Předmět: NMTM101 Matematická analýza I

Typ výuky: Cvičení

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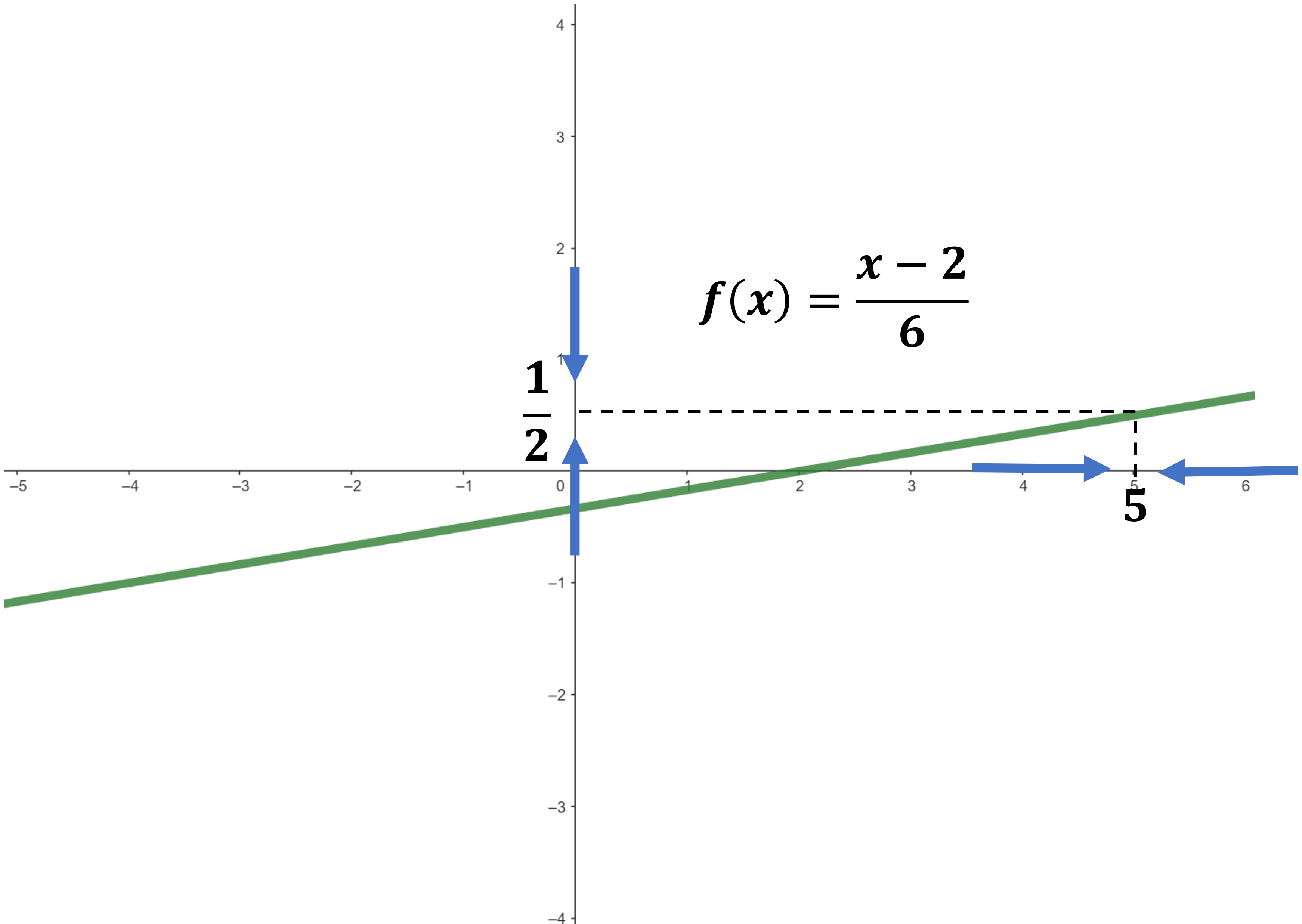
Limita funkce

Při řešení limity funkcí můžeme díky Heineho větě využívat již známých vět jako jsou:

Věta o aritmetice limit, věty o policajtech, nulová · omezená apod.

$$\lim_{x\to 5}\frac{x-2}{6} = \frac{5-2}{6} = \frac{1}{2}$$



$$\lim_{x\to\pi}\frac{\cos x}{x} = \frac{\cos\pi}{\pi} = \frac{-1}{\pi}$$

$$\lim_{x\to\infty}\frac{4x^2+\log x}{3x^2} = \lim_{x\to\infty}\frac{x^2\left(4+\frac{\log x}{x^2}\right)}{3x^2} = \frac{4}{3}$$

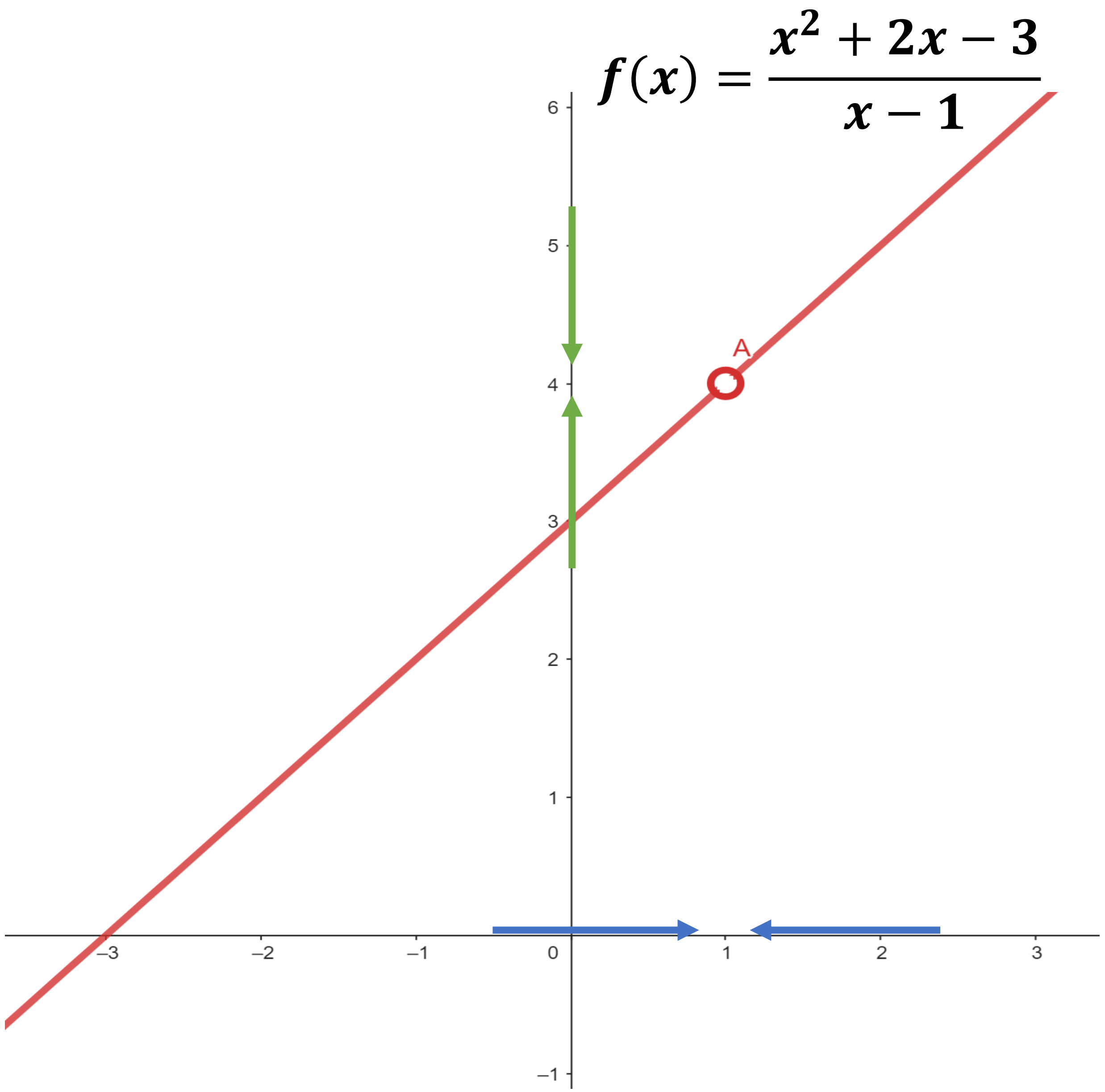
($\frac{\log x}{x^2} \longrightarrow 0$)

$$\lim_{x\to\infty}(2^x-3^x) = \lim_{x\to\infty}3^x\left(\left(\frac{2}{3}\right)^x-1\right) = \lim_{x\to\infty}3^x\lim_{x\to\infty}\left(\left(\frac{2}{3}\right)^x-1\right) = \infty(0-1) = -\infty$$

($\left(\frac{2}{3}\right)^x \longrightarrow 0$)

$$\lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1} \qquad \frac{0}{0}$$

$$\lim_{x \to 1} \frac{(x - 1)(x + 3)}{x - 1} = 4$$



$$\lim_{x\to\infty} \frac{\log(1+2^x)}{3x} \qquad \frac{\infty}{\infty}$$

$$\log a.b = \log a + \log b$$

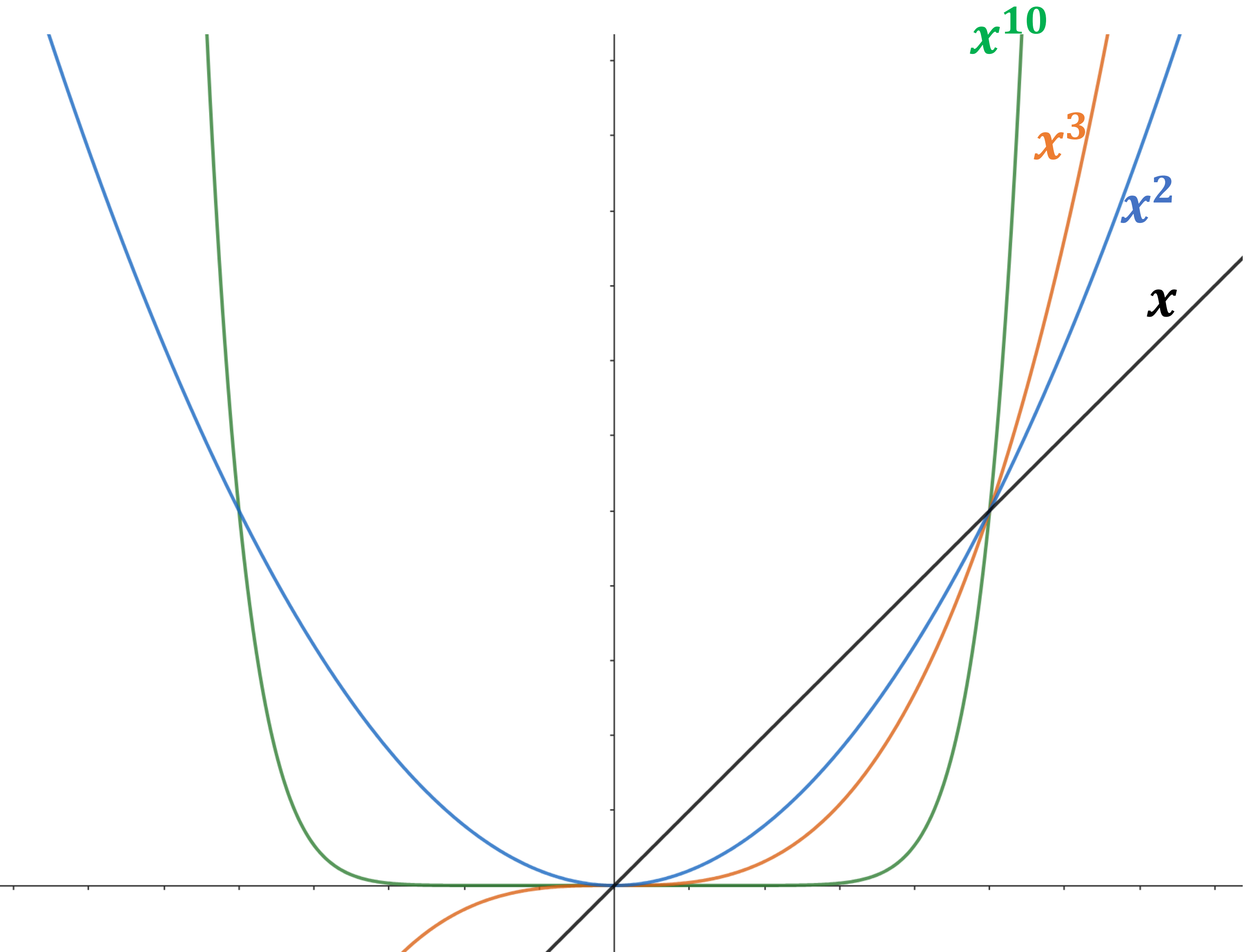
$$\log a^x = x \log a$$

$$\lim_{x\to\infty} \frac{\log(1+2^x)}{3x} = \lim_{x\to\infty} \frac{\log(2^x(\frac{1}{2^x}+1))}{3x} = \lim_{x\to\infty} \frac{\log 2^x + \log(\frac{1}{2^x}+1)}{3x}$$

$$= \lim_{x\to\infty} \frac{x \log 2}{3x} + \lim_{x\to\infty} \frac{\log(\frac{1}{2^x}+1)}{3x} = \frac{\log 2}{3} + \frac{\lim_{x\to\infty} \log(\frac{1}{2^x}+1)}{\lim_{x\to\infty} 3x} = \frac{\log 2}{3} + 0 = \frac{\log 2}{3}$$

$$\lim_{x\to\infty} \frac{x^{10} - 2x^3 + x^2 + 3x}{2x^{10} + 2x} = \frac{1}{2}$$

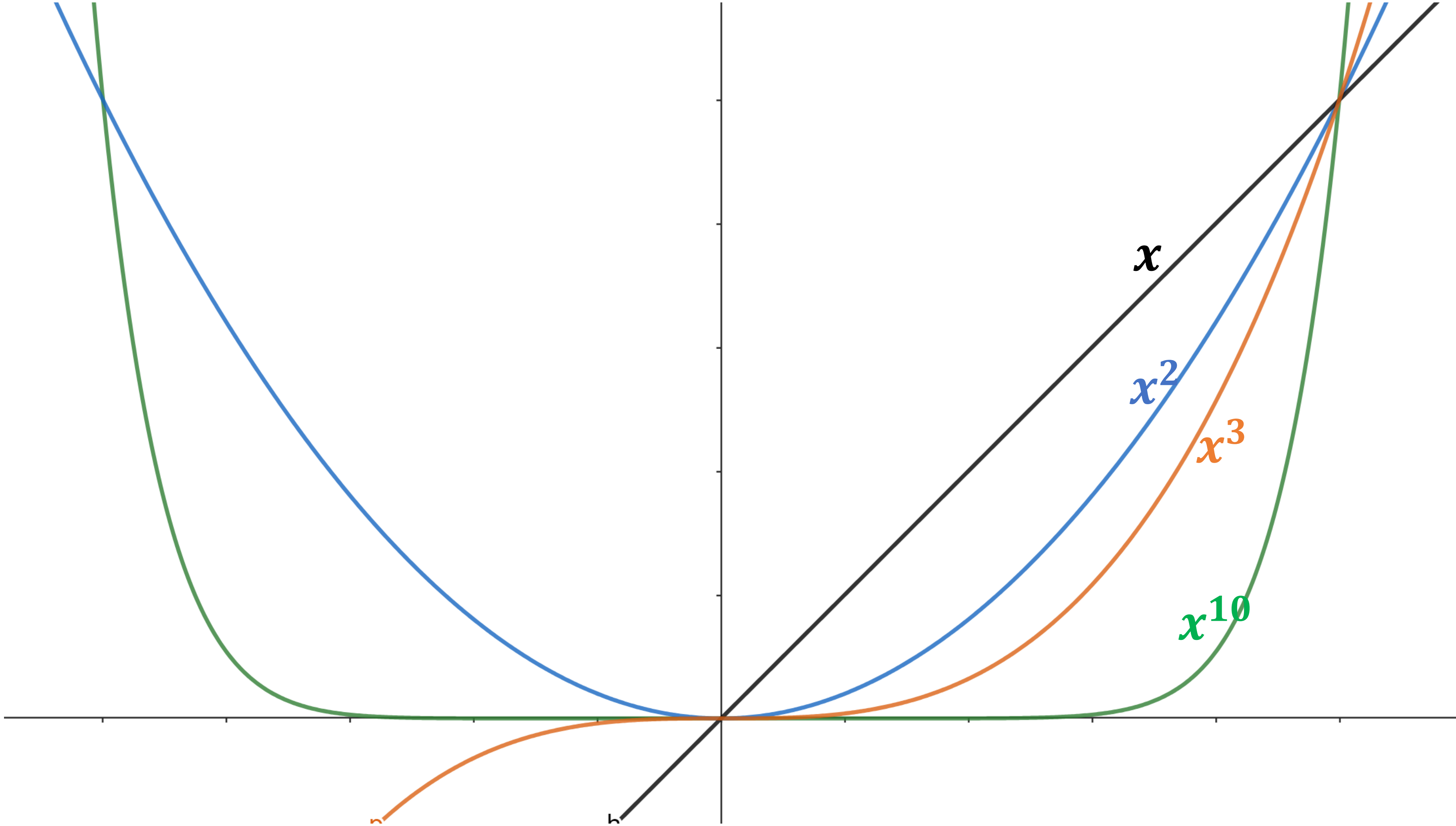
$$\lim_{x\to 0} \frac{x^{10} - 2x^3 + x^2 + 3x}{2x^{10} + 2x}$$



$$\lim_{x \to 0} \frac{x^{10} - 2x^3 + x^2 + 3x}{2x^{10} + 2x} \rightarrow \boxed{\frac{3}{2}}$$

$$\lim_{x \to 0} \frac{\cancel{x}(x^9 - 2x^2 + x + 3)}{\cancel{x}(2x^9 + 2)}$$

$$= \frac{3}{2}$$



$$\lim_{x\to 0}\frac{(1+x)(1+2x)(1+3x)-1}{x} \qquad \frac{0}{0}$$

$$\lim_{x\to 0}\frac{Ax^3+Bx^2+Cx+1}{x}=\lim_{x\to 0}\frac{x(Ax^2+Bx+C)}{x}=C=6$$

$C=?$

$$(1+x)(1+2x)(1+3x)\quad\begin{cases} x.1.1=x \\ 1.2x.1=2x \\ 1.1.3x=3x \end{cases}\qquad x+2x+3x=6x$$

$C=6$

$$\lim_{x \to 0} \frac{(1+x)(1+2x)(1+3x)\ldots(1+nx)-1}{x} = \frac{n(n+1)}{2}$$

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\lim_{x\to 1}\frac{x^3-3x+2}{x^4-4x+3} \quad \frac{0}{0}$$

$(x-1)$

$$x^3-3x+2 \div (x-1) = x^2+x-2$$

$$\begin{array}{l} x^3-3x+2 \\ -\;\underline{x^3-x^2} \\ x^2-3x+2 \\ -\;\underline{x^2-x} \\ -2x+2 \\ -\;\underline{-2x+2} \\ 0 \end{array}$$

$$x^3-3x+2=(x-1)(x^2+x-2)$$

$$x^4-4x+3=(x-1)(x^3+x^2+x-3)$$

$$x^3 - 3x + 2 = (x-1)(x^2 + x - 2)$$

$$x^4 - 4x + 3 = (x-1)(x^3 + x^2 + x - 3)$$

$$\lim_{x\to 1} \frac{x^3 - 3x + 2}{x^4 - 4x + 3} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2 + x - 2)}{\cancel{(x-1)}(x^3 + x^2 + x - 3)} \qquad \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+2)}{\cancel{(x-1)}(x^2 + 2x + 3)} = \frac{3}{6} = \frac{1}{2}$$

$$\lim_{x \to -7} \frac{x+7}{x^2+14x+49} \qquad \frac{0}{0}$$

$$= \lim_{x \to -7} \frac{x+7}{(x+7)^2} = \lim_{x \to -7} \frac{1}{x+7} \quad \textbf{neexistuje}$$

$$(x-1)$$

$$\lim_{x\to 1}\frac{x^{100}-2x+1}{x^{50}-2x+1} \qquad \frac{0}{0}$$

$$x^{100}-2x+1 = (x-1)(x^{99}+x^{98}+x^{97}+\cdots+x-1)$$

$$x^{50}-2x+1 = (x-1)(x^{49}+x^{48}+x^{47}+\cdots+x-1)$$

$$=\lim_{x\to 1}\frac{\cancel{(x-1)}(x^{99}+x^{98}+x^{97}+\cdots+x-1)}{\cancel{(x-1)}(x^{49}+x^{48}+x^{47}+\cdots+x-1)} = \frac{99-1}{49-1} = \frac{98}{48} = \frac{49}{24}$$

$$(x-1)$$

$$\lim_{x\to 1}\frac{x^n-1}{x^m-1} \qquad \frac{0}{0}$$

$$A^n - B^n = (A-B)(A^{n-1}B^0 + A^{n-2}B^1 + \ldots + A^1B^{n-2} + A^0B^{n-1})$$

$$x^n - 1 = (x-1)\left(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1\right)$$

$$x^m - 1 = (x-1)\left(x^{m-1} + x^{m-2} + x^{m-3} + \cdots + x + 1\right)$$

$$= \lim_{x\to 1}\frac{\cancel{(x-1)}\left(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1\right)}{\cancel{(x-1)}\left(x^{m-1} + x^{m-2} + x^{m-3} + \cdots + x + 1\right)} = \frac{n}{m}$$

$$(x-1)$$

$$\lim_{x\to 1}\frac{x+x^2+x^3+\cdots+x^n-n}{x-1} \qquad \frac{0}{0}$$

$$=\lim_{x\to 1}\frac{x+x^2+x^3+\cdots+x^n-1-1-1\ldots-1}{x-1}$$

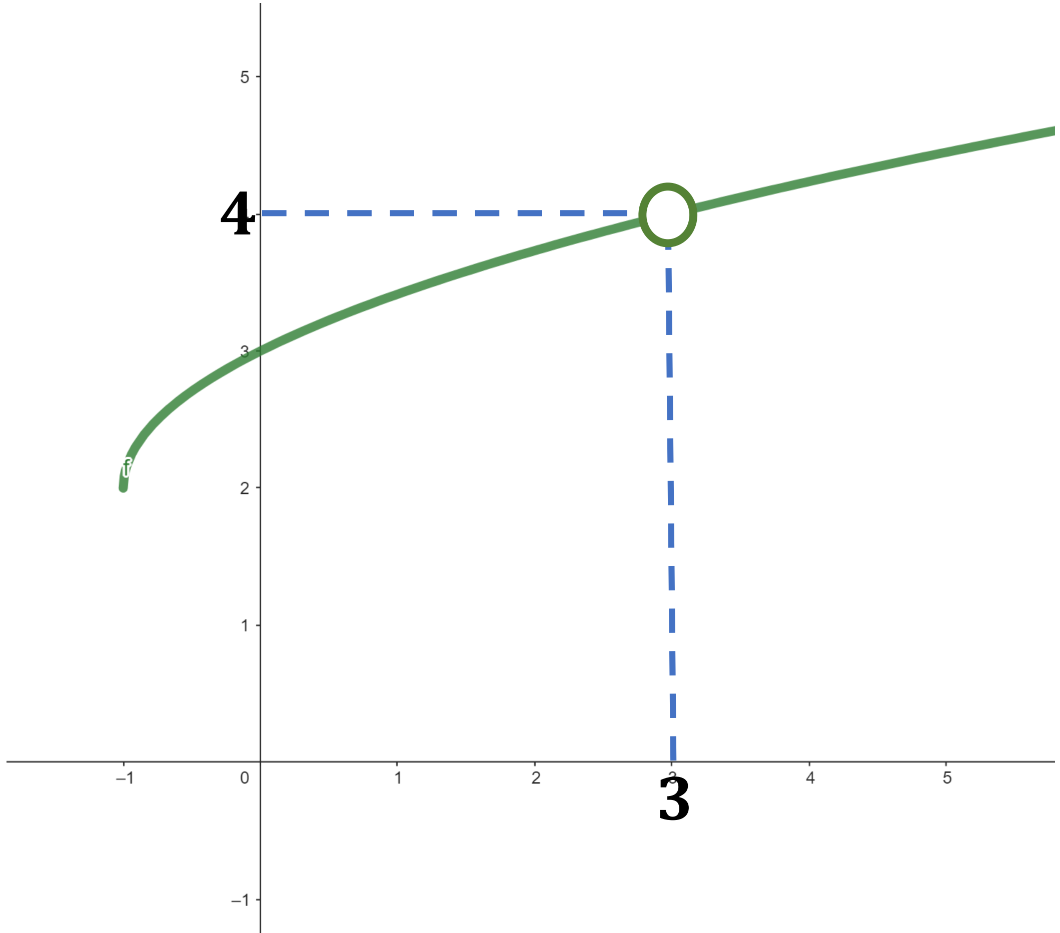
$$=\lim_{x\to 1}\frac{(x-1)+(x^2-1)+(x^3-1)\ldots+(x^n-1)}{x-1}$$

$$=\lim_{x\to 1}\frac{\cancel{(x-1)}+\cancel{(x-1)}(x+1)+\cancel{(x-1)}(x^2+x+1)\ \ldots+\cancel{(x-1)}(x^{n-1}+\cdots+x+1)}{\cancel{x-1}}$$

$$=1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2} \qquad \frac{0}{0} \qquad f(x)=\frac{x-3}{\sqrt{x+1}-2} \qquad g(x)=\sqrt{x+1}+2$$

$$=\lim_{x\to 3}\frac{(x-3)(\sqrt{x+1}+2)}{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}=\lim_{x\to 3}\frac{\cancel{(x-3)}(\sqrt{x+1}+2)}{\cancel{x+1-4}}=4$$



$$\lim_{x\to 4}\frac{\sqrt{2x+1}-3}{\sqrt{x}-2} \qquad \frac{0}{0}$$

$$=\lim_{x\to 4}\frac{(\sqrt{2x+1}-3)(\sqrt{2x+1}+3)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)(\sqrt{2x+1}+3)}$$

$$=\lim_{x\to 4}\frac{\overset{2(x-4)}{(2x+1-9)}(\sqrt{x}+2)}{(x-4)(\sqrt{2x+1}+3)}=\frac{2\times 4}{3+3}=\frac{8}{6}=\frac{4}{3}$$

$$\lim_{x \to 16} \frac{\sqrt[4]{x} - 2}{\sqrt{x} - 4} \qquad \frac{0}{0}$$

$$= \lim_{x \to 16} \frac{(\sqrt[4]{x} - 2)}{(\sqrt[4]{x} - 2)(\sqrt[4]{x} + 2)} = \frac{1}{2 + 2} = \frac{1}{4}$$

$$\lim_{x\to 7}\frac{\sqrt{x+2}-\sqrt[3]{x+20}}{\sqrt[4]{x+9}-2}$$

$\frac{0}{0}$

$(x-7)$

$\sqrt{x+2}=A$ $\sqrt[3]{x+20}=B$ $A^6-B^6=(A-B)(A^5+A^4B+A^3B^2+\cdots+B^5)$

$$(A-B)(A^5+A^4B+A^3B^2+\cdots+B^5)=A^6-B^6=(\sqrt{x+2})^6-(\sqrt[3]{x+20})^6$$

$$=(x+2)^3-(x+20)^2=x^3+5x^2-28x-392=(x-7)(x^2+12x+56)$$

$\sqrt[4]{x+9}=C$ $2=D$ $C^4-D^4=(C-D)(C^3+C^2D+CD^2+D^3)$

$$(C-D)(C^3+C^2D+CD^2+D^3)=C^4-D^4=(\sqrt[4]{x+9})^4-(2)^4$$

$$=x+9-16=x-7$$

$$\sqrt{x+2} = A \quad \sqrt[3]{x+20} = B \quad \sqrt[4]{x+9} = C \quad 2 = D$$

$$x = 7 \Rightarrow A = 3, B = 3 \qquad x = 7 \Rightarrow C = 2$$

$$\lim_{x\to 7} \frac{(\sqrt{x+2} - \sqrt[3]{x+20})(A^5 + A^4B + \cdots + B^5)(C^3 + C^2D + \cdots + D^3)}{(\sqrt[4]{x+9} - 2)(A^5 + A^4B + \cdots + B^5)(C^3 + C^2D + \cdots + D^3)}$$

$$\lim_{x\to 7} \frac{(A^6 - B^6)(C^3 + C^2D + \cdots + D^3)}{(C^4 - D^4)(A^5 + A^4B + \cdots + B^5)} = \lim_{x\to 7} \frac{(x-7)(x^2 + 12x + 56)(C^3 + C^2D + \cdots + D^3)}{(x-7)(A^5 + A^4B + \cdots + B^5)}$$

$$(49 + 12 \times 7 + 56) \quad (8 + 8 + 8 + 8)$$

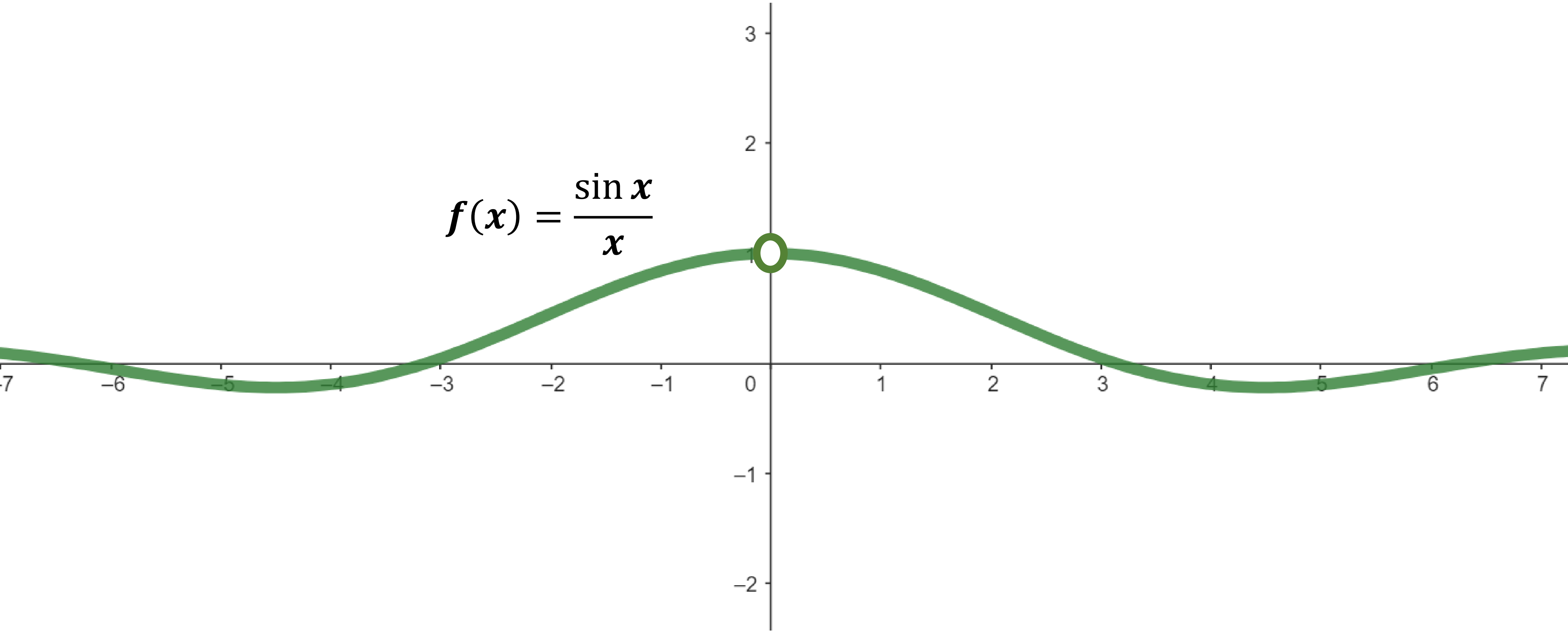
$$(3^5 + 3^5 + \cdots + 3^5)$$

$$(27)$$

$$= \frac{7 \times (7 + 12 + 8) \times 4 \times 8}{6 \times 3^5} = \frac{112}{27}$$

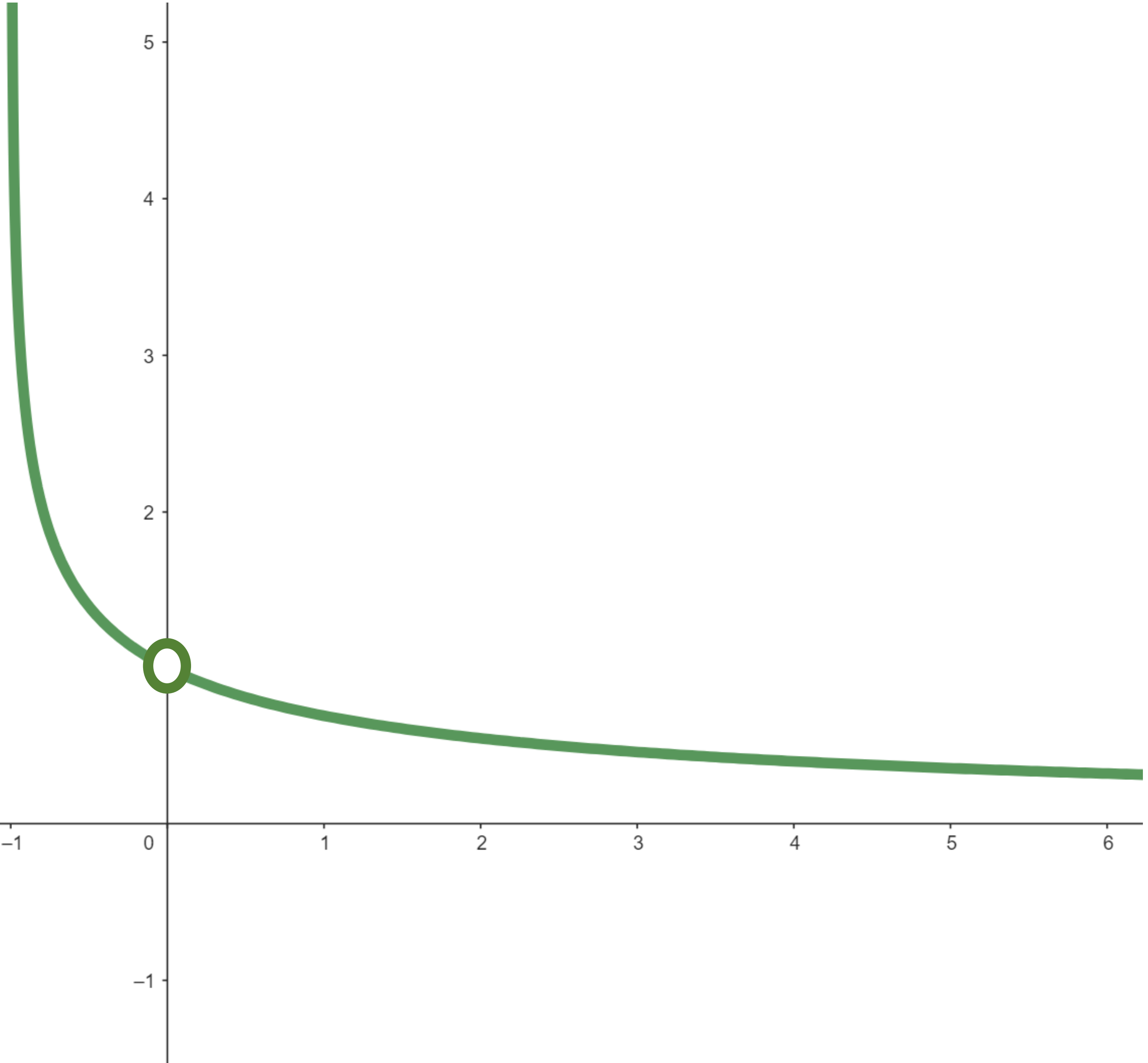
Poučka. Při vyhodnocení limit funkcí se často využívají následující výsledky. Jde o tzv. známé limity:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

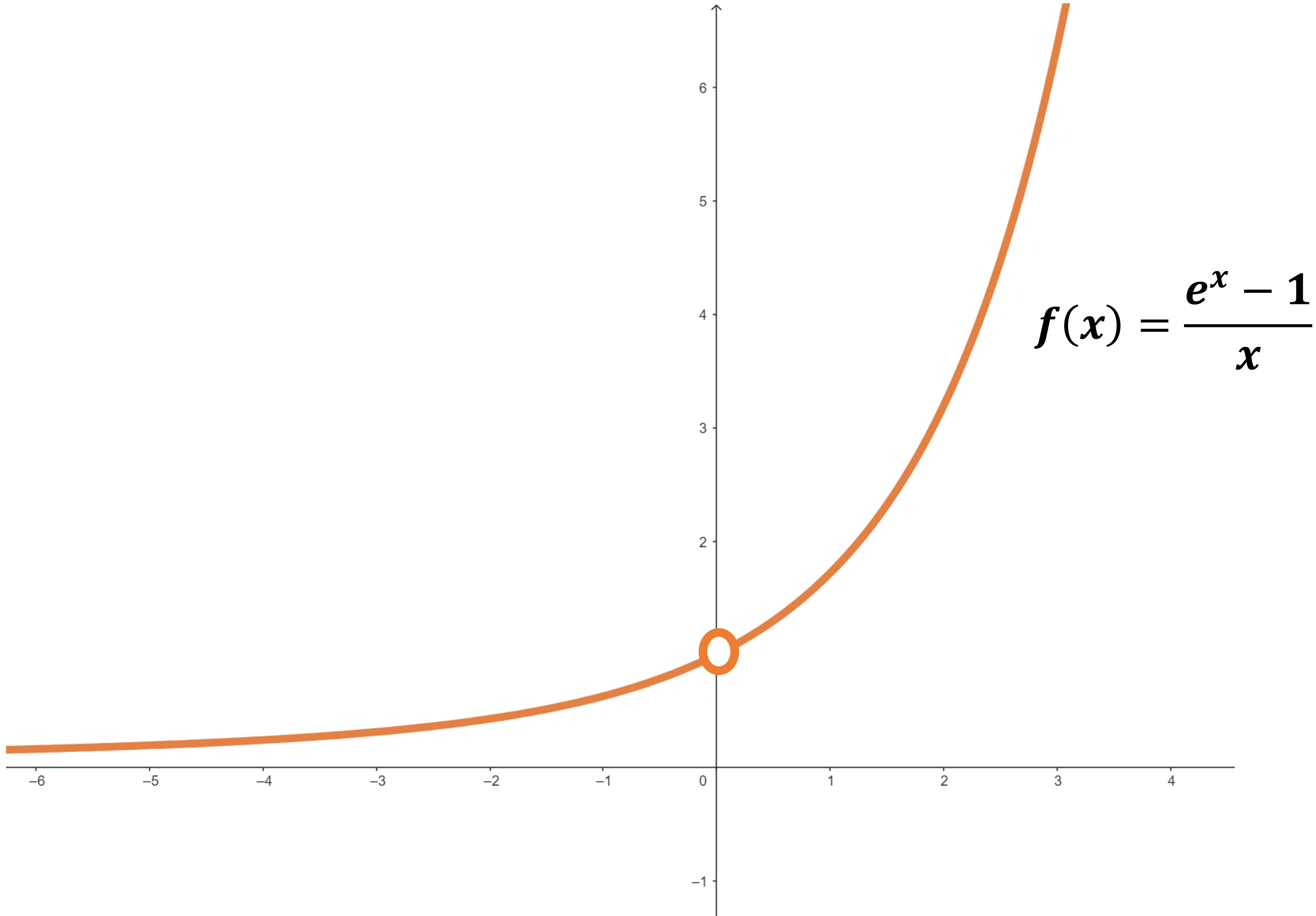


$$\lim_{x \to 0} \frac{\ln(x+1)}{x} = 1$$

$$f(x) = \frac{\ln(x+1)}{x}$$



$$\lim_{x\to 0} \frac{e^x-1}{x} = 1$$



$$\lim_{x\to 0} \frac{\sin 2x}{2x} \qquad \frac{0}{0}$$

$$f(g(x)) = \frac{\sin 2x}{2x} \qquad f(x) = \frac{\sin x}{x} \qquad g(x) = 2x$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Věta (Věta o limitě složené funkce s vnitřní funkcí „vyhýbající se své limitě“).

(a) $\lim_{x\to c} g(x) = B$ a $\lim_{y\to B} f(y) = A$;

$$\lim_{x\to 0} 2x = 0 \qquad \lim_{y\to 0} \frac{\sin y}{y} = 1$$

(b) $\exists \delta > 0\ \forall x \in P(c,\delta) : g(x) \neq B$

$$\delta > 0 \quad \forall x \in (-\delta, \delta) \setminus \{0\}: \quad 2x \neq 0$$

Potom platí: $\lim_{x\to c} f(g(x)) = A$.

$$\lim_{x\to 0} \frac{\sin 2x}{2x} = 1$$

$$\lim_{x\to 0}\frac{\ln\left(\frac{x}{e^x}+1\right)}{\frac{x}{e^x}} \qquad \frac{0}{0}$$

$$f(g(x)) = \frac{\ln\left(\frac{x}{e^x}+1\right)}{\frac{x}{e^x}} \quad \to \quad f(x) = \frac{\ln(x+1)}{x}, \quad g(x) = \frac{x}{e^x}$$

$$\lim_{x\to 0}\frac{\ln(x+1)}{x} = 1$$

Věta (Věta o limitě složené funkce s vnitřní funkcí „vyhýbající se své limitě").

(a) $\lim_{x\to c} g(x) = B$ a $\lim_{y\to B} f(y) = A$;

$$\lim_{x\to 0}\frac{x}{e^x} = 0 \qquad \lim_{y\to 0}\frac{\ln(y+1)}{y} = 1$$

(b) $\exists\delta > 0\ \forall x \in P(c,\delta) : g(x) \neq B$

$$\delta > 0\ \ \forall x \in (-\delta, 0) \cup (0, \delta):\ \ \frac{x}{e^x} \neq 0$$

Potom platí: $\lim_{x\to c} f(g(x)) = A$.

$$\lim_{x\to 0}\frac{\ln\left(\frac{x}{e^x}+1\right)}{\frac{x}{e^x}} = 1$$

$$\lim_{x \to 0} \frac{\sin 3x}{x} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{3 \sin 3x}{3x} = 3 \lim_{x \to 0} \frac{\sin 3x}{3x} = 3 \times 1 = 3$$

$3x \to 0$

Vnitřní funkce

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

Vnější funkce

Věta o limitě složené funkce

$$\lim_{x\to 0}\frac{\sin 4x - \sin 5x}{x} \qquad \frac{0}{0} \qquad \sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$= \lim_{x\to 0}\frac{2\cos\frac{9x}{2}\sin(\frac{-x}{2})}{x} = \lim_{x\to 0} 2\cos\frac{9x}{2}\lim_{x\to 0}\frac{\frac{-1}{2}\sin(\frac{-x}{2})}{\frac{-x}{2}}$$

$$= (2\cos 0)(\frac{-1}{2})\lim_{x\to 0}\frac{\sin(\frac{-x}{2})}{\frac{-x}{2}} = 2\times\frac{-1}{2}\times 1 = -1$$

$$\frac{-x}{2}\to 0$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Věta o limitě složené funkce

$$\lim_{x\to 0}\frac{\sqrt{1-\cos^2 x}}{1-\cos x}\qquad \frac{0}{0}$$

$$1-\cos 2\theta = 2\sin^2\theta \qquad 1-\cos^2\theta=\sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$2\left|\sin\frac{x}{2}\right|\left|\cos\frac{x}{2}\right|$$

$$=\lim_{x\to 0}\frac{\sqrt{\sin^2 x}}{2\sin^2\frac{x}{2}}=\lim_{x\to 0}\frac{|\sin x|}{2\sin^2\frac{x}{2}}=\lim_{x\to 0}\frac{\left|2\sin\frac{x}{2}\cos\frac{x}{2}\right|}{2\left|\sin^2\frac{x}{2}\right|}=\infty$$

$$2\left|\sin\frac{x}{2}\right|\left|\sin\frac{x}{2}\right|$$

$$\lim_{x\to 0}\frac{5^x-1}{x} \qquad \frac{0}{0}$$

$$a^x = e^{\ln a^x} = e^{x\ln a}$$

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1$$

$$= \lim_{x\to 0}\frac{\left(e^{x\ln 5}-1\right)\ln 5}{x\ln 5} = \ln 5 \lim_{x\to 0}\frac{e^{x\ln 5}-1}{x\ln 5} = \ln 5 \times 1 = \ln 5$$

Věta o limitě složené funkce

$$x\ln 5 \to 0$$

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1$$

$$\lim_{x\to 0}\frac{\ln(\cos x)}{3x^2} \qquad \frac{0}{0} \qquad \cos x = -2\sin^2\frac{x}{2}+1$$

$$= \lim_{x\to 0}\frac{\ln(-2\sin^2\frac{x}{2}+1)\,(-2\sin^2\frac{x}{2})}{(-2\sin^2\frac{x}{2})\,3x^2}$$

$$= \lim_{x\to 0}\frac{\ln(-2\sin^2\frac{x}{2}+1)}{-2\sin^2\frac{x}{2}}\lim_{x\to 0}\frac{-2\sin^2\frac{x}{2}}{3x^2} = 1\times\left(\frac{-2}{3}\right)\lim_{x\to 0}\frac{\sin\frac{x}{2}\sin\frac{x}{2}\,(\frac{1}{2}\cdot\frac{1}{2})}{\frac{x}{2}\,\frac{x}{2}}$$

$$-2\sin^2\frac{x}{2}\to 0$$

$$\frac{x}{2}\to 0$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$\lim_{x\to 0}\frac{\ln(x+1)}{x}=1$$

$$= \frac{-2}{3}\times 1\times 1\times\frac{1}{4} = -\frac{1}{6}$$

$$\lim_{x\to 0}\frac{\ln(x^2+e^{2x^2})}{\ln(x^2+e^{3x^2})} \qquad \frac{0}{0}$$

$$\lim_{x\to 0}\frac{\ln(x+1)}{x}=1$$

$$=\lim_{x\to 0}\frac{\ln\left(e^{2x^2}\left(\frac{x^2}{e^{2x^2}}+1\right)\right)}{\ln\left(e^{3x^2}\left(\frac{x^2}{e^{3x^2}}+1\right)\right)}=\lim_{x\to 0}\frac{\ln e^{2x^2}+\ln\left(\frac{x^2}{e^{2x^2}}+1\right)}{\ln e^{3x^2}+\ln\left(\frac{x^2}{e^{3x^2}}+1\right)}=\lim_{x\to 0}\frac{x^2\left(2+\frac{\ln\left(\frac{x^2}{e^{2x^2}}+1\right)}{x^2}\right)}{x^2\left(3+\frac{\ln\left(\frac{x^2}{e^{3x^2}}+1\right)}{x^2}\right)}$$

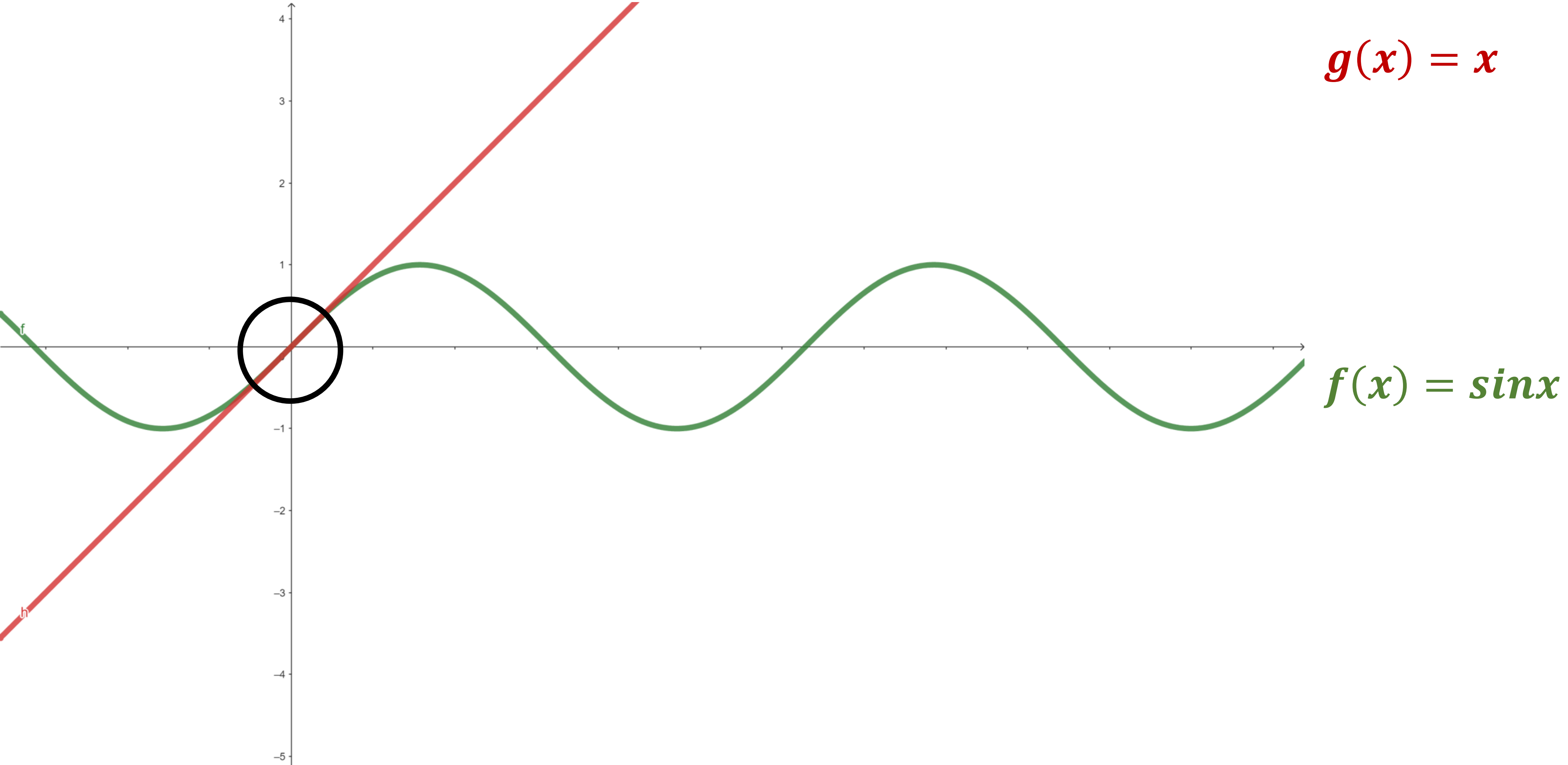
$\ln e^{2x^2} \to 2x^2$, $\ln e^{3x^2} \to 3x^2$

$$\frac{x^2}{e^{2x^2}}\to 0 \qquad \frac{x^2}{e^{3x^2}}\to 0$$

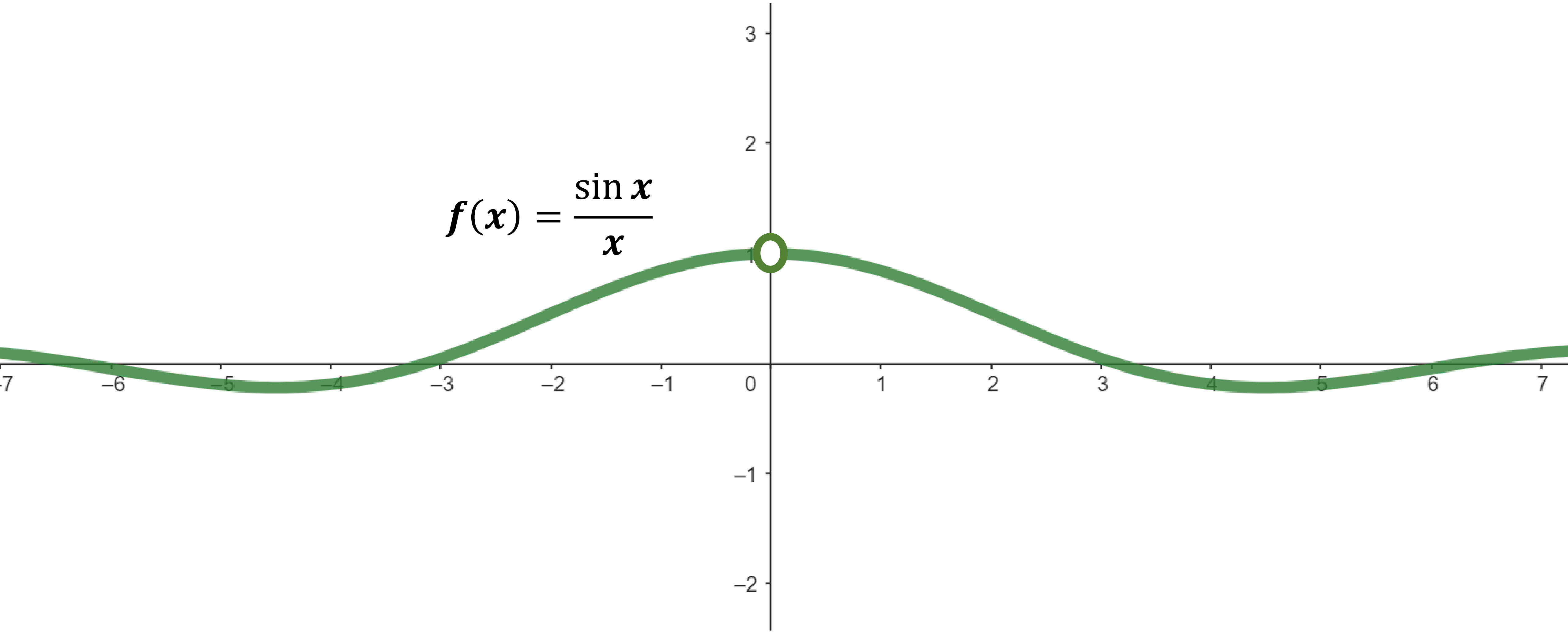
$$=\frac{\lim\limits_{x\to 0}2+\lim\limits_{x\to 0}\frac{\ln\left(\frac{x^2}{e^{2x^2}}+1\right)}{\frac{x^2}{e^{2x^2}}}e^{2x^2}}{\lim\limits_{x\to 0}3+\lim\limits_{x\to 0}\frac{\ln\left(\frac{x^2}{e^{3x^2}}+1\right)}{\frac{x^2}{e^{3x^2}}}e^{3x^2}}=\frac{2+\lim\limits_{x\to 0}\frac{\ln\left(\frac{x^2}{e^{2x^2}}+1\right)}{\frac{x^2}{e^{2x^2}}}\cdot\lim\limits_{x\to 0}\frac{1}{e^{2x^2}}}{3+\lim\limits_{x\to 0}\frac{\ln\left(\frac{x^2}{e^{3x^2}}+1\right)}{\frac{x^2}{e^{3x^2}}}\cdot\lim\limits_{x\to 0}\frac{1}{e^{3x^2}}}=\frac{2+1}{3+1}=\frac{3}{4}$$

Poučka. Při vyhodnocení limit funkcí se často využívají následující výsledky. Jde o tzv. známé limity:

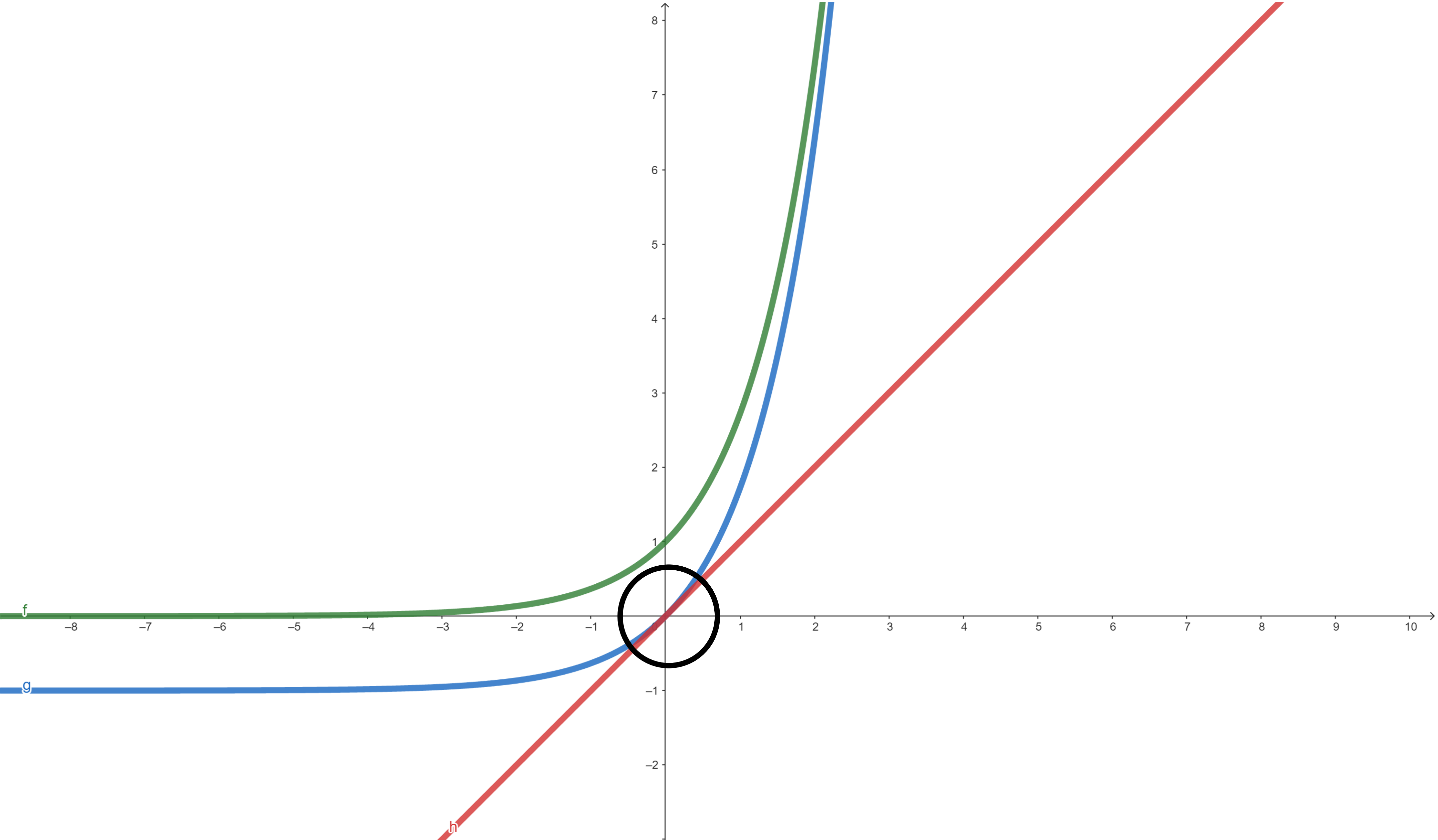
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$



$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$



$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

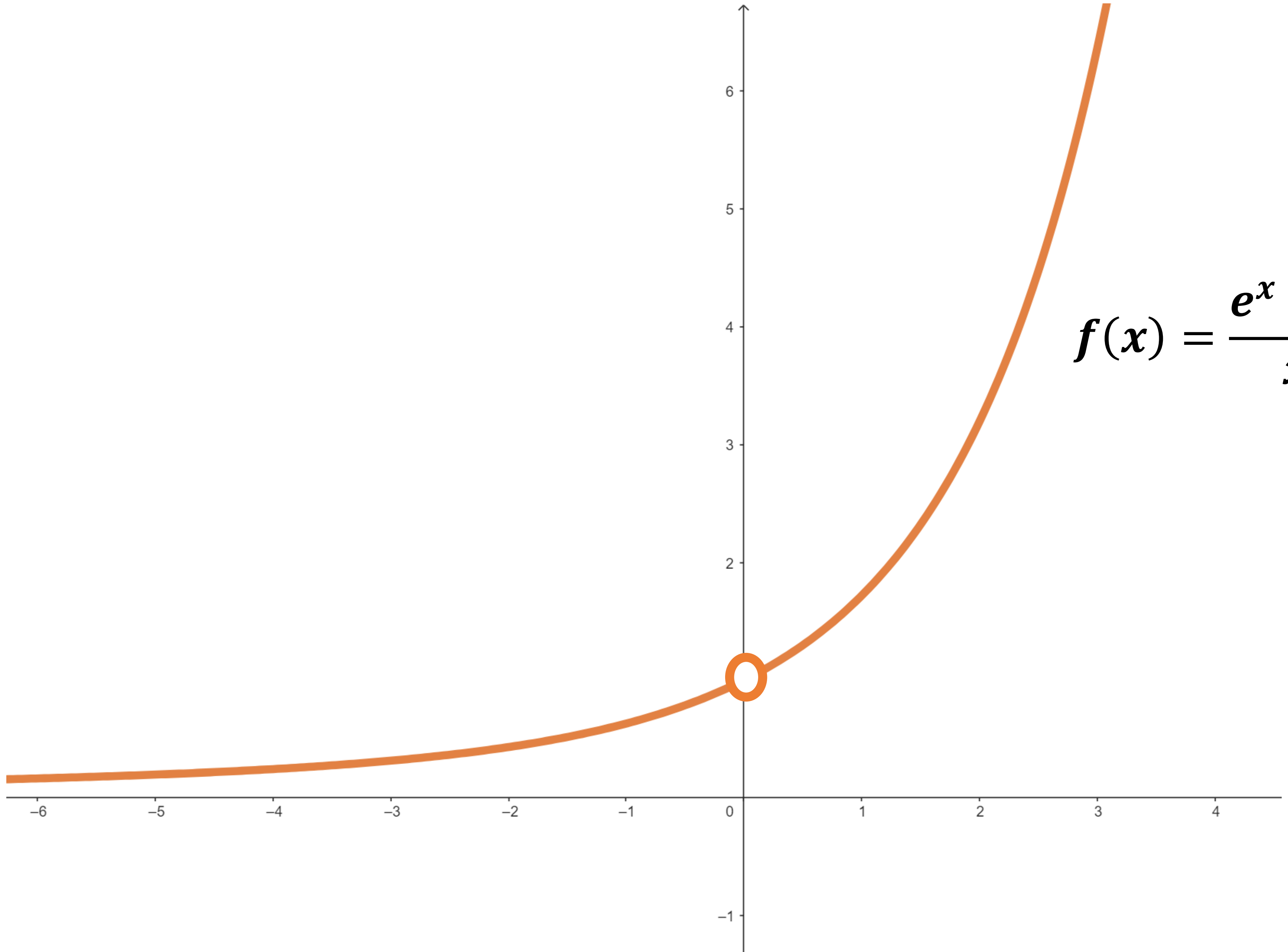


$$h(x) = x$$

$$g(x) = e^x - 1$$

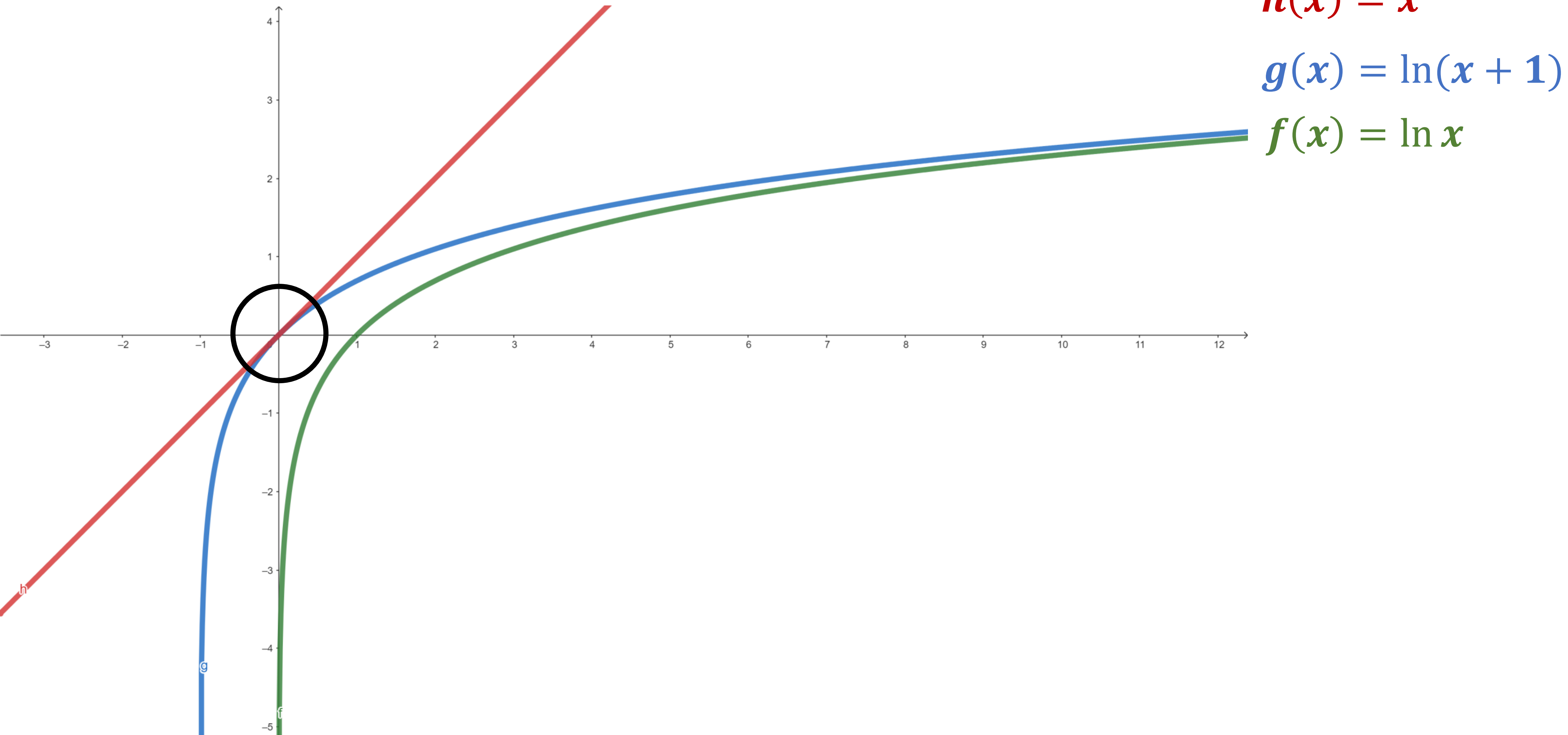
$$f(x) = e^x$$

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$



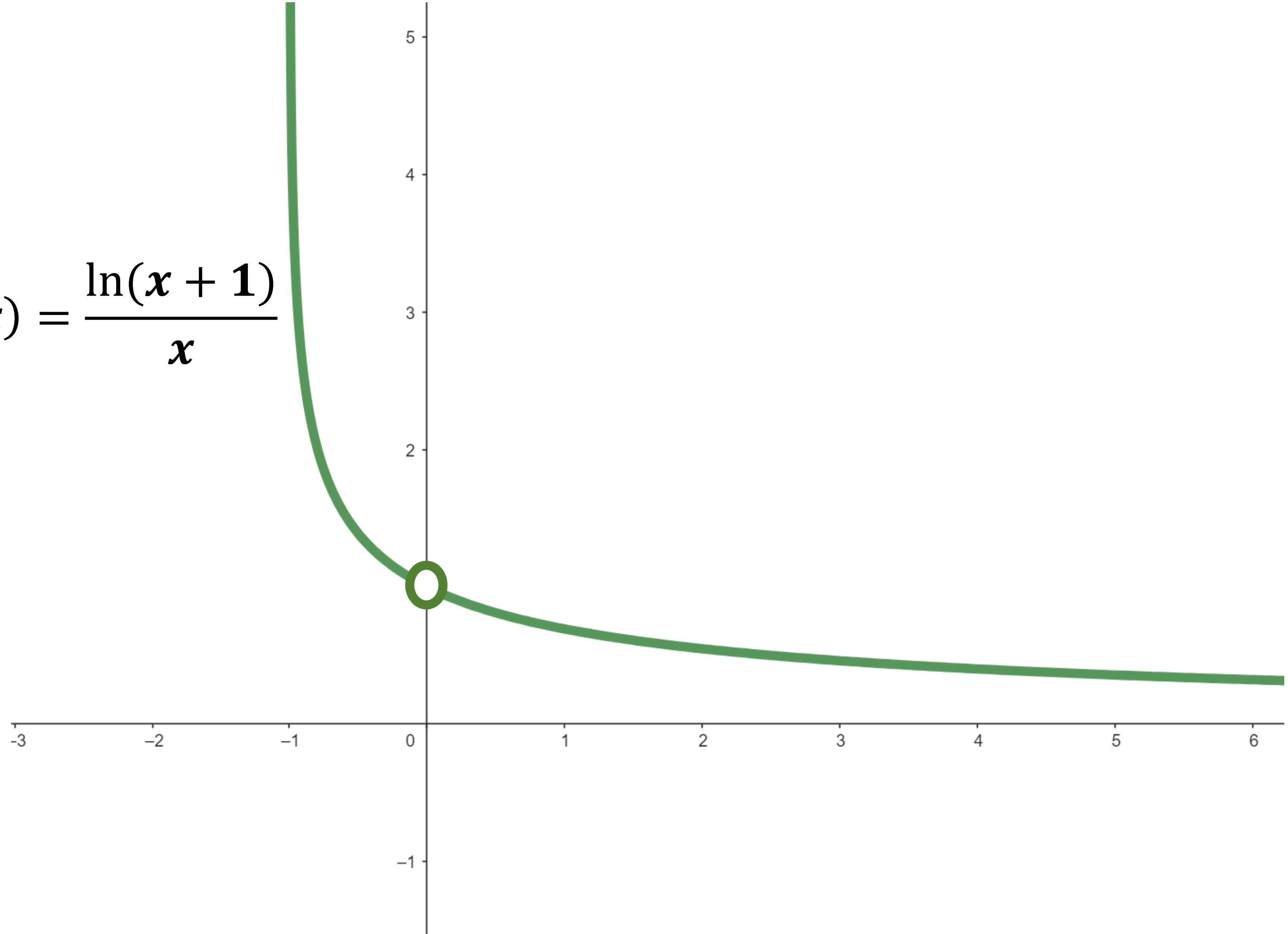
$$f(x)=\frac{e^x-1}{x}$$

$$\lim_{x \to 0} \frac{\ln(x+1)}{x} = 1$$



$$\lim_{x \to 0} \frac{\ln(x+1)}{x} = 1$$

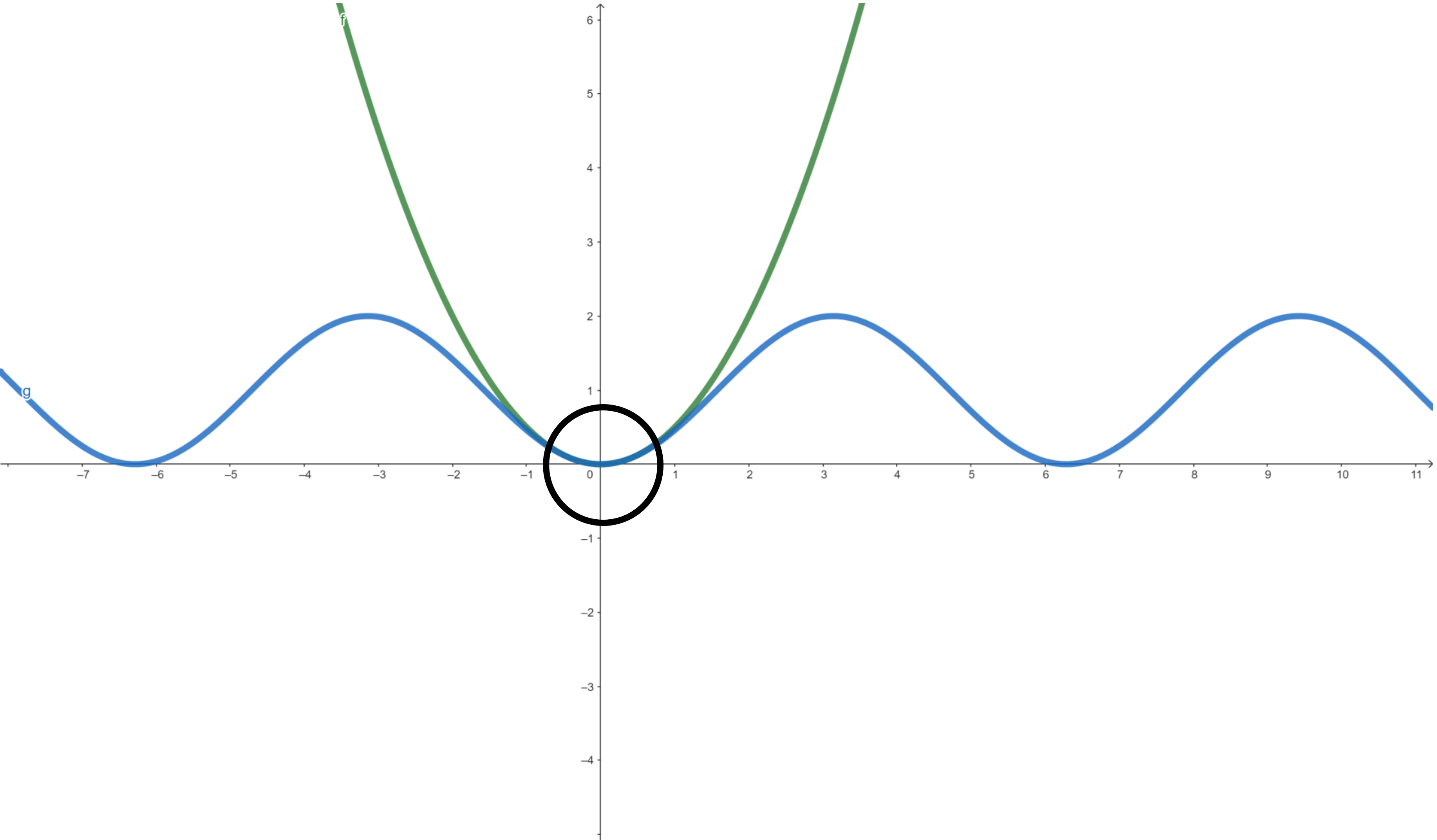
$$f(x) = \frac{\ln(x+1)}{x}$$



$$\lim_{x \to 0} \frac{\ln(\boldsymbol{x} + \mathbf{1})}{\boldsymbol{x}} = \mathbf{1}$$

$$\boxed{\lim_{y \to 1} \frac{\ln \boldsymbol{y}}{\boldsymbol{y} - \mathbf{1}} = \mathbf{1}}$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$



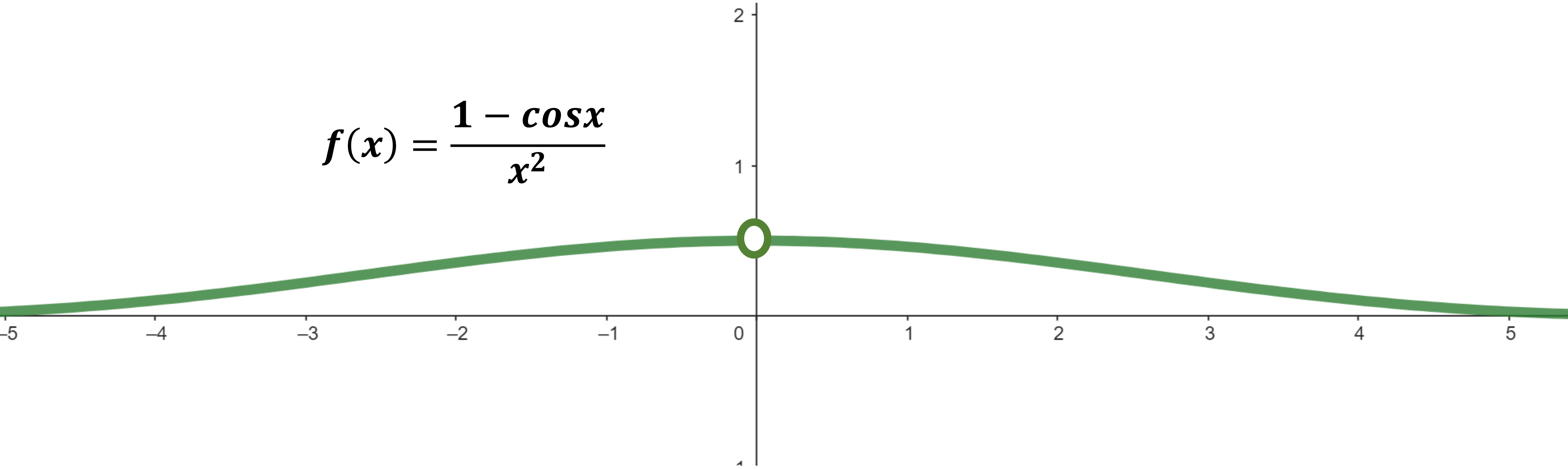
$$x \to 0 \quad \frac{1 - \cos x}{x^2} \approx \frac{1}{2}$$

$$x \to 0 \quad 1 - \cos x \approx \frac{x^2}{2}$$

$$g(x) = \frac{x^2}{2}$$

$$f(x) = 1 - \cos x$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$



$$\lim_{x\to 1}\frac{\ln(x)}{x-1}=1 \qquad \lim_{x\to 0}\frac{1-cosx}{x^2}=\frac{1}{2}$$

$$\lim_{x\to 0}\frac{\ln(\cos x)}{3x^2} \qquad \frac{0}{0}$$

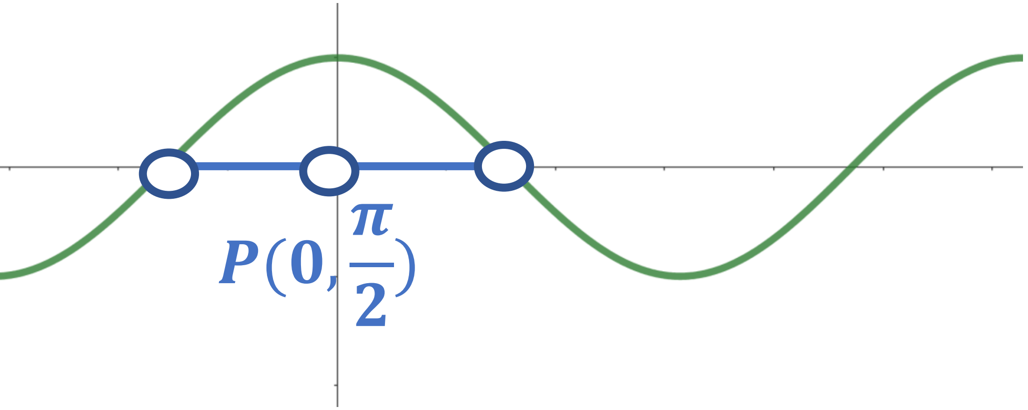
$$=\frac{1}{3}\lim_{x\to 0}\frac{\ln(cos\,x)\,(cosx\ -1)}{(cosx\ -1)\ x^2}=\frac{1}{3}\lim_{x\to 0}\frac{\ln(cos\,x)}{cosx\ -1}\lim_{x\to 0}\frac{cosx\ -1}{x^2}=\frac{1}{3}(1)(-\frac{1}{2})=-\frac{1}{6}$$

$$f(g(x))=\frac{\ln(\cos x)}{cosx-1}$$

$$f(y)=\frac{ln(y)}{y-1} \qquad \lim_{y\to 1}f(y)=1$$

$$y=g(x)=cosx \qquad \lim_{x\to 0}cosx=1$$

$$\lim_{y\to 1}\frac{\ln(\cos x)}{cosx-1}=1$$



$$\delta=\frac{\pi}{2}\ \forall x\in P(0,\frac{\pi}{2})\ :\ cosx\neq 1$$

$$\lim_{x\to 1}\frac{\ln(x)}{x-1}=1$$

$$\lim_{x\to 0}\frac{\ln(x^2+e^{2x^2})}{\ln(x^2+e^{3x^2})}=\lim_{x\to 0}\frac{\dfrac{\ln(x^2+e^{2x^2})}{x^2+e^{2x^2}-1}(x^2+e^{2x^2}-1)}{\dfrac{\ln(x^2+e^{3x^2})}{x^2+e^{3x^2}-1}(x^2+e^{3x^2}-1)}$$

$$\overset{VOAL}{=}\frac{\lim_{x\to 0}\dfrac{\ln(x^2+e^{2x^2})}{x^2+e^{2x^2}-1}}{\lim_{x\to 0}\dfrac{\ln(x^2+e^{3x^2})}{x^2+e^{3x^2}-1}}\lim_{x\to 0}\frac{x^2+e^{2x^2}-1}{x^2+e^{3x^2}-1}=\left(\frac{1}{1}\right)\lim_{x\to 0}\frac{\cancel{x^2}\left(1+\dfrac{e^{2x^2}-1}{x^2}\right)}{\cancel{x^2}\left(1+\dfrac{e^{3x^2}-1}{x^2}\right)}$$

$$\overset{VOAL}{=}\frac{\lim_{x\to 0}1+\lim_{x\to 0}\dfrac{e^{2x^2}-1}{2\,x^2}2}{\lim_{x\to 0}1+\lim_{x\to 0}\dfrac{e^{3x^2}-1}{3\,x^2}3}=\frac{1+2}{1+3}=\frac{3}{4}$$

$$\lim_{x\to 0}\frac{\ln(x^2+e^{2x^2})}{x^2+e^{2x^2}-1}$$

$$f(g(x)) = \frac{\ln(x^2+e^{2x^2})}{x^2+e^{2x^2}-1}$$

$$f(y) = \frac{ln(y)}{y-1} \qquad \lim_{y\to 1} f(y) = 1$$

$$y = g(x) = x^2 + e^{2x^2} \qquad \lim_{x\to 0}(x^2+e^{2x^2}) = 1$$

$$x^2 + e^{2x^2} \geq e^{2x^2} > e^0 = 1$$

$$x \neq 0$$

Vypočtěte limity složených funkcí:

$$\lim_{x\to 0} e^{-\frac{x^2}{2}} \qquad f(g(x)) = e^{-\frac{x^2}{2}} \qquad f(y) = e^y \qquad y = g(x) = -\frac{x^2}{2}$$

Věta o limitě složené funkce se spojitou vnější funkcí

(a) $\lim_{x\to c} g(x) = B$;

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}\left(-\frac{x^2}{2}\right) = 0 = B$$

(b) funkce f je v bodě B spojitá.

$$f(y) = e^y \qquad B = 0 \qquad \lim_{y\to 0} e^y = e^0 = 1$$

Potom platí: $\lim_{x\to c} f(g(x)) = f(B)$.

$$\lim_{x\to c} f(g(x)) = \lim_{x\to 0} e^{-\frac{x^2}{2}} = \overset{f}{e}^{\overbrace{\lim_{x\to 0}\left(-\frac{x^2}{2}\right)}^{B}} = e^0 = 1$$

$$\lim_{x\to\infty} \ln \frac{x-1}{x+1}$$

$$f(g(x)) = \ln \frac{x-1}{x+1}$$

$$f(y) = \ln y$$

$$y = g(x) = \frac{x-1}{x+1}$$

$$\lim_{x\to\infty} f(g(x)) = \lim_{x\to\infty} ln \frac{x-1}{x+1} = \ln(\lim_{x\to\infty} \frac{x-1}{x+1}) = \ln 1 = 0$$

Vypočtěte limity složených funkcí:

$$\lim_{x\to\infty} \sqrt[3]{\frac{x^7 - 12x^2 + 3}{27x^7 - 12x^2 + 3}}$$

$$f(g(x)) = \sqrt[3]{\frac{x^7 - 12x^2 + 3}{27x^7 - 12x^2 + 3}}$$

$$f(y) = \sqrt[3]{y}$$

$$g(x) = \frac{x^7 - 12x^2 + 3}{27x^7 - 12x^2 + 3}$$

$$\lim_{x\to\infty} f(g(x)) = \lim_{x\to\infty} \sqrt[3]{\frac{x^7 - 12x^2 + 3}{27x^7 - 12x^2 + 3}} = \sqrt[3]{\lim_{x\to\infty} \frac{x^7 - 12x^2 + 3}{27x^7 - 12x^2 + 3}} = \sqrt[3]{\frac{1}{27}} = \frac{1}{3}$$

Limity typu „1^∞":

$$\underbrace{f(x)}_{\to 1}^{\overbrace{g(x)}^{\to\infty}} = e^{\ln f(x)^{g(x)}} = e^{g(x)\ln f(x)} = exp(g(x)\ln f(x))$$

$$\lim_{x\to\infty}(1+\frac{1}{x})^x = e$$

$$\lim_{x\to\infty}(1+\frac{1}{x})^x = \lim_{x\to\infty} exp(x\ln(1+\frac{1}{x})) = exp(\lim_{x\to\infty} x\ln(1+\frac{1}{x}))$$

$$= exp(\lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}})$$

$$= exp(1) = e^1 = e$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x \qquad 1^\infty$$

$$\lim_{x\to 0}\frac{ln(1+x)}{x}=1$$

$$=\lim_{x\to\infty} exp\left(x\,ln\left(1+\frac{a}{x}\right)\right)=exp\left(\lim_{x\to\infty}\frac{x\,ln\left(1+\frac{a}{x}\right)\ \frac{a}{x}}{\frac{a}{x}}\right)$$

$$=exp\left(\lim_{x\to\infty}\frac{ln\left(1+\frac{a}{x}\right)}{\frac{a}{x}}\lim_{x\to\infty}\frac{ax}{x}\right)=exp(1.a)=e^a$$

$$\lim_{x\to\infty}\left(1+\frac{1}{3x-1}\right)^{2x} \qquad 1^{\infty}$$

$$\boxed{\lim_{x\to 0}\frac{\ln(1+x)}{x}=1}$$

$$=\lim_{x\to\infty} exp\left(2x\, ln\left(1+\frac{1}{3x-1}\right)\right)=exp\left(\lim_{x\to\infty} 2x ln\left(1+\frac{1}{3x-1}\right)\right)$$

$$=exp\left(\lim_{x\to\infty}\frac{2x\, ln\left(1+\frac{1}{3x-1}\right)\left(\frac{1}{3x-1}\right)}{\frac{1}{3x-1}}\right) \text{ VOAL } = exp\left(\lim_{x\to\infty}\frac{ln\left(1+\frac{1}{3x-1}\right)}{\frac{1}{3x-1}}\lim_{x\to\infty}\frac{2x}{3x-1}\right)$$

$$=exp\left(1.\frac{2}{3}\right)=exp\left(\frac{2}{3}\right)=e^{\frac{2}{3}}$$

$$\lim_{x\to\infty}\frac{1}{3x-1}=0$$

$$\lim_{y\to 0}\frac{ln(1+y)}{y}=1 \qquad \delta>0\,\forall x\in P(\infty,\delta)=\left(\frac{1}{\delta},\infty\right):\frac{1}{3x-1}\neq 0$$

$$\lim_{x\to\pi}(-\cos x)^{cotg^2 x} = \lim_{x\to\pi} exp(cotg^2 x\ \ln(-\cos x)) = exp(\lim_{x\to\pi} cotg^2 x \ln(-\cos x))$$

$x - \pi = y$

$x \to \pi \Longrightarrow y \to 0$

$x = \pi + y$

$$= exp(\lim_{y\to 0} \frac{cos^2(\pi + y)}{sin^2(\pi + y)} \ln(-\cos(\pi + y)))$$

cos^2 a sin^2 jsou π-priodické fce.

$cos^2(\pi + y) = cos^2(y)$

$sin^2(\pi + y) = sin^2(y)$

$$= exp(\lim_{y\to 0} \frac{cos^2 y}{sin^2 y} \ln(\underbrace{\cos y}_{\to 1}))$$

$$= exp(\lim_{y\to 0} \frac{cos^2 y}{sin^2 y} \frac{ln(cos\, y)}{\cos y - 1}(cos\, y - 1))$$

$cos(\pi + y) = -cosy$

$$= exp(\lim_{y\to 0} cos^2 y \lim_{y\to 0} \frac{y^2}{sin^2 y} \frac{ln(cos\, y)}{\cos y - 1} \frac{(cos\, y - 1)}{y^2})$$

$$= exp(1 \cdot (\lim_{y\to 0} \frac{y}{siny})^2 . \lim_{y\to 0} \frac{ln(cos\, y)}{\cos y - 1} . \lim_{y\to 0} \frac{(cos\, y - 1)}{y^2}) = exp(\frac{-1}{2}) = e^{\frac{-1}{2}}$$

$$\lim_{x\to 0}\left(\frac{1+tgx}{1+sinx}\right)^{\frac{1}{sin^3x}} \qquad 1^{\infty}$$

$$= \lim_{x\to 0} exp\left(\frac{1}{sin^3x}\ ln\frac{1+tgx}{1+sinx}\right)$$

VOLSF(S)

$$= exp\left(\lim_{x\to 0}\left(\frac{1x^3}{sin^3x}\ \frac{ln\frac{1+tgx}{1+sinx}}{\frac{1+tgx}{1+sinx}-1}\ \frac{\frac{1+tgx}{1+sinx}-1}{x^3}\right)\right)$$

$$\frac{\frac{1+tgx-1-sinx}{1+sinx}}{x^3}$$

$$\frac{\frac{sinx}{cosx}-sinx}{x^3(1+sinx)}$$

$$\frac{sinx\left(\frac{1}{cosx}-1\right)}{x^3(1+sinx)}$$

VOAL

$$= exp\left(\left(\lim_{x\to 0}\frac{x}{sinx}\right)^3 \lim_{x\to 0}\frac{ln\frac{1+tgx}{1+sinx}}{\frac{1+tgx}{1+sinx}-1}\lim_{x\to 0}\frac{sinx\left(\frac{1}{cosx}-1\right)}{x^3(1+sinx)}\right)$$

$$= exp\left((1)^3\, 1\ \lim_{x\to 0}\frac{sinx}{x}\lim_{x\to 0}\frac{1}{1+sinx}\lim_{x\to 0}\frac{\frac{1-cosx}{cosx}}{x^2}\right)$$

$$= exp\left(\lim_{x\to 0}\frac{1-cosx}{x^2}\lim_{x\to 0}\frac{1}{cosx}\right) = exp\left(\frac{1}{2}.1\right) = exp\left(\frac{1}{2}\right) = e^{\frac{1}{2}}$$

$$\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^{x} = \lim_{y\to 0^+}(\sin y+\cos y)^{\frac{1}{y}} = \lim_{y\to 0^+} exp\left(\frac{1}{y}\ln(\underbrace{\sin y+\cos y}_{\to 1})\right)$$

VOLSF(S)

$$= exp\left(\lim_{y\to 0^+}\left(\frac{1}{y}\frac{\ln(\sin y+\cos y)}{\sin y+\cos y-1}(\sin y+\cos y-1)\right)\right)$$

$$= exp\left(\lim_{y\to 0^+}\frac{\ln(\sin y+\cos y)}{\sin y+\cos y-1}\left(\lim_{y\to 0^+}\frac{\sin y}{y}+\lim_{y\to 0^+}\frac{\cos y-1}{y2}y\right)\right)$$

$$= exp\left(1\left(1+\lim_{y\to 0^+}\frac{\cos y-1}{y^2}\lim_{y\to 0} y\right)\right) = exp\left(1\left(1-\frac{1}{2}.0\right)\right) = exp(1) = e^1 = e$$

$$\lim_{x\to 0}\left(\frac{3^x+5^x}{2}\right)^{\frac{1}{x}} = \lim_{x\to 0} exp\left(\frac{1}{x}\ln\left(\frac{3^x+5^x}{2}\right)\right) = exp\left(\lim_{x\to 0}\frac{1}{x}\ln\left(\underbrace{\frac{3^x+5^x}{2}}_{\to 1}\right)\right)$$

$$= exp\left(\lim_{x\to 0}\frac{\frac{3^x+5^x}{2}-1}{x}\,\frac{\ln\frac{3^x+5^x}{2}}{\frac{3^x+5^x}{2}-1}\right) = exp\left(\lim_{x\to 0}\frac{3^x+5^x-2}{2x}\ \lim_{x\to 0}\frac{\ln\frac{3^x+5^x}{2}}{\frac{3^x+5^x}{2}-1}\right)$$

$$\lim_{x\to 0}\frac{\ln\frac{3^x+5^x}{2}}{\frac{3^x+5^x}{2}-1} = 1$$

$$= exp\left(\frac{1}{2}\lim_{x\to 0}\frac{3^x-1+5^x-1}{x}\right) = exp\left(\frac{1}{2}\lim_{x\to 0}\left(\frac{3^x-1}{x}+\frac{5^x-1}{x}\right)\right)$$

$$= exp\left(\frac{1}{2}\left(\lim_{x\to 0}\frac{e^{x\,ln\,3}-1}{ln\,3\,.\,x}\,ln\,3+\lim_{x\to 0}\frac{e^{x\,ln\,5}-1}{ln\,5\,.\,x}\,ln\,5\right)\right)$$

$$= exp\left(\frac{1}{2}(\ln 3+\ln 5)\right) = exp\left(\frac{1}{2}\ln 15\right) = exp(ln\sqrt{15}) = \sqrt{15}$$

Limity posloupností (Heineho věta):

$$\lim_{n\to\infty} n(\sqrt[n]{2}-1) = \lim_{n\to\infty} n(2^{\frac{1}{n}}-1) \overset{\text{H.V}}{=} \lim_{x\to\infty} x(2^{\frac{1}{x}}-1)$$

$$= \lim_{x\to\infty} \frac{2^{\frac{1}{x}}-1}{\frac{1}{x}} = \lim_{x\to\infty} \frac{e^{\frac{1}{x}ln2}-1}{\frac{1}{x}ln2}\, ln2$$

$\longrightarrow 1$

$$= ln2$$