) $\in$ int K, the body K contains a ball $B^n(\alpha)$, $\alpha > 0$. For given $\varepsilon \in (0, 1)$, by (a) here is some $P \in \mathcal{P}^n$, $P \subset K$, such that $d(K, P) < \alpha\varepsilon/2$. Note that P depends on ε and α. Hence

$$h_P(u) \geq h_K(u) - \frac{\alpha\varepsilon}{2} \geq \alpha\left(1 - \frac{\varepsilon}{2}\right) > 0, \quad u \in \mathbb{S}^{n-1},$$

and therefore $\alpha(1 - \varepsilon/2)B^n \subset P$. This shows that

$$P \subset K \subset P + \frac{\alpha\varepsilon}{2}B^n \subset P + \frac{\alpha\varepsilon}{2}\frac{1}{\alpha(1-\varepsilon/2)}P = \left(1 + \frac{\varepsilon/2}{1-\varepsilon/2}\right)P \subset (1+\varepsilon)P.$$

Hence we obtain (c) and also get

$$\|h_{(1+\varepsilon)P} - h_K\| \leq \varepsilon\|h_P\| \leq \varepsilon\|h_K\|.$$

To deduce (b), we may assume that $0 \in$ int K. Let $\varepsilon \in (0, 1)$ be given. Define $\varepsilon' := \varepsilon/(1 + \|h_K\|) \in (0, 1)$. Then by what we have already shown there exists a polytope P' such that $0 \in$ int P', $P' \subset K \subset (1 + \varepsilon')P'$ and

$$\|h_{(1+\varepsilon')P'} - h_K\| \leq \varepsilon'\|h_K\| \leq \varepsilon.$$

Thus the polytope $(1 + \varepsilon')P'$ satisfies all requirements.

Exercises and Supplements for Sect. 3.1

1. For $K, L \in \mathcal{K}^n$ show that

$$d(K, L) = \min\{\varepsilon \geq 0 : K \subset L + B^n(\varepsilon), L \subset K + B^n(\varepsilon)\},$$

   that is, the infimum in the definition of $d(K, L)$ is attained.
2. Let $\mathcal{K}^n_c$ denote the set of all $K \in \mathcal{K}^n$ for which there is some point $c \in \mathbb{R}^n$ such that $K - c = -(K - c)$. Show that for each $K \in \mathcal{K}^n_c$ such a point c is uniquely determined. It is denoted by $c(K)$ and called the *centre of symmetry* of K. Show that the map $c : \mathcal{K}^n_c \to \mathbb{R}^n$ is continuous.
3. Let $K, L, M \in \mathcal{K}^n$. Show that if $K + L \subset M + L$, then $K \subset M$ (generalized cancellation rule). Can you avoid the use of support functions?
   More generally, let A, B, C be sets in $\mathbb{R}^n$. Suppose that A is nonempty and bounded, that C is closed and convex, and that $A + B \subset A + C$. Show that then $B \subset C$.
4. Let $K, L, M \in \mathcal{K}^n$. Let $u \in \mathbb{S}^{n-1}$. Show that $K(u) + M(u) = (K + M)(u)$. Can you avoid the use of support functions?

5. Let $(K_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{K}^n$ and $K \in \mathcal{K}^n$. Show that $K_i \to K$ (in the Hausdorff metric) if and only if the following two conditions are satisfied:

   (a) Every $x \in K$ is a limit point of a suitable sequence $(x_i)_{i\in\mathbb{N}}$ with $x_i \in K_i$ for $i \in \mathbb{N}$.
   (b) For each sequence $(x_i)_{i\in\mathbb{N}}$ with $x_i \in K_i$, for $i \in \mathbb{N}$, every accumulation point lies in K.

6. Let $K_i, K \in \mathcal{K}^n$, $i \in \mathbb{N}$. Then the following statements are equivalent.

   (i) If $U \subset \mathbb{R}^n$ is open and $K \cap U \neq \emptyset$, then $K_i \cap U \neq \emptyset$ for almost all $i \in \mathbb{N}$.
   (ii) For $x \in K$, there are $x_i \in K_i$, $i \in \mathbb{N}$, such that $x_i \to x$ as $i \to \infty$.

7. Let $K_i, K \in \mathcal{K}^n$, $i \in \mathbb{N}$. Suppose that $\operatorname{int} K \neq \emptyset$. Then the following statements (a), (b), and (c) are equivalent.

   (a) $K_i \to K$ as $i \to \infty$.
   (b) (i) If $x \in \operatorname{int} K$, then $x \in \operatorname{int} K_i$ for almost all $i \in \mathbb{N}$,
   (ii) If $I \subset \mathbb{N}$ with $|I| = \infty$, $x_i \in K_i$ for $i \in I$, and $x_i \to x$, as $i \to \infty$, then $x \in K$.
   (c) (i') If $x \in \operatorname{int} K$, then $x \in K_i$ for almost all $i \in \mathbb{N}$,
   (ii) If $I \subset \mathbb{N}$ with $|I| = \infty$, $x_i \in K_i$ for $i \in I$, and $x_i \to x$, as $i \to \infty$, then $x \in K$.

8.* (a) Let $K, M \in \mathcal{K}^n$ be convex bodies, which cannot be separated by a hyperplane (i.e., there is no hyperplane $\{f = \alpha\}$ with $K \subset \{f \leq \alpha\}$ and $M \subset \{f \geq \alpha\}$). Further, let $(K_i)_{i\in\mathbb{N}}$ and $(M_i)_{i\in\mathbb{N}}$ be sequences in $\mathcal{K}^n$. Show that

$$K_i \to K,\ M_i \to M \Longrightarrow K_i \cap M_i \to K \cap M.$$

   (b) Let $K \in \mathcal{K}^n$ be a convex body, and let $E \subset \mathbb{R}^n$ be an affine subspace with $E \cap \operatorname{int} K \neq \emptyset$. Further, let $(K_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{K}^n$. Show that

$$K_i \to K \Longrightarrow E \cap K_i \to E \cap K.$$

   Hint: Use Exercise 3.1.5.

9. Let $K \subset \mathbb{R}^n$ be compact. Show that:

   (a) There is a unique Euclidean ball $B_c(K)$ of smallest diameter with $K \subset B_c(K)$ (which is called the *circumball* of K). The map $K \mapsto B_c(K)$ is continuous.
   (b) If $\operatorname{int} K \neq \emptyset$, then there exists a Euclidean ball $B_i(K)$ of maximal diameter with $B_i(K) \subset K$ (which is called an *inball* of K). In general, inballs of convex bodies are not uniquely determined.

10. A body $K \in \mathcal{K}^n$ is *strictly convex*, if it does not contain any segments in the boundary.

(a) Show that the set of all strictly convex bodies in $\mathbb{R}^n$ is a G_δ-set in $\mathcal{K}^n$, that is, an intersection of countably many open sets in $\mathcal{K}^n$.
(b) Show that the set of all strictly convex bodies in $\mathbb{R}^n$ is dense in $\mathcal{K}^n$.

11. Let $(K_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{K}^n$ for which the support functions $h_{K_i}(u)$ converge to the values $h(u)$ of a function $h : \mathbb{S}^{n-1} \to \mathbb{R}$, for each $u \in \mathbb{S}^{n-1}$. Show that h is the support function of a convex body and that $h_{K_i} \to h$ uniformly on $\mathbb{S}^{n-1}$.
12. Let P be a convex polygon in $\mathbb{R}^2$ with int $P \neq \emptyset$. Show that:

(a) There is a polygon P_1 and a triangle (or a segment) Δ with $P = P_1 + \Delta$.
(b) P has a representation $P = \Delta_1 + \cdots + \Delta_m$, with triangles (segments) Δ_j which are pairwise not homothetic.
(c) P is a triangle if and only if $m = 1$.

13. A body $K \in \mathcal{K}^n, n \geq 2$, is *indecomposable* if $K = M + L$ implies that $M = \alpha K + x$ and $L = \beta K + y$, for some $\alpha, \beta \geq 0$ and $x, y \in \mathbb{R}^n$. Show that:

(a) If $P \in \mathcal{K}^n$ is a polytope and all 2-faces of P are triangles, then P is indecomposable.
(b) For $n \geq 3$, the set of indecomposable convex bodies is a dense G_δ-set in $\mathcal{K}^n$.

14. Let $\mathcal{I}^n$ be the set of convex bodies $K \in \mathcal{K}^n$ which are strictly convex **and** indecomposable. Let $n \geq 3$. Show that $\mathcal{I}^n$ is dense in $\mathcal{K}^n$.
It seems to be an open problem to explicitly construct an element of $\mathcal{I}^n$.
15.* (Simultaneous approximation of convex sets, their unions and nonempty intersections) Let $K_1, \ldots, K_m \in \mathcal{K}^n$ be convex bodies such that $K := K_1 \cup \cdots \cup K_m$ is convex. Let $\varepsilon > 0$. Then there are polytopes $P_1, \ldots, P_m \in \mathcal{P}^n$ with $K_i \subset P_i \subset K_i + \varepsilon B^n$ for $i = 1, \ldots, m$ and such that $P := P_1 \cup \cdots \cup P_m$ is convex.
16. Let $K \in \mathcal{K}^n$ with $\dim K = n$. Suppose that $H(u, t) = \{x \in \mathbb{R}^n : \langle x, u\rangle = t\}$, $u \in \mathbb{R}^n \setminus \{0\}$ and $t \in \mathbb{R}$, is a supporting hyperplane of K with $K \subset H^-(u, t)$. Show that

$$\lim_{s \nearrow t}(H(u, s) \cap K) = K \cap H(u, t),$$

where the convergence is with respect to the Hausdorff metric.
17. Let $K \in \mathcal{K}^2$. A convex body M is called a rotation average of K if there is some $m \in \mathbb{N}$ and there are proper rotations $\rho_1, \ldots, \rho_m \in SO(2)$ such that $M = \frac{1}{m}(\rho_1 K + \cdots + \rho_m K)$. Show that there is a sequence of rotation averages of K which converges to a two-dimensional ball (a disc) (which might be a point).