j} \mathbf{f}_j(\mathbf{u}) = \mathbf{0} \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \tag{1.1}$$

where $x \in \mathbb{R}^d, t > 0$. Here $\mathbf{u} = (u_1, \dots, u_s)$ stands for a density of the investigated quantities and $\mathbf{f} = (\mathbf{f}_1, \dots, \mathbf{f}_d)$ is the flux vector. Since the equation (1.1) is the subject of many mathematical considerations, we will state precisely in which sense this equation is understood.

Let $\mathcal{O} \subseteq \mathbb{R}^s$ be a domain and let $\mathbf{f}_j, 1 \leq j \leq d$, be smooth functions mapping $\mathcal{O}$ into $\mathbb{R}^s$. We look for solutions $\mathbf{u} : \mathbb{R}^+ \times \mathbb{R}^d \to \mathcal{O}$ of the system (1.1). In the sequel we restrict ourselves to hyperbolic systems, which were defined in Chapter 1. For the sake of completeness, we repeat here the definition: for every $j = 1, \dots, d$ we denote by $\mathbf{J}_j$ the Jacobian matrix of the function $\mathbf{f}_j$, i.e.,

$$[\mathbf{J}_j(\mathbf{u})]_{ik} = \frac{\partial f_{ji}}{\partial u_k}(\mathbf{u})\,, \qquad i, k = 1, \dots, s\,.$$

The system (1.1) is called **hyperbolic** if for every $\mathbf{u} \in \mathcal{O}$ and every vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ the matrix

$$\mathbf{J}(\mathbf{u}, \boldsymbol{\alpha}) = \alpha_j\, \mathbf{J}_j(\mathbf{u})$$

has s real eigenvalues

$$\lambda_1(\mathbf{u}, \boldsymbol{\alpha}) \leq \cdots \leq \lambda_s(\mathbf{u}, \boldsymbol{\alpha})$$

and is diagonalizable. Recall that $\lambda_1 < \cdots < \lambda_s$ implies that the corresponding matrix $\mathbf{J}$ is diagonalizable. Such systems are sometimes called **strictly hyperbolic**.

In this chapter we will investigate **scalar conservation laws**, i.e., the case $s = 1$. Note that in this case the equation (1.1) is trivially hyperbolic since the Jacobian matrices $\mathbf{J}_j(\mathbf{u})$ are of dimension 1×1. For scalar conservation laws we consider the Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} + \operatorname{div} \mathbf{f}(u) = 0 \qquad & \text{in } \mathbb{R}^+ \times \mathbb{R}^d \,, \\ u(0,\cdot) = u_0 \qquad & \text{in } \mathbb{R}^d \,, \end{aligned} \tag{1.2}$$

where $\mathbf{f} = (f_1, \ldots, f_d)$, $f_j \in C^1(\mathbb{R})$ and $u_0 : \mathbb{R}^d \to \mathbb{R}$ are given functions. To solve this problem we use the **method of vanishing viscosity**, the basic idea of which is as follows: for $\varepsilon > 0$ we study the parabolic perturbation of the Cauchy problem (1.2) given by

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + \operatorname{div} \mathbf{f}(u^\varepsilon) - \varepsilon \Delta u^\varepsilon = 0 \qquad & \text{in } \mathbb{R}^+ \times \mathbb{R}^d \,, \\ u^\varepsilon(0,\cdot) = u_0 \qquad & \text{in } \mathbb{R}^d \,. \end{aligned} \tag{1.3}$$

After having obtained the classical solutions u^ε of (1.3) for $\varepsilon > 0$, which turn out to be smooth for smooth data, one tries to find a function u solving (1.2) as a limit of u^ε as $\varepsilon \to 0+$.

Let us note that the vanishing viscosity method is not the only one used to study the solutions of the Cauchy problem (1.2). Another interesting approach was presented in the recent preprint by Cockburn, Gripenberg and Londen [1995], where the evolution integral equation

$$\frac{\partial}{\partial t}\big(k * (u - u_0)\big) + \operatorname{div} \mathbf{f}(u) = 0 \tag{1.4}$$

was studied instead of (1.2). Here,

$$\big(k * u\big)(t,x) \equiv \int_0^t k(t-s) u(s,x)\, ds \,.$$

Formally, for $k(t)\,dt = \delta_0$ (the Dirac measure at the origin), (1.4) reduces to (1.2), while for $k \in L^1_{\text{loc}}(\mathbb{R}^+)$, a positive non-increasing function, one obtains solutions u^k and tries to find a function u solving (1.2) as a limit of u^k when k approaches δ_0. For more details see Cockburn, Gripenberg and Londen [1995].

In the first part of this chapter we want to prove the existence, uniqueness and also some qualitative properties of the weak solution of the parabolic equation (1.3). Employing the vanishing viscosity method we prove existence and uniqueness of the so-called weak entropy solution of the Cauchy problem (1.2). All main ideas and concepts presented in the first part of this chapter go back to the fundamental paper of Kružkov [1970]. However, our approach is very close to Godlewski and Raviart [1991]; in only a few points do we differ or are more explicit.

The second part contains an overview of analogous results for bounded domains proved recently by Otto [1992, 1993].

2.2 Parabolic perturbation to scalar conservation laws

The aim of this section is to study the existence, uniqueness and further properties of a weak solution to the Cauchy problem (1.3). After introducing the notion of a weak solution, we prove in Lemma 2.3 its existence and uniqueness under the assumptions that $u_0 \in L^2(\mathbb{R}^d)$ and that $\mathbf{f} \in C^1(\mathbb{R})^d$ is globally Lipschitz continuous. If moreover $u_0 \in L^\infty(\mathbb{R}^d)$ then we show in Theorem 2.9 a uniform L^∞-estimate of u^ε which will allow us to consider general nonlinearities $\mathbf{f} \in C^1(\mathbb{R})^d$. Using then the regularity theorems for linear parabolic equations, we get for smoother u_0 and $\mathbf{f}$ in Lemma 2.16 the corresponding regularity results for parabolic perturbation (1.3). This will help us to derive in Theorem 2.29 the estimates of u^ε and its spatial and time derivatives in L^1 norms. These will play the key role in the limiting process as $\varepsilon \to 0+$ in Section 2.4.

Let us start with the definition of weak solutions to the Cauchy problem (1.3). Define for $T \leq \infty$ the space

$$W(T) \equiv \left\{ u \in L^2(0,T;W^{1,2}(\mathbb{R}^d)), \frac{\partial u}{\partial t} \in L^2(0,T;W^{-1,2}(\mathbb{R}^d)) \right\}.$$

Recall that by Lemma 2.45 in Chapter 1 the space $W(T)$ is continuously imbedded into the space $C(0,T;L^2(\mathbb{R}^d))$. In particular, $\lim_{t\to 0+} v(t) = v(0)$ is a well-defined element of the space $L^2(\mathbb{R}^d)$ for all $v \in W(T)$.

Definition 2.1 *Let $u_0 \in L^2(\mathbb{R}^d)$. A function $u^\varepsilon : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ is called a weak solution to (1.3) if and only if $u^\varepsilon \in W(T)$ for all $T < \infty$, $u^\varepsilon(0) = u_0$, and the identity*

$$\left\langle \frac{\partial u^\varepsilon}{\partial t}(t), \varphi \right\rangle + \varepsilon \int_{\mathbb{R}^d} \nabla u^\varepsilon(t) \nabla \varphi \, dx = \int_{\mathbb{R}^d} \mathbf{f}(u^\varepsilon(t)) \nabla \varphi \, dx \quad (2.2)$$

is fulfilled for almost all $t > 0$ and all $\varphi \in W^{1,2}(\mathbb{R}^d)$. Here $\langle \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_{W^{1,2}(\mathbb{R}^d)}$ denotes the duality between $W^{1,2}(\mathbb{R}^d)$ and its dual space, $W^{-1,2}(\mathbb{R}^d)$.

Note that for general $u_0 \in L^2(\mathbb{R}^d)$ and $\mathbf{f} \in C^1(\mathbb{R})^d$ the integral on the right-hand side of (2.2) need not exist. However, this integral is finite if, e.g., $\mathbf{f}$ is globally Lipschitz continuous and $\mathbf{f}(0) = \mathbf{0}$. The following lemma shows that the assumption of global Lipschitz continuity of $\mathbf{f}$ is strong enough to ensure the existence of a unique weak solution u^ε to (1.3).

In order to simplify the notation, we drop the superscript ε in all remaining proofs of this section.

Lemma 2.3 *Let $u_0 \in L^2(\mathbb{R}^d)$ and let $\mathbf{f} \in C^1(\mathbb{R})^d$ be globally Lipschitz continuous, i.e., there exists $M > 0$ such that*

$$|\mathbf{f}(\xi_1) - \mathbf{f}(\xi_2)| \le M|\xi_1 - \xi_2| \qquad \forall\, \xi_1, \xi_2 \in \mathbb{R}. \tag{2.4}$$

Then there exists exactly one $u^\varepsilon \in W(T)$ for all $T > 0$, satisfying (2.2) and $u^\varepsilon(0) = u_0$.

Proof : Without loss of generality we can assume that $\mathbf{f}(0) = \mathbf{0}$ (otherwise we define $\widetilde{\mathbf{f}}(u) \equiv \mathbf{f}(u) - \mathbf{f}(0)$, which gives $\widetilde{\mathbf{f}}(0) = \mathbf{0}$, $\operatorname{div} \widetilde{\mathbf{f}}(u) = \operatorname{div} \mathbf{f}(u)$). We seek $u \in W(T)$ satisfying

$$\Big\langle \frac{\partial u}{\partial t}(t), \varphi \Big\rangle + \varepsilon \int_{\mathbb{R}^d} \nabla u(t) \nabla \varphi \, dx = \int_{\mathbb{R}^d} \mathbf{f}(u(t)) \nabla \varphi \, dx \tag{2.5}$$

for almost every $t > 0$ and all $\varphi \in W^{1,2}(\mathbb{R}^d)$. Let $\lambda > 0$ be a fixed parameter. Then using $\varphi = \psi e^{-\lambda t}$ in (2.5) and setting $u = v e^{\lambda t}$ we can rewrite (2.5) as

$$\begin{aligned} \Big\langle \frac{\partial v}{\partial t}(t), \psi \Big\rangle &+ \varepsilon \int_{\mathbb{R}^d} \nabla v(t) \nabla \psi \, dx \\ &+ \int_{\mathbb{R}^d} \lambda v(t) \psi \, dx = \int_{\mathbb{R}^d} e^{-\lambda t} \mathbf{f}(v(t) e^{\lambda t}) \nabla \psi \, dx, \end{aligned} \tag{2.6}$$

where we used the fact that $e^{\lambda t}$ is a multiplicator in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$. Now we want to use the Banach fixed point theorem. Thus, let $v \in L^2(0, \infty; L^2(\mathbb{R}^d))$ be a given function and let us look for $w \in W(\infty)$ solving the linearized problem

$$\begin{aligned} \Big\langle \frac{\partial w}{\partial t}(t), \psi \Big\rangle &+ \varepsilon \int_{\mathbb{R}^d} \nabla w(t) \nabla \psi dx \\ &+ \int_{\mathbb{R}^d} \lambda w(t) \psi dx = \int_{\mathbb{R}^d} e^{-\lambda t} \mathbf{f}(v(t) e^{\lambda t}) \nabla \psi \, dx \, , \\ w(0) = v(0) \, . & \end{aligned} \tag{2.7}$$

We know that $e^{-\lambda t} \operatorname{div} \mathbf{f}(ve^{\lambda t})$ belongs to $L^2(0,\infty; W^{-1,2}(\mathbb{R}^d))$, as

$$\begin{aligned}\Big| \int_0^\infty \int_{\mathbb{R}^d} & e^{-\lambda t}\mathbf{f}(ve^{\lambda t})\nabla\psi \, dx \, dt \Big| \\ &\leq \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} |\mathbf{f}(ve^{\lambda t}) - \mathbf{f}(0)||\nabla\psi| \, dx \, dt \\ &\leq M \int_0^\infty \int_{\mathbb{R}^d} |v||\nabla\psi| \, dx \, dt\end{aligned}$$

holds for all $\psi \in L^2(0,\infty; W^{1,2}(\mathbb{R}^d))$. Thus we get the existence of a $w \in W(\infty)$ solving (2.7) from Theorem 2.2 in the Appendix. Let us define the mapping $F_\lambda : L^2(0,\infty; L^2(\mathbb{R}^d)) \to W(\infty)$ by

$$F_\lambda(v) \equiv w.$$

We want to show that there is a $\lambda_0 > 0$ such that F_{λ_0} is a contraction in the space $L^2(0,\infty; L^2(\mathbb{R}^d))$. Let $v_i, i = 1, 2$, be two elements of $L^2(0,\infty; L^2(\mathbb{R}^d))$ and let $w_i \equiv F_\lambda(v_i)$. Setting $w \equiv w_1 - w_2$ we obtain for almost every $t > 0$ and all $\psi \in W^{1,2}(\mathbb{R}^d)$ the identity

$$\begin{aligned}\Big\langle \frac{\partial w}{\partial t}(t), \psi \Big\rangle + \varepsilon \int_{\mathbb{R}^d} \nabla w(t) \nabla\psi \, dx + \lambda \int_{\mathbb{R}^d} w(t)\psi \, dx \\ = e^{-\lambda t} \int_{\mathbb{R}^d} \big[\mathbf{f}(e^{\lambda t} v_1(t)) - \mathbf{f}(e^{\lambda t} v_2(t))\big] \nabla\psi \, dx.\end{aligned} \tag{2.8}$$

At this point we choose $\psi = w(t)$ and integrate (2.8) over an interval $(0, t)$. Using $w(0) = 0$ and (2.4) we obtain

$$\begin{aligned}\frac{1}{2}\|w(t)\|^2_{L^2(\mathbb{R}^d)} + \varepsilon \int_0^t \|\nabla w(s)\|^2_{L^2(\mathbb{R}^d)} \, ds + \lambda \int_0^t \|w(s)\|^2_{L^2(\mathbb{R}^d)} \, ds \\ \leq M \int_0^t \|v_1(s) - v_2(s)\|_{L^2(\mathbb{R}^d)} \|\nabla w(s)\|_{L^2(\mathbb{R}^d)} \, ds.\end{aligned}$$

Dropping the first term and applying Young's inequality, we get

$$\int_0^t \|w(s)\|^2_{L^2(\mathbb{R}^d)} ds \leq \frac{M^2}{4\lambda\varepsilon} \int_0^t \|v_1(s) - v_2(s)\|^2_{L^2(\mathbb{R}^d)} ds.$$

Notice that $v_1 - v_2 \in L^2(0,\infty; L^2(\mathbb{R}^d))$ and the left-hand side is a monotonous continuous function of t. Therefore, letting $t \to \infty$ we obtain

$$\|w_1 - w_2\|_{L^2(0,\infty;L^2(\mathbb{R}^d))} \leq \frac{M}{2\sqrt{\lambda\varepsilon}} \|v_1 - v_2\|_{L^2(0,\infty;L^2(\mathbb{R}^d))} .$$

We see that F_{λ_0} is a contraction for λ_0 large enough and therefore, has a unique fixed point $v \in L^2(0,\infty; L^2(\mathbb{R}^d))$. Consequently, v solves (2.6) and belongs to the space $W(\infty)$. Now, setting

$$u \equiv v\, e^{\lambda t},$$

we obtain a unique weak solution to our problem (1.3) belonging to $W(T)$ for all $T > 0$. Note that $e^{\lambda t} \notin L^\infty(0,\infty)$ and therefore in general $u \notin W(\infty)$. ■

The next theorem shows that under an additional assumption on u_0, namely $u_0 \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we can avoid the assumption of global Lipschitz continuity of $\mathbf{f}$ since a uniform L^∞-estimate of u^ε will be derived. Note that also in this case the integral on the right-hand side of (2.2) is finite.

Theorem 2.9 *Let $u_0 \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $\mathbf{f} \in C^1(\mathbb{R})^d$. Then the Cauchy problem (1.3) has a unique weak solution $u^\epsilon \in W(T) \cap L^\infty((0,T) \times \mathbb{R}^d)$ for every $T > 0$. This solution satisfies*

$$\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)} \quad \text{for a.a. } t \in (0,T) \tag{2.10}$$

and for any bounded open set $Q \subset \mathbb{R}^+ \times \mathbb{R}^d$,

$$\|\sqrt{\varepsilon}\, \nabla u^\varepsilon\|_{L^2(Q)} \le c(Q)\,, \tag{2.11}$$

where the constant $c(Q)$ is independent of ε.

Proof : We will use Lemma 2.3 and therefore we cut off the nonlinearity $\mathbf{f}$. Let $\zeta \in \mathcal{D}(\mathbb{R})$ be such that

$$\zeta(r) \equiv \begin{cases} 1\,, & r \le \|u_0\|_{L^\infty(\mathbb{R}^d)}\,, \\ 0\,, & r \ge \|u_0\|_{L^\infty(\mathbb{R}^d)} + 1. \end{cases}$$

Define

$$\begin{aligned} g_j &\equiv \zeta\, f_j\,, \qquad 1 \le j \le d\,, \\ \mathbf{g} &\equiv (g_1, \dots, g_d)\,. \end{aligned}$$

Thus, $\mathbf{g}$ is a C^1-function with compact support and the assumption (2.4) is fulfilled for $\mathbf{g}$ with some constant M. Then Lemma 2.3 gives the existence of a unique $u \in W(T)$ for all $T > 0$, satisfying

$u(0) = u_0$ and

$$\Big\langle \frac{\partial u}{\partial t}(t), \varphi \Big\rangle + \varepsilon \int_{\mathbb{R}^d} \nabla u(t) \nabla \varphi \, dx = - \int_{\mathbb{R}^d} g_j'(u(t)) \frac{\partial u}{\partial x_j} \varphi \, dx \tag{2.12}$$

for almost all $t > 0$ and all $\varphi \in W^{1,2}(\mathbb{R}^d)$. We want to show that u satisfies also (2.2). We define

$$v \equiv u - \|u_0\|_{L^\infty(\mathbb{R}^d)} \tag{2.13}$$

and will show that $v^+ = 0$. Since $v^+ \in W^{1,2}(\mathbb{R}^d)$ due to Lemma 1.7 in the Appendix, we can use v^+ as a test function in (2.12). Using (1.16) from the Appendix and the fact that $\nabla v^+ = 0$ on the set $\{v \le 0\}$, while $\nabla v^+ = \nabla v = \nabla u$ and $v^+ = v$ elsewhere, we obtain

$$\frac{1}{2}\|v^+(t)\|^2_{L^2(\mathbb{R}^d)} - \frac{1}{2}\|v^+(0)\|^2_{L^2(\mathbb{R}^d)} + \varepsilon \int_0^t \|\nabla v^+(s)\|^2_{L^2(\mathbb{R}^d)} \, ds \le M \int_0^t \|\nabla v^+(s)\|_{L^2(\mathbb{R}^d)} \|v^+(s)\|_{L^2(\mathbb{R}^d)} \, ds.$$

Further, Young's inequality and $v^+(0) = 0$ imply

$$\|v^+(t)\|^2_{L^2(\mathbb{R}^d)} \le \frac{M^2}{2\varepsilon} \int_0^t \|v^+(s)\|^2_{L^2(\mathbb{R}^d)} \, ds.$$

Now, Gronwall's lemma implies

$$v^+(t) = \big(u(t) - \|u_0\|_{L^\infty(\mathbb{R}^d)}\big)^+ = 0 \tag{2.14}$$

for almost all $t \in (0, T)$. In the same way it can also be shown that

$$\big(- u(t) - \|u_0\|_{L^\infty(\mathbb{R}^d)}\big)^+ = 0. \tag{2.15}$$

However, (2.14) and (2.15) mean nothing other than

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)} \qquad \text{a.e. in } (0, T).$$

Hence, the definition of $\mathbf{g}$ implies that for our particular u

$$\mathbf{f}(u) = \mathbf{g}(u),$$

and therefore u satisfies (2.2) and belongs to $W(T) \cap L^\infty((0, T) \times \mathbb{R}^d)$ for all $T > 0$.

Now we show that our solution is unique. Let $u_1, u_2 \in W(T) \cap L^\infty((0,T)\times\mathbb{R}^d)$ be two solutions satisfying (2.2) and $u_i(0) = u_0$. Cutting off the nonlinearity $\mathbf{f}$ outside the ball

$$B_{1,2} \equiv \left\{u \in \mathbb{R},\ |u| \le \max_{i=1,2}\left(\|u_i\|_{L^\infty((0,T)\times\mathbb{R}^d)}\right)\right\}$$

we get a function $\mathbf{g}$, which is globally Lipschitz continuous and therefore u_1 and u_2 satisfy (2.12) while $u_1(0) = u_2(0) = u_0$. But Lemma 2.3 gives the uniqueness of such solutions in the class $W(T)$. Thus $u_1 = u_2$.

It remains to show (2.11). Let Q be any bounded open set in $\mathbb{R}^+ \times \mathbb{R}^d$. Define $\chi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$ such that $\chi \ge 0$ on $\mathbb{R}^+ \times \mathbb{R}^d$, $\chi = 1$ on Q and $\chi(t,x) = 0$ for $\operatorname{dist}((t,x),Q) > 1$. Using $u\chi^2$ as a test function in (2.2), we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}(u\chi)^2\,dx + \int_{\mathbb{R}^d}|\sqrt{\varepsilon}\,\nabla(u\chi)|^2\,dx = -\int_{\mathbb{R}^d}\frac{\partial f_i(u)}{\partial x_i}u\chi^2\,dx + \int_{\mathbb{R}^d}u^2\chi\frac{\partial\chi}{\partial t}\,dx + \varepsilon\int_{\mathbb{R}^d}u^2|\nabla\chi|^2\,dx.$$

Using the fact that $u \in W(T) \cap L^\infty((0,T)\times\mathbb{R}^d)$ for all $T > 0$ and that χ has compact support, one easily obtains (2.11). ∎

After this general existence result we will show which qualitative properties the solution u^ε has if the data $\mathbf{f}$ and u_0 are more regular. The results are based on the regularity theory for linear parabolic equations (see, e.g., the Appendix).

Lemma 2.16 *Let $\mathbf{f} \in C^m(\mathbb{R})^d$ and let $u_0 \in W^{m,2}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, $m \in \mathbb{N}$. Then the weak solution u^ε of the problem (1.3) satisfies, for all $T > 0$,*

$$u^\varepsilon \in L^2(0,T;W^{m+1,2}(\mathbb{R}^d)) \cap C(0,T;W^{m,2}(\mathbb{R}^d))\,. \quad (2.17)$$

Further, for $k \in \mathbb{N}$, $2k \le m$ we have

$$\frac{\partial^k u^\varepsilon}{\partial t^k} \in L^2(0,T;W^{m+1-2k,2}(\mathbb{R}^d)) \cap C(0,T;W^{m-2k,2}(\mathbb{R}^d))$$

while for $2k = m+1$ we have

$$\frac{\partial^k u^\varepsilon}{\partial t^k} \in L^2(0,T;L^2(\mathbb{R}^d)).$$

PROOF : By Theorem 2.9, a weak solution u solves for almost all $t \in (0,T)$ the equation

$$\frac{\partial u(t)}{\partial t} - \varepsilon \Delta u(t) = -\operatorname{div} \mathbf{f}\big(u(t)\big)\,, \tag{2.18}$$

understood as the equation in $\big(W^{1,2}(\mathbb{R}^d)\big)^*$. However, since $u \in L^2\big(0,T;W^{1,2}(\mathbb{R}^d)\big) \cap L^\infty\big((0,T)\times\mathbb{R}^d\big)$, one sees that

$$\operatorname{div} \mathbf{f}(u) = f_j(u)\frac{\partial u}{\partial x_j} \in L^2\big(0,T;L^2(\mathbb{R}^d)\big)\,,$$

and the regularity theory for linear parabolic equations can be applied. Thus, putting $m = 1$ in Theorem 2.4 in the Appendix, we obtain

$$\begin{aligned} &u \in L^2(0,T;W^{2,2}(\mathbb{R}^d)) \cap C(0,T;W^{1,2}(\mathbb{R}^d))\,,\\ &\frac{\partial u}{\partial t} \in L^2(0,T;L^2(\mathbb{R}^d))\,, \end{aligned} \tag{2.19}$$

which is the statement of Lemma 2.16 for $m = 1$. Moreover, u satisfies the equation

$$\frac{\partial u}{\partial t} - \varepsilon\Delta u = -\operatorname{div}\mathbf{f}(u) \tag{2.20}$$

for almost all $(t,x) \in (0,T)\times\mathbb{R}^d$.

Let us prove (2.17) for $m = 2$. We differentiate formally (2.20) with respect to t and denote $v \equiv \frac{\partial u}{\partial t}$. We get

$$\frac{\partial v}{\partial t} - \varepsilon\Delta v = -\operatorname{div}\Big(\mathbf{f}'(u)\frac{\partial u}{\partial t}\Big). \tag{2.21}$$

Since $u \in L^\infty\big((0,T)\times\mathbb{R}^d\big)$ and $\frac{\partial u}{\partial t} \in L^2\big((0,T)\times\mathbb{R}^d\big)$, the right-hand side of (2.21) belongs to $L^2(0,T;W^{-1,2}(\mathbb{R}^d))$. In agreement with (2.20), let us demand

$$v(0) = \varepsilon\Delta u_0 - \operatorname{div}\mathbf{f}(u_0)\,. \tag{2.22}$$

Because $u_0 \in W^{2,2}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we see that $v(0) \in L^2(\mathbb{R}^d)$. Therefore, we can use Theorem 2.2 from the Appendix on the existence of a weak solution to the Cauchy problem for linear parabolic equations. We obtain

$$v \in L^2\big(0,T;W^{1,2}(\mathbb{R}^d)\big) \cap C\big(0,T;L^2(\mathbb{R}^d)\big)\,. \tag{2.23}$$

The standard argument implies that $v = \frac{\partial u}{\partial t}$ and we obtain from (2.19) and (2.23) that

$$\frac{\partial u}{\partial x_i} \in W^{1,2}((0,T) \times \mathbb{R}^d)\,, \qquad i = 1, \dots, d\,.$$

The Sobolev imbedding theorem (see Theorem 2.17 in Chapter 1) implies

$$\frac{\partial u}{\partial x_i} \in L^{\frac{2(d+1)}{d-1}}\big((0,T) \times \mathbb{R}^d\big)\,, \qquad i = 1, \dots, d\,, \tag{2.24}$$

for $d > 1$ and

$$\frac{\partial u}{\partial x} \in L^r\big((0,T) \times \mathbb{R}\big)\,, \qquad \forall\, r < \infty\,, \tag{2.25}$$

if $d = 1$.

Now, we compute formally the derivative of (2.20) with respect to x_i and obtain

$$\begin{aligned} \frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial x_i}\Big) - \varepsilon \Delta \frac{\partial u}{\partial x_i} &= -f_j''(u) \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i}\,, \\ \frac{\partial u}{\partial x_i}(0) &= \frac{\partial u_0}{\partial x_i}. \end{aligned} \tag{2.26}$$

Again, we will use the regularity result for linear parabolic equations, this time with a view to $\frac{\partial u}{\partial x_i}$. The assertion will follow immediately, once we prove that the right-hand side of (2.26) belongs to $L^2\big((0,T) \times \mathbb{R}^d\big)$, i.e., if

$$\frac{\partial u}{\partial x_i} \in L^4\big((0,T) \times \mathbb{R}^d\big)\,, \qquad i = 1, \dots, d\,. \tag{2.27}$$

Let us consider two cases. If $d \le 3$, then (2.27) is a consequence of (2.24), (2.25). Indeed, if $d = 3$, then $\frac{2(d+1)}{d-1} = 4$ and if $1 < d < 3$, we get (2.27) by interpolation between $L^{\frac{2(d+1)}{d-1}}\big((0,T) \times \mathbb{R}^d\big)$ and $L^2\big((0,T) \times \mathbb{R}^d\big)$. The case $d = 1$ is trivial.

If $d > 3$, then we use the L^q-theory of linear parabolic equations applied to the equation (2.18). Using Theorem 2.3 in Giga and Sohr [1991] and (2.24), we obtain for $p_0 \equiv \frac{2(d+1)}{d-1}$,

$$\begin{aligned} u &\in L^{p_0}\big(0,T; W^{2,p_0}(\mathbb{R}^d)\big)\,, \\ \frac{\partial u}{\partial t} &\in L^{p_0}\big(0,T; L^{p_0}(\mathbb{R}^d)\big)\,. \end{aligned} \tag{2.28}$$

It is known (see SOLONNIKOV [1977]) that if a function u satisfies (2.28), then

$$\nabla u \in L^{p_1}\big(0,T;L^{p_1}(\mathbb{R}^d)^d\big)\,,$$

where

$$p_1 = \frac{(d+2)p_0}{(d+2)-p_0}\,.$$

Iterating this process, we can conclude that

$$\nabla u \in L^{p_k}\big(0,T;L^{p_k}(\mathbb{R}^d)^d\big)\,,$$

with

$$p_k = \frac{(d+2)p_0}{(d+2)-kp_0}\,.$$

Obviously, there exists a $k_0 \in \mathbb{N}$ such that $\tilde{p} \equiv p_{k_0} > 4$ (if not, then $p_0 < \frac{4(d+2)}{(d+2)+4k} \to 0$ as $k \to \infty$, which is a contradiction). Interpolating between $L^{\tilde{p}}\big(0,T;L^{\tilde{p}}(\mathbb{R}^d)\big)$ and $L^2\big(0,T;L^2(\mathbb{R}^d)\big)$ we get (2.27) and consequently (2.17) for $m=2$.

For $m>2$ one uses the same procedure. The proof of Lemma 2.16 is complete. ■

Theorem 2.29 *Let* $\mathbf{f} \in C^m(\mathbb{R})^d$ *and let* $u_0 \in W^{m,2}(\mathbb{R}^d) \cap W^{2,1}(\mathbb{R}^d)$ *for some* $m > \max\{\frac{d}{2}+2, 3\}$. *Put*

$$M \equiv \sup\{|\mathbf{f}'(\zeta)|, |\zeta| \le \|u_0\|_{L^\infty(\mathbb{R}^d)}\}. \tag{2.30}$$

Then for all $t>0$ *the solution* u^ε *of the problem (1.3) satisfies*

$$\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}\,, \tag{2.31}$$

$$\int_{\mathbb{R}^d} u^\varepsilon(t,x)\,dx = \int_{\mathbb{R}^d} u_0(x)\,dx\,, \tag{2.32}$$

$$\Big\|\frac{\partial u^\varepsilon}{\partial x_i}(t)\Big\|_{L^1(\mathbb{R}^d)} \le \Big\|\frac{\partial u_0}{\partial x_i}\Big\|_{L^1(\mathbb{R}^d)}\,, \tag{2.33}$$

$$\Big\|\frac{\partial u^\varepsilon}{\partial t}(t)\Big\|_{L^1(\mathbb{R}^d)} \le M\|\nabla u_0\|_{L^1(\mathbb{R}^d)^d} + \varepsilon\|\Delta u_0\|_{L^1(\mathbb{R}^d)} \tag{2.34}$$

and for $t \in [0,T]$ *we have*

$$\|u^\varepsilon(t)\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)} + MT\|\nabla u_0\|_{L^1(\mathbb{R}^d)^d}. \tag{2.35}$$

To prove this theorem we need some technical results. Let $\eta : \mathbb{R}^+ \to \mathbb{R}$ be a non-increasing C^2 function satisfying

$$\eta(s) = \begin{cases} 1 & \text{for } s \in [0, 1/2]\,, \\ \text{non-negative polynomial} & \text{for } s \in [1/2, 1]\,, \\ \exp(-s) & \text{for } s \geq 1. \end{cases} \tag{2.36}$$

For $R > 0$ we put

$$\eta_R(x) = \eta\left(\frac{|x|}{R}\right), \qquad x \in \mathbb{R}^d. \tag{2.37}$$

Lemma 2.38 *There is a constant c such that*

$$\begin{aligned} |\nabla \eta_R(x)| &\leq \frac{c}{R}\eta_R(x)\,, \\ |\Delta \eta_R(x)| &\leq \frac{c}{R^2}\eta_R(x)\,. \end{aligned} \tag{2.39}$$

PROOF : From (2.37) it follows that

$$\begin{aligned} \frac{\partial}{\partial x_i}\eta_R(x) &= \frac{1}{R}\eta'\left(\frac{|x|}{R}\right)\frac{x_i}{|x|}\,, \\ \Delta \eta_R(x) &= \frac{1}{R^2}\,\eta''\left(\frac{|x|}{R}\right) + \frac{d-1}{R\,|x|}\,\eta'\left(\frac{|x|}{R}\right)\,. \end{aligned} \tag{2.40}$$

Further, from (2.36) it is clear that there is some constant c_0 such that

$$|\eta'(s)| \leq c_0\eta(s)\,, \qquad |\eta''(s)| \leq c_0\eta(s).$$

Thus $(2.39)_1$ is proved. Since $\eta'(|x|/R)$ is non-zero only if $\frac{1}{|x|} < \frac{2}{R}$, we get $(2.39)_2$ as well. ∎

Lemma 2.41 *Let η_R be defined by (2.36)–(2.37). Then we have for all $v \in W^{2,2}(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} \Delta v \;\mathrm{sgn}\,(v)\eta_R \,dx \leq \int_{\mathbb{R}^d} |v|\Delta\eta_R \,dx. \tag{2.42}$$

PROOF : For $\theta > 0$ we put

$$j_\theta(s) = \begin{cases} -1 & \text{if } s \leq -\theta\,, \\ s/\theta & \text{if } -\theta \leq s \leq \theta\,, \\ 1 & \text{if } s \geq \theta. \end{cases} \tag{2.43}$$

Thus we get

$$\int_{\mathbb{R}^d} j_\theta(v)\Delta v\,\eta_R\,dx = -\int_{\mathbb{R}^d} j'_\theta(v)|\nabla v|^2\eta_R\,dx - \int_{\mathbb{R}^d} j_\theta(v)\nabla v\nabla\eta_R\,dx$$

and due to the fact that $j'_\theta(s) \geq 0$ almost everywhere,

$$\int_{\mathbb{R}^d} \Delta v\, j_\theta(v)\eta_R\,dx \leq -\int_{\mathbb{R}^d} j_\theta(v)\nabla v\nabla\eta_R\,dx\,.$$

The Lebesgue dominated convergence theorem enables us to pass with θ to $0+$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^d} \Delta v\ \mathrm{sgn}(v)\eta_R\,dx &\leq -\int_{\mathbb{R}^d} \mathrm{sgn}\,(v)\nabla v\nabla\eta_R\,dx\\ &= -\int_{\mathbb{R}^d} \nabla|v|\nabla\eta_R\,dx\\ &= \int_{\mathbb{R}^d} |v|\Delta\eta_R\,dx,\end{aligned}$$

where we also used Remark 1.12 from the Appendix. ∎

Proof (of Theorem 2.29): First we note that in our situation $W^{m,2}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. Therefore the assumptions of Theorem 2.9 and Lemma 2.16 are satisfied. Thus we have the existence of a unique weak solution to problem (1.3) satisfying for all $T > 0$

$$\begin{aligned} u &\in L^2(0,T;W^{m+1,2}(\mathbb{R}^d)) \cap C(0,T;W^{m,2}(\mathbb{R}^d))\,,\\ \frac{\partial u}{\partial t} &\in L^2(0,T;W^{m-1,2}(\mathbb{R}^d)) \cap C(0,T;W^{m-2,2}(\mathbb{R}^d))\,,\\ \frac{\partial^2 u}{\partial t^2} &\in L^2(0,T;L^2(\mathbb{R}^d))\end{aligned} \tag{2.44}$$

and inequality (2.31). Let us differentiate (1.3) with respect to x_i. We obtain

$$\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial x_i}\Big) + \frac{\partial^2 f_j(u)}{\partial x_i \partial x_j} - \varepsilon\Delta\frac{\partial u}{\partial x_i} = 0 \tag{2.45}$$

almost everywhere in $(0,T) \times \mathbb{R}^d$. The following calculations are motivated by the fact that we are going to multiply (2.45) by $\mathrm{sgn}\,\big(\frac{\partial u}{\partial x_i}\big)$. Due to (2.44), ∇u is an element of $W^{1,2}((0,T) \times \mathbb{R}^d)$

and we can use formula (1.13) from the Appendix, which gives†

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x_i}\right)\operatorname{sgn}\left(\frac{\partial u}{\partial x_i}\right)=\frac{\partial}{\partial t}\Big|\frac{\partial u}{\partial x_i}\Big|,$$

$$\begin{aligned}\frac{\partial^2 f_j(u)}{\partial x_i\,\partial x_j}\operatorname{sgn}\left(\frac{\partial u}{\partial x_i}\right)&=f_j''(u)\frac{\partial u}{\partial x_j}\Big|\frac{\partial u}{\partial x_i}\Big|+f_j'(u)\frac{\partial}{\partial x_j}\Big|\frac{\partial u}{\partial x_i}\Big|\\&=\frac{\partial}{\partial x_j}\Big(f_j'(u)\Big|\frac{\partial u}{\partial x_i}\Big|\Big).\end{aligned}$$

Therefore, multiplying (2.45) by $\operatorname{sgn}\left(\frac{\partial u}{\partial x_i}\right)$ we get

$$\frac{\partial}{\partial t}\Big|\frac{\partial u}{\partial x_i}\Big|+\frac{\partial}{\partial x_j}\Big(f_j'(u)\Big|\frac{\partial u}{\partial x_i}\Big|\Big)-\varepsilon\Delta\frac{\partial u}{\partial x_i}\operatorname{sgn}\left(\frac{\partial u}{\partial x_i}\right)=0.$$

This we multiply by $\eta_R(x)$, integrate over $\mathbb{R}^d$ and obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\Big|\frac{\partial u}{\partial x_i}\Big|\eta_R\,dx&=\int_{\mathbb{R}^d}f_j'(u)\,\Big|\frac{\partial u}{\partial x_i}\Big|\,\frac{\partial \eta_R}{\partial x_j}\,dx\\&\quad+\varepsilon\int_{\mathbb{R}^d}\Delta\frac{\partial u}{\partial x_i}\operatorname{sgn}\Big(\frac{\partial u}{\partial x_i}\Big)\eta_R\,dx\\&\le\int_{\mathbb{R}^d}f_j'(u)\,\Big|\frac{\partial u}{\partial x_i}\Big|\,\frac{\partial \eta_R}{\partial x_j}\,dx\\&\quad+\varepsilon\int_{\mathbb{R}^d}\Big|\frac{\partial u}{\partial x_i}\Big|\,\Delta\eta_R\,dx\,,\end{aligned}\tag{2.46}$$

where we used Lemma 2.41. But Lemma 2.38 with $R>1$ gives

$$\frac{d}{dt}\int_{\mathbb{R}^d}\Big|\frac{\partial u}{\partial x_i}\Big|\eta_R\,dx\le\frac{c}{R}\int_{\mathbb{R}^d}\Big|\frac{\partial u}{\partial x_i}\Big|\eta_R\,dx.$$

Using Gronwall's lemma we obtain

$$\int_{\mathbb{R}^d}\Big|\frac{\partial u}{\partial x_i}(t,x)\Big|\eta_R(x)\,dx\le\exp\left(\frac{ct}{R}\right)\int_{\mathbb{R}^d}\Big|\frac{\partial u_0}{\partial x_i}(x)\Big|\,\eta_R(x)\,dx\,.$$

The function $\eta(s)$ is non-increasing and therefore the Lebesgue monotone convergence theorem gives, as $R\to\infty$,

$$\int_{\mathbb{R}^d}\Big|\frac{\partial u}{\partial x_i}(t,x)\Big|\,dx\le\int_{\mathbb{R}^d}\Big|\frac{\partial u_0}{\partial x_i}(x)\Big|\,dx\,.$$

This proves (2.33).

† In the following two formulae we do not sum over an index i.

In order to prove (2.34) we differentiate (1.3) with respect to t and multiply by $\operatorname{sgn}\left(\frac{\partial u}{\partial t}\right)\eta_R$. Since $\frac{\partial u}{\partial t} \in W^{1,2}((0,T)\times\mathbb{R}^d)$, we can proceed in a similar way as before to obtain

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_{L^1(\mathbb{R}^d)} \leq \left\|\frac{\partial u}{\partial t}(0)\right\|_{L^1(\mathbb{R}^d)} . \tag{2.47}$$

On the other hand the regularity result (2.44) allows us to rewrite equation (1.3) as

$$\frac{\partial u}{\partial t}(0) = \varepsilon\Delta u_0 - f_i'(u_0)\frac{\partial u_0}{\partial x_i} .$$

Substituting this equation into (2.47) and using the assumption (2.30), one gets (2.34). Finally, we multiply (1.3) by $\operatorname{sgn}(u)\,\eta_R$ and integrate over $\mathbb{R}^d$. Thus,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}|u|\eta_R\,dx + \int_{\mathbb{R}^d} f_j'(u)\frac{\partial u}{\partial x_j}\operatorname{sgn}(u)\,\eta_R\,dx \\ = \varepsilon\int_{\mathbb{R}^d}\Delta u\ \operatorname{sgn}(u)\,\eta_R\,dx .\end{aligned}$$

In the same way as before, using (2.42), (2.39) and (2.30), one obtains

$$\frac{d}{dt}\int_{\mathbb{R}^d}|u|\eta_R\,dx \leq M\int_{\mathbb{R}^d}|\nabla u|\eta_R\,dx + \frac{c\,\varepsilon}{R^2}\int_{\mathbb{R}^d}|u|\eta_R\,dx .$$

Integrating the last inequality between 0 and $t \in (0,T]$, employing the Lebesgue monotone convergence theorem as $R \to \infty$ and using (2.33), one gets (2.35).

It remains to show (2.32). We multiply (1.3) by η_R, integrate over $\mathbb{R}^d$, and after partial integration we arrive at

$$\frac{d}{dt}\int_{\mathbb{R}^d}u\eta_R\,dx = \int_{\mathbb{R}^d} f_j(u)\frac{\partial\eta_R}{\partial x_j}\,dx + \varepsilon\int_{\mathbb{R}^d}u\Delta\eta_R\,dx . \tag{2.48}$$

Now, we integrate this over $(0,t)$ and use relation (2.40). Letting $R \to \infty$ and using again the Lebesgue monotone convergence theorem, one gets (2.32). The proof of Theorem 2.29 is complete. ■

2.3 The concept of entropy

In the previous section we have shown that for any $\varepsilon > 0$ there exists a unique solution $u^\varepsilon \in W(T)\cap L^\infty(\mathbb{R}^+\times\mathbb{R}^d)$ to the parabolic perturbation (1.3) of the original hyperbolic problem (1.2). We are interested in the limiting process as $\varepsilon \to 0+$.

It is well known that problem (1.2) cannot, in general, have a classical solution, i.e. $u \in C^1(\mathbb{R}^+ \times \mathbb{R}^d)$, even for infinitely smooth data. This is a consequence of the fact that a classical solution has to be constant along the characteristics (see for example SMOLLER [1983] or ZACHMANOGLOU AND THOE [1986]). In the nonlinear case, the characteristics can intersect in finite time for suitable chosen smooth initial data u_0. At the intersection point the solution need not be continuous (for details see, e.g., SMOLLER [1983, Chapter 15]). In such a way, one is led to introduce the concept of a weak solution.

Let us denote by $C_C^n(\mathbb{R} \times \mathbb{R}^d)$ the set of n-times continuously differentiable functions with compact support (note that $\varphi(0,x)$ is in general non-zero for $\varphi \in C_C^n(\mathbb{R} \times \mathbb{R}^d)$). This choice of test functions allows us to include the initial condition u_0 into the weak formulation of the problem (see (3.2)).

Definition 3.1 *Let $u_0 \in L^\infty(\mathbb{R}^d)$. A function $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ is called a weak solution to the problem (1.2) if, for all $\varphi \in C_C^1(\mathbb{R} \times \mathbb{R}^d)$, the following integral identity is fulfilled:*

$$\int_0^\infty \int_{\mathbb{R}^d} \Big(u \frac{\partial \varphi}{\partial t} + f_j(u) \frac{\partial \varphi}{\partial x_j} \Big)\, dx\, dt + \int_{\mathbb{R}^d} u_0(x)\, \varphi(0,x)\, dx = 0. \tag{3.2}$$

Remark 3.3

1. From Definition 3.1 it is obvious that a weak solution of (1.2) is the distributional solution on $\mathbb{R}^+ \times \mathbb{R}^d$.
2. We say that $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ is a **piecewise C^1 function**, if there exists a finite number of smooth oriented d-dimensional surfaces in $\mathbb{R}^+ \times \mathbb{R}^d$ outside of which u is a C^1 function and across which u has a jump discontinuity. If Γ is such a surface and $\mathbf{n} = (n_t, n_1, \ldots, n_d)$ a normal vector to Γ, we define for $(t,x) \in \Gamma$: $u_\pm(t,x) \equiv \lim_{\delta \to 0+} u\,((t,x) \pm \delta \mathbf{n})$. Then, $[u] \equiv u_+ - u_-$ and $[f_j(u)] = f_j(u_+) - f_j(u_-)$.

   If $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ is a piecewise C^1 function, then it can be shown using (3.2) (see SMOLLER [1983]) that the following two statements are equivalent:
   - u is a weak solution to (1.2);
   - u is a classical solution of (1.2) in domains where u is a C^1

function, and

$$n_t[u] + n_j[f_j(u)] = 0 \tag{3.4}$$

holds for all points which lie on exactly one discontinuity surface.

The condition (3.4) is often referred to as the **Rankine-Hugoniot condition** and can be easily obtained if one uses (3.2) with φ compactly supported in a small ball centred at a point belonging to just one discontinuity surface. Then, employing the smoothness of u on both sides of the discontinuity surface, (3.4) follows from Green's theorem.

3. Note that the validity of (3.4) is not required at points in which the surfaces of discontinuity intersect. However, it can be supposed that the $(d+1)$-dimensional Lebesgue measure of the set of such points is zero.
4. Without loss of generality, one can assume that $(n_1, \ldots, n_d) \neq 0$ on the discontinuity surface Γ. Indeed, $(n_1, \ldots, n_d) = 0$ would imply $n_t \neq 0$ and then (3.4) would give $[u] = 0$ on Γ, which contradicts the assumptions. Therefore, $\mathbf{n}$ can be rescaled in such a way that $|\boldsymbol{\nu}| = 1$ for $\boldsymbol{\nu} = (n_1, \ldots, n_d)$. Further, denoting $s \equiv -n_t$, we have $\mathbf{n} = (-s, \boldsymbol{\nu})$, $|\boldsymbol{\nu}| = 1$ and (3.4) reads

$$s[u] = \nu_j[f_j(u)]\,. \tag{3.5}$$

Particularly, in one space dimension, $\mathbf{n} = (-s, 1)$ on the curve of discontinuity and the Rankine-Hugoniot condition takes the form of

$$s[u] = [f(u)]\,. \tag{3.6}$$

In this case, if Γ is a smooth oriented discontinuity curve parametrized by $\big(t, x(t)\big)$, we see that $\big(1, \frac{dx}{dt}\big)$ is the tangent and $\big(-\frac{dx}{dt}, 1\big)$ the normal vector to Γ. By comparison, we have for s in (3.6) that $s = \frac{dx}{dt} = 1/\frac{dt}{dx}$. Therefore, s represents the reciprocal of the slope of the discontinuity curve Γ considered in the x–t plane rather than in the t–x one.

Unfortunately, Definition 3.1 permits nonuniqueness of weak solutions even in simple cases, as shown by the following example.

Example 3.7 (inviscid Burgers equation) Let us consider the equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \tag{3.8}$$

with initial condition

$$u_0(x) = \begin{cases} u_\ell\,, & x < 0\,, \\ u_r\,, & x > 0, \end{cases} \tag{3.9}$$

where $u_\ell \neq u_r$. The Rankine-Hugoniot condition (3.6) for $s \in \mathbb{R}$ reads

$$s = \frac{u_+ + u_-}{2} \tag{3.10}$$

whenever u jumps from the value u_- to the value u_+ over a curve in the x–t plane with slope $1/s$. Therefore, defining

$$u(t,x) \equiv \begin{cases} u_\ell\,, & x < \frac{u_\ell + u_r}{2} t\,, \\ u_r\,, & x > \frac{u_\ell + u_r}{2} t\,, \end{cases} \tag{3.11}$$

we see that u satisfies the Rankine-Hugoniot condition on the line of discontinuity $x = \frac{u_\ell + u_r}{2} t$. Outside the line, u is constant and therefore satisfies (1.2) in the classical sense. By Remark 3.3, u is a weak solution to (3.8), (3.9). On the other hand, there is a one-parameter family of weak solutions given by

$$u_a(t,x) = \begin{cases} u_\ell\,, & x < s_1 t\,, \\ -a\,, & s_1 t < x < 0\,, \\ a\,, & 0 < x < s_2 t\,, \\ u_r\,, & x > s_2 t\,, \end{cases} \tag{3.12}$$

where

$$s_1 = \frac{u_\ell - a}{2}\,, \qquad s_2 = \frac{u_r + a}{2}\,.$$

Note that $s_1 < 0$ and $s_2 > 0$ if one chooses

$$a > \max\,(u_\ell, -u_r)$$

and the whole construction makes sense. All the solutions u_a satisfy the Rankine-Hugoniot condition on the lines of discontinuity and by the same argument as above, all of them are weak solutions to (3.8)–(3.9) in the sense of Definition 3.1.

Further, if $u_\ell < u_r$, then the characteristics $t = \frac{x - x_0}{u_\ell}$ emanating from $(0, x_0)$, $x_0 < 0$, do not intersect the characteristics $t = \frac{x - x_0}{u_r}$ emanating from $(0, x_0)$, $x_0 > 0$. Moreover, there is a domain $G \equiv \{(t,x), u_\ell t < x < u_r t\}$—an angle in the x–t plane—which does not contain any of the characteristics. This 'gap' G can be 'filled' with a smooth function in such a way that the resulting function u_c is a weak solution to (3.8)–(3.9), which is continuous for $t > 0$. Namely, defining

$$u_c(t,x) = \begin{cases} u_\ell\,, & x \le u_\ell t\,, \\ \frac{x}{t}\,, & u_\ell t < x < u_r t\,, \\ u_r\,, & x \ge u_r t\,, \end{cases} \tag{3.13}$$

we see that inside G, $\frac{\partial u_c}{\partial t} + u_c \frac{\partial u_c}{\partial x} = -\frac{x}{t^2} + \frac{x}{t}\frac{1}{t} = 0$ and u_c solves (3.8) in a classical sense inside G. The same holds outside G, where u_c is constant. The Rankine-Hugoniot condition on border lines of G is satisfied trivially, since u_c exhibits no jump there. Consequently, u_c is a weak solution to (3.8)–(3.9), continuous for $t > 0$.

Note that in the case $u_\ell > u_r$ the characteristics carrying the value of u_ℓ intersect with those carrying the value of u_r and no continuous solution exists.

Now, it is natural to ask if there is a criterion which would, from the mathematical viewpoint, ensure the uniqueness of a weak solution, and, from the physical viewpoint, select the 'correct' physical solution among all weak ones. One of the possibilities is to consider an additional conservation law which on the one hand would be automatically satisfied by any smooth solution u of (1.2), but on the other would play the role of a selector for weak solutions. Hence, we want to find conditions for η, $\mathbf{q}$, $\mathbf{q} = (q_1, \ldots, q_d)$, $\eta, q_j \in C^1(\mathbb{R})$, such that a smooth solution u will automatically satisfy an additional conservation law

$$\frac{\partial}{\partial t}\eta(u) + \operatorname{div} \mathbf{q}(u) = 0\,. \tag{3.14}$$

Let us multiply (1.2) by $\eta'(u)$ and use the chain rule. We obtain

$$\frac{\partial}{\partial t}\eta(u) + \eta'(u) f_j'(u) \frac{\partial u}{\partial x_j} = 0. \tag{3.15}$$

We see that if $\eta, q_j \in C^1(\mathbb{R})$ satisfy the following compatibility conditions

$$\eta'(u) f_j'(u) = q_j'(u) \qquad \forall\, u \in \mathbb{R},\ 1 \le j \le d, \tag{3.16}$$

then (3.14) is automatically fulfilled for any smooth solution u of (1.2).

Suppose now that u is only a piecewise C^1 solution to (1.2) in the weak sense. Then (3.14) is satisfied only in domains where u is C^1. Proceeding in the same way as in Remark 3.3 it can be shown that the weak form of equation (3.14) implies the corresponding Rankine-Hugoniot condition across discontinuities,

$$s[\eta(u)] = \nu_j[q_j(u)] \tag{3.17}$$

for all $\eta, q_j \in C^1(\mathbb{R})$ satisfying (3.16).

However, the condition (3.17) is in general not compatible with (3.5). Indeed, let us consider again the problem (3.8)–(3.9) and put $u_\ell = 1$, $u_r = 0$ for simplicity. Then (3.10) implies $s = \frac{1}{2}$ for the solution of the type (3.11). On the other hand, taking $\eta(u) = u^k$ and $q(u) = \frac{k}{k+1}u^{k+1}$ (satisfying (3.16)), one obtains $s = \frac{k}{k+1} \neq \frac{1}{2}$ for all $k \neq 1$. In this way, there is no weak solution to our problem of the type (3.11) satisfying an additional conservation law (3.14) for polynomial η in a weak sense. The same conclusion can be drawn for solutions of the type (3.12).

Therefore, it turns out that the demand that a solution $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ should satisfy an additional conservation law (3.14) in a weak sense is too restrictive. We will see that replacing the equality in (3.14) by a proper inequality will result in a so-called entropy inequality which will play the role of a proper selector. In the rest of this section, we present a heuristic derivation of this inequality.

Let us return to the vanishing viscosity method. If we multiply the parabolic perturbation (1.3) of (1.2) by $\eta'(u^\varepsilon)$ and use the chain rule—recall that u^ε are smooth enough—we get

$$\frac{\partial}{\partial t}\eta(u^\varepsilon) + \eta'(u^\varepsilon)f_j'(u^\varepsilon)\frac{\partial u^\varepsilon}{\partial x_j} = \varepsilon\eta'(u^\varepsilon)\Delta u^\varepsilon\,. \tag{3.18}$$

Now, using the compatibility conditions (3.16), we obtain for $\eta \in C^2(\mathbb{R})$,

$$\frac{\partial}{\partial t}\eta(u^\varepsilon) + \operatorname{div}\mathbf{q}(u^\varepsilon) = \varepsilon\Delta\eta(u^\varepsilon) - \varepsilon\eta''(u^\varepsilon)|\nabla u^\varepsilon|^2.$$

At this point, to be able to control the sign on the right, we introduce the additional requirement that η has to be convex, and we obtain

$$\frac{\partial}{\partial t}\eta(u^\varepsilon) + \operatorname{div}\mathbf{q}(u^\varepsilon) \le \varepsilon\Delta\eta(u^\varepsilon) \tag{3.19}$$

for every smooth solution u^ε of (1.3). Now, (3.19) can be viewed as a parabolic perturbation of an additional conservation inequality

$$\frac{\partial}{\partial t}\eta(u) + \operatorname{div} \mathbf{q}(u) \le 0\,; \tag{3.20}$$

compare (1.2), (1.3) with (3.19), (3.20). Indeed, if we assume for a moment that some kind of convergence of u^ε to u has already been proved, for example, that

$$\begin{aligned} u^\varepsilon \to u \qquad &\text{a.e. in } \mathbb{R}^+ \times \mathbb{R}^d\,, \\ \|u^\varepsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^d)} &\le c\,, \end{aligned} \tag{3.21}$$

we will be able to show that (3.19) implies (3.20) in the sense of distributions (see Theorem 4.22 below). Moreover, it will turn out that there is just one weak solution to (1.2) in the sense of Definition 3.1 satisfying the conservation inequality (3.20) in a weak sense (see Theorem 5.1).

Our considerations motivate the following definitions.

Definition 3.22 *Let $\eta \in C^1(\mathbb{R})$ be a convex function. If there exist functions $q_j \in C^1(\mathbb{R}), 1 \le j \le d$, such that for all $u \in \mathbb{R}$*

$$\eta'(u) f_j'(u) = q_j'(u)\,, \qquad 1 \le j \le d\,, \tag{3.23}$$

then $(\eta, \mathbf{q})$ is called an **entropy-entropy flux pair** *of the conservation law (1.2).*

Definition 3.24 *A weak solution u to the problem (1.2) is called* **entropy solution** *if for every entropy-entropy flux pair $\eta, \mathbf{q}$ of (1.2) the so-called* **entropy inequality**

$$\frac{\partial}{\partial t}\eta(u) + \operatorname{div} \mathbf{q}(u) \le 0 \tag{3.25}$$

is fulfilled in the following sense: for all $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d), \varphi \ge 0$, it holds

$$\int_0^\infty \int_{\mathbb{R}^d} \big(\eta(u) - \eta(u_0)\big)\frac{\partial \varphi}{\partial t} + q_j(u)\frac{\partial \varphi}{\partial x_j}\, dx\, dt \ge 0. \tag{3.26}$$

Note that

$$\begin{aligned} -\int_0^\infty \int_{\mathbb{R}^d} \eta(u_0)\frac{\partial \varphi}{\partial t}\, dx\, dt &= -\int_{\mathbb{R}^d} \eta(u_0) \int_0^\infty \frac{\partial \varphi}{\partial t}\, dt\, dx \\ &= \int_{\mathbb{R}^d} \eta(u_0)\varphi(0)\, dx\,, \end{aligned}$$

which shows the equivalence of (3.25) and (3.26) for smooth u.

In the following sections we will show that the solution constructed with the help of the parabolic perturbation (1.3) is a unique entropy solution to the problem (1.2) in the sense of Definition 3.24.

Example 3.27 Let us show that the entropy inequality (3.25) selects just one among all weak solutions obtained in Example 3.7. We have observed that if a weak solution to (1.2) is a piecewise C^1 function, (3.25) is satisfied with the equality sign in domains where u is C^1. On a discontinuity surface Γ, one can use again the Rankine-Hugoniot approach to obtain

$$s[\eta(u)] \geq \nu_j[q_j(u)] \tag{3.28}$$

for all η, $q_j \in C^1(\mathbb{R})$ satisfying (3.16), η convex. Since we are dealing with an inequality now, it is important to recall that on Γ the normal vector $\mathbf{n} = (-s, \boldsymbol{\nu})$ points in 'the domain of u_+', i.e., $u_+(t,x) = \lim_{\delta\to 0+} u((t,x) + \delta\mathbf{n})$ on Γ.

For the particular case of an inviscid Burgers equation in one space dimension (3.8), we will show the following fact: if a piecewise C^1 solution u to (3.8) jumps from the value U_L to the value U_R over a smooth curve Γ for some fixed time t, as x increases, then (3.28) implies $U_L \geq U_R$. Indeed, considering (3.10) together with (3.28), and using the fact that $q'(u) = u\eta'(u)$ in the case of the equation (3.8), one obtains that

$$\frac{1}{2}(U_L + U_R)\big(\eta(U_R) - \eta(U_L)\big) \geq \int_{U_L}^{U_R} \lambda\, \eta'(\lambda)\, d\lambda$$

should hold for all convex $\eta \in C^1(\mathbb{R})$. Now, if $U_L < U_R$, one chooses $\eta(u) \equiv (u - c_0)^2$, where $c_0 = \frac{U_L+U_R}{2}$ so that $\eta(U_L) = \eta(U_R)$. On the other hand, the integral on the right-hand side equals $\frac{1}{6}(U_R - U_L)^3 > 0$ which is a contradiction.

This fact can be formulated as follows: 'For the particular case of Burgers equation, (3.28) implies that no jump from U_L to U_R, $U_L < U_R$, as x increases, is allowed for piecewise C^1 weak solutions.' In such a way we see that (3.28) selects a uniquely determined weak solution within the class of solutions constructed in Example 3.7. More precisely, for $u_\ell > u_r$ the proper solution is of the type (3.11), while for $u_\ell < u_r$ one gets the continuous solution u_c defined in (3.13). This is of course by no means a uniqueness proof even in the case of Burgers equation, since not all possible weak solutions were taken into account. Nevertheless, these considerations give us a hope that we are on the right track.

Remark 3.29 If we deal only with a scalar conservation law, every convex function $\eta \in C^1(\mathbb{R})$ is an entropy. Indeed, if we define the entropy flux as

$$q_j(u) = \int_0^u \eta'(\lambda) f_j'(\lambda)\, d\lambda\,,$$

relation (3.23) is trivially fulfilled.

Another important entropy-entropy flux pair in the case of one equation is

$$\eta(u) = |u-k|\,, \tag{3.30}$$
$$q_j(u) = \operatorname{sgn}(u-k)(f_j(u) - f_j(k)). \tag{3.31}$$

$k \in \mathbb{R}$. Of course, in this case η and q_j are not smooth enough, but one can overcome this difficulty by an appropriate smoothing—see Lemma 5.3.

For a general system of conservation laws, the entropy is defined by compatibility conditions analogous to (3.23) (see Godlewski and Raviart [1991] or Smoller [1983]). However, in general it can be difficult to fulfill these conditions and the question arises if there is at least one entropy flux pair for a general system of conservation laws. The situation becomes better if the system is symmetrizable, which is typical for physically motivated models. In that case one can always find an entropy having a physical interpretation, as for example in the case of Euler equations (for details see Schochet [1989], Godlewski and Raviart [1991], Rokyta [1992]).

2.4 Existence of an entropy solution

In this section we will show that an entropy solution to the Cauchy problem (1.2) exists under the only assumptions that $u_0 \in L^\infty(\mathbb{R}^d)$ and $\mathbf{f} \in C^1(\mathbb{R})$. We will use the method of vanishing viscosity, combined here with a regularization of the initial condition and the nonlinearity $\mathbf{f}$.

Let $\rho \in \mathcal{D}(\mathbb{R}^p)$ be a symmetric **mollifier**, i.e.,

$$\begin{aligned} \rho(x) \geq 0\,, \quad &\operatorname{supp}\rho \subseteq B_1(0)\,, \\ \int_{\mathbb{R}^p} \rho(x)\,dx = 1\,, \quad &\rho(-x) = \rho(x)\,. \end{aligned} \tag{4.1}$$

Put

$$\rho_\varepsilon(x) \equiv \frac{1}{\varepsilon^p}\rho\Big(\frac{x}{\varepsilon}\Big) \tag{4.2}$$

and denote for any $g \in L^1_{\mathrm{loc}}(\mathbb{R}^p)$

$$g_\varepsilon(x) \equiv (g * \rho_\varepsilon)(x) \equiv \int_{\mathbb{R}^p} g(x-y)\rho_\varepsilon(y)\,dy\,. \tag{4.3}$$

In particular, we define

$$\begin{aligned} &u_{0\varepsilon} \equiv u_0 * \rho_\varepsilon\,, \\ &f_{j\varepsilon}(\xi) \equiv (f_j * \rho_\varepsilon)(\xi)\,, \qquad \mathbf{f}_\varepsilon \equiv (f_{1\varepsilon}, \ldots, f_{d\varepsilon})\,. \end{aligned} \tag{4.4}$$

We will study the following approximation of the problem (1.2):

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + \operatorname{div}\mathbf{f}_\varepsilon(u^\varepsilon) - \varepsilon\Delta u^\varepsilon = 0 \qquad &\text{in } \mathbb{R}^+ \times \mathbb{R}^d\,, \\ u^\varepsilon(0,\cdot) = u_{0\varepsilon} \qquad &\text{in } \mathbb{R}^d\,. \end{aligned} \tag{4.5}$$

In what follows, we give estimates of u^ε in various norms by means of nonregularized initial condition u_0. Prior to this, let us recall that for $g \in L^1(\mathbb{R}^d)$, the total variation of g is defined as

$$TV(g) \equiv \sup_{\substack{\boldsymbol{\varphi} \in C^1_{\mathrm{C}}(\mathbb{R}^d)^d \\ \|\boldsymbol{\varphi}\|_\infty \leq 1}} \int_{\mathbb{R}^d} g \operatorname{div}\boldsymbol{\varphi}\,dx\,.$$

Here $C^1_{\mathrm{C}}(\mathbb{R}^d)$ denotes the space of continuously differentiable functions with compact support in $\mathbb{R}^d$. Denote by

$$BV(\mathbb{R}^d) \equiv \{g \in L^1(\mathbb{R}^d)\,, TV(g) < \infty\}\,,$$

the space of functions with bounded variation. See Lemma 2.50 in Chapter 1 for more details on these spaces.

Lemma 4.6 *Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$. Then for any $m \in \mathbb{N}$ the regularization $u_{0\varepsilon} \equiv u_0 * \rho_\varepsilon$ belongs to the space $W^{m,2}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and satisfies*

$$\|u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)}\,, \tag{4.7}$$

$$\int_{\mathbb{R}^d} u_{0\varepsilon}\,dx = \int_{\mathbb{R}^d} u_0\,dx\,, \tag{4.8}$$

$$\|u_{0\varepsilon}\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}\,, \tag{4.9}$$

$$\|\nabla u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)^d} \le TV(u_0)\,, \tag{4.10}$$

$$\|\Delta u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)} \le \frac{c}{\varepsilon} TV(u_0)\,, \tag{4.11}$$

where the constant c does not depend on ε.

Proof : The first part of the lemma (up to (4.9)) is standard (see, e.g., Kufner, John and Fučík [1977, Section 2.5]) and therefore we omit the proofs. We will show (4.10), (4.11). Let $\boldsymbol{\varphi} \in C^1_C(\mathbb{R}^d)^d$ be such that $\|\boldsymbol{\varphi}\|_{L^\infty(\mathbb{R}^d)^d} \le 1$. We have for $\boldsymbol{\varphi}_\varepsilon \equiv \rho_\varepsilon * \boldsymbol{\varphi}$ that $\|\boldsymbol{\varphi}_\varepsilon\|_{L^\infty(\mathbb{R}^d)^d} \le 1$ and

$$\begin{aligned}\int_{\mathbb{R}^d} u_{0\varepsilon}\operatorname{div}\boldsymbol{\varphi}\,dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho_\varepsilon(y)u_0(x-y)\operatorname{div}_x \boldsymbol{\varphi}(x)\,dy\,dx\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_0(x-y)\operatorname{div}_x(\rho_\varepsilon(y)\boldsymbol{\varphi}(x))\,dy\,dx\,.\end{aligned}$$

Now we use the substitution $x' = x - y$, $y' = -y$, the symmetry property $\rho_\varepsilon(-y') = \rho_\varepsilon(y)$ and finally replace x' by x, y' by y. We get that the last integral is equal to

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_0(x)\operatorname{div}_x(\rho_\varepsilon(y)\boldsymbol{\varphi}(x-y))\,dy\,dx\\ &\qquad= \int_{\mathbb{R}^d} u_0(x)\operatorname{div}_x\left(\int_{\mathbb{R}^d}\rho_\varepsilon(y)\boldsymbol{\varphi}(x-y)\,dy\right)dx\\ &\qquad= \int_{\mathbb{R}^d} u_0(x)\operatorname{div}_x(\boldsymbol{\varphi}_\varepsilon)(x)\,dx \le TV(u_0)\,.\end{aligned}$$

Therefore, $TV(u_{0\varepsilon}) \le TV(u_0)$. However,

$$\int_{\mathbb{R}^d} |\nabla u_{0\varepsilon}|\,dx = TV(u_{0\varepsilon})$$

and inequality (4.10) follows. Similarly, for $0 \ne \psi \in L^\infty(\mathbb{R}^d)$ we get

$$\int_{\mathbb{R}^d} \Delta u_{0\varepsilon} \psi \, dx = \int_{\mathbb{R}^d} u_0 \operatorname{div}(\nabla \psi_\varepsilon) \, dx$$

and therefore

$$\begin{aligned} \|\Delta u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)} &= \sup_{0 \neq \psi \in L^\infty(\mathbb{R}^d)} \frac{\left|\int_{\mathbb{R}^d} \Delta u_{0\varepsilon} \psi \, dx\right|}{\|\psi\|_{L^\infty(\mathbb{R}^d)}} \\ &= \sup_{0 \neq \psi \in L^\infty(\mathbb{R}^d)} \frac{\left|\int_{\mathbb{R}^d} u_0 \operatorname{div}(\nabla \psi_\varepsilon) \, dx\right|}{\|\psi\|_{L^\infty(\mathbb{R}^d)}} \qquad (4.12) \\ &\leq TV(u_0) \sup_{0 \neq \psi \in L^\infty(\mathbb{R}^d)} \frac{\|\nabla \psi_\varepsilon\|_{L^\infty(\mathbb{R}^d)^d}}{\|\psi\|_{L^\infty(\mathbb{R}^d)}} \ . \end{aligned}$$

We know that $\nabla \psi_\varepsilon = \psi * \nabla \rho_\varepsilon$ and

$$\begin{aligned} \|\nabla \psi_\varepsilon\|_{L^\infty(\mathbb{R}^d)^d} &\leq \|\nabla \rho_\varepsilon\|_{L^1(\mathbb{R}^d)^d} \|\psi\|_{L^\infty(\mathbb{R}^d)} \\ &\leq \frac{c}{\varepsilon} \|\psi\|_{L^\infty(\mathbb{R}^d)} \ , \end{aligned}$$

which together with (4.12) implies (4.11). ∎

Lemma 4.13 *Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$. Then there exists a unique C^∞ solution u^ε to the problem (4.5), which satisfies for all $t \geq 0$*

$$\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)} \ , \qquad (4.14)$$

$$\int_{\mathbb{R}^d} u^\varepsilon(t,x) \, dx = \int_{\mathbb{R}^d} u_0(x) \, dx \ , \qquad (4.15)$$

$$\|\nabla u^\varepsilon(t)\|_{L^1(\mathbb{R}^d)^d} \leq TV(u_0) \ , \qquad (4.16)$$

$$\left\|\frac{\partial u^\varepsilon}{\partial t}(t)\right\|_{L^1(\mathbb{R}^d)} \leq c \, TV(u_0) \qquad (4.17)$$

and for $t \in [0,T]$

$$\|u^\varepsilon(t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)} + cT \cdot TV(u_0) \ , \qquad (4.18)$$

where c does not depend on ε.

Proof : According to Lemma 4.6 and (4.4), all assumptions of Lemma 2.16 are satisfied. Therefore, we have the existence of a unique solution u^ε to the problem (4.5), satisfying

$$\frac{\partial^k u^\varepsilon}{\partial t^k} \in C(0,T;W^{n,2}(\mathbb{R}^d))\,,$$
$$u^\varepsilon \in C(0,T;W^{n,2}(\mathbb{R}^d))$$

for every n, $k \in \mathbb{N}$. However, this means that $u^\varepsilon \in C^\infty((0,T)\times\mathbb{R}^d)$. On the other hand, Lemma 4.6 and Theorem 2.29 imply

$$\begin{aligned}
\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R}^d)} &\le \|u_{0\varepsilon}\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}\,,\\
\int_{\mathbb{R}^d} u^\varepsilon(t,x)\,dx &= \int_{\mathbb{R}^d} u_{0\varepsilon}(x)\,dx = \int_{\mathbb{R}^d} u_0(x)\,dx\,,\\
\|\nabla u^\varepsilon(t)\|_{L^1(\mathbb{R}^d)^d} &\le \|\nabla u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)^d} \le TV(u_0)\,, \\
\Big\|\frac{\partial u^\varepsilon}{\partial t}(t)\Big\|_{L^1(\mathbb{R}^d)} &\le M_\varepsilon\|\nabla u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)^d} + \varepsilon\|\Delta u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)}\\
&\le M_\varepsilon\, TV(u_0) + c\, TV(u_0)\ ,
\end{aligned} \tag{4.19}$$

where

$$M_\varepsilon = \sup\left\{|\mathbf{f}_\varepsilon'(\xi)|, |\xi| \le \|u_{0\varepsilon}\|_{L^\infty(\mathbb{R}^d)}\right\}\ .$$

We know that

$$f_{j\varepsilon}'(\xi) = \int_{\mathbb{R}} f_j'(\xi-\theta)\rho_\varepsilon(\theta)\,d\theta\,,$$

which gives

$$M_\varepsilon \le \sup\left\{|\mathbf{f}'(\xi)|, |\xi| \le \|u_0\|_{L^\infty} + \varepsilon\right\} \le \widetilde{c}\ .$$

This together with $(4.19)_4$ implies

$$\Big\|\frac{\partial u^\varepsilon}{\partial t}(t)\Big\|_{L^1(\mathbb{R}^d)} \le (c+\widetilde{c})\,TV(u_0)\,, \tag{4.20}$$

which is (4.17).

Finally, from (2.35) and Lemma 4.6 we have

$$\begin{aligned}
\|u^\varepsilon(t)\|_{L^1(\mathbb{R}^d)} &\le \|u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)} + M_\varepsilon T\|\nabla u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)^d}\\
&\le \|u_0\|_{L^1(\mathbb{R}^d)} + \widetilde{c}T\cdot TV(u_0)\ .
\end{aligned} \tag{4.21}$$

Formulae (4.19)–(4.21) prove the lemma. ■

Now we are ready to show the existence of an entropy solution to the Cauchy problem (1.2). The following existence theorem is formulated under slightly stronger assumptions on u_0 than will be finally needed. The general situation is studied in Theorem 4.71 below, where the existence is obtained for $u_0 \in L^\infty(\mathbb{R}^d)$ only.

The key point in proving both theorems is to obtain an L^∞-estimate and almost everywhere convergence of a subsequence of $\{u^\varepsilon\}$ (see (4.26), (4.27), cf. the proof of Theorem 4.71). Both proofs then differ only in the way they establish the above-mentioned properties of $\{u^\varepsilon\}$.

Theorem 4.22 *Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ and let $\mathbf{f} \in C^1(\mathbb{R})^d$. Then the problem (1.2) has an entropy solution u, belonging to the space $L^\infty((0,\infty) \times \mathbb{R}^d) \cap C(0,T; L^1(\mathbb{R}^d))$ for all $T > 0$ and satisfying, for almost every $t > 0$,*

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)} . \tag{4.23}$$

Furthermore, $u(t) \in BV(\mathbb{R}^d)$ for all $t \ge 0$ and

$$TV(u(t)) \le TV(u_0)\,, \tag{4.24}$$

$$\int_{\mathbb{R}^d} |u(t_1,x) - u(t_2,x)|\, dx \le c\, TV(u_0)\, |t_2 - t_1|\,, \tag{4.25}$$

for all t, t_1, $t_2 \ge 0$.

Proof : Let $\{u^\varepsilon\}$ be the sequence of solutions to the approximate problem (4.5). Due to Lemma 4.13 this sequence is bounded in the space $L^\infty((0,\infty) \times \mathbb{R}^d) \cap W^{1,1}_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^d)$. Let $\{K_n\}$ be an increasing compact smooth covering of $[0,\infty) \times \mathbb{R}^d$. Due to the compact imbedding of $W^{1,1}(K_n)$ into $L^1(K_n)$ it is possible to select a subsequence, still denoted u^ε, converging strongly in $L^1(K_n)$ and almost everywhere. Using a diagonalization process we obtain a sequence $\{u^\varepsilon\}$ such that

$$\|u^\varepsilon\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)} \tag{4.26}$$

and

$$\begin{aligned} u^\varepsilon &\to u \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)\,, \\ u^\varepsilon &\to u \quad \text{a.e. in } \mathbb{R}^+ \times \mathbb{R}^d. \end{aligned} \tag{4.27}$$

In what follows we will show that the limit function u is an entropy solution to the problem (1.2). We have that

$$u_{0\varepsilon} \to u_0 \quad \text{in } L^1(\mathbb{R}^d)\,, \tag{4.28}$$

$$\mathbf{f}_\varepsilon \overset{\mathrm{loc}}{\rightrightarrows} \mathbf{f} \quad \text{in } \mathbb{R}\,, \tag{4.29}$$

and due to (4.26) we also get

$$\|\mathbf{f}_\varepsilon(u^\varepsilon) - \mathbf{f}(u^\varepsilon)\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^d)} \to 0\,,$$
$$\|\mathbf{f}(u^\varepsilon)\|_{L^1_{\text{loc}}(\mathbb{R}^+\times\mathbb{R}^d)} \le c\,.$$

Thus the Lebesgue dominated convergence theorem gives

$$\mathbf{f}(u^\varepsilon) \to \mathbf{f}(u) \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d) \tag{4.30}$$

and altogether we have

$$\mathbf{f}_\varepsilon(u^\varepsilon) \to \mathbf{f}(u) \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d). \tag{4.31}$$

The weak formulation of (4.5) for $\varphi \in C^1_C(\mathbb{R} \times \mathbb{R}^d)$ reads

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^d} u^\varepsilon \frac{\partial\varphi}{\partial t} + f_{j\varepsilon}(u^\varepsilon)\frac{\partial\varphi}{\partial x_j}\,dx\,dt + \int_{\mathbb{R}^d} u_{\varepsilon 0}(x)\varphi(0,x)\,dx \\ - \varepsilon \int_0^\infty \int_{\mathbb{R}^d} \nabla u^\varepsilon \nabla\varphi\,dx\,dt = 0\,. \end{aligned} \tag{4.32}$$

From (4.27), (4.28), (4.31) and (4.16), it immediately follows that u is a weak solution to the problem (1.2).

Now, let η be a C^2 entropy. Multiplying (4.5) by $\eta'(u^\varepsilon)$ we get

$$\frac{\partial}{\partial t}\eta(u^\varepsilon) + \frac{\partial}{\partial x_j}q_j^\varepsilon(u^\varepsilon) = \varepsilon\eta'(u^\varepsilon)\Delta u^\varepsilon\ ,$$

where $\mathbf{q}^\varepsilon$ is the entropy flux corresponding to η, $\mathbf{f}_\varepsilon$, and is defined by the compatibility conditions (3.16). The right-hand side can be rewritten as

$$\varepsilon\eta'(u^\varepsilon)\Delta u^\varepsilon = \varepsilon\Delta\eta(u^\varepsilon) - \varepsilon\eta''(u^\varepsilon)|\nabla u^\varepsilon|^2\,.$$

Then, the convexity of η gives

$$\frac{\partial}{\partial t}\eta(u^\varepsilon) + \frac{\partial}{\partial x_j}q_j^\varepsilon(u^\varepsilon) \le \varepsilon\Delta\eta(u^\varepsilon)\,. \tag{4.33}$$

Similarly as in (4.30) we obtain

$$\eta(u^\varepsilon) \to \eta(u) \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)\,. \tag{4.34}$$

Further,

$$\Big|\int_0^{u^\varepsilon} \eta'(s) f'_{j\varepsilon}(s)\,ds - \int_0^u \eta'(s) f'_j(s)\,ds\Big| \\ \le \int_{-K}^{K} |\eta'(s)||f'_{j\varepsilon}(s) - f'_j(s)|\,ds \\ + \Big|\int_u^{u^\varepsilon} \eta'(s) f'_{j\varepsilon}(s)\,ds\Big|,$$

where $K = \|u_0\|_{L^\infty(\mathbb{R}^d)}$, and thus

$$q_j^\varepsilon(u^\varepsilon) = \int_0^{u^\varepsilon} \eta'(s) f'_{j\varepsilon}(s)\,ds \to \int_0^u \eta'(s) f'_j(s)\,ds = q_j(u)$$

a.e. in $\mathbb{R}^+ \times \mathbb{R}^d$. This implies, again by the Lebesgue dominated convergence theorem,

$$q_j^\varepsilon(u^\varepsilon) \to q_j(u) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)\,. \tag{4.35}$$

Since

$$\int_0^\infty \int_{\mathbb{R}^d} \varepsilon\Delta\eta(u^\varepsilon)\,\varphi\,dx\,dt = -\varepsilon \int_0^\infty \int_{\mathbb{R}^d} \eta'(u^\varepsilon)\,\nabla u^\varepsilon\,\nabla\varphi\,dx\,dt\,, \tag{4.36}$$

and (4.16) holds, we see that

$$\int_0^\infty \int_{\mathbb{R}^d} \varepsilon\Delta\eta(u^\varepsilon)\,\varphi\,dx\,dt \to 0\,, \qquad \text{as } \varepsilon \to 0+\,, \tag{4.37}$$

for all $\varphi \in C^1_{\mathrm{C}}(\mathbb{R} \times \mathbb{R}^d)$. The limiting process in the weak formulation of (4.33) then follows from (4.34)–(4.37) in the same way as before. Thus, we have proved that u is an entropy solution.

Now we want to show that for all $T > 0$,

$$u^\varepsilon \to u \quad \text{in } C(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^d)). \tag{4.38}$$

The estimates (4.16)–(4.18) give that

$$u^\varepsilon \text{ is bounded in the norm of } C(0,T; W^{1,1}(\mathbb{R}^d))\,,$$
$$\frac{\partial u^\varepsilon}{\partial t} \text{ is bounded in the norm of } C(0,T; L^1(\mathbb{R}^d)).$$

Let K be a bounded smooth set in $\mathbb{R}^d$. For $0 \le t_1 \le t_2 \le T$ and all $x \in K$ we can write

$$u^\varepsilon(t_2,x) - u^\varepsilon(t_1,x) = \int_{t_1}^{t_2} \frac{\partial u^\varepsilon}{\partial t}(s,x)\,ds$$

and due to (4.17) we get

$$\begin{aligned} \|u^\varepsilon(t_2) - u^\varepsilon(t_1)\|_{L^1(K)} &\le \int_{t_1}^{t_2} \Big\| \frac{\partial u^\varepsilon}{\partial t}(s) \Big\|_{L^1(K)} \, ds \\ &\le c\, TV(u_0)(t_2 - t_1) \ . \end{aligned} \tag{4.39}$$

Thanks to the compact imbedding of $W^{1,1}(K)$ into $L^1(K)$, the sequence $\{u^\varepsilon\}$ lies in a compact set of $L^1(K)$ and one can apply the Arzelà-Ascoli theorem to obtain (4.38). Due to (4.18) we have for every bounded K:

$$\|u^\varepsilon(t)\|_{L^1(K)} \le c_0 \equiv \|u_0\|_{L^1(\mathbb{R}^d)} + cT \cdot TV(u_0)$$

and therefore we have $u(t) \in L^1(\mathbb{R}^d)$. Since the right-hand side of (4.39) does not depend on ε and K, we have

$$\|u(t_2) - u(t_1)\|_{L^1(\mathbb{R}^d)} \le c\, TV(u_0)(t_2 - t_1) \ ,$$

which proves on the one hand $u \in C(0,T; L^1(\mathbb{R}^d))$ and on the other hand (4.25).

It remains to show (4.24). Let $\boldsymbol{\varphi} \in C^1_C(\mathbb{R}^d)^d$, $\|\boldsymbol{\varphi}\|_{L^\infty(\mathbb{R}^d)^d} \le 1$. Using (4.16) we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} u^\varepsilon(t) \operatorname{div} \boldsymbol{\varphi} \, dx &= - \int_{\mathbb{R}^d} \nabla u^\varepsilon(t) \boldsymbol{\varphi} \, dx \\ &\le TV(u_0) \|\boldsymbol{\varphi}\|_{L^\infty(\mathbb{R}^d)^d} \ . \end{aligned}$$

Now (4.38) gives (4.24) and proof of the theorem is complete. ∎

Remark 4.40 In the rest of this section we want to prove the existence of an entropy solution u under a weaker assumption on the initial data, namely that $u_0 \in L^\infty(\mathbb{R}^d)$ only. In this case, the estimate (4.16) cannot be used and therefore the limiting processes in (4.32) and in the integral (4.36) cannot be justified. However, supposing that $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$ in (4.32) and (4.36), we can write

$$-\varepsilon \int_0^\infty \int_{\mathbb{R}^d} \nabla u^\varepsilon \, \nabla \varphi \, dx \, dt = \varepsilon \int_0^\infty \int_{\mathbb{R}^d} u^\varepsilon \Delta \varphi \, dx \, dt$$

and

$$\varepsilon \int_0^\infty \int_{\mathbb{R}^d} \Delta \eta(u^\varepsilon) \, \varphi \, dx \, dt = \varepsilon \int_0^\infty \int_{\mathbb{R}^d} \eta(u^\varepsilon) \Delta \varphi \, dx \, dt \, .$$

Using these formulae we see that the limiting processes as $\varepsilon \to 0+$ in both (4.32) and the weak form of (4.33) can be justified in the sense of distributions under the assumptions that $u^\varepsilon \to u$ almost everywhere and $\|u^\varepsilon\|_\infty \le c$.

Our goal in the rest of this section is firstly to prove the existence of solutions u^ε of the parabolic perturbations for $u_0 \in L^\infty(\mathbb{R}^d)$, and secondly to show that u^ε converge to u almost everywhere in $\mathbb{R}^+ \times \mathbb{R}^d$.

To do this, we need some technical results.

Let $B_r \equiv \{x \in \mathbb{R}^d; |x| < r\}$. For $v \in L^1(B_r)$ we denote[‡]

$$\omega_r(h) = \sup_{|z|\le h} \int_{B_r} |v(x+z) - v(x)|\, dx \ .$$

It is clear that for all $r > 0$

$$\lim_{h\to 0+} \omega_r(h) = 0 \ . \tag{4.41}$$

Lemma 4.42 *Let $v \in L^1(B_{r+2\lambda})$. Then for $0 < \varepsilon < h$ we have*

$$\omega_r^\varepsilon(\lambda) \equiv \sup_{|z|\le\lambda} \int_{B_r} |v_\varepsilon(x+z) - v_\varepsilon(x)|\, dx \le \omega_{r+h}(\lambda)\,, \tag{4.43}$$

$$\int_{B_r} |\,|v| - v(\operatorname{sgn} v)_\varepsilon|\, dx \le 2\omega_r(\varepsilon)\,. \tag{4.44}$$

Proof : We have

$$\begin{aligned}
&\int_{B_r} |v_\varepsilon(x+z) - v_\varepsilon(x)|\, dx \\
&\qquad = \int_{B_r} \left| \int_{B_\varepsilon(0)} \big(v(x-y+z) - v(x-y)\big)\rho_\varepsilon(y)\, dy \right| dx \\
&\qquad \le \int_{B_1(0)} \rho(y) \int_{B_r} |v(x-\varepsilon y+z) - v(x-\varepsilon y)|\, dx\, dy \\
&\qquad \le \int_{B_1(0)} \rho(y)\, dy \int_{B_{r+h}} |v(x+z) - v(x)|\, dx
\end{aligned}$$

which immediately gives (4.43). In order to prove (4.44) we note that

$$\begin{aligned}
&\int_{B_r} \big|\,|v| - v(\operatorname{sgn} v)_\varepsilon \big|\, dx \\
&\qquad = \int_{B_r} \int_{B_\varepsilon(0)} \rho_\varepsilon(y) \big|\,|v(x)| - v(x) \operatorname{sgn} v(x-y)\big|\, dy\, dx
\end{aligned}$$

‡ We define $v \equiv 0$ outside B_r.

and use

$$\big|\,|v| - v\,\mathrm{sgn}\,w\big| = \big|\,|v| - |w| - (v-w)\,\mathrm{sgn}\,w\big|$$
$$\leq 2|v-w|.$$

■

Let $\mathbf{a} = (a_1, \ldots, a_d) \in C([0,t_0] \times \mathbb{R}^d)^d$ and let us denote

$$K \equiv \sup_{[0,t_0]\times\mathbb{R}^d} |\mathbf{a}(t,x)| + 1.$$

For some smooth function q defined in $[0,t_0]\times\mathbb{R}^d$ we can introduce the operator

$$L(q) \equiv \frac{\partial q}{\partial t} + a_i \frac{\partial q}{\partial x_i} + \varepsilon \Delta q. \tag{4.45}$$

Then we have the following version of the maximum principle:

Lemma 4.46 *Let $q \in C^2([0,t_0]\times\mathbb{R}^d)$ for which $L(q) \geq 0$. Assume the existence of q_0 and $r_0 > 0$ such that*

$$\begin{aligned} &|q(t,x)| \leq q_0 && \text{on } [0,t_0]\times\mathbb{R}^d\ , \\ &q(t_0,x) = 0 && \text{if } |x| \geq r_0\ . \end{aligned}$$

Then for $0 \leq t \leq t_0$, $|x| \geq r_0 + K(t_0 - t)$ we have the estimate

$$q(t,x) \leq q_0 \exp\left(\frac{1}{\varepsilon}(K(t_0-t) + r_0 - |x|)\right)\ . \tag{4.47}$$

Proof : Let us denote the right-hand side of (4.47) by q_ε. Then we have for all $(t,x) \in [0,t_0]\times\mathbb{R}^d$,

$$\begin{aligned} L(q_\varepsilon) &= q_0 \exp\left(\frac{1}{\varepsilon}(K(t_0-t)+r_0-|x|)\right) \times \\ &\qquad \times \left[-\frac{K}{\varepsilon} - a_i \frac{x_i}{|x|}\frac{1}{\varepsilon} + \varepsilon\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\Delta|x|\right)\right] \\ &\leq q_0 \exp\left(\frac{1}{\varepsilon}(K(t_0-t)+r_0-|x|)\right) \times \\ &\qquad \times \left[\frac{-K+1}{\varepsilon} - \frac{1}{\varepsilon} + \frac{K-1}{\varepsilon} + \frac{1}{\varepsilon} - \Delta|x|)\right] \\ &\leq 0 \end{aligned}$$

and

$$\begin{aligned} &q_\varepsilon(t,x) = q_0 \quad \forall\,|x| = r_0 + K(t_0-t) \\ &q_\varepsilon(t_0,x) \geq 0 \quad \forall\,|x| \geq r_0\ . \end{aligned}$$

Now the statement of the lemma follows from the usual parabolic maximum principle (cf. PROTTER AND WEINBERGER [1967]). ■

In the following theorem we keep $\varepsilon > 0$ fixed and consider the solution u^ε to the parabolic perturbation (1.3) of the problem (1.2) under the assumptions $\mathbf{f} \in C^1(\mathbb{R})^d$, $u_0 \in L^\infty(\mathbb{R}^d)$ only.

Theorem 4.48 *Let $u_0 \in L^\infty(\mathbb{R}^d)$ and let $\mathbf{f} \in C^1(\mathbb{R})^d$. Then there exists a solution $u^\varepsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ of (1.3), such that*

$$\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)} \quad \text{for a.a. } t > 0 \tag{4.49}$$

and

$$\int_0^\infty \int_{\mathbb{R}^d} u^\varepsilon \frac{\partial \varphi}{\partial t} + \varepsilon u^\varepsilon \Delta\varphi + f(u^\varepsilon)\nabla\varphi \, dx\, dt + \int_{\mathbb{R}^d} u_0(x)\varphi(0,x)\, dx = 0 \tag{4.50}$$

for all $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$.

PROOF : For simplicity we drop the index ε. Let $n \in \mathbb{N}$ and let χ_n be the characteristic function of the ball $B_n(0) \subset \mathbb{R}^d$. We regularize the initial data as follows:

$$u_0^n \equiv (u_0\, \chi_n) * \rho_{1/n} \ .$$

From Theorem 2.9 and Lemma 2.16 we now obtain the existence of some $u^n \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d) \cap L^2(0,T; W^{2,2}(\mathbb{R}^d)) \cap C(0,T; W^{1,2}(\mathbb{R}^d))$ with $\frac{\partial u^n}{\partial t} \in L^2((0,T) \times \mathbb{R}^d)$ for all $T > 0$, solving the problem

$$\begin{aligned} \frac{\partial u^n}{\partial t} - \varepsilon \Delta u^n + \operatorname{div} \mathbf{f}(u^n) &= 0\,, \\ u^n(0,\cdot) &= u_0^n\,. \end{aligned} \tag{4.51}$$

We denote $w(t,x) \equiv u^n(t,x) - u^k(t,x)$, $k, n \in \mathbb{N}$. We see that $w \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$. For w we have the equation

$$\frac{\partial w}{\partial t} - \varepsilon \Delta w + \operatorname{div}(\mathbf{a}\, w) = 0 \ , \tag{4.52}$$

where

$$a_i(t,x) = \int_0^1 f_i'(u^k(t,x) + \alpha(u^n(t,x) - u^k(t,x)))\, d\alpha \ .$$

We multiply (4.52) by $g \in C^1(\mathbb{R}; C^2_C(\mathbb{R}^d))$ and after integration over $\mathbb{R}^d$ and between 0 and $t_0 > 0$ we obtain (see (4.45))

$$
\begin{aligned}
\int_{\mathbb{R}^d} w(t_0) g(t_0)\, dx - \int_0^{t_0} \int_{\mathbb{R}^d} L(g) w \, dx\, dt& \\
= \int_{\mathbb{R}^d} w(0) g(0)\, dx\,.&
\end{aligned} \tag{4.53}
$$

Now, we will construct a particular test function g. Let $r > 1$ and $0 < h \ll 1$ be arbitrary. Put

$$
\beta(x) \equiv \begin{cases} \operatorname{sgn} w(t_0, x) & |x| \le r - h\,, \\ 0 & |x| > r - h\,, \end{cases}
$$

and $\beta_h \equiv \beta * \rho_h$. Let q_h be the solution of

$$
\begin{aligned}
L(q_h) &= 0 \quad \text{on } (0, t_0) \times \mathbb{R}^d\,, \\
q_h(t_0, x) &= \beta_h(x)\,.
\end{aligned} \tag{4.54}
$$

Let $\widetilde{\rho}$ be the mollifier in $\mathbb{R}$. Set

$$
\eta_m(\lambda) \equiv 1 - \int_{-\infty}^{\lambda} \widetilde{\rho}(s - m)\, ds
$$

for $m \in \mathbb{N}$, and use

$$
g(t, x) \equiv q_h(t, x) \eta_m(|x|)
$$

as a test function in (4.53). After partial integration, using (4.54), we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^d} w(t_0) q_h(t_0) \eta_m \, dx = - \int_0^{t_0} \int_{\mathbb{R}^d} \Big[\widetilde{\rho}(|x| - m) \frac{x_i}{|x|} a_i w + \varepsilon \Delta \eta_m w& \\
- 2\varepsilon \frac{\partial w}{\partial x_i} \frac{x_i}{|x|} \widetilde{\rho}(|x| - m) \Big] q_h \, dx\, dt& \\
+ \int_{\mathbb{R}^d} w(0) q_h(0) \eta_m \, dx\,.&
\end{aligned} \tag{4.55}
$$

By the usual maximum principle applied to equation (4.54) we get that $|q_h(t, x)| \le 1$ for all $(t, x) \in [0, t_0] \times \mathbb{R}^d$. Moreover, for $|x| > r + K(t_0 - t)$ we have from Lemma 4.46 that

$$
|q_h(t, x)| \le e^{\frac{1}{\varepsilon}(K(t_0 - t) + r - |x|)} \le c\, e^{-|x|}\,,
$$

where c is independent of $\varepsilon < 1$. Altogether we get

$$|q_h(t,x)| \le c_0 e^{-|x|} \quad \forall\,(t,x) \in [0,t_0] \times \mathbb{R}^d \ . \tag{4.56}$$

Now, letting m tend to infinity in (4.55) and using $\widetilde{\rho}(|x| - m) \to 0, \eta_m \to 1$, as $m \to \infty$, we get

$$\begin{aligned} \int_{\mathbb{R}^d} w(t_0) q_h(t_0)\, dx &= \int_{\mathbb{R}^d} w(0) q_h(0)\, dx \\ &\le c_0 \int_{\mathbb{R}^d} e^{-|x|} |w(0)|\, dx\,. \end{aligned} \tag{4.57}$$

Letting $h \to 0+$ and using (4.56) we find:

$$\int_{B_r} |w(t_0)|\, dx \le c_0 \int_{\mathbb{R}^d} e^{-|x|} |w(0)|\, dx \ . \tag{4.58}$$

From the definition of u_0^n it is clear that

$$u_0^n \to u_0 \quad \text{a.e. in } \mathbb{R}^d$$

which gives, together with (4.58), that for all $r > 1$ and $t_0 \in [0,T]$, $u^n(t_0,x)$ forms a Cauchy sequence in $L^1(\mathbb{R}^d)$. Therefore,

$$u^n(t,x) \to u(t,x) \quad \text{a.e. in } \mathbb{R}^+ \times \mathbb{R}^d. \tag{4.59}$$

Moreover, Theorem 2.9 gives

$$\|u^n(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0^n\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}, \tag{4.60}$$

which implies (4.49). If we now multiply (4.51) by some test function $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$, we get

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^d} u^n \frac{\partial \varphi}{\partial t} + \varepsilon u^n \Delta\varphi + f(u^n) \nabla\varphi\, dx\, dt \\ = - \int_{\mathbb{R}^d} u_0^n(x) \varphi(0,x)\, dx \ . \end{aligned} \tag{4.61}$$

The limiting process in (4.61) based on (4.59) and (4.60) uses the same ideas as in the proof of Theorem 4.22 and is therefore left to the reader. ■

The following theorem gives us the desired almost everywhere convergence of the sequence u^ε under the assumption $u_0 \in L^\infty(\mathbb{R}^d)$ only.

Theorem 4.62 *Let $u_0 \in L^\infty(\mathbb{R}^d)$, $\mathbf{f} \in C^1(\mathbb{R})^d$ and let $u^\varepsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ be the solutions of (1.3) constructed in Theorem 4.48. Then there is a subsequence still denoted u^ε and a function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ such that*

$$u^\varepsilon \to u \quad \text{a.e. in } \mathbb{R}^+ \times \mathbb{R}^d \qquad \text{as } \varepsilon \to 0+ \ . \tag{4.63}$$

Proof : Let $T > 0$ and $R > 1$ be arbitrary. In order to prove (4.63), we will use the compactness criterion for $\{u^\varepsilon\}$ on $L^1((0,T)\times B_R(0))$. By (4.49) we know that the set $\{u^\varepsilon\}$ is uniformly bounded in $L^1((0,T)\times B_R(0))$. It remains to show that for any $m \in \mathbb{N}$ there exists a $\delta > 0$ such that for all $|z| \le \delta$, $|s| \le \delta$,

$$\int_{B_R} |u^\varepsilon(t+s, x+z) - u^\varepsilon(t,x)|\, dx \le \frac{1}{m} \tag{4.64}$$

independently of ε and uniformly in $t \in [0,T]$. Due to (4.58) we obtain

$$\int_{B_R} |u^k(t,x) - u^n(t,x)|\, dx \le c_0 \int_{\mathbb{R}^d} e^{-|x|} |u_0^k(x) - u_0^n(x)|\, dx\,.$$

Letting $k \to \infty$ and using (4.59) we get

$$\int_{B_R} |u^\varepsilon(t,x) - u^n(t,x)|\, dx \le c_0 \int_{\mathbb{R}^d} e^{-|x|} |u_0(x) - u_0^n(x)|\, dx\,. \tag{4.65}$$

It is clear that the right-hand side of (4.65) tends to zero for $n \to \infty$. Now, for

$$w(t,x) \equiv u^n(t, x+z) - u^n(t,x)$$

one obtains, in the same way as in the proof of Theorem 4.48, that

$$\int_{B_R} |u^n(t,x+z) - u^n(t,x)|\, dx \le c_0 \int_{\mathbb{R}^d} e^{-|x|} |u_0^n(x+z) - u_0^n(x)|\, dx \ . \tag{4.66}$$

Altogether we have

$$\begin{aligned}\int_{B_R} & |u^\varepsilon(t,x+z) - u^\varepsilon(t,x)|\,dx \\ & \le c_0 \int_{\mathbb{R}^d} \Big(|u_0(x+z) - u_0^n(x+z)|\Big)\, e^{-|x|}\,dx \\ & \quad + \int_{\mathbb{R}^d} \Big(|u_0^n(x+z) - u_0^n(x)|\Big)\, e^{-|x|}\,dx \\ & \quad + \int_{\mathbb{R}^d} \Big(|u_0^n(x) - u_0(x)|\Big)\, e^{-|x|}\,dx\,.\end{aligned}$$

Note that due to (4.65) there is an n_0 such that for all $n \ge n_0$ the first and the third integral can be made less than $\frac{1}{6m}$. In order to treat the second one, we choose an R_0 such that

$$\begin{aligned}c_0 & \int_{\mathbb{R}^d \setminus B_{R_0}} e^{-|x|} |u_0^n(x+z) - u_0^n(x)|\,dx \\ & \le 2c_0 \|u_0\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d \setminus B_{R_0}} e^{-|x|}\,dx \le \frac{1}{12m}\ .\end{aligned}$$

For this R_0, n_0 it follows from (4.43) that

$$c_0 \int_{B_{R_0}} e^{-|x|} |u_0^n(x+z) - u_0^n(x)|\,dx \le c_0\, \omega_{R_0+1/n_0}(|z|)\,.$$

Now we choose a δ_1 such that $|z| \le \delta_1$ implies that the right-hand side of the last inequality is less than $\frac{1}{12m}$. Therefore we have proved that for all $m \in \mathbb{N}$ there is a δ_1 such that for $|z| \le \delta_1$,

$$\int_{B_R} |u^\varepsilon(t,x+z) - u^\varepsilon(t,x)|\,dx \le \frac{1}{2m} \tag{4.67}$$

uniformly in $t \in [0,T]$ and independently of ε.

Let $t_0 < T$ and $s \ll 1$ be fixed. Denote

$$v(x) \equiv u^n(t_0+s,x) - u^n(t_0,x)$$

and put

$$\beta(x) \equiv \begin{cases} \operatorname{sgn} v\,, & |x| \le R\,, \\ 0\,, & |x| > R\,. \end{cases}$$

Clearly, the mollification $\beta_h \equiv \beta * \rho_h$ satisfies

$$\beta_h \le 1\ , \quad |\nabla \beta_h| \le \frac{c}{h}\ , \quad |\Delta \beta_h| \le \frac{c}{h^2}\ .$$

Multiplying (4.51) by β_h we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} v\beta_h \, dx &= \int_{\mathbb{R}^d} \big(u^n(t_0+s,x) - u^n(t_0,x)\big)\beta_h(x)\, dx \\ &= \int_{t_0}^{t_0+s} \int_{\mathbb{R}^d} \varepsilon u^n \Delta\beta_h + \mathbf{f}(u^n)\nabla\beta_h \, dx\, dt \\ &\le s\, c_1\big(\|u_0\|_{L^\infty(\mathbb{R}^d)}\big)\left(\frac{1}{h} + \frac{1}{h^2}\right).\end{aligned}$$

For $|x| < R$ we have $v\beta_h = v\,(\operatorname{sgn} v)_h$ and it follows from (4.44) that

$$\begin{aligned}\int_{B_R} |v|\, dx \le \int_{B_R} v\beta_h\, dx + 2\omega_R(h) &\le \int_{\mathbb{R}^d} v\beta_h\, dx + 2\omega_R(h) \\ &\le s\, c_1\left(\frac{1}{h} + \frac{1}{h^2}\right) + 2\omega_R(h)\,,\end{aligned} \tag{4.68}$$

where

$$\begin{aligned}\omega_R(h) = \sup_{|z|\le h} \int_{B_R} \Big| &\big(u^n(t_0+s, x+z) - u^n(t_0, x+z)\big) \\ &- \big(u^n(t_0+s,x) - u^n(t_0,x)\big)\Big|\, dx\,.\end{aligned}$$

According to (4.66),

$$\omega_R(h) \le 2\, c_0 \sup_{|z|\le h} \int_{\mathbb{R}^d} |u_0^n(x+z) - u_0^n(x)| e^{-|x|}\, dx\,,$$

and therefore $\omega_R(h)$ can be made arbitrarily small for small enough h uniformly in t_0 and s. Thus we have

$$\begin{aligned}\int_{B_R} &|u^\varepsilon(t_0+s,x) - u^\varepsilon(t_0,x)|\, dx \\ &\le \int_{B_R} |u^\varepsilon(t_0+s,x) - u^n(t_0+s,x)|\, dx \\ &\quad + \int_{B_R} |u^n(t_0,x) - u^\varepsilon(t_0,x)|\, dx \\ &\quad + \int_{B_R} |u^n(t_0+s,x) - u^n(t_0,x)|\, dx\,.\end{aligned} \tag{4.69}$$

The first two integrals can be made smaller than $\frac{1}{6m}$ due to (4.65) for all $n \ge n_0$. For this n_0 we find an h_0 such that $2\omega_R(h_0) \le \frac{1}{12m}$.

Further, for this h_0 we find a δ_2 such that for all $|s| < \delta_2$,

$$s\,c_0\Big(\frac{1}{h_0} + \frac{1}{h_0^2}\Big) \le \frac{1}{12m}\,.$$

Therefore we get that for all $|s| \le \delta_2$,

$$\int_{B_R} |u^\varepsilon(t_0+s,x) - u^\varepsilon(t_0,x)|\,dx \le \frac{1}{2m} \tag{4.70}$$

uniformly in ε and t_0. Now for $\delta = \min(\delta_1,\delta_2)$ we immediately get (4.64) from (4.67) and (4.70). Thus, $\{u^\varepsilon\}$ is uniformly 1-mean equicontinuous and a uniformly bounded set in $L^1((0,T)\times B_R(0))$, and (4.63) on $(0,T)\times B_R(0)$ follows at least for a subsequence. Now, since T and R were arbitrary, (4.63) follows. Finally, the limit function u belongs to $L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ due to (4.49), (4.63). ■

Theorem 4.71 *Let $u_0 \in L^\infty(\mathbb{R}^d)$ and $\mathbf{f} \in C^1(\mathbb{R})^d$. Then the problem (1.2) has an entropy solution $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ such that, for almost all $t > 0$,*

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}\,. \tag{4.72}$$

Proof : Theorem 4.48 and Theorem 4.62 imply the existence of weak solutions $u^\varepsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ of the problem (1.3) and a function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ such that

$$\begin{aligned} &\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)} \qquad \text{for a.a. } t \in \mathbb{R}^+\,,\\ &u^\varepsilon(t,x) \to u(t,x) \qquad \text{for a.a. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d\,. \end{aligned}$$

Using the same argument as in the proof of Theorem 4.22 together with Remark 4.40, we get that u is an entropy weak solution to (1.2). ■

2.5 Uniqueness of the entropy solution

The aim of this section is to prove the following uniqueness theorem:

Theorem 5.1 *Let $u_0 \in L^\infty(\mathbb{R}^d)$ and $\mathbf{f} \in C^1(\mathbb{R}^d)$. Then there exists exactly one entropy solution to the problem (1.2).*

We will prove this theorem in several steps. Firstly, we will show that the validity of the entropy inequality (3.26) for all smooth entropy-entropy flux pairs η, $\mathbf{q}$ is equivalent to the validity of (3.26) for the following family of non-smooth entropy-entropy flux pairs:

$$\begin{aligned} \eta(u) &= |u-k|\,, \\ q_j(u) &= \operatorname{sgn}(u-k)(f_j(u)-f_j(k))\,, \end{aligned} \qquad k \in \mathbb{R}, \tag{5.2}$$

see Lemma 5.3 and Remark 5.7. In Lemma 5.9 the local version of entropy inequality for these non-smooth entropy-entropy flux pairs is derived.

Secondly, having in mind the aim to prove the uniqueness, we will find an inequality (5.22) for the difference of two entropy solutions u, v, corresponding to the initial data u_0, v_0, respectively. This and Lemma 5.12 will help us to obtain in Theorem 5.38 the crucial L^1-contraction inequality

$$\int_{B_R} |u(t,x)-v(t,x)|\,dx \leq \int_{B_{R+Mt}} |u_0(x)-v_0(x)|\,dx\,,$$

holding for all B_R, $R>0$ and for almost every $t>0$. Here, M denotes the constant of Lipschitz continuity of $\mathbf{f}$. The uniqueness result will then follow almost immediately using $u_0 = v_0$ in the last inequality and can be found at the very end of this section.

Let us start with the following lemma.

Lemma 5.3 *The entropy solution u of (1.2) satisfies for all $k \in \mathbb{R}$ and for all $\varphi \geq 0$, $\varphi \in C^1_C(\mathbb{R}^+ \times \mathbb{R}^d)$, the inequality*

$$\int_0^\infty \int_{\mathbb{R}^d} |u-k| \frac{\partial \varphi}{\partial t} + \operatorname{sgn}(u-k)(f_j(u)-f_j(k)) \frac{\partial \varphi}{\partial x_j}\,dx\,dt \geq 0\,, \tag{5.4}$$

and for all $\varphi \geq 0$, $\varphi \in C^1_C(\mathbb{R} \times \mathbb{R}^d)$ the inequality

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^d} \big(|u-k| - |u_0-k|\big) \frac{\partial \varphi}{\partial t} & \\ + \operatorname{sgn}(u-k)\big(f_j(u)-f_j(k)\big) \frac{\partial \varphi}{\partial x_j}\,dx\,dt &\geq 0\,. \end{aligned} \tag{5.5}$$

PROOF : Let $G \in C^\infty(\mathbb{R})$ be such that

$$\begin{aligned} G(x) &= |x| \quad \forall\,|x| \geq 1\,, \\ G'(0) &= 0\,, \quad G'' \geq 0\,. \end{aligned}$$

For $k \in \mathbb{R}$ fixed we put

$$G_\varepsilon(x) \equiv \varepsilon\, G\Big(\frac{x-k}{\varepsilon}\Big)$$

which immediately implies that

$$G_\varepsilon(x) \to |x-k| \qquad \text{as } \varepsilon \to 0+ \ .$$

The function G_ε is convex and smooth and therefore $(G_\varepsilon, \mathbf{q}_\varepsilon)$ is an entropy-entropy flux pair provided that $\mathbf{q}_\varepsilon \equiv (q_{1\varepsilon}, \dots, q_{d\varepsilon})$ is defined by

$$q_{j\varepsilon}(u) \equiv \int_k^u G'_\varepsilon(v) f'_j(v)\, dv \, .$$

Since u is an entropy solution of (1.2) in the sense of distributions on $\mathbb{R}^+ \times \mathbb{R}^d$, we have for all $\varphi \geq 0$, $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$,

$$\int_0^\infty \int_{\mathbb{R}^d} (G_\varepsilon(u) - G_\varepsilon(u_0)) \frac{\partial \varphi}{\partial t} + q_{j\varepsilon} \frac{\partial \varphi}{\partial x_j}\, dx\, dt \geq 0 \, . \tag{5.6}$$

It is a simple matter to check that for $\varepsilon \to 0+$ we get the pointwise convergences

$$\begin{aligned} G'_\varepsilon(v) &\to \operatorname{sgn}(v-k)\, , \\ \mathbf{q}_\varepsilon(v) &\to \mathbf{f}(k) - \mathbf{f}(v) \qquad \text{if } v \geq k\, , \\ \mathbf{q}_\varepsilon(v) &\to \mathbf{f}(v) - \mathbf{f}(k) \qquad \text{if } v \leq k\, . \end{aligned}$$

Since φ has compact support, we obtain (5.5) from the limiting process as $\varepsilon \to 0+$ in (5.6). Note that for $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$ the term obtaining $G_\varepsilon(u_0)$ vanishes, which immediately gives (5.4). ■

Remark 5.7 Let us notice that a weak solution u satisfying (5.5) for all $k \in \mathbb{R}$ is an entropy solution. This follows from the fact that every convex function belongs to the convex hull of the set of all affine functions and all functions of the form

$$x \mapsto |x-k|\, , \quad k \in \mathbb{R}\ .$$

For the proof of uniqueness we will need a local version of (5.5), valid on the characteristic 'cone' of equation (1.2).

Let $M \in \mathbb{R}^+$ be fixed.[§] For $R > 0$, $t_0 > 0$ we define

$$\begin{aligned} C_R &\equiv \{(t,x) \in [0,\infty) \times \mathbb{R}^d, |x| + Mt \leq R\}\, , \\ C_{R,t_0} &\equiv \{(t,x) \in C_R\, , t < t_0\}. \end{aligned} \tag{5.8}$$

[§] In what follows, M will usually denote the constant of Lipschitz continuity of $\mathbf{f}$, restricted to an appropriate ball, cf. (2.30).

Lemma 5.9 *Let $T > 0$ and $k \in \mathbb{R}$. Let $u_0 \in L^\infty(\mathbb{R}^d)$, $\mathbf{f} \in C^1(\mathbb{R})^d$ and M be given by (2.30). Then for all $t_0 \in (0,T), R > 0$, the entropy solution u of (1.2) satisfies*

$$\begin{aligned}\int_{B_{R-Mt_0}} |u(t_0) - k|\varphi(t_0)\,dx \le \iint_{C_{R,t_0}} |u-k|\frac{\partial\varphi}{\partial t}\,dx\,dt \\ + \iint_{C_{R,t_0}} \operatorname{sgn}(u-k)(f_j(u) - f_j(k))\frac{\partial\varphi}{\partial x_j}\,dx\,dt \\ + \int_{B_R} |u_0 - k|\varphi(0)\,dx\end{aligned} \tag{5.10}$$

for all $\varphi \ge 0$, $\varphi \in \mathcal{D}(C_R)$.

PROOF : Let ρ_h be the mollifier defined for $p = 1$ in (4.1). Put

$$\psi_h(\lambda) \equiv 1 - \int_{-\infty}^{\lambda} \rho_h(s)\,ds\,.$$

Hence,

$$\lim_{h\to 0+} \psi_h(\lambda) = \begin{cases} 0\,, & \lambda > 0\,, \\ 1\,, & \lambda < 0\,. \end{cases}$$

For given $\varphi \ge 0$, $\varphi \in \mathcal{D}(C_R)$, we define

$$\varphi_h(t,x) \equiv \varphi(t,x)\,\psi_h(t-t_0)\ .$$

Therefore, we have

$$\begin{aligned}\frac{\partial\varphi_h}{\partial t}(t,x) &= \frac{\partial\varphi}{\partial t}(t,x)\,\psi_h(t-t_0) - \varphi(t,x)\,\rho\Big(\frac{t-t_0}{h}\Big)\frac{1}{h}\,, \\ \frac{\partial\varphi_h}{\partial x_j}(t,x) &= \frac{\partial\varphi}{\partial x_j}(t,x)\,\psi_h(t-t_0)\,, \\ \varphi_h(0,x) &= \varphi(0,x) \quad \text{for } h \le t_0\,.\end{aligned} \tag{5.11}$$

If we use φ_h instead of φ in (5.5), we get

$$\begin{aligned}&\iint_{C_R} |u-k|\frac{\partial\varphi}{\partial t}\,\psi_h(t-t_0)\,dx\,dt - \iint_{C_R} |u_0-k|\frac{\partial\varphi_h}{\partial t}\,dx\,dt \\ &- \iint_{C_R} |u-k|\varphi(t,x)\,\rho\Big(\frac{t-t_0}{h}\Big)\frac{1}{h}\,dx\,dt \\ &+ \iint_{C_R} \operatorname{sgn}(u-k)(f_j(u)-f_j(k))\frac{\partial\varphi}{\partial x_j}\,\psi_h(t-t_0)\,dx\,dt \ge 0\,.\end{aligned}$$

Note that

$$-\iint_{C_R} |u_0 - k| \frac{\partial \varphi_h}{\partial t}\, dx\, dt = \int_{B_R} |u_0 - k| \varphi(0)\, dx \ .$$

Letting $h \to 0+$ and using the fact that $\frac{1}{h}\rho\left(\frac{t-t_0}{h}\right)$ is an approximation of the Dirac distribution, we obtain

$$\iint_{C_{R,t_0}} |u-k|\frac{\partial \varphi}{\partial t} + \operatorname{sgn}(u-k)(f_j(u) - f_j(k))\frac{\partial \varphi}{\partial x_j}\, dx\, dt$$
$$+ \int_{B_R} |u_0 - k|\varphi(0)\, dx \geq \int_{B_{R-Mt_0}} |u(t_0) - k|\varphi(t_0)\, dx \ ,$$

which is (5.10). ∎

Let us now give a further characterization in which the initial condition is fulfilled.

Lemma 5.12 *Let $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ be an entropy solution to the problem (1.2) corresponding to initial data $u_0 \in L^\infty(\mathbb{R}^d)$, $\mathbf{f} \in C^1(\mathbb{R})^d$. Then for all $R > 0$,*

$$\lim_{t\to 0+} \int_{B_{R-Mt}} |u(t) - u_0|\, dx = 0 \ ,$$

where M is given by (2.30).

Proof : Let $x_0 \in \mathbb{R}^d$, $\ell > 0$ be such that $|x_0| + 2\ell < R$. Further let $\omega \in C^\infty(\mathbb{R})$ be such that

$$\begin{aligned} &\omega' \leq 0\,, \\ &\omega = 1 \text{ on } (-\infty, \ell]\,, \\ &\omega = 0 \text{ on } [2\ell, \infty)\,. \end{aligned} \tag{5.13}$$

Now we define $\varphi \in C(C_R)$, $\varphi \geq 0$ by

$$\varphi(t,x) \equiv \omega(Mt + |x - x_0|) \ , \tag{5.14}$$

where M is given by (2.30). Thus, we have

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(t,x) &= \omega'(Mt + |x-x_0|)M \ , \\ \frac{\partial \varphi}{\partial x_i}(t,x) &= \omega'(Mt + |x-x_0|)\frac{(x-x_0)_i}{|x-x_0|} \end{aligned}$$

and therefore it holds

$$\frac{\partial\varphi}{\partial t}(t,x) + M|\nabla\varphi(t,x)| = 0\,. \tag{5.15}$$

Due to (5.10) and (5.15) we obtain for $k = u_0(x_0)$

$$\begin{aligned}
&\int_{B_{R-Mt_0}} |u(t_0) - u_0(x_0)|\varphi(t_0)\,dx \\
&\qquad \le \int_{B_R} |u_0 - u_0(x_0)|\varphi(0)\,dx + \iint_{C_{R,t_0}} |u - u_0(x_0)|\frac{\partial\varphi}{\partial t} \\
&\qquad\quad + \operatorname{sgn}(u - u_0(x_0))(f_j(u) - f_j(u_0(x_0)))\frac{\partial\varphi}{\partial x_j}\,dx\,dt \\
&\qquad \le \int_{B_{2\ell}(x_0)} |u_0 - u_0(x_0)|\,dx \\
&\qquad\quad + \iint_{C_{R,t_0}} |u - u_0(x_0)|\frac{\partial\varphi}{\partial t} + M|u - u_0(x_0)|\,|\nabla\varphi|\,dx\,dt \\
&\qquad = \int_{B_{2\ell}(x_0)} |u_0(x) - u_0(x_0)|\,dx\,.
\end{aligned} \tag{5.16}$$

But for the left-hand side of (5.16) we have

$$\begin{aligned}
&\int_{B_{R-Mt_0}} |u(t_0) - u_0(x_0)|\varphi(t_0)\,dx \\
&\qquad \ge \int_{B_{\ell-Mt_0}(x_0)} |u(t_0) - u_0(x_0)|\,dx\ ,
\end{aligned} \tag{5.17}$$

where we used (5.13). Thus,

$$\begin{aligned}
&\int_{B_{\ell-Mt_0}(x_0)} |u(t_0,x) - u_0(x)|\,dx \\
&\qquad \le \int_{B_{\ell-Mt_0}(x_0)} |u(t_0,x) - u_0(x_0)|\,dx \\
&\qquad\quad + \int_{B_{\ell-Mt_0}(x_0)} |u_0(x) - u_0(x_0)|\,dx \\
&\qquad \le 2\int_{B_{2\ell}(x_0)} |u_0(x) - u_0(x_0)|\,dx\,.
\end{aligned} \tag{5.18}$$

We will integrate the last inequality over $B_{R-2\ell}(0)$ with respect to x_0. Then Fubini's theorem applied to the left-hand side implies

$$\begin{aligned}
&\int_{B_{R-2\ell}(0)} \int_{B_{\ell-Mt_0}(x_0)} |u(t_0,x)-u_0(x)|\,dx\,dx_0 \\
&= \int_{B_{\ell-Mt_0}(0)} \int_{B_{R-2\ell}(x)} |u(t_0,x_0)-u_0(x_0)|\,dx_0\,dx \\
&\geq \int_{B_{\ell-Mt_0}(0)} \int_{B_{R-3\ell+Mt_0}(0)} |u(t_0,x_0)-u_0(x_0)|\,dx_0\,dx \\
&= \operatorname{meas}(B_{\ell-Mt_0}(0)) \int_{B_{R-3\ell+Mt_0}(0)} |u(t_0)-u_0|\,dx \ ,
\end{aligned} \tag{5.19}$$

where we used the fact that $|x| \leq \ell - Mt_0$, $|x_0 - x| \leq R - 2\ell$ implies $|x_0| \leq R - 3\ell + Mt_0$ for all x. Further we have

$$\begin{aligned}
&\int_{B_{R-2\ell}(0)} \int_{B_{2\ell}(x_0)} |u_0(x)-u_0(x_0)|\,dx\,dx_0 \\
&= \int_{B_{2\ell}(0)} \int_{B_{R-2\ell}(0)} |u_0(x_0)-u_0(x_0+x)|\,dx_0\,dx \\
&\leq \operatorname{meas}(B_{2\ell}(0)) \sup_{|z|\leq 2\ell} \int_{B_{R-2\ell}(0)} |u_0(x)-u_0(x+z)|\,dx
\end{aligned} \tag{5.20}$$

and therefore we obtain

$$\frac{\widehat{c}}{2} \int_{B_{R-3\ell+Mt_0}} |u(t_0)-u_0|\,dx \leq \sup_{|z|\leq 2\ell} \int_{B_{R-2\ell}} |u_0(x)-u_0(x+z)|\,dx$$

for $\widehat{c} = \left(\frac{\ell - Mt_0}{2\ell}\right)^d$. Putting $\ell \equiv 2Mt_0$ we get

$$\begin{aligned}
&\int_{B_{R-5Mt_0}} |u(t_0)-u_0|\,dx \\
&\qquad \leq 2^{2d+1} \sup_{|z|\leq 4Mt_0} \int_{B_{R-4Mt_0}} |u_0(x)-u_0(x+z)|\,dx \ .
\end{aligned}$$

But this implies

$$\begin{aligned}\int_{B_{R-Mt_0}} & |u(t_0,x) - u_0(x)|\,dx \\ & \leq 2^{2d+1} \sup_{|z|\leq 4Mt_0} \int_{B_{R-4Mt_0}} |u_0(x) - u_0(x+z)|\,dx \\ & \quad + \int_{B_{R-Mt_0}\setminus B_{R-5Mt_0}} |u(t_0,x) - u_0(x)|\,dx \ .\end{aligned}$$

Note that the first integral converges to 0 for $t_0 \to 0+$ because of (4.41) and the second integral converges to 0, because the integrand is bounded in L^∞ and $\mathrm{meas}(B_{R-Mt_0} \setminus B_{R-5Mt_0}) \to 0$ for $t_0 \to 0+$.

■

The following lemma is of great importance for the proof of uniqueness of an entropy solution.

Lemma 5.21 *Let $\mathbf{f} \in C^1(\mathbb{R})^d$. Let $u, v \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ be two entropy solutions of (1.2) corresponding to the initial data $u_0, v_0 \in L^\infty(\mathbb{R}^d)$, respectively. Then we have*

$$\frac{\partial}{\partial t}|u - v| + \mathrm{div}\,\big(\mathrm{sgn}(u-v)(\mathbf{f}(u) - \mathbf{f}(v))\big) \leq 0 \qquad (5.22)$$

in the sense of distributions on $\mathbb{R}^+ \times \mathbb{R}^d$.

PROOF : From (5.4) we get for $k, \ell \in \mathbb{R}$ and all $\varphi = \varphi(s,x), \psi = \psi(t,y) \in \mathcal{D}((0,\infty) \times \mathbb{R}^d)$, $\psi, \varphi \geq 0$,

$$\int_0^\infty \int_{\mathbb{R}^d} |u-k|\frac{\partial \varphi}{\partial s} + \mathrm{sgn}(u-k)(f_j(u) - f_j(k))\frac{\partial \varphi}{\partial x_j}\,dx\,ds \geq 0 \qquad (5.23)$$

and

$$\int_0^\infty \int_{\mathbb{R}^d} |v-\ell|\frac{\partial \psi}{\partial t} + \mathrm{sgn}(v-\ell)(f_j(v) - f_j(\ell))\frac{\partial \psi}{\partial y_j}\,dy\,dt \geq 0 \ . \qquad (5.24)$$

Put

$$k = v(t,y) \ , \quad \ell = u(s,x) \qquad (5.25)$$

and choose φ, ψ in the following way. Let $\phi = \phi(\theta, \lambda) \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$,

$\phi \geq 0$, and let $\rho_\varepsilon \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$ be a symmetric mollifier. Put

$$\begin{aligned} \varphi(s,x) = \varphi_{t,y}(s,x) = \psi(t,y) = \psi_{s,x}(t,y) \\ \equiv \phi\Big(\frac{s+t}{2}, \frac{x+y}{2}\Big)\rho_\varepsilon\Big(\frac{s-t}{2}, \frac{x-y}{2}\Big) \ . \end{aligned} \tag{5.26}$$

For this special choice of φ, ψ we integrate (5.23) with respect to t, y and (5.24) with respect to s, x and add up the results. Thus we obtain

$$\begin{aligned} 0 \leq \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & |u(s,x) - v(t,y)| \\ & \times \left(\frac{\partial \varphi_{t,y}}{\partial s}(s,x) + \frac{\partial \psi_{s,x}}{\partial t}(t,y)\right) \\ + \operatorname{sgn}\big(u(s,x) - v(t,y)\big) & \big(f_j(u(s,x)) - f_j(v(t,y))\big) \\ & \times \left(\frac{\partial \varphi_{t,y}}{\partial x_j}(s,x) + \frac{\partial \psi_{s,x}}{\partial y_j}(t,y)\right) dx\, dy\, ds\, dt \ . \end{aligned} \tag{5.27}$$

From (5.26) it follows that

$$\begin{aligned} \frac{\partial}{\partial s}\varphi_{t,y}(s,x) + \frac{\partial}{\partial t}\psi_{s,x}(t,y) & \\ &= \frac{\partial \phi}{\partial \theta}\left(\frac{s+t}{2}, \frac{x+y}{2}\right)\rho_\varepsilon\left(\frac{s-t}{2}, \frac{x-y}{2}\right) , \\ \frac{\partial}{\partial x_j}\varphi_{t,y}(s,x) + \frac{\partial}{\partial y_j}\psi_{s,x}(t,y) & \\ &= \frac{\partial \phi}{\partial \lambda_j}\left(\frac{s+t}{2}, \frac{x+y}{2}\right)\rho_\varepsilon\left(\frac{s-t}{2}, \frac{x-y}{2}\right) \ . \end{aligned}$$

Let us introduce new coordinates

$$\begin{aligned} \xi &= \frac{x+y}{2} , & \eta &= \frac{x-y}{2} , \\ \sigma &= \frac{s+t}{2} , & \tau &= \frac{s-t}{2} , \end{aligned}$$

and therefore, we now integrate in (5.27) over

$$\Omega = \big\{(\sigma, \tau, \xi, \eta) \in \mathbb{R}^2 \times (\mathbb{R}^d)^2 \ , \ \sigma + \tau \geq 0 \ , \ \sigma - \tau \geq 0\big\} \ .$$

If we denote

$$\begin{aligned} G(\sigma,\tau,\xi,\eta) \equiv \big|u(\sigma+\tau,\xi+\eta) - v(\sigma-\tau,\xi-\eta)\big| \frac{\partial\phi}{\partial\theta}(\sigma,\xi) \\ + \operatorname{sgn}(u(\sigma+\tau,\xi+\eta) - v(\sigma-\tau,\xi-\eta)) \\ \times\big(f_j(u(\sigma+\tau,\xi+\eta)) - f_j(v(\sigma-\tau,\xi-\eta))\big)\frac{\partial}{\partial\lambda_j}\phi(\xi,\sigma)\,, \end{aligned} \tag{5.28}$$

we can write (5.27) as

$$J_\varepsilon \equiv \int_\Omega G(\sigma,\tau,\xi,\eta)\rho_\varepsilon(\tau,\eta)\,d\xi\,d\eta\,d\sigma\,d\tau \geq 0\ . \tag{5.29}$$

To prove (5.22), it is enough to show

$$J_\varepsilon \to J_0 \equiv \int_0^\infty \int_{\mathbb{R}^d} G(\sigma,0,\xi,0)\,d\xi\,d\sigma \tag{5.30}$$

for $\varepsilon \to 0+$. Indeed, if (5.30) holds, then one has for all $\psi \in \mathcal{D}((0,\infty)\times\mathbb{R}^d)$, $\psi \geq 0$,

$$J_0 = \int_0^\infty \int_{\mathbb{R}^d} |u-v|\frac{\partial\psi}{\partial\theta} + \operatorname{sgn}(u-v)(f_j(u)-f_j(v))\frac{\partial\psi}{\partial\lambda_j}\,d\xi\,d\sigma \geq 0\,,$$

which is exactly (5.22). Let us show that (5.30) is true. Denote

$$\chi(\sigma,\tau) = \begin{cases} 1 & \text{if } \sigma+\tau\geq 0,\ \sigma-\tau\geq 0\,, \\ 0 & \text{elsewhere.} \end{cases}$$

Then (5.29) can be rewritten as

$$J_\varepsilon = \int_{\mathbb{R}^{2d+2}} G(\sigma,\tau,\xi,\eta)\chi(\sigma,\tau)\rho_\varepsilon(\tau,\eta)\,d\xi\,d\eta\,d\sigma\,d\tau\,.$$

But $\rho_\varepsilon(\tau,\eta)$ is a mollifier (see (4.1)) and therefore we have

$$J_0 = \int_{\mathbb{R}^{2d+2}} G(\sigma,0,\xi,0)\chi(\sigma,0)\rho_\varepsilon(\tau,\eta)\,d\xi\,d\eta\,d\sigma\,d\tau\ .$$

Let us denote $K = \operatorname{supp}\phi$, $C_\varepsilon = \operatorname{supp}\rho_\varepsilon$. Note that

$$C_\varepsilon \subseteq \big\{(\tau,\eta)\ ,\ |\tau|\leq\varepsilon\ ,\ |\eta|\leq\varepsilon\big\}\ .$$

Thus, we have that $|J_\varepsilon - J_0|$ is less than or equal to

$$\begin{aligned} \int_K\bigg\{\int_{C_\varepsilon} |G(\sigma,\tau,\xi,\eta)\chi(\sigma,\tau) - G(\sigma,0,\xi,0)\chi(\sigma,0)| \\ \times\rho_\varepsilon(\tau,\eta)\,d\eta\,d\tau\bigg\}\,d\xi\,d\sigma \leq\ A_\varepsilon + B_\varepsilon\ , \end{aligned} \tag{5.31}$$

where A_ε and B_ε are defined by

$$\int_K \left\{ \int_{C_\varepsilon} |G(\sigma,\tau,\xi,\eta) - G(\sigma,0,\xi,0)| \rho_\varepsilon(\tau,\eta)\, d\eta\, d\tau \right\} d\xi\, d\sigma\,,$$

and

$$\int_K \left\{ |G(\sigma,0,\xi,0)| \int_{C_\varepsilon} |\chi(\sigma,\tau) - \chi(\sigma,0)| \rho_\varepsilon(\tau,\eta)\, d\eta\, d\tau \right\} d\xi\, d\sigma\,,$$

respectively.

Let us recall that, for $r_0 \equiv \max(\|u_0\|_\infty, \|v_0\|_\infty)$, $\mathbf{f}$ satisfies the Lipschitz condition

$$|\mathbf{f}(u) - \mathbf{f}(v)| \le M|u - v|\,, \qquad \forall\, |u|, |v| \le r_0\,.$$

Then we have for any $|u_i|, |v_i| \le r_0$, $i = 1, 2$, and $j = 1, \ldots, d$,

$$\begin{aligned} &|\operatorname{sgn}(u_1 - v_1)(f_j(u_1) - f_j(v_1)) - \operatorname{sgn}(u_2 - v_2)(f_j(u_2) - f_j(v_2))| \\ &\qquad \le M(|u_1 - u_2| + |v_1 - v_2|)\,. \end{aligned} \tag{5.32}$$

Indeed, defining

$$g(\zeta,\omega) \equiv \operatorname{sgn}(\zeta - \omega)(f_j(\zeta) - f_j(\omega))\,,$$

one gets for almost all $|\zeta|$, $|\omega| < r_0$

$$\begin{aligned} \frac{\partial}{\partial\zeta} g(\zeta,\omega) &= \operatorname{sgn}(\zeta - \omega)\, f_j'(\zeta)\,, \\ \frac{\partial}{\partial\omega} g(\zeta,\omega) &= -\operatorname{sgn}(\zeta - \omega)\, f_j'(\omega) \end{aligned}$$

and the statement (5.32) follows. In particular, putting $f_j \equiv \mathrm{Id}$ we obtain (for all u_i, $v_i \in \mathbb{R}$)

$$\big|\, |u_1 - v_1| - |u_2 - v_2|\, \big| \le |u_1 - u_2| + |v_1 - v_2|\,. \tag{5.33}$$

Using (5.28), (5.32) and (5.33) we conclude that

$$\begin{aligned} |G(\sigma,\tau,\xi,\eta) - G(\sigma,0,\xi,0)| &\le c_1(M,\phi)(|u(\sigma+\tau,\xi+\eta) - u(\sigma,\xi)| \\ &\quad + |v(\sigma-\tau,\xi-\eta) - v(\sigma,\xi)|)\,. \end{aligned} \tag{5.34}$$

Note that

$$|\rho_\varepsilon| \le c_2 \varepsilon^{-(d+1)}\,, \tag{5.35}$$

which together with (5.34) gives

$$A_\varepsilon \le c_3\varepsilon^{-(d+1)} \int_K \left\{ \int_{C_\varepsilon} |u(\sigma+\tau,\xi+\eta)-u(\sigma,\xi)| \right.$$
$$\left. + |v(\sigma-\tau,\xi-\eta)-v(\sigma,\xi)|\, d\eta\, d\tau \right\} d\xi\, d\sigma \ .$$

But $u \in L^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^d)$, and thus for almost every $(\sigma,\xi) \in (0,\infty)\times\mathbb{R}^d$ we have

$$\lim_{\varepsilon\to 0+} \frac{1}{\mathrm{meas}(C_\varepsilon)} \int_{C_\varepsilon} |u(\sigma+\tau,\xi+\eta)-u(\sigma,\xi)|\, d\eta\, d\tau = 0$$

and therefore also

$$\lim_{\varepsilon\to 0+} \varepsilon^{-(d+1)} \int_K \int_{C_\varepsilon} |u(\sigma+\tau,\xi+\eta)-u(\sigma,\xi)|\, d\eta\, d\tau\, d\xi\, d\sigma = 0 \ .$$

In the same way we obtain

$$\lim_{\varepsilon\to 0+} \varepsilon^{-(d+1)} \int_K \int_{C_\varepsilon} |v(\sigma-\tau,\xi-\eta)-v(\sigma,\xi)|\, d\eta\, d\tau\, d\xi\, d\sigma = 0 \ .$$

Therefore we proved

$$\lim_{\varepsilon\to 0+} A_\varepsilon = 0 \ . \tag{5.36}$$

Due to (5.35) we have

$$\int_{C_\varepsilon} |\chi(\sigma,\tau)-\chi(\sigma,0)|\, \rho_\varepsilon(\tau,\eta)\, d\eta\, d\tau \le c_4 \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} |\chi(\sigma,\tau)-\chi(\sigma,0)|\, d\tau$$

and therefore

$$B_\varepsilon \le \frac{c_5}{\varepsilon} \int_{\mathbb{R}} \int_{-\varepsilon}^{\varepsilon} |\chi(\sigma,\tau)-\chi(\sigma,0)|\, d\tau\, d\sigma \, .$$

From the properties of the function χ it follows that

$$\int_{\mathbb{R}} \int_{-\varepsilon}^{\varepsilon} |\chi(\sigma,\tau)-\chi(\sigma,0)|\, d\tau\, d\sigma \ = \int_{-\varepsilon}^{\varepsilon} \int_0^{|\tau|} d\sigma\, d\tau = \varepsilon^2$$

and thus

$$\lim_{\varepsilon\to 0+} B_\varepsilon = 0 \, . \tag{5.37}$$

The statement of the lemma follows from (5.31), (5.36), (5.37). ■

The following theorem is due to Kružkov [1970].

Theorem 5.38 (Kružkov) *Let $\mathbf{f} \in C^1(\mathbb{R})^d$. Let u and $v \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ be two entropy solutions of the problem (1.2) for initial conditions u_0 and $v_0 \in L^\infty(\mathbb{R}^d)$, respectively. Put*

$$M \equiv \max\{|\mathbf{f}'(\xi)|, |\xi| \le \max\{\|u_0\|_\infty, \|v_0\|_\infty\}\}. \tag{5.39}$$

Then for all $R > 0$ and almost every $t > 0$

$$\int_{B_R} |u(t,x) - v(t,x)|\, dx \le \int_{B_{R+Mt}} |u_0(x) - v_0(x)|\, dx. \tag{5.40}$$

PROOF : We define for a given $R, T > 0$

$$D_{R,T} \equiv \{(t,x) \in [0,T] \times \mathbb{R}^d, |x| \le R + M(T-t)\}\ .$$

Let us start with an approximation of the characteristic function of the set $D_{R,T}$. Again we use the following approximation of the Heaviside function:

$$Y_\varepsilon(\lambda) \equiv \int_{-\infty}^{\lambda} \rho_\varepsilon(s)\, ds\ ,$$

where ρ_ε is a mollifier with support in $[-\varepsilon, \varepsilon]$. For arbitrary but fixed $\delta > \varepsilon > 0$, $\theta > 0$, $T > \delta$, we put

$$\varphi(t,x) \equiv \big(Y_\varepsilon(t-\delta) - Y_\varepsilon(t-T)\big)\big(1 - Y_\theta(|x| - R - M(T-t))\big)\ .$$

It is obvious that $\varphi \in \mathcal{D}((0,\infty) \times \mathbb{R}^d)$, $\varphi \ge 0$, and

$$\varphi(t,x) = \begin{cases} 1 & \text{for } |x| \le R + M(T-t) - \theta \\ & \quad \text{and } \varepsilon + \delta \le t \le T - \varepsilon\,, \\ 0 & \text{for } |x| \ge R + M(T-t) + \theta \\ & \quad \text{or } t \le \delta - \varepsilon \text{ or } t \ge T + \varepsilon\,. \end{cases} \tag{5.41}$$

For such a φ the weak form of the inequality (5.22) reads

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} |u - v|(\rho_\varepsilon(t-\delta) - \rho_\varepsilon(t-T)) \\ &\qquad\qquad \times (1 - Y_\theta(|x| - R - M(T-t)))\, dx\, dt \\ -&\int_0^\infty \int_{\mathbb{R}^d} (Y_\varepsilon(t-\delta) - Y_\varepsilon(t-T)) \\ &\qquad\qquad \times \left\{ M|u - v| + \operatorname{sgn}(u-v)(f_j(u) - f_j(v))\frac{x_j}{|x|} \right\} \\ &\qquad\qquad \times \rho_\theta(|x| - R - M(T-t))\, dx\, dt \ge 0\,. \end{aligned} \tag{5.42}$$

We have

$$Y_\varepsilon(t-\delta) - Y_\varepsilon(t-T) \geq 0$$

for $t \geq \delta$ and

$$\left|\operatorname{sgn}(u-v)(f_j(u)-f_j(v))\frac{x_j}{|x|}\right| \leq M|u-v| \ .$$

Thus the second integral in (5.42) is non-negative and therefore we get

$$\int_0^\infty \int_{\mathbb{R}^d} |u-v|(\rho_\varepsilon(t-\delta) - \rho_\varepsilon(t-T)) \\ \times (1 - Y_\theta(|x| - R - M(T-t)))\, dx\, dt \geq 0 \ .$$

We now let θ tend to $0+$. From the Lebesgue dominated convergence theorem it follows that (see (5.41))

$$\int_{D^\varepsilon_{R,T}} |u-v|(\rho_\varepsilon(t-\delta) - \rho_\varepsilon(t-T))\, dx\, dt \geq 0 \ , \qquad (5.43)$$

where

$$D^\varepsilon_{R,T} \equiv \{(t,x), |x| \leq R + M(T-t), \delta - \varepsilon < t \leq T + \varepsilon\} \ .$$

Denoting for fixed $t_0 \in [0,T]$

$$S_{t_0} \equiv \{(t_0,x), |x| \leq R + M(T-t_0)\} = \{(t,x) \in D_{R,T}, t = t_0\} \, ,$$

we can rewrite (5.43) as

$$\int_0^\infty (\rho_\varepsilon(t-\delta) - \rho_\varepsilon(t-T)) \int_{S_t} |u(t,x) - v(t,x)|\, dx\, dt \geq 0 \ . \qquad (5.44)$$

Introducing the function $w : \mathbb{R}^+ \to \mathbb{R}$ by

$$w(t) \equiv \int_{S_t} |u(t,x) - v(t,x)|\, dx \, ,$$

we see that (5.44) is nothing else than

$$w_\varepsilon(\delta) - w_\varepsilon(T) \geq 0 \ .$$

Now, w belongs to $L^\infty(\mathbb{R}^+)$ and therefore $w_\varepsilon \to w$ in all Lebesgue points, i.e. almost everywhere in $\mathbb{R}^+$. Thus we get in Lebesgue points δ, T:¶

$$\int_{S_\delta} |u(\delta,x) - v(\delta,x)|\, dx \geq \int_{B_R} |u(T,x) - v(T,x)|\, dx\,.$$

The triangle inequality and Lemma 5.12 imply

$$\int_{B_R} |u(T,x) - v(T,x)|\, dx \leq \int_{B_{R+MT}} |u_0(x) - v_0(x)|\, dx$$

for all Lebesgue points. ∎

Now we are ready to prove Theorem 5.1.

PROOF (of Theorem 5.1): By applying (5.40) to $u_0(x) = v_0(x)$ one gets $u(t,x) = v(t,x)$ for all Lebesgue points $t \in \mathbb{R}^+$, which gives $u = v$ almost everywhere. ∎

Remark 5.45 Uniqueness of the entropy solution to scalar conservation law (1.2) can be shown under weaker assumptions on $\mathbf{f}$. Namely, suppose that $u_0 \in L^\infty(\mathbb{R}^d)$ and that $\mathbf{f}$ satisfies the following **generalized Lipschitz continuity** condition:

$$|f_i(u) - f_i(v)| \leq \omega_i\big(|u-v|\big) \qquad i = 1, \ldots, d\,, \tag{5.46}$$

for all u, $v \in [-K, K]$, where $K \equiv \|u_0\|_\infty$ and $\omega_i \in C\,([0, 2K])$ are convex functions, $\omega_i(0) = 0$, such that

$$\prod_{i=1}^{d} \omega_i(u) \leq c_\omega u^{d-1}\,.$$

Then it is shown in KRUŽKOV AND PANOV [1991] that there exists a unique entropy solution to scalar conservation law (1.2). Easily, any $\mathbf{f} \in C^{0,1}(\mathbb{R})^d$ satisfies (5.46).

Moreover, the condition (5.46) is sharp in the following sense: there exist f_1, $f_2 \in C(\mathbb{R})$ which do not satisfy (5.46) and a function $u_0 \in L^\infty(\mathbb{R}^2)$ such that there exists a one-parameter family of weak entropy solutions to corresponding scalar conservation law (1.2) in two space dimensions. Such an example is explicitly constructed in KRUŽKOV AND PANOV [1991, Example 2].

¶ Note that $S_T = B_R \times \{t = T\}$.