① $$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{1-\cos\frac{1}{n}}\left(\sqrt{1-\frac{1}{n^2}}-\sqrt{1+\frac{1}{n^2}}\right) \overset{\text{H.V.}}{\underset{\text{(míše)}}{=}}$$

$$= \lim_{x\to 0} \frac{\sin x}{1-\cos x}\left(\sqrt{1-x^2}-\sqrt{1+x^2}\right) =$$

$$= \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{\to 1}\cdot x\cdot \underbrace{\frac{x^2}{1-\cos x}}_{\to \left(\frac{1}{2}\right)^{-1}=2}\cdot\frac{1}{x^2}\cdot\frac{1-x^2-(1+x^2)}{\sqrt{1-x^2}+\sqrt{1+x^2}} \underset{=}{\text{VOAL}}$$

(známé lim.)

$$= 1\cdot 2\cdot \lim_{x\to 0}\frac{1}{x}\cdot\frac{-2x^2}{\sqrt{1-x^2}+\sqrt{1+x^2}} = -4\cdot\lim_{x\to 0}\frac{x}{\sqrt{1-x^2}+\sqrt{1+x^2}} =$$

$$= -4\cdot\frac{0}{\sqrt{1-0^2}+\sqrt{1+0^2}} = -4\cdot\frac{0}{1+1} = \underline{\underline{0}}$$

<u>Heineho věta:</u> $f(x) = \frac{\sin x}{1-\cos x}\left(\sqrt{1-x^2}-\sqrt{1+x^2}\right)$

$a_n = \frac{1}{n^2}$ (H1) $\lim_{n\to\infty} a_n = \boxed{0}$ ✓

(H2) $\forall n\in\mathbb{N}: a_n \neq \boxed{0}$ ✓

Pak $\underbrace{\lim_{n\to\infty} f(a_n)}_{\text{limita ze zadání}} = \lim_{x\to \boxed{0}} f(x) = \ldots$ vyšlo nám $=0$

(2.) $\lim\limits_{x\to-\infty}\left(\sqrt{\frac{3x^2+1}{3x^2+2}}\right)^{x^2} = \lim\limits_{x\to-\infty} e^{x^2\cdot\ln\sqrt{\frac{3x^2+1}{3x^2+2}}} \overset{\text{VOLSF(S)}}{=} \underline{\underline{e^{-\frac{1}{6}}}}$

$$\lim_{x\to-\infty} x^2\cdot\ln\sqrt{\frac{3x^2+1}{3x^2+2}} = \lim_{x\to-\infty} x^2\cdot\frac{1}{2}\ln\frac{3x^2+1}{3x^2+2} =$$

$$= \frac{1}{2}\lim_{x\to-\infty} x^2\cdot\frac{\ln\left(1-\frac{1}{3x^2+2}\right)}{-\frac{1}{3x^2+2}}\cdot\frac{-1}{3x^2+2} \overset{\text{VOLSF + VOAL}}{=}$$

$$= \frac{1}{2}\cdot 1\cdot\lim_{x\to-\infty}\frac{-x^2}{x^2\left(3+\frac{2}{x^2}\right)} = -\frac{1}{2}\lim_{x\to-\infty}\frac{1}{3+\frac{2}{x^2}} =$$

$$= -\frac{1}{2}\cdot\frac{1}{3+0} = -\frac{1}{6}$$

<u>VOLSF</u>: vnější: $f(y) = \frac{\ln(1+y)}{y}$, $\lim\limits_{y\to 0} f(y) = 1$ (známá lim.)

vnitřní: $g(x) = \frac{-1}{3x^2+2}$, $\lim\limits_{x\to-\infty} g(x) = \underline{\underline{0}}$

(P): zřejmě $g \neq 0$ na D_g, takže např. pro $\delta = 1$ platí $\forall x \in \underbrace{P(-\infty,\delta)}_{=(-\infty,-\frac{1}{\delta})=(-\infty,-1)}: g(x) \neq 0$.

3. $$\sum_{n=1}^{\infty}\left(\frac{5n+1}{n}\right)^n \cdot \sin\left(7^{-n}\right) =$$

$$= \sum_{n=1}^{\infty}\underbrace{\left(5+\frac{1}{n}\right)^n}_{\leq 6^n} \cdot \underbrace{\sin\left(\frac{1}{7^n}\right)}_{\approx \frac{1}{7^n}}$$

Srovnáme s řadou $\sum_{n=1}^{\infty} b_n$, kde $b_n = \dfrac{6^n}{7^n}$

(ta konverguje, dokonce známe hodnotu součtu):

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\left(5+\frac{1}{n}\right)^n}{6^n}\cdot\underbrace{\frac{\sin\frac{1}{7^n}}{\frac{1}{7^n}}}_{\to 1} \overset{\text{H.V. + VOAL}}{=}$$

1. $$\lim_{n\to\infty}\left(\frac{5+\frac{1}{n}}{6}\right)^n = \underline{0}.$$

$$0 \leq \left(\frac{5+\frac{1}{n}}{6}\right)^n \overset{n\geq 2}{\leq} \left(\frac{5+\frac{1}{2}}{6}\right)^n = \left(\frac{11}{12}\right)^n$$

$0 \to 0$, $\left(\frac{5+\frac{1}{n}}{6}\right)^n \to 0$ PODLE LO2P, $\left(\frac{11}{12}\right)^n \xrightarrow{n\to\infty} 0$

Podle LSK: ($\sum b_n$ K. $\Rightarrow$ $\sum a_n$ K.)

Protože $\sum b_n$ K., $\sum a_n$ K.

(Protože $\forall n \in \mathbb{N}: a_n \geq 0$, $a_n = |a_n|$,

a $\sum |a_n|$ K., to jest, $\sum a_n$ AK.)

4. $f(x) = (x^2+3x+2)\cdot e^{|x-3|+3}$

1 $D_f = \mathbb{R}$, spojitá na $\mathbb{R}$, symetrie nejsou

1 $\lim\limits_{x\to\pm\infty} f(x) = \infty$, asymptoty nejsou $\left(\lim\limits_{x\to\pm\infty} \frac{f(x)}{x} = \pm\infty\right)$

1 $f'(x) = (2x+3)\cdot e^{|x-3|+3} + (x^2+3x+2)\cdot e^{|x-3|+3}\cdot \operatorname{sgn}(x-3)$

$= e^{|x-3|+3}\left(2x+3+\operatorname{sgn}(x-3)(x^2+3x+2)\right)$, tj.

1 pro $x>3$: $f'(x) = e^{|x-3|+3}\left(x^2+5x+5\right) =$

$\frac{-5\pm\sqrt{25-20}}{2} = \begin{cases} \frac{-5+\sqrt{5}}{2} \\ \frac{-5-\sqrt{5}}{2} \end{cases}$

$= \underbrace{e^x}_{>0}(x^2+5x+5) > 0 \iff x^2+5x+5>0$

$\iff x \in \left(-\infty, \frac{-5-\sqrt{5}}{2}\right) \cup \left(\frac{-5+\sqrt{5}}{2}, \infty\right)$

Tedy (protože $\frac{-5+\sqrt{5}}{2} < 0 < 3$) f je rostoucí na $(3,\infty)$

1 pro $x<3$: $f'(x) = e^{-x+3+3}\left(2x+3-x^2-3x-2\right) =$

$\frac{-1\pm\sqrt{1+4}}{2} = \begin{cases} \frac{-1+\sqrt{5}}{2} \\ \frac{-1-\sqrt{5}}{2} \end{cases}$

$= e^{6-x}\left(-x^2-x+1\right) > 0 \iff x^2+x-1<0$

$\iff x \in \left(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right) \subseteq (-\infty, 3)$

2

	$\left(-\infty, \frac{-1-\sqrt{5}}{2}\right)$	$\left(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$	$\left(\frac{-1+\sqrt{5}}{2}, 3\right)$	$(3,\infty)$
f'	$-$	$+$	$-$	$+$
f	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

$x=3$: $f'_+(3) \overset{\text{spojitost, V. o lim. der.}}{=} \lim\limits_{x\to 3_+} f'(x) =$

2 $= \lim\limits_{x\to 3_+} e^x(x^2+5x+5) = e^3(9+15+5) = \underline{29e^3}$

$f'_-(3) = \lim\limits_{x\to 3_-} f'(x) = \lim\limits_{x\to 3_-} e^{6-x}(-x^2-x+1) = \underline{\underline{-11e^3}}$

Lokální minima:

$f\left(\frac{-1-\sqrt{5}}{2}\right) = (2-\sqrt{5})\,e^{\sqrt{5}/2+13/2} \doteq$

1 $\doteq -480{,}3$ (globální min.)

$f(3) = 20\cdot e^3 \doteq 401{,}7$

Lok. maximum:

$f\left(\frac{-1+\sqrt{5}}{2}\right) = (2+\sqrt{5})\cdot e^{\frac{13-\sqrt{5}}{2}} \doteq$

1 $\doteq 921{,}1$

2 $f''(x) = \left(e^{|x-3|+3}\left(2x+3+\operatorname{sgn}(x-3)(x^2+3x+2)\right)\right)' =$

$= e^{|x-3|+3} \cdot \operatorname{sgn}(x-3)\left(2x+3+\operatorname{sgn}(x-3)(x^2+3x+2)\right) +$

$+ e^{|x-3|+3} \cdot \left(2+\operatorname{sgn}(x-3)(2x+3)\right) =$

$= e^{|x-3|+3} \cdot \left(\operatorname{sgn}(x-3)(4x+6) + x^2+3x+4\right)$

$\underline{x>3}$: $f''(x) = e^x\left(x^2+7x+10\right)$ $\qquad \dfrac{-7\pm\sqrt{49-40}}{2} = \dfrac{-7\pm 3}{2} \left\langle \begin{matrix} -2 \\ -5 \end{matrix}\right.$

$\underline{x<3}$: $f''(x) = e^{6-x}\left(x^2-x-2\right)$ $\qquad \dfrac{1\pm\sqrt{1+8}}{2} = \dfrac{1\pm 3}{2} \left\langle \begin{matrix} 2 \\ -1 \end{matrix}\right.$

2

	$(-\infty,-1)$	$(-1,2)$	$(2,3)$	$(3,\infty)$
f''	+	−	+	+
f	∪	∩	∪	∪

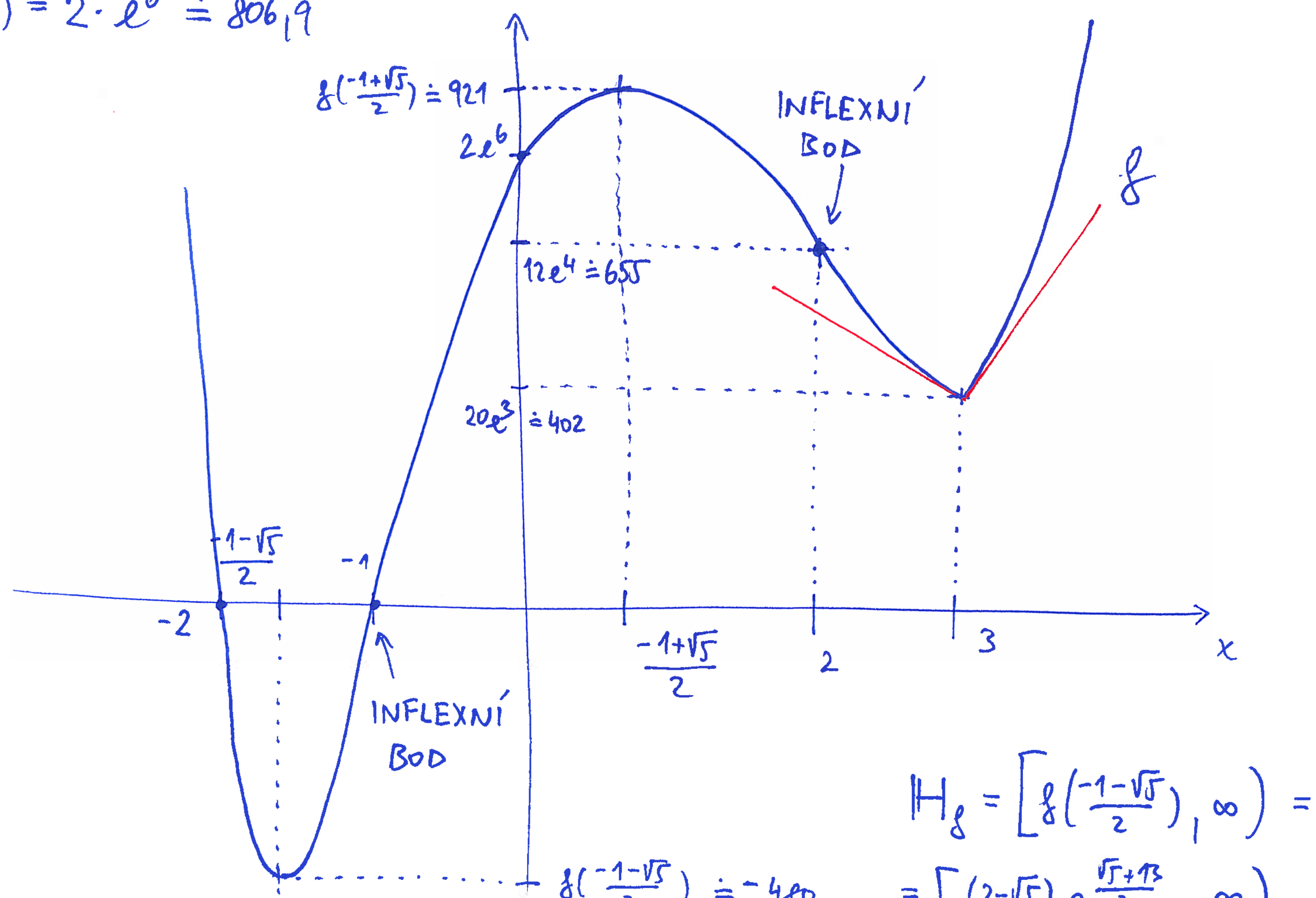
Inflexní body: $\underline{-1, 2}$.

$f(-1) = 0$, $f(2) = e^4 \cdot 12 \doteq 655{,}2$

$f(x) = 0 \iff (x^2+3x+2) = 0 \iff (x+1)(x+2) = 0 \iff x = -1 \vee x = -2$

$f(0) = 2\cdot e^6 \doteq 806{,}9$

5



$H_f = \left[f\left(\frac{-1-\sqrt{5}}{2}\right), \infty\right) =$

$= \left[(2-\sqrt{5})\,e^{\frac{\sqrt{5}+13}{2}}, \infty\right).$