4g) $$\sum_{n=1}^{\infty} \underbrace{\frac{2\cdot 5\cdot 8\cdot \ldots \cdot (3n-1)}{3\cdot 8\cdot 13\cdot \ldots \cdot (5n-2)}}_{a_n}$$

d'Alembertovo kr.:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\cancel{2\cdot 5\cdot 8}\cdot \ldots \cdot \cancel{(3n-1)}(3n+2)}{\cancel{3\cdot 8\cdot 13}\cdot \ldots \cdot \cancel{(5n-2)}(5n+3)} \cdot \frac{\cancel{3\cdot 8\cdot 13}\cdot \ldots \cdot \cancel{(5n-2)}}{\cancel{2\cdot 5\cdot 8}\cdot \ldots \cdot \cancel{(3n-1)}} = \lim \frac{3n+2}{5n+3} = \frac{3}{5} < 1$$

LPK $\Rightarrow$ $\sum a_n$ K.

Fce 1o) $$\lim_{x\to 1} \frac{x^n-1}{x^m-1} \underset{\left(\frac{0}{0}\right)}{\overset{L'H}{=}} \lim_{x\to 1} \frac{n x^{n-1}}{m x^{m-1}} = \frac{n}{m}$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x^{n-1}+x^{n-2}+\ldots+x+1)}{\cancel{(x-1)}(x^{m-1}+x^{m-2}+\ldots+x+1)} =$$

$$= \lim_{x\to 1} \frac{(\ldots)}{(\ldots)} = \frac{n}{m}.$$

3h) $$\lim_{x\to\infty} x\left(2^{\frac{1}{x}} - 2^{-x}\right) = \quad \left| \begin{array}{l} \frac{1}{x} = y \\ x = \frac{1}{y} \\ y \to 0_+ \end{array} \right| \text{ VOLSF}$$

$$= \lim_{x\to\infty} x\left(2^{\frac{1}{x}} - e^{-x\ln 2}\right) =$$

$$= \lim_{y\to 0_+} \underbrace{\frac{1}{y}}_{\to\infty}\Big(\underbrace{2^{y}}_{\to 1} - \underbrace{2^{\overbrace{-\frac{1}{y}}^{\to -\infty}}}_{\to 0}\Big) = \infty\cdot(1-0) = \infty$$

3h alt) $$\lim_{x\to\infty} \frac{2^{\frac{1}{x}} - \underbrace{2^{-x}}_{\to 0}}{x} =$$

$$= \lim_{x\to\infty} \frac{2^{\frac{1}{x}}}{x} - \lim_{x\to\infty} \frac{2^{-x}}{x} =$$

$$= -\frac{0}{\infty} + \frac{2^0}{\infty} = 0$$

3h alhalh)

$$\lim_{x\to\infty} x\cdot\left(2^{\frac{1}{x}}-2^{-\frac{1}{x}}\right) = \lim_{x\to\infty}\frac{2^{\frac{1}{x}}-2^{-\frac{1}{x}}}{\frac{1}{x}} = .$$

$$\lim_{y\to 0_+}\frac{2^y-2^{-y}}{y} = \underbrace{\lim_{y\to 0_+}\frac{2^y-1}{y}}_{(*)} + \underbrace{\lim_{y\to 0}\frac{1-2^{-y}}{y}}_{(+)}$$

$$(*) = \lim_{y\to 0_+}\frac{e^{y\ln 2}-1}{y\cdot\ln 2}\cdot\ln 2 = \ln 2$$

$$(+) = \lim_{y\to 0_+}\frac{2^{-y}-1}{-y} = \lim\frac{e^{-y\ln 2}-1}{-y\ln 2}\cdot\ln 2 = \ln 2$$

$$= 2\ln 2 = \underline{\underline{\ln 4}}.$$

$$\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}$$

$y\to 0$

VOLSF $\quad f(y)=\frac{\sin y}{y} \quad \lim_{y\to 0} f(y) = 1 \quad$ (známá)

$$g(x)=\frac{1}{x} \quad \lim_{x\to\infty} g(x) = 0$$

(P) $\exists\,\delta>0\ \forall x\in P(\infty,\delta):\ g(x)\neq 0$

Zde triviální, protože $g\neq 0$ na D_g.

$\delta=2 \ \ldots\ P(\infty,\delta)=\left(\frac{1}{2},\infty\right)$, kde $g\neq 0$.

$$\lim_{x\to 0}\frac{\ln(1+\sin x)}{\sin x}$$

$$\lim_{x\to 0} \frac{\ln(1+\sin x)}{\sin x} = \lim f(y) = 1$$

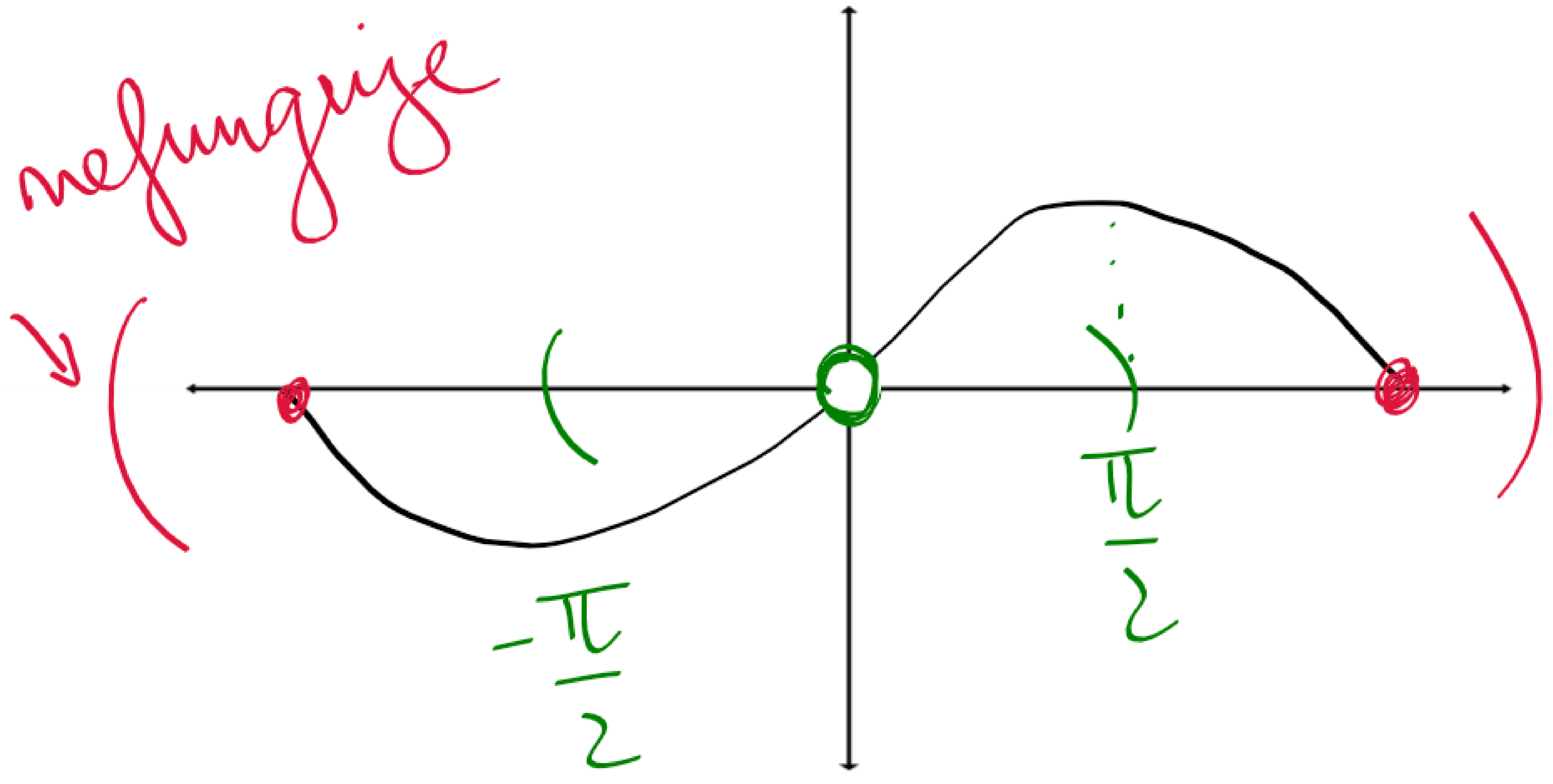
$\sin x \to y$

VOLSF: $f(y) = \frac{\ln(1+y)}{y}$ $\quad \lim_{y\to 0} f(y) = 1$

$g(x) = \sin x$ $\quad \lim_{x\to 0} g(x) = 0$

(P) $\exists \delta > 0 \ \forall x \in P(0,\delta): g(x) \neq 0$.

δ nefunguje



(P) $\sin'(0) = 1 \implies \sin$ je v 0 rostoucí, tedy podle def. (P) platí.

$$\lim_{x\to 0} \frac{1-\cos(\sqrt[3]{x})}{x^{\frac{2}{3}}}$$

VOLSF: $f(y) = \frac{1-\cos y}{y^2}$ $\quad \lim_{y\to 0} f(y) = \frac{1}{2}$

$g(x) = \sqrt[3]{x}$ $\quad \lim_{x\to 0} g(x) = 0$

(P) vnitřní fce g je rostoucí (a tedy prostá) na celém $D_g = \mathbb{R}$.

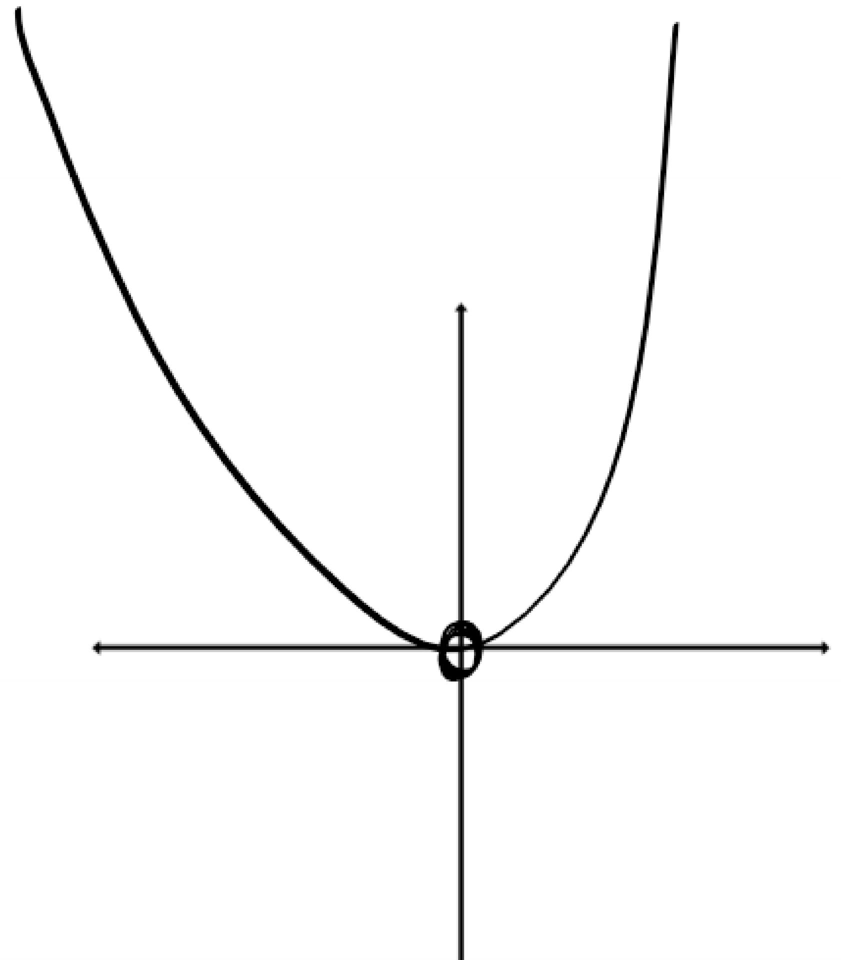
$$\lim_{x\to 0} \frac{e^{x^2}-1}{x^2} \qquad y \to 0$$

VOLSF $\quad f(y) = \dfrac{e^y - 1}{y} \qquad \lim_{y\to 0} f(y) = 1$

$$g(x) = x^2 \qquad \lim_{x\to 0} g(x) = 0$$

(P) $\quad x^2 = 0 \iff x = 0$, což nám nevadí

$\delta = 667 \qquad \forall x \in P(0,\delta): \ x^2 \neq 0$.



3g) $$\lim_{x\to 0} \frac{\log(x^2+e^{2x^2})}{\log(x^2+e^{3x^2})} =$$

$$= \lim_{x\to 0} \underbrace{\frac{\log(x^2+e^{2x^2})}{x^2+e^{2x^2}-1}}_{(*)} \cdot \frac{x^2+e^{2x^2}-1}{x^2+e^{3x^2}-1} \cdot \underbrace{\frac{x^2+e^{3x^2}-1}{\log(x^2+e^{3x^2})}}_{(+)}$$

$$= \lim_{x\to 0} (*) \cdot \lim_{x\to 0} (+) \cdot \lim_{x\to 0} \frac{e^{2x^2}-1+x^2}{e^{3x^2}-1+x^2} =$$

$$= 1 \cdot 1 \cdot \lim_{x\to 0} \frac{\frac{e^{2x^2}-1}{2x^2}\cdot 2\cancel{x^2} + \cancel{x^2}}{\frac{e^{3x^2}-1}{3x^2}\cdot 3\cancel{x^2} + \cancel{x^2}} = \frac{1\cdot 2 + 1}{1 \cdot 3 + 1}$$

$$\sum_{n=1}^{\infty} \frac{\log n}{n^2}$$

od jistého n_0

$$\frac{\log n}{n^2} \boxed{<} \frac{n}{n^2} = \frac{1}{n} \quad D.$$

$$\frac{\log n}{n^2} \boxed{<} \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}} \quad K.$$

$$\frac{\log n}{n^2} =: a_n$$

Zkusme srovnat s $b_n = \frac{1}{n^2}$

LSK: $\lim \frac{a_n}{b_n} = \lim \log n = \infty$

LSK $\Rightarrow$ $\left(\sum a_n \; K \Rightarrow \sum b_n \; K\right)$

nevíme nic.

$$\Updownarrow$$

$\left(\sum b_n \; D \Rightarrow \sum a_n \; D\right)$

nevíme nic.

Musíme být skromnější

$$c_n := \frac{1}{n^{3/2}}.$$

$$\lim \frac{a_n}{c_n} = \lim \frac{\log n}{\sqrt{n}} = 0$$

LSK: $\left(\sum c_n \; K \Rightarrow \sum a_n \; K.\right)$

Ale $\sum c_n \; K$. Takže $\sum a_n \; K$. $\square$

log (řady))

$$\sum_{n=1}^{\infty} \underbrace{\log\left(\frac{3^n+7^n}{7^n}\right)\cdot\left(1+\frac{1}{n}\right)^{n^2}}_{a_n}$$

$$a_n = \log\left(1+\frac{3^n}{7^n}\right)\cdot\left(1+\frac{1}{n}\right)^{n^2}$$

LSK: $b_n = \frac{3^n}{7^n}\cdot\left(1+\frac{1}{n}\right)^{n^2}$

$$\lim \frac{a_n}{b_n} = \lim \frac{\log\left(1+\frac{3^n}{7^n}\right)}{\frac{3^n}{7^n}} = 1 \in (0,\infty)$$

LSK $\Rightarrow$ ($\sum a_n$ K $\Leftrightarrow$ $\sum b_n$ K).

Nyní nás zajímá (jednodušší řada)

$$\sum b_n = \sum \frac{3^n}{7^n}\cdot\left(1+\frac{1}{n}\right)^{n^2} \quad b_n$$

$$\frac{\log(1+y)}{y} \xrightarrow{y\to 0} 1$$

$$\lim \sqrt[n]{b_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{3}{7}\right)^n}\cdot\sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} =$$

$$= \lim \frac{3}{7}\cdot\left(1+\frac{1}{n}\right)^n = \frac{3}{7}\cdot e < 1$$

$$\Leftrightarrow \frac{7}{3} > e$$

$7:3 = 2{,}3\ldots > e \Rightarrow \frac{3}{7}e < 1$,

10

a tedy $\sum b_n$ K.

Podle LSK $\sum a_n$ K.

3k) $\sum_{n=1}^{\infty} n^{\alpha} \sin\frac{1}{n^2+3} \cdot \underbrace{\left(\sqrt{n+4}-\sqrt{n+3}\right)}_{}$

$$= \frac{1}{\sqrt{n+4}+\sqrt{n+3}}$$

Srovnávací řada $b_n := \boxed{n^{\alpha} \cdot \frac{1}{n^2} \cdot \frac{1}{\sqrt{n}}}$

$$= n^{\alpha-2-\frac{1}{2}} = \frac{1}{n^{5/2-\alpha}}$$

$$\lim \frac{a_n}{b_n} = \lim \frac{\cancel{n^{\alpha}} \cdot \sin\frac{1}{n^2+3} \cdot \frac{1}{\cancel{\sqrt{n}}\left(\sqrt{1+\frac{4}{n}}+\sqrt{1+\frac{3}{n}}\right)}}{\cancel{n^{\alpha}} \cdot \frac{1}{n^2} \cdot \frac{1}{\cancel{\sqrt{n}}}}$$

$$= \underbrace{\lim \frac{1}{\sqrt{1+\frac{4}{n}}+\sqrt{1+\frac{3}{n}}}}_{\frac{1}{\sqrt{1+0}+\sqrt{1+0}} = \frac{1}{2}} \cdot \underbrace{\lim \frac{\sin\frac{1}{n^2+3}}{\frac{1}{n^2+3}}}_{=1} \cdot \boxed{\frac{\frac{1}{n^2+3}}{\frac{1}{n^2}}}$$

$$\frac{1}{2} \cdot 1 \cdot \lim \boxed{\frac{n^2}{n^2+3}} = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2} \in (0,\infty)$$

LSK $\Rightarrow$ $\left(\sum a_n \text{ K} \iff \sum b_n \text{ K}\right)$

Pro která $\alpha \in \mathbb{R}$ K. $\sum b_n$?

$$\sum b_n = \sum \frac{1}{n^{\frac{5}{2}-\alpha}} \text{ K.} \iff$$

$$\iff \frac{5}{2}-\alpha > 1 \iff \alpha < \frac{3}{2}$$

Celkem: $\sum a_n \text{ K} \overset{\text{LSK}}{\iff} \sum b_n \text{ K} \iff \alpha < \frac{3}{2}$

Čili $\sum a_n \text{ K} \iff \alpha < \frac{3}{2}$.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\underbrace{n \cdot \sin\frac{1}{n}}_{a_n}}$$

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{\overbrace{(-1)^{2n}}^{1}}{2n \cdot \sin\frac{1}{2n}} =$$

$$\lim \frac{\frac{1}{2n}}{\sin\frac{1}{2n}} = \lim \left(\frac{\sin\frac{1}{2n}}{\frac{1}{2n}}\right)^{-1}$$

$$= \left(\lim \frac{\sin\cdots}{\cdots}\right)^{-1} = (1)^{-1} = 1 \neq 0$$

nesplněná NPK $\Rightarrow \sum a_n$ D.

$$\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n^2}\right)$$

$$\sum |a_n| = \sum \sin\frac{1}{n^2}$$

Srovnej s. $b_n = \frac{1}{n^2}$ $\quad y \to 0$

$$\lim \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}} = 1 \in (0,\infty)$$

LSK $\Rightarrow \left(\sum |a_n| \text{ K.} \iff \sum b_n \text{ K.}\right)$

ale $\sum b_n$ K $\Rightarrow \sum |a_n|$ K.

$\Rightarrow \sum a_n$ AK $\Rightarrow \sum a_n$ K

Leibniz: Pokud $\varepsilon_n \downarrow 0$, pak

$\sum (-1)^n \varepsilon_n$ K.

5d)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\log(\log n)} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\log \log n}$$

$$\lim \frac{1}{\log \log n} = \frac{1}{\infty} = 0$$

Ale jmenovatel $\log \log n$ je rostoucí $\Rightarrow$

$$\frac{1}{\log \log n} \downarrow 0.$$

Tedy $\sum (-1)^n \frac{1}{\log \log n}$ K.

AK: Srovnej s $\frac{1}{n} = b_n$

$$\lim \frac{|a_n|}{b_n} = \lim \frac{\frac{1}{\log \log(n)}}{\frac{1}{n}} = \lim \frac{n}{\log \log n} = \infty$$

LSK $\Rightarrow$ ($\sum |a_n|$ K $\Longrightarrow$ $\sum b_n$ K)

Ale $\sum b_n = \sum \frac{1}{n}$ D.

Takže $\sum |a_n|$ D.

Celkem: K $\wedge$ $\neg$AK, tj. RK.