1.
$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + o(y^3), \quad y \to 0$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^5), \quad x \to 0$$

$$e^{\cos x - 1} = 1 + \left(-\frac{x^2}{2} + \frac{x^4}{4!} + o(x^5)\right) + \frac{1}{2}\left(\frac{x^4}{4} + o(x^5)\right) + \frac{1}{3!}\left(o(x^5)\right), \quad x \to 0 =$$

$$= 1 - \frac{x^2}{2} + x^4\left(\frac{1}{4!} + \frac{1}{8}\right) + o(x^5), \quad x \to 0$$

$$\lim_{x \to 0} \frac{e^{\cos x - 1} - \cos x - \frac{x^4}{8}}{x^5} =$$

$$= \lim_{x \to 0} \frac{1}{x^5} \cdot \left(1 - \frac{x^2}{2} + x^4\left(\frac{1}{4!} + \frac{1}{8}\right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{4!}\right) - \frac{x^4}{8} + o(x^5)\right) =$$

$$= \lim_{x \to 0} \frac{o(x^5)}{x^5} = 0.$$

2.
$$\int x^{\uparrow 2} \operatorname{arctg}^{\downarrow}(x+1)\,dx = \frac{x^3}{3}\operatorname{arctg}(x+1) - \int \frac{\frac{1}{3}x^3}{1+(x+1)^2}\,dx = I$$

$$\int \frac{x^3}{1+(x+1)^2}\,dx \leftarrow \begin{bmatrix} y = x+1 \\ dy = dx \end{bmatrix} = \int \frac{(y-1)^3}{1+y^2}\,dy = \int \frac{y^3 - 3y^2 + 3y - 1}{1+y^2}\,dy =$$

$$(y^3 - 3y^2 + 3y - 1) : (y^2 + 1) = y - 3 + \frac{2y+2}{y^2+1} = y - 3 + \frac{2y}{y^2+1} + \frac{2}{y^2+1}$$

$$\begin{array}{l} -(y^3 + y) \\ \hline \quad -3y^2 + 2y - 1 \\ \quad -(-3y^2 - 3) \\ \hline \quad\quad 2y + 2 \end{array}$$

$$= \int \left(y - 3 + \frac{2y}{y^2+1} + 2 \cdot \frac{1}{y^2+1}\right) dy \stackrel{c}{=}$$

$$\stackrel{c}{=} \frac{y^2}{2} - 3y + \ln(y^2+1) + 2\operatorname{arctg} y =$$

$$= \frac{1}{2}(x+1)^2 - 3(x+1) + \ln((x+1)^2+1) + 2\operatorname{arctg}(x+1)$$

$$I \stackrel{c}{=} \frac{x^3}{3}\operatorname{arctg}(x+1) - \frac{1}{6}(x+1)^2 + (x+1) - \frac{1}{3}\ln((x+1)^2+1) - \frac{2}{3}\operatorname{arctg}(x+1)$$

3.) Délku křivky $y = 1 - \log(\cos x)$, $0 \le x \le \frac{\pi}{4}$.

$$l = \int_0^{\frac{\pi}{4}} \sqrt{1 + \left((1-\log(\cos x))'\right)^2}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{\sin x}{\cos x}\right)^2}\, dx =$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\frac{1}{\cos^2 x}}\, dx \stackrel{(*)}{=} \int_0^{\frac{\pi}{4}} \frac{1}{\cos x}\, dx.$$

[$\underline{(*):}$ $\sqrt{\cos^2 x} = |\cos x| = \cos x$ pro $x \in [0, \frac{\pi}{4}]$, neboť pro tato x je $\cos x > 0$.]

Spočteme primitivní fci a dosadíme do N.-L. fle:

$$\int \frac{1}{\cos x}\, dx = \int \frac{\cos x}{\cos^2 x}\, dx = \int \frac{\cos x}{1 - \sin^2 x}\, dx \hookleftarrow \begin{bmatrix} y = \cancel{\cos}\ \sin x \\ dy = \cos x\, dx \end{bmatrix} =$$

$$= \int \frac{dy}{1-y^2} = \frac{1}{2}\left(\int \frac{dy}{1+y} + \int \frac{dy}{1-y}\right) \stackrel{c}{=} \frac{1}{2}\left(\ln|1+y| - \ln|1-y|\right) =$$

$$\left[\ \frac{1}{1-y^2} = \frac{1}{(1-y)(1+y)} = \frac{A}{1-y} + \frac{B}{1+y} = \frac{A + Ay + B - By}{1-y^2}\ \right.$$

$$\left.\Rightarrow \begin{array}{l} A + B = 1 \\ A - B = 0 \end{array} \Rightarrow A = B = \frac{1}{2}\ \right]$$

$$= \frac{1}{2} \ln\left|\frac{1+\sin x}{1-\sin x}\right|.$$ <u>Tedy</u>:

$$l = \int_0^{\frac{\pi}{4}} \frac{1}{\cos x}\, dx = \left[\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|\right]_0^{\frac{\pi}{4}} =$$

$$= \frac{1}{2}\left(\ln\left|\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}\right| - \ln 1\right) = \frac{1}{2}\ln\left(\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}\right)$$