1. $$\lim_{n\to\infty} \frac{\sqrt[n]{7\cdot 3^n}-3}{n\cdot\left(\cos\frac{1}{n}-1\right)} = \lim_{n\to\infty}\frac{3\cdot\left(\sqrt[n]{7}-1\right)}{n\cdot\left(\cos\frac{1}{n}-1\right)} \overset{H.V.}{=}$$

$$3\cdot\lim_{x\to\infty} \frac{7^{\frac{1}{x}}-1}{\frac{1}{x}}\cdot\frac{1}{x}\cdot\frac{1}{x\left(\cos\frac{1}{x}-1\right)} = 3\lim_{x\to\infty}\frac{e^{\frac{1}{x}\cdot\ln 7}-1}{\frac{1}{x}\cdot\ln 7}\cdot\frac{\ln 7}{x^2}\cdot\frac{1}{\cos\frac{1}{x}-1}$$

$$\overset{VOLSF}{\underset{VOAL}{=}} -3\cdot\ln 7\cdot\left(\lim_{x\to\infty}\frac{1-\cos\frac{1}{x}}{\frac{1}{x^2}}\right)^{-1} \overset{VOLSF}{=} -3\cdot\ln 7\cdot\left(\frac{1}{2}\right)^{-1} = -3\cdot 2\ln 7 = -6\ln 7$$

VOLSF: • vnitřní funkce: $g(x)=\frac{1}{x}$, $\lim_{x\to\infty} g(x)=0$, splňuje podmínku (P): dokonce $\forall x\in(0,\infty): g(x)\neq 0$.

• vnější funkce 1: $f_1(y)=\frac{e^y-1}{y}$, $\lim_{y\to 0} f_1(y)=1$ (známá).

• vnější funkce 2: $f_2(y)=\frac{1-\cos y}{y^2}$, $\lim_{y\to 0} f_2(y)=\frac{1}{2}$ (známá)

2. $$\lim_{x\to 4}\frac{\arcsin(x-4)}{\sqrt{x^2+20}-\sqrt{4x+20}} = \lim_{x\to 4}\frac{\arcsin(x-4)}{x-4}\cdot\frac{(x-4)\cdot\left(\sqrt{x^2+20}+\sqrt{4x+20}\right)}{x^2+20-(4x+20)}$$

$$\overset{VOLSF}{\underset{VOAL}{=}} 1\cdot\lim_{x\to 4}\left(\sqrt{x^2+20}+\sqrt{4x+20}\right)\cdot\lim_{x\to 4}\frac{x-4}{x^2-4x} =$$

$$= 1\cdot\left(\sqrt{36}+\sqrt{36}\right)\cdot\lim_{x\to 4}\frac{x-4}{x(x-4)} = 12\cdot\lim_{x\to 4}\frac{1}{x} = \frac{12}{4} = 3$$

VOLSF vnější $f(y)=\frac{\arcsin y}{y}$ $\lim_{y\to 0} f(y)=1$ (zn.)

vnitřní $g(x)=x-4$ $\lim_{x\to 4} g(x)=0$

(P) g je prostá (rostoucí) $\Rightarrow$ podmínka splněna.

(3.) $$\sum_{n=1}^{\infty} \operatorname{arctg} n \left(\ln\left((n+1)!+1\right) - \ln\left((n+1)!\right) \right) =$$

$$= \sum_{n=1}^{\infty} \operatorname{arctg} n \cdot \ln\left(\frac{(n+1)!+1}{(n+1)!} \right) =$$

$$= \sum_{n=1}^{\infty} \underbrace{\operatorname{arctg} n}_{\to \frac{\pi}{2}} \cdot \underbrace{\ln\left(1+\frac{1}{(n+1)!}\right)}_{\approx \frac{1}{(n+1)!}}.$$

Srovnáme s řadou $\sum b_n$, kde $b_n = \frac{1}{(n+1)!}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\operatorname{arctg} n \cdot \ln\left(1+\frac{1}{(n+1)!}\right)}{\frac{1}{(n+1)!}} \overset{\text{VOAL}}{\underset{\text{HEINE}}{=}}$$

$$= \lim_{n\to\infty} \operatorname{arctg} n \cdot \lim_{y\to 0} \frac{\ln(1+y)}{y} = \frac{\pi}{2} \cdot 1 \in (0,\infty).$$

HEINE: funkce $f(y) = \frac{\ln(1+y)}{y}$ $\lim_{y\to 0} f(y) = 1$

posloupnost $\left\{\frac{1}{(n+1)!}\right\}_{n=1}^{\infty}$ splňuje (H1): $\lim \frac{1}{(n+1)!} = 0$

i (H2), $\forall n \in \mathbb{N}: \frac{1}{(n+1)!} \neq 0$

Podle LSK: ($\sum a_n$ K. $\Leftrightarrow$ $\sum b_n$ K.)

Ale $\sum b_n = \sum \frac{1}{(n+1)!}$ K., např. pomocí srov. krit.

$\frac{1}{(n+1)!} = \frac{1}{n+1} \cdot \frac{1}{n} \cdot \frac{1}{(n-1)!} < \frac{1}{n^2}$ $(n>2)$ a $\sum \frac{1}{n^2}$ K. Tedy $\sum a_n$ K.

(4.) $2\cos^2 x + \sin^2(2x) = f(x)$

• $D_f = \mathbb{R}$, 1b spojitá na $\mathbb{R}$, sudá 1b, $\lim_{x\to\pm\infty} f(x)$ neexistují

• $f(x+\pi) = 2\cos^2(x+\pi) + \sin^2(2x+2\pi) = 2(-\cos x)^2 + \sin^2 2x =$

$= 2\cos^2 x + \sin^2 2x = f(x)$.

2b Tedy f je π - periodická $\Rightarrow$ stačí vyšetřit na $[0,\pi]$

(2π-per $\Rightarrow$ 1b.)

(dokonce, díky sudosti, stačí na $[0,\frac{\pi}{2}]$) (resp. $[0,\pi)$)

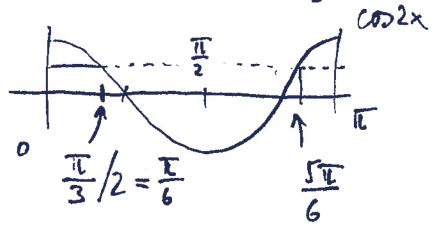
• $f'(x) = \left(2\cos^2 x + \sin^2(2x)\right)' = 2\cdot 2\cdot \cos x\,(-\sin x) + 2\sin(2x)\cos(2x)\cdot 2$

2b $= 4\sin 2x\cdot\cos 2x - 4\sin x\cos x =$

$= 4\sin 2x\cdot\cos 2x - 2\sin 2x = 2\sin 2x\,(2\cos 2x - 1)$

• $\sin 2x = 0$, $x\in[0,\pi)$: $x\in\{0,\frac{\pi}{2}\}$

• $\cos 2x = \frac{1}{2}$, $x\in[0,\pi)$: $x\in\{\frac{\pi}{6},\frac{5\pi}{6}\}$



4b

	$[0,\frac{\pi}{6}]$	$[\frac{\pi}{6},\frac{\pi}{2}]$	$[\frac{\pi}{2},\frac{5\pi}{6}]$	$[\frac{5\pi}{6},\pi]$
f'	$\oplus$	$\ominus$	$\oplus$	$\ominus$
f	↗	↘	↗	↘

$\Rightarrow$ lok max : $\frac{\pi}{6}$, $\frac{5\pi}{6}$ $(+k\pi,\ k\in\mathbb{Z})$

lok min : 0, $\frac{\pi}{2}$

• $f(\frac{\pi}{6}) = 2\cos^2\frac{\pi}{6} + \sin^2\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{9}{4}$

2b $f(\frac{5\pi}{6}) = f(-\frac{\pi}{6}) = f(\frac{\pi}{6}) = \frac{9}{4}$

• $f(0) = 2\cos^2 0 + \sin^2 0 = 2$

• $f(\frac{\pi}{2}) = 2\cos^2\frac{\pi}{2} + \sin^2\pi = 0$

$\Big\}\ H_f = [0,\frac{9}{4}]$

$f''(x) = \left(2\cdot\sin 2x\,(2\cos 2x - 1)\right)' = 2\cdot\Big(2\cos 2x\,(2\cos 2x - 1) +$

2b+ $\cancel{2}$ $\sin 2x\cdot 2\cdot 2\cdot(-\sin 2x)\Big) =$

$[\cos 2t = \cos^2 t - \sin^2 t$

$= 4\cdot\left(2\cos^2 2x - \cos 2x\ \cancel{+} - 2\sin^2 2x\right) =$

$= 4\cdot\left(2\cos 4x - \cos 2x\right) = 4\left(-\cos 2x + 4\cos^2 2x - 2\right)$

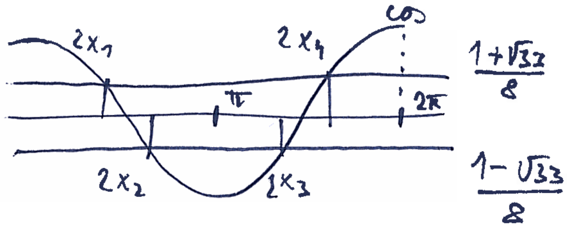
$4\cos^2 2x - \cos 2x - 2 = 0 \rightsquigarrow$ SUBST. $\rightsquigarrow 4a^2 - a - 2 = 0$

$$a_{1,2} = \frac{1 \pm \sqrt{1+32}}{8} = \begin{cases} \frac{1+\sqrt{33}}{8} \\ \frac{1-\sqrt{33}}{8} \end{cases}$$

$$\left[4\left(-\left(\cos^2 x - \sin^2 x\right) + 4\cos^2 2x - 2\right) = 4\left(1 - 2\cos^2 x + 4\cos^2 2x - 2\right) = 16\cos^2 2x - 8\cos^2 x - 4 \right]$$

$$\frac{1+\sqrt{33}}{8} \in \left(\frac{1+\sqrt{25}}{8}, \frac{1+\sqrt{36}}{8}\right) = \left(\frac{6}{8}, \frac{7}{8}\right), \quad \frac{1-\sqrt{33}}{8} \in \left(\frac{1-\sqrt{36}}{8}, \frac{1-\sqrt{25}}{8}\right) = \left(\frac{-5}{8}, -\frac{1}{2}\right)$$

$$\cos 2x = \frac{1+\sqrt{33}}{8}$$



1b $2x_1 = \arccos \frac{1+\sqrt{33}}{8}$

$2x_4 = 2\pi - \arccos \frac{1+\sqrt{33}}{8}$

$2x_2 = \arccos \frac{1-\sqrt{33}}{8}$

$2x_3 = 2\pi - \arccos \frac{1-\sqrt{33}}{8}$

x_1, x_2, x_3, x_4 inflexní body

5b