1. $\lim\limits_{x \to 0} \dfrac{e^{\sin x} - e^x}{x \ln(1+x^2)} =: L$

$$\ln(1+y) = y - \frac{y^2}{2} + o(y^2), \quad y \to 0$$

$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + o(x^4), \quad x \to 0$$

$$x \ln(1+x^2) = x^3 + o(x^4), \quad x \to 0$$

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + o(y^3), \quad y \to 0$$

$$\sin x = x - \frac{x^3}{6} + o(x^3), \quad x \to 0$$

$$e^{\sin x} = 1 + \left(x - \frac{x^3}{6}\right) + \frac{1}{2}\left(x - \frac{x^3}{6}\right)^2 + \frac{1}{3!}\left(x - \frac{x^3}{6}\right)^3 +$$

$$\underbrace{+\, o(x^3) \quad + o(\sin^3 x)}_{= o(x^3),\ x \to 0,\ \text{protože:}}, \quad x \to 0$$

$\sin x \sim x$, $x \to 0$, tedy i $\sin^3 x \sim x^3$, $x \to 0$,

odkud to plyne: $f(x) = o(x^3), x \to 0 \iff f(x) = o(\sin^3 x), x \to 0$

$$= 1 + x - \frac{x^3}{6} + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3), \quad x \to 0$$

$$= 1 + x + \frac{1}{2}x^2 + o(x^3), \quad x \to 0.$$

$$L = \lim_{x \to 0} \frac{1 + x + \frac{x^2}{2} + o(x^3) - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)\right)}{x^3 + o(x^3)}$$

$$= \lim_{x \to 0} \frac{-\frac{x^3}{6} + o(x^3)}{x^3 + o(x^3)} = \lim_{x \to 0} \frac{x^3\left(-\frac{1}{6} + \frac{o(x^3)}{x^3}\right)}{x^3\left(1 + \frac{o(x^3)}{x^3}\right)}$$

$$\overset{\text{VOAL}}{=} \frac{-\frac{1}{6} + \lim\limits_{x \to 0} \frac{o(x^3)}{x^3}}{1 + \lim\limits_{x \to 0} \frac{o(x^3)}{x^3}} \underset{\text{„}o\text{"}}{\overset{\text{def.}}{=}} \frac{-\frac{1}{6} + 0}{1 + 0} = -\frac{1}{6}.$$

2. $$\int \frac{\sin x \cos x}{4\cos^2 x + 9\sin^2 x}\,dx =: I =$$

$$R(x,y) = \frac{xy}{4y^2+9x^2} \quad \dots \quad R(-x,-y) = \dots = R(x,y)$$

Volíme tedy substituci $y = \operatorname{tg} x$. Tj.

$$y^2 = \frac{\sin^2 x}{\cos^2 x}, \quad y^2\cos^2 x = 1 - \cos^2 x,$$

$$\cos^2 x\,(y^2+1) = 1, \quad \cos^2 x = \frac{1}{y^2+1}.$$

$$dy = (\operatorname{tg} x)'\,dx = \frac{1}{\cos^2 x}\,dx.$$

$$= \int \frac{\frac{\sin x}{\cos x}\cdot\cos^2 x\cdot\cos^2 x}{4\cos^2 x + 9 - 9\cos^2 x}\cdot\frac{dx}{\cos^2 x} = \int \frac{y\cdot\left(\frac{1}{y^2+1}\right)^2}{9 - \frac{5}{y^2+1}}\,dy$$

$$= \int \frac{\frac{y}{(y^2+1)^2}}{\frac{9y^2+4}{y^2+1}}\,dy = \int \underbrace{\frac{y}{(y^2+1)(9y^2+4)}}\,dy$$

PARCIÁLNÍ ZLOMKY: $$= \frac{Ay+B}{y^2+1} + \frac{Cy+D}{9y^2+4} =$$

$$= \frac{1}{\dots}\left[(Ay+B)(9y^2+4) + (Cy+D)(y^2+1)\right] =$$

$$= \frac{1}{\dots}\left[9Ay^3 + 9By^2 + 4Ay + 4B + Cy^3 + Dy^2 + Cy + D\right]$$

$$9Ay^3 + 9By^2 + 4Ay + 4B + Cy^3 + Dy^2 + Cy + D = y$$

$\underline{y^3}:\quad 9A + C = 0 \qquad\qquad \underline{y^2}:\quad 9B + D = 0$

$\underline{y}:\quad 4A + C = 1 \qquad\qquad \underline{1}:\quad 4B + D = 0$

$$\left(\begin{array}{cccc|c} 9 & 0 & 1 & 0 & 0 \\ 0 & 9 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 1 \\ 0 & 4 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 9 & 0 & 1 & 0 & 0 \\ 0 & 9 & 0 & 1 & 0 \\ -5 & 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 & 0 \end{array}\right)$$

$$B = 0, \quad A = -\frac{1}{5}, \quad D = 0, \quad C = \frac{9}{5}.$$

$$I = -\frac{1}{5}\int \frac{y}{y^2+1}\,dy + \frac{9}{5}\int \frac{y}{9y^2+4}\,dy =$$

$$= -\frac{1}{10}\int \frac{2y\,dy}{y^2+1} + \frac{9}{5}\cdot\frac{1}{18}\int \frac{18y\,dy}{9y^2+4} \stackrel{c}{=}$$

$$\stackrel{c}{=} -\frac{1}{10}\ln\left|y^2+1\right| + \frac{1}{10}\ln\left|9y^2+4\right|$$

$$= \frac{1}{10}\ln\frac{9\,tg^2x+4}{tg^2x+1} = \frac{1}{10}\ln\frac{\frac{4+5\sin^2x}{\cos^2x}}{\frac{1}{\cos^2x}} = \frac{1}{10}\ln\left(4+5\sin^2x\right)$$

$$\left[9\,tg^2x + 4 = \frac{9\sin^2x + 4\cos^2x}{\cos^2x} = \frac{4+5\sin^2x}{\cos^2x}\right]$$

③ $y = x^{3/2}$, $0 \leq x \leq 1$. Délka:

$$\int_0^1 \sqrt{1 + \left((x^{3/2})'\right)^2}\, dx = \int_0^1 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2}\, dx =$$

$$= \int_0^1 \sqrt{1 + \frac{9}{4}x}\, dx = \int_1^{\frac{13}{4}} \sqrt{y} \cdot \frac{4}{9}\, dy =$$

SUBSTITUCE: $y = 1 + \frac{9}{4}x \qquad dy = \frac{9}{4}dx$

x	0	1
y	1	$\frac{13}{4}$

$$= \frac{4}{9}\left[\frac{2}{3}y^{\frac{3}{2}}\right]_1^{\frac{13}{4}} = \frac{8}{27}\left(\left(\frac{13}{4}\right)^{\frac{3}{2}} - 1\right) = \frac{13^{\frac{3}{2}} - 8}{27}$$