① $y' = (1+y^2)\,\mathrm{tg}\,x \quad , \quad y(0) = 1$

$$\frac{y'}{1+y^2} = \frac{\sin x}{\cos x} \qquad / \int (\dots)\,dx$$

$$\mathrm{arctg}\, y = -\ln|\cos x| + C$$

NEPOVINNÉ:

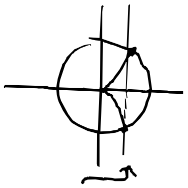
$\hookrightarrow H_{\mathrm{arctg}} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow$ musí platit

$$-\frac{\pi}{2} < -\ln|\cos x| + C < \frac{\pi}{2}$$

$$C + \frac{\pi}{2} > \ln|\cos x| > C - \frac{\pi}{2} \qquad \underline{K := e^C}$$

$$Ke^{-\frac{\pi}{2}} < |\cos x| < K\cdot e^{\frac{\pi}{2}}$$

- $K\cdot e^{-\frac{\pi}{2}} \geq 1$ ... nemá řešení: $x \in \emptyset$.
  $\left[K \geq e^{\frac{\pi}{2}}\right]$ ...

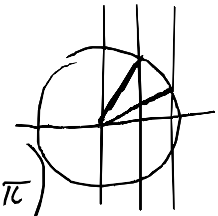


- $K\cdot e^{-\frac{\pi}{2}} < 1$ & $Ke^{\frac{\pi}{2}} > 1$:
  $\left[K \in \left(e^{-\frac{\pi}{2}}, e^{\frac{\pi}{2}}\right)\right]$: $|\cos x| > K\cdot e^{-\frac{\pi}{2}}$

$$x \in \left(-\arccos\left(Ke^{-\frac{\pi}{2}}\right) + k\pi, \arccos\left(Ke^{-\frac{\pi}{2}}\right) + k\pi\right)$$

- $K\cdot e^{\frac{\pi}{2}} \leq 1$: $\left[K \in \left(0, e^{-\frac{\pi}{2}}\right]\right]$:

$$Ke^{-\frac{\pi}{2}} < |\cos x| < Ke^{\frac{\pi}{2}}$$



$$x \in \left(-\arccos\left(Ke^{-\frac{\pi}{2}}\right) + k\pi, \arccos\left(Ke^{-\frac{\pi}{2}}\right) + k\pi\right) \setminus \left[-\arccos\left(Ke^{\frac{\pi}{2}}\right) + k\pi, \arccos\left(Ke^{\frac{\pi}{2}}\right) + k\pi\right],$$

$k \in \mathbb{Z}$.

$$\operatorname{arctg} y = -\ln|\cos x| + C$$

$$y = \operatorname{tg}\left(-\ln|\cos x| + C\right)$$

$$y(x) = -\operatorname{tg}\left(\ln|\cos x| - C\right)$$

Počáteční podmínka $y(0) = 1$:

$$1 = y(0) = -\operatorname{tg}\left(\ln|\cos 0| - C\right)$$

$$= -\operatorname{tg}(-C) = \operatorname{tg}(C)$$

$$C \in \left\{ \frac{\pi}{4} + k\pi : k \in \mathbb{Z} \right\} \qquad C = \frac{\pi}{4}$$

$$\boxed{y(x) = -\operatorname{tg}\left(\ln|\cos x| - \frac{\pi}{4}\right)},$$

$$x \in \left(-\arccos\left(e^{\frac{\pi}{4} - \frac{\pi}{2}}\right) + k\pi, \arccos\left(e^{\frac{\pi}{4} - \frac{\pi}{2}}\right) + k\pi\right),$$

kde ovšem $k \in \mathbb{Z}$ musíme volit tak, aby příslušný interval obsahoval bod $\underline{\underline{x = 0}}$ (protože hledané řešení splňuje $y(\underline{\underline{0}}) = 1$).

Tedy $k = 0$, a $\boxed{x \in \left(-\arccos\left(e^{-\frac{\pi}{4}}\right), \arccos\left(e^{-\frac{\pi}{4}}\right)\right)}$.

(2.) $y' = \sqrt[3]{\operatorname{tg} y} \cdot \sqrt{y^4 - y^2} =$

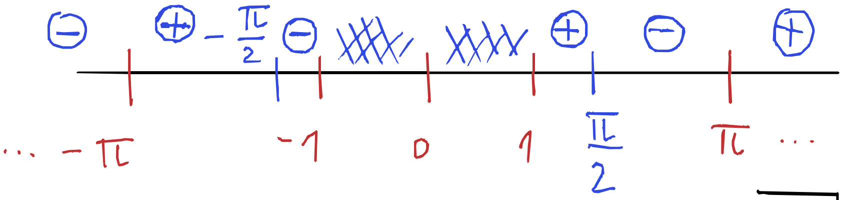
$$= \sqrt[3]{\operatorname{tg} y} \cdot \sqrt{y^2 (y-1)(y+1)} =: g(y)$$

STACIONÁRNÍ ŘEŠENÍ:

$$\left[y \equiv k\pi,\ k \in \mathbb{Z}, \quad y \equiv 1,\ y \equiv -1. \right]$$

$\mathbb{D}_g = \mathbb{D}_{tg} \setminus \big((-1,0) \cup (0,1) \big)$ (viz obr.)

monotonie:



[Pro $y \in D_g$: $\operatorname{sgn}(g(y)) = \operatorname{sgn}(\operatorname{tg}(y))$, protože 2. činitel v def. $g(y)$, tj. $\sqrt[2]{\dots}$, je vždy nezáporný]

Lepení: • na $0_\pm$ nepadá v úvahu.

• „1_+": $\int_1^{\frac{3}{2}} \frac{1}{g}$ K. ?

LSK: srovnáme s $h(y) := \dfrac{1}{\sqrt{y-1}}$.

$$\lim_{y \to 1_+} \frac{\frac{1}{g(y)}}{h(y)} = \lim_{y \to 1_+} \frac{1}{\sqrt[3]{\operatorname{tg} y} \cdot \sqrt{y^2 (y+1)}} \in (0, \infty)$$

Tedy LSK $\Longrightarrow \left[\int_1^{\frac{3}{2}} \frac{1}{g} \text{ K.} \Longleftrightarrow \int_1^{\frac{3}{2}} h \text{ K.} \right]$

Ovšem $\displaystyle\int_1^{\frac{3}{2}} h = \int_1^{\frac{3}{2}} \frac{dy}{\sqrt{y-1}} = \int_0^{\frac{1}{2}} \frac{dy}{\sqrt{y}}$ K.

Tedy lze lepit na 1_+; analog 1_-.

• „$k\pi_+$" : $\displaystyle\int_{k\pi}^{k\pi+\varepsilon} \frac{1}{g}$ K.

$k \neq 0$

Srovnáme s funkcí $h(y) = \dfrac{1}{\sqrt[3]{y-k\pi}}$.

PROČ TATO VOLBA h: $\operatorname{tg}(k\pi) = 0$, přičemž na okolí $k\pi$ se tg „chová lineárně". Tj.

$$\operatorname{tg} y \sim y - k\pi, \quad y \to k\pi, \quad \text{a tedy}$$

$$\sqrt[3]{\operatorname{tg} y} \sim \sqrt[3]{y-k\pi}, \quad y \to k\pi \quad \text{atd.}$$

Ovšem další činitelé v předpisu pro g nehrají v bodě $k\pi$ $(k \neq 0)$ významnou roli, majíce v tom bodě nenulové vlastní limity.

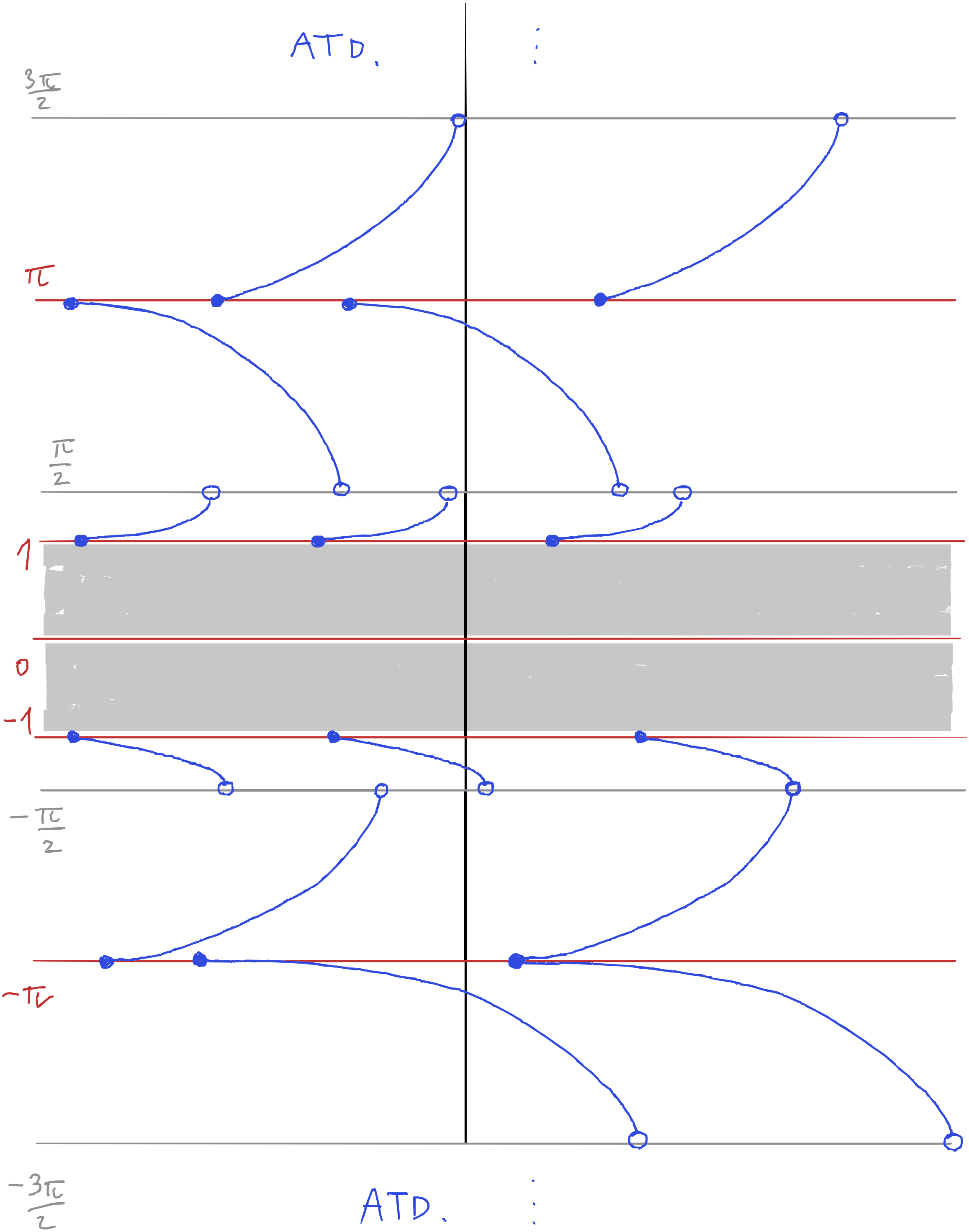
LSK:
$$\lim_{y \to k\pi_+} \frac{\frac{1}{g(y)}}{h(y)} = \lim_{y \to k\pi_+} \frac{\sqrt[3]{y-k\pi}}{\sqrt[3]{\operatorname{tg} y} \cdot \sqrt{y^4 - y^2}} =$$

$$= \frac{1}{\sqrt{(k\pi)^4 - (k\pi)^2}} \cdot \left(\lim_{x \to k\pi_+} \frac{\operatorname{tg} y}{y - k\pi} \right)^{-\frac{1}{3}} = \dots \in (0, \infty).$$

Tedy opět: „oba integrály se chovají stejně".

Ovšem $\displaystyle\int_{k\pi}^{k\pi+\varepsilon} h = \int_{k\pi}^{k\pi+\varepsilon} (y-k\pi)^{-\frac{1}{3}}\, dy \overset{\text{SUBST.}}{=} \int_0^{\varepsilon} \frac{1}{y^{1/3}}\, dy$

je konečný. Tedy lepit lze.



ATD.
3π/2
π
π/2
1
0
-1
-π/2
-π
-3π/2
ATD.

③ $y' + xy = e^{-\frac{1}{2}x^2}$

$\left[p(x) = x, \quad P(x) = \frac{x^2}{2}, \quad \text{I.F.} = e^{\frac{x^2}{2}} \right]$

$$y' e^{\frac{x^2}{2}} + y \cdot e^{\frac{x^2}{2}} \cdot x = 1$$

$$\left(y e^{\frac{x^2}{2}} \right)' = 1 \quad \Big/ \int (\dots)\, dx$$

$$y e^{\frac{x^2}{2}} = x + C$$

$$y(x) = (x + c)\, e^{-\frac{x^2}{2}}, \quad C \in \mathbb{R},\ x \in \mathbb{R}$$

$$y''' + 3y'' + 3y' + y = 0$$

CHAR. POL.: $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = \chi(\lambda)$.

$$0 = \chi(\lambda) = (\lambda + 1)^3 \iff$$

$\lambda = -1$ ... 3-násobný kořen

Tedy FUND. SYSTÉM: $\{ e^{-t}, t e^{-t}, t^2 e^{-t} \}$.

Libovolné max. řešení rce je tedy tvaru:

$$y(x) = C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t}, \quad t \in \mathbb{R},$$

$$C_1, C_2, C_3 \in \mathbb{R}.$$