ZOBECNĚNÝ R. INT.

Definice: $f : [a,b] \to \mathbb{R}$ omezená (kde $a, b \in \mathbb{R}$, $a < b$).

- D dělení $[a,b]$
- $s(f,D)$, $S(f,D)$ ... součty
- $\underline{\int_a^b} f = \sup\{ s(f,D) : D \text{ děl. } [a,b]\}$

  $\overline{\int_a^b} f = \inf \{ S(f,D) : \quad \dots \}$
- Pokud $\overline{\int_a^b} f = \underline{\int_a^b} f$, definujeme

  $$\int_a^b f = \int_a^b f(x)\,dx = \overline{\int_a^b} f.$$

Otázky:

- Co neomezené fce?

  $$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

  $f : (0,1] \to \mathbb{R}$ je neomezená
- Co neomezené intervaly?

  $$\int_1^\infty \frac{1}{x^2}\,dx$$
- Co obojí?

  $$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$$

Opakování: • PF … antiderivace

• Newtonova – Leibnizova fle:

nechť $f : [a,b] \to \mathbb{R}$ je spojitá.

Je-li F prim. fce k f na $[a,b]$,

pak $\displaystyle\int_a^b f = F(b) - F(a) = [F]_a^b$.

Příklad: $\boxed{\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx}$ … počítejme

$(\varepsilon > 0)$

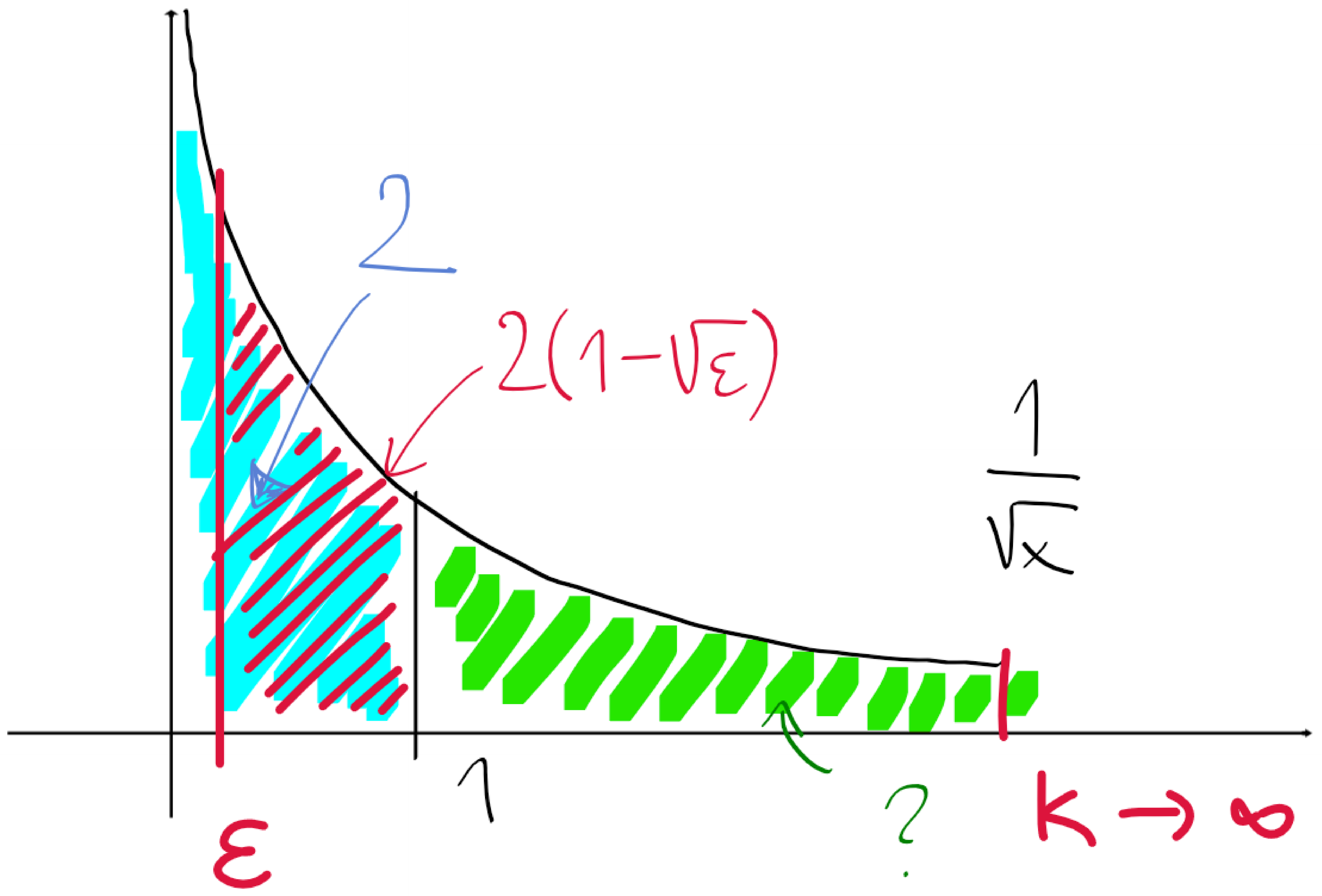
$$\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx \overset{N.-L.}{=} \left[2\sqrt{x}\right]_\varepsilon^1 =$$

$$= 2\sqrt{1} - 2\sqrt{\varepsilon} = 2(1-\sqrt{\varepsilon}) \xrightarrow{\varepsilon \to 0_+} 2$$

My definujeme $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx \overset{def.}{=}$

$$\overset{def.}{=} \lim_{\varepsilon \to 0_+} \int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx = \dots =$$

$$= \lim_{\varepsilon \to 0_+} \left(2(1-\sqrt{\varepsilon})\right) = \underline{\underline{2}}$$



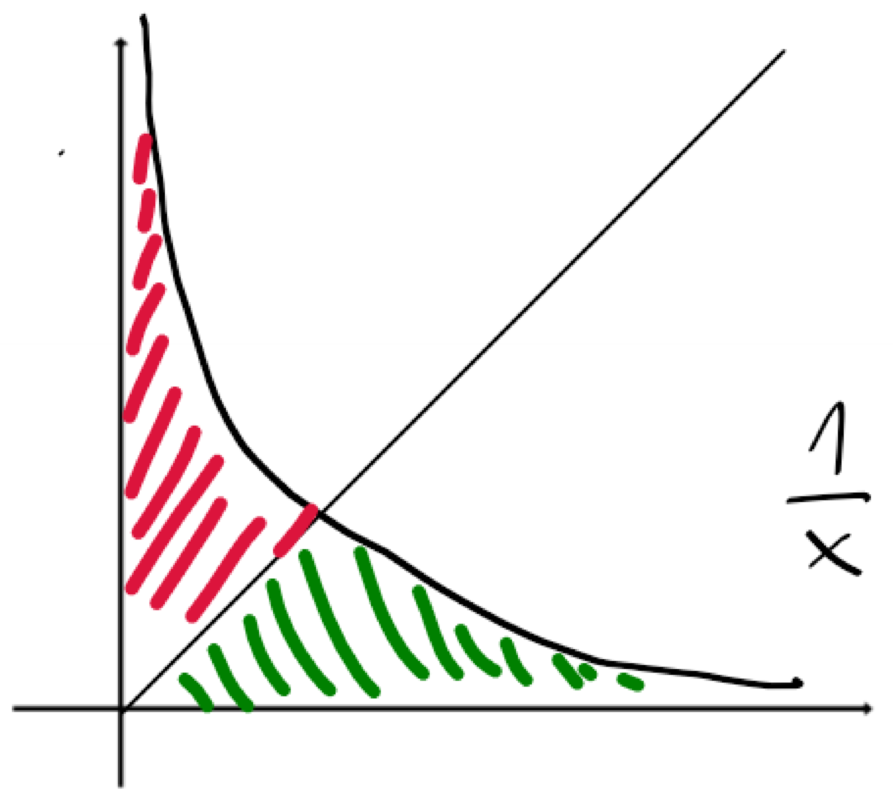
$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx \overset{def}{=} \lim_{K\to\infty} \int_1^{K} \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{k\to\infty} \left[2\sqrt{x}\right]_1^k = \lim_{k\to\infty} \left(2\sqrt{k} - 2\sqrt{1}\right)$$

$$= 2\cdot\infty - 2 = \infty$$

Příklad:

$$\int_1^\infty \frac{1}{x}\,dx \overset{def}{=} \lim_{k\to\infty} \int_1^k \frac{1}{x}\,dx$$



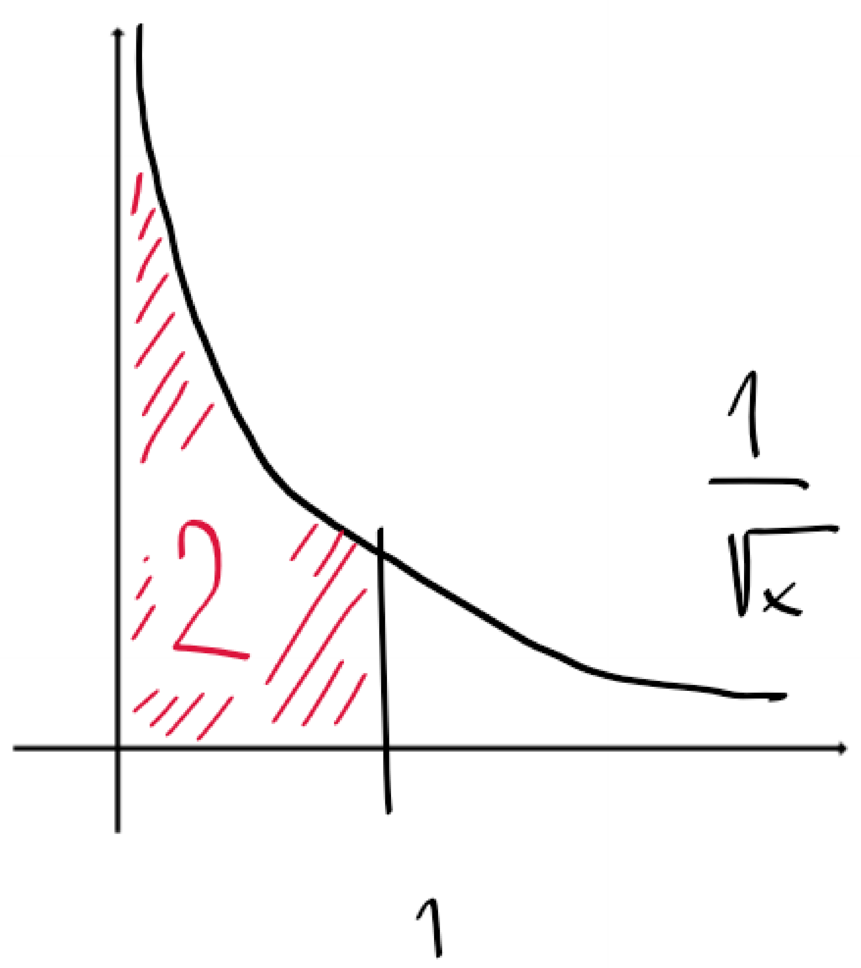
$$= \lim_{k\to\infty} \left[\ln x\right]_1^k = \lim_{k\to\infty} \left(\ln k - \ln 1\right) = \infty$$

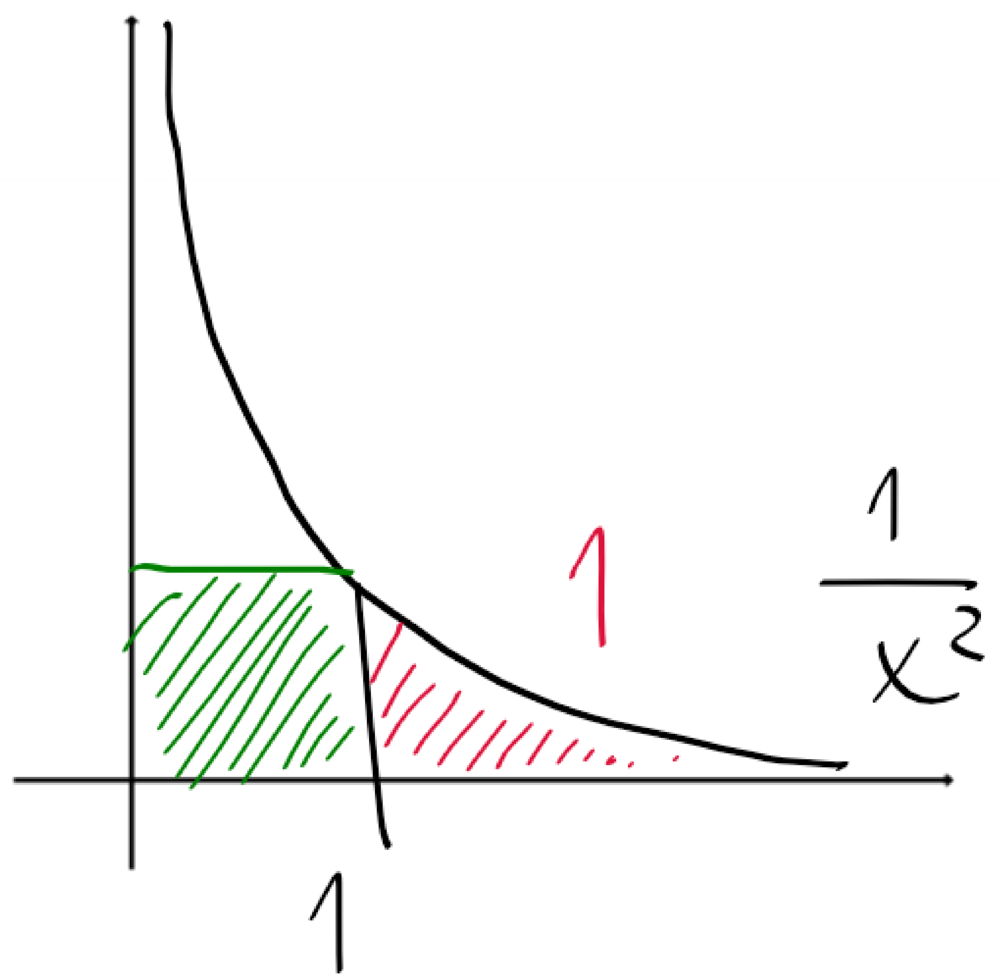
Př.:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{k\to\infty} \int_1^k \frac{1}{x^2}\,dx =$$

$$= \lim_{k\to\infty} \left[\boxed{-\frac{1}{x}}\right]_1^k =$$

$$= \lim_{k\to\infty} \left(\left(\boxed{-\frac{1}{k}}\right) - \left(\boxed{-\frac{1}{1}}\right) \right) = 1$$





Příklad:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx \overset{\text{def.}}{=}$$

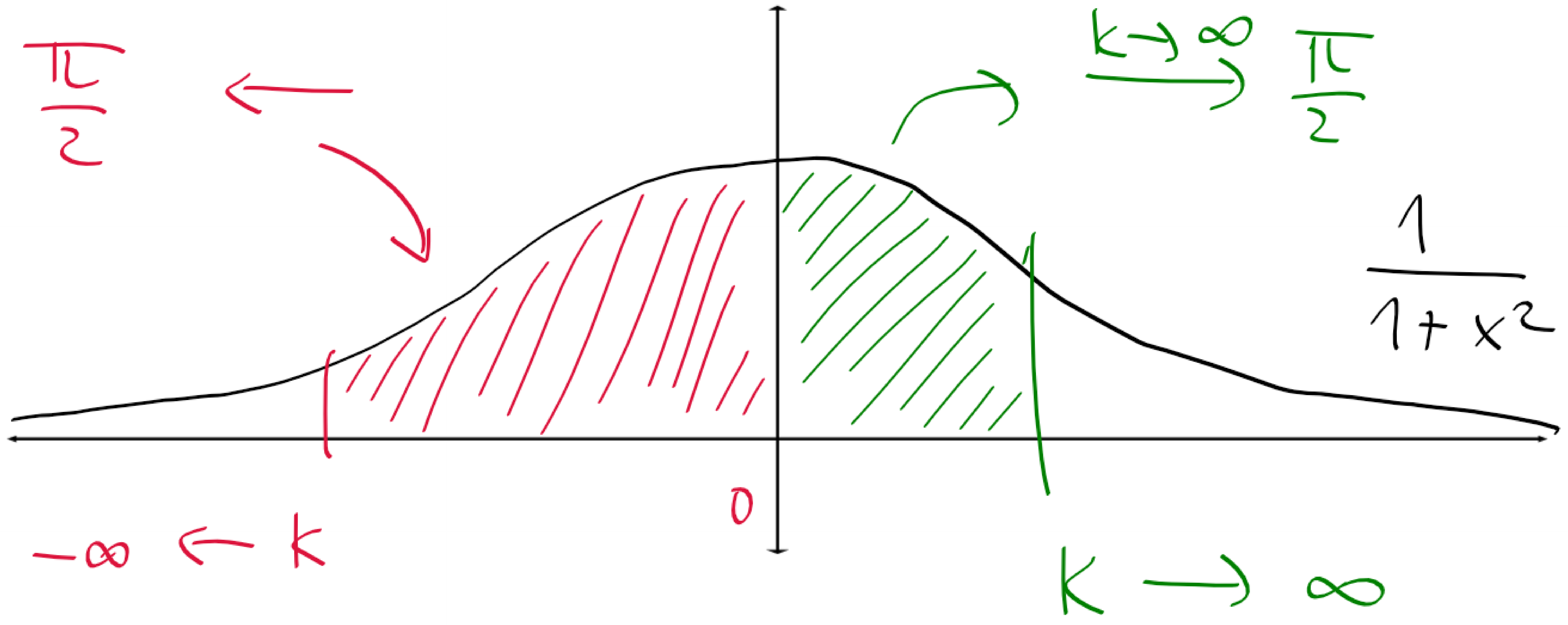
$$\overset{\text{def.}}{=} \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx \overset{\text{def.}}{=}$$

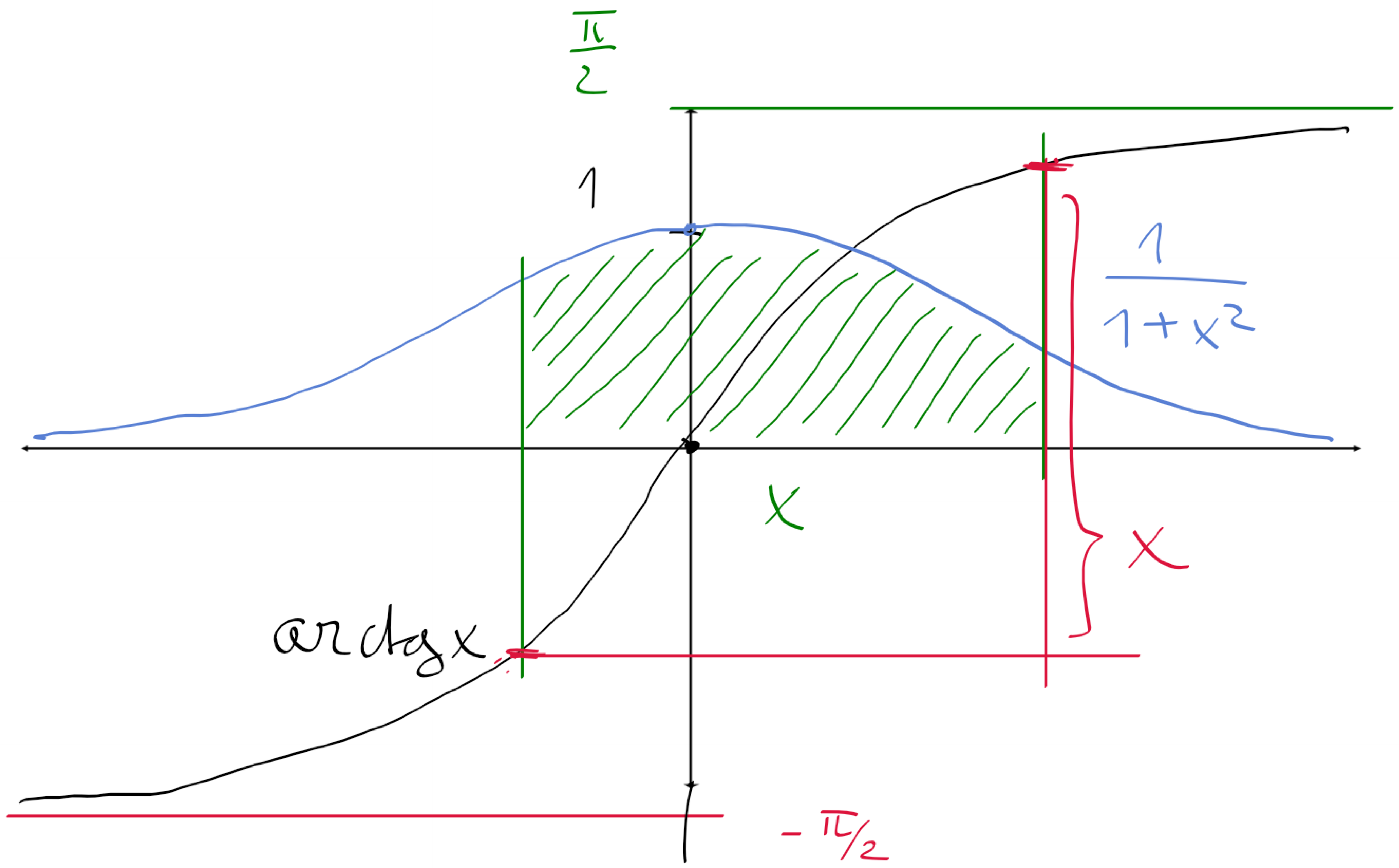
$$= \lim_{k\to-\infty} \int_{k}^{0} \frac{1}{1+x^2}\,dx + \lim_{k\to\infty} \int_{0}^{k} \frac{1}{1+x^2}\,dx$$

$$= \lim_{k\to-\infty} \Big[\operatorname{arctg} x\Big]_{k}^{0} + \lim_{k\to\infty} \Big[\operatorname{arctg} x\Big]_{0}^{k}$$

$$= \lim_{k\to-\infty} \big(0 - \operatorname{arctg} k\big) + \lim_{k\to\infty} \big(\operatorname{arctg} k - 0\big)$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \underline{\underline{\pi}}$$





$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,dx \quad \dots \text{ nejdříve PF:}$$

$$\int \frac{1}{\sqrt{x}(x+1)}\,dx \longleftarrow \left[\begin{array}{l} y=\sqrt{x} \\ dy = \frac{1}{2\sqrt{x}}\,dx \end{array}\right] = \int \frac{2}{y^2+1}\,dy$$

$$\stackrel{c}{=} 2\cdot \operatorname{arctg} y = 2\operatorname{arctg}\sqrt{x}.$$

$$= \lim_{\varepsilon\to 0_+} \int_\varepsilon^1 \frac{1}{\sqrt{x}(x+1)}\,dx + \lim_{K\to\infty}\int_1^K \frac{1}{\sqrt{x}(x+1)}\,dx$$

$$= \lim_{\varepsilon\to 0_+}\left[2\operatorname{arctg}\sqrt{x}\right]_\varepsilon^1 + \lim_{K\to\infty}\left[2\operatorname{arctg}\sqrt{x}\right]_1^K$$

$$= \lim_{\varepsilon\to 0_+}\left(2\operatorname{arctg}1 - 2\operatorname{arctg}\sqrt{\varepsilon}\right) + \lim_{K\to\infty}\left(2\operatorname{arctg}\sqrt{K} - 2\operatorname{arctg}1\right) =$$

$$= -2\operatorname{arctg}0 + 2\cdot\frac{\pi}{2} = \underline{\underline{\pi}}$$

Bylo:

$$\sum_{n=1}^\infty \frac{1}{\sqrt{n}} = \infty \qquad \int_1^\infty \frac{1}{\sqrt{x}} = \infty$$

$$\sum_{n=1}^\infty \frac{1}{n^2} < \infty \qquad \int_1^\infty \frac{1}{x^2} < \infty$$

„**Hraniční**": $$\sum_{n=1}^\infty \frac{1}{n} = \infty \qquad \int_1^\infty \frac{1}{x} = \infty$$

Integrální kritérium konvergence řad

Pokud $f: [1,\infty) \to [0,\infty)$ je nerostoucí, pak $\int_1^\infty f(x)\,dx < \infty \iff \sum_{n=1}^\infty f(n) < \infty$