A1 $\lim\limits_{x\to 1} \dfrac{\ln^2 x^3}{\sin(x-1)(x^2-1)} = \lim\limits_{x\to 1} \dfrac{x-1}{\sin(x-1)} \cdot \dfrac{1}{(x-1)(x-1)(x+1)} \cdot (\ln x^3)^2 =$

$$= \lim_{x\to 1}\left(\frac{\sin(x-1)}{x-1}\right)^{-1} \cdot \lim \frac{1}{x+1} \cdot \lim\left(\frac{\ln x^3}{x^3-1}\right)^2 \cdot \frac{(x^3-1)^2}{(x-1)^2} =$$

$$= 1^{-1} \cdot \frac{1}{1+1} \cdot 1^2 \cdot \left(\lim \frac{x^3-1}{x-1}\right)^2 = \frac{1}{2} \cdot \left(\lim_{x\to 1}(x^2+x+1)\right)^2 = \frac{1}{2}\cdot 3^2 = \frac{9}{2}$$

VOLSF : • $\dfrac{\sin(x-1)}{x-1} \to 1$ jasné. (vnitřní fce lineární $\Rightarrow$ (P) ok)

• vnější $f(y) = \dfrac{\ln y}{y-1}$ $\quad \lim\limits_{y\to 1} f(y) = 1$ (známá)

vnitřní $g(x) = x^3$ $\quad \lim\limits_{x\to 1} g(x) = 1$

(P) g (vnitřní fce) je prostá. $\Rightarrow \lim\limits_{x\to 1} \dfrac{\ln x^3}{x^3-1} = 1$

A2 $\left(5 \cdot \dfrac{\cos^3 \ln^2 x}{x} + 1337\right)' = 5 \cdot \left(\dfrac{\left(\cos\left((\ln x)^2\right)\right)^3}{x}\right)' + 0 =$

$$= 5 \cdot \frac{\left[\left(\cos\left((\ln x)^2\right)\right)^3\right]' \cdot x - \cos^3 \ln^2 x \cdot 1}{x^2},$$

kde $\left[\left(\cos\left((\ln x)^2\right)\right)^3\right]' = \underbrace{3\left(\cos \ln^2 x\right)^2}_{\text{der. }(\cdot)^3} \cdot \underbrace{\left(-\sin\left(\ln^2 x\right)\right)}_{\text{der. cos}} \cdot \underbrace{2\ln x}_{\text{der }(\cdot)^2} \cdot \overbrace{\frac{1}{x}}^{\text{der. ln}x}.$

A1 jinak $\lim\limits_{x\to 1} \dfrac{(\ln x^3)^2}{\sin(x-1)(x-1)(x+1)} = \lim\limits_{x\to 1} \dfrac{3\ln x \cdot 3\ln x}{(x-1)\cdot(x-1)\cdot \frac{\sin(x-1)}{x-1} \cdot (x+1)}$

$$= 9 \cdot \left(\lim_{x\to 1} \frac{\ln x}{x-1}\right)^2 \cdot \lim_{x\to 1} \frac{x-1}{\sin(x-1)} \cdot \lim_{x\to 1} \frac{1}{x+1} = 9 \cdot 1^2 \cdot \frac{1}{1} \cdot \frac{1}{2} = \frac{9}{2}$$

A3 $\displaystyle\sum_{n=3}^{\infty} (-1)^n \sin\left(n^{-\frac{2}{3}}\right)\cdot\left(\sqrt{n}-\sqrt{n-3}\right) =: \sum_{n=3}^{\infty} a_n$

$$a_n = (-1)^n \underbrace{\sin\Big(\underbrace{\frac{1}{n^{2/3}}}_{\in(0,1]}\Big)}_{>0}\cdot \underbrace{\frac{3}{\sqrt{n}+\sqrt{n-3}}}_{>0}$$

$$|a_n| = \underbrace{\sin\left(\frac{1}{n^{2/3}}\right)}_{\approx \frac{1}{n^{2/3}}}\cdot\underbrace{\frac{3}{\sqrt{n}}}_{\frac{3}{\sqrt{n}}}\cdot\underbrace{\frac{1}{1+\sqrt{1-\frac{3}{n}}}}_{\to \frac{1}{1+1}}$$

Pro vyšetření AK (tj. konvergence řady $\sum |a_n|$) tedy volíme srovnání s řadou $\sum b_n$, kde $b_n = \frac{1}{n^{2/3}}\cdot\frac{1}{\sqrt{n}}$, tj. $b_n = \frac{1}{n^{7/6}}$, a $\sum b_n$ tedy K.

$$\lim_{n\to\infty}\frac{|a_n|}{b_n} = \lim_{n\to\infty}\frac{\sin\left(\frac{1}{n^{2/3}}\right)\cdot\frac{3}{\cancel{\sqrt{n}}}\cdot\frac{1}{1+\sqrt{1-\frac{3}{n}}}}{\frac{1}{n^{2/3}}\cdot\frac{1}{\cancel{\sqrt{n}}}} =$$

$$= 1\cdot 3\cdot\lim_{n\to\infty}\frac{1}{1+\sqrt{1-\frac{3}{n}}} = \frac{3}{2}\in(0,\infty).$$

LSK v této situaci říká: $\left(\sum |a_n| \text{ K} \iff \sum b_n \text{ K}\right)$

Ale $\sum b_n$ K (jak víme). Tedy $\sum |a_n|$ K.

Řada tedy AK (a tedy i K.).

B1

$$\lim_{x\to\infty}\left(\frac{x^5+3x^3}{x^5+x^3}\right)^{3x^2+5x} = \lim_{x\to\infty} e^{(3x^2+5x)\cdot\ln\left(\frac{x^5+3x^3}{x^5+x^3}\right)}$$

$$\lim_{x\to\infty}(3x^2+5x)\cdot\ln\left(1+\frac{2x^3}{x^5+x^3}\right)=$$

$$=\lim_{x\to\infty}(3x^2+5x)\cdot\underbrace{\frac{\ln\left(1+\frac{2}{x^2+1}\right)}{\frac{2}{x^2+1}}}_{\to 1 \text{ (VOLSF + známá lim.)}}\cdot\frac{2}{x^2+1} = 1\cdot\lim_{x\to\infty}\frac{6x^2+10x}{x^2+1}$$

$$=\lim_{x\to\infty}\frac{6+\frac{10}{x}}{1+\frac{1}{x^2}} = \frac{6+0}{1+0} = 6$$

Celkem: $\lim\ldots = \underline{\underline{e^6}}$

VOLSF: $f(y)=\frac{\ln(1+y)}{y}$, $\lim_{y\to 0} f(y) = 1$

vnitřní: $g(x)=\frac{2}{x^2+1}$, $\lim_{x\to\infty} g(x)=0$

(P) splněna triv., protože $\forall x\in\mathbb{R}: g(x)\neq 0$.

B2

$$\left[\operatorname{arctg}\left(\frac{x^2-1}{x^2+1}\right)+\ln\left(\sin\left((2x+3)^4\right)\right)\right]'=$$

$$=\frac{1}{1+\left(\frac{x^2-1}{x^2+1}\right)^2}\cdot\frac{2x(x^2+1)-(x^2-1)\cdot 2x}{(x^2+1)^2}+\frac{1}{\sin\left((2x+3)^4\right)}\cdot\cos\left((2x+3)^4\right)\cdot$$

$$\cdot 4(2x+3)^3\cdot 2.$$

B3 $\displaystyle\sum_{n=1}^{\infty} \frac{4^n \cdot n^{4n}}{\left(\frac{3+n}{2+n}\right)^{n^2} \cdot (7n+2n^4)^n} =: \sum_{n=1}^{\infty} a_n .$

Cauchy: $\displaystyle\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{4 \cdot n^4}{\left(1+\frac{1}{2+n}\right)^n \cdot (7n+2n^4)}$

$\displaystyle = 4 \cdot \lim_{n\to\infty} \frac{\left(1+\frac{1}{2+n}\right)^2}{\left(1+\frac{1}{2+n}\right)^{2+n}} \cdot \lim_{n\to\infty} \frac{n^4}{7n+2n^4} =$

$\displaystyle = 4 \cdot \frac{1^2}{e} \cdot \frac{1}{2} = \frac{2}{e} < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \ K.$

podle Cauchyova odmocninového kritéria.

C1 $\lim_{x \to 1} \frac{\sin(x^4 - x^3 - x + 1)}{x^2 - 2x + 1} =$

$= \lim_{x \to 1} \frac{\sin(x^4 - x^3 - x + 1)}{x^4 - x^3 - x + 1} \cdot \frac{x^4 - x^3 - x + 1}{(x-1)^2} \underline{\underline{=}}$ VOLSF + VOAL

VOLSF: vnější $f(y) = \frac{\sin y}{y}$, $\lim_{y \to 0} f(y) = 1$

vnitřní $g(x) = x^4 - x^3 - x + 1$, $\lim_{x \to 1} g(x) = 0$

(P): g je polynom stupně 4 $\Rightarrow$ má nejvýše 4 kořeny, z nichž jeden je přímo bod 1.

Najdeme $\delta > 0$ tak malé aby $P(1, \delta)$ neobsahovalo žádný ze zbývajících (nejvýše) 3 kořenů polynomu.

Pak $\forall x \in P(1, \delta): g(x) \neq \lim_{x \to 1} g(x) = 0$.

TENTO ARGUMENT LZE POUŽÍT PRO LIB. POLYNOM

$= 1 \cdot \lim_{x \to 1} \frac{x^4 - x^3 - x + 1}{(x-1)^2} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x^3-1)}{(x-1)^{\cancel{2}}} = \lim_{x \to 1} \frac{x^3 - 1}{x - 1} =$

$(x^4 - x^3 - x + 1) : (x - 1) = x^3 - 1$

$-(x^4 - x^3)$

$-x + 1$

Tedy $x^4 - x^3 - x + 1 = (x-1)(x^3-1)$ *

$= \lim_{x \to 1} \frac{(x-1)(x^2 + x + 1)}{x - 1} = \lim_{x \to 1} (x^2 + x + 1) = 1 + 1 + 1 = \underline{\underline{3}}$

C2 $\quad f(x) = \left(\operatorname{tg}^2 x\right)^{6x^3 + \operatorname{arccotg} x} = e^{(6x^3+\operatorname{arccotg} x)\cdot \ln(\operatorname{tg}^2 x)}$

$$f'(x) = e^{(6x^3+\operatorname{arccotg} x)\cdot \ln(\operatorname{tg}^2 x)} \cdot \Big((6x^3+\operatorname{arccotg} x)\cdot 2\ln(\operatorname{tg} x)\Big)'$$

$$= \left(\operatorname{tg}^2 x\right)^{6x^3+\operatorname{arccotg} x} \cdot \left[\left(18x^2 + \frac{-1}{1+x^2}\right)\cdot 2\ln(\operatorname{tg} x) + (6x^3+\operatorname{arccotg} x)\cdot 2\cdot \frac{1}{\operatorname{tg} x}\cdot\frac{1}{\cos^2 x}\right].$$

C3 $\quad \displaystyle\sum_{n=1}^{\infty} \underbrace{\operatorname{arctg} n}_{\to \frac{\pi}{2} \dots \text{ nehraje roli}} \cdot \underbrace{\ln\left(1+\frac{1}{n^4}\right)}_{\approx \frac{1}{n^4}} \underbrace{\left(\sqrt{n^4+n^{-4}} - \sqrt{n^3+n^{-4}}\right)}_{(*)\ \approx \sqrt{n^4}}$

(∗): Zde není potřeba rozšiřovat součtem $\sqrt{\dots}+\sqrt{\dots}$, protože tyto odmocniny nejsou „stejně rychlé", tj. je mezi nimi jasný převládající člen. (a to $\sqrt{n^4+n^{-4}}$).

Srovnáme s řadou $\sum b_n$, kde $b_n = \frac{1}{n^4}\cdot\sqrt{n^4} = \frac{1}{n^2}$.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\operatorname{arctg} n \cdot \ln\left(1+\frac{1}{n^4}\right)\cdot \cancel{\sqrt{n^4}}\left(\sqrt{1+n^{-8}} - \sqrt{n^{-1}+n^{-8}}\right)}{\frac{1}{n^4}\cdot\cancel{\sqrt{n^4}}}$$

$\to 1$

$$= \frac{\pi}{2}\cdot 1\cdot \lim_{n\to\infty}\left(\sqrt{1+n^{-8}} - \sqrt{n^{-1}+n^{-8}}\right) = \frac{\pi}{2}\cdot\left(\sqrt{1+0}-\sqrt{0+0}\right) = \frac{\pi}{2} \in (0,\infty).$$

LSK tedy dává, že $\left(\sum a_n \text{ K.} \iff \sum b_n \text{ K.}\right)$.

Ovšem $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ konverguje.

Tedy $\displaystyle\sum_{n=1}^{\infty} a_n$ K.

D1

$$\lim_{x\to 0}\frac{3x^3+2x^2}{e^{\cos x-1}-1}=\lim_{x\to 0}(3x+2)\cdot x^2\cdot\frac{\cos x-1}{e^{\cos x-1}-1}\cdot\frac{1}{\cos x-1}$$

$$=(3\cdot 0+2)\cdot\left(\lim_{x\to 0}\frac{e^{\cos x-1}-1}{\cos x-1}\right)^{-1}\cdot\left(-\lim_{x\to 0}\frac{1-\cos x}{x^2}\right)^{-1}=$$

$$=2\cdot(1)^{-1}\cdot\left(-\frac{1}{2}\right)^{-1}=-4$$

VOLSF: • vnější $f(y)=\dfrac{e^y-1}{y}$, $\lim_{y\to 0}f(y)=1$ (známá)

vnitřní $g(x)=\cos x-1$, $\lim_{x\to 0}g(x)=0$

(P): Protože na $P(0,\frac{\pi}{2})$ je $\cos x\neq 1$, jest na $P(0,\frac{\pi}{2})$ $g(x)\neq 0$. ✓

D2

$$\left(\arcsin\left(\cos(3x^4+6)\right)\cdot(7x+11)^5\right)'=$$

$$=\underbrace{\frac{1}{\sqrt{1-\left(\cos(3x^4+6)\right)^2}}}_{\text{der. }\arcsin(\cdot)}\cdot\underbrace{\left(-\sin(3x^4+6)\right)}_{\text{der. }\cos(\cdot)}\cdot\underbrace{(12x^3)}_{(3x^4+6)'}\cdot(7x+11)^5+$$

$$+\arcsin\left(\cos(3x^4+6)\right)\cdot\underbrace{5(7x+11)^4}_{\text{der }(\cdot)^5}\cdot\underbrace{7}_{(7x+11)'}$$

D3 $\sum_{n=1}^{\infty} \underbrace{\left(\sqrt[n]{9}-1\right)}_{(*)} \underbrace{(n-\sqrt{n})}_{\approx n} \cdot \underbrace{\sin\frac{1}{n^2}}_{\approx \frac{1}{n^2}}$

$(*)\ \sqrt[n]{9}-1 = 9^{\frac{1}{n}}-1 = e^{\frac{1}{n}\ln 9}-1 = \underbrace{\frac{e^{\frac{1}{n}\ln 9}-1}{\frac{1}{n}\ln 9}}_{\approx 1} \cdot \frac{1}{n}\ln 9$

Srovnáme s řadou $\sum_{n=1}^{\infty} b_n$, kde $b_n = \frac{1}{n}\cdot n \cdot \frac{1}{n^2}$

LSK: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(e^{\frac{1}{n}\ln 9}-1\right)(n-\sqrt{n})\cdot \sin\frac{1}{n^2}}{\frac{1}{n}\cdot n \cdot \frac{1}{n^2}}$

$= \lim\left(\ln 9 \cdot \frac{e^{\frac{1}{n}\ln 9}-1}{\frac{1}{n}\cdot \ln 9}\right) \cdot \lim \frac{n-\sqrt{n}}{n} \cdot \lim \frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}} = 1\cdot 1\cdot 1\cdot \ln 9$

$\hookrightarrow \in (0,\infty)$

Tedy $\left(\sum_{n=1} a_n \text{ K} \iff \sum b_n \text{ K.}\right)$

Avšem $\sum_{n=1}^{\infty} b_n = \sum \frac{1}{n}\cdot n \cdot \frac{1}{n^2} = \sum \frac{1}{n^2}$ konverguje.

Tedy $\sum a_n$ K. (a tedy AK, neboť $a_n \geq 0$).